MORPHISMS AND PARTITIONS OF V-SETS

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Introduction

The set theoretic connection between functions and partitions is not worthy of further remark. Nevertheless, this connection turns out to have deep consequences for the theory of the Ershov numbering of lambda terms and thus for the connection between lambda calculus and classical recursion theory. Under the traditional understanding of lambda terms as function definitions , there are morphisms of
the Ershov numbering of lambda terms which are not definable.
This appears to be a serious incompleteness in the lambda calculus. However, we believe, instead, that this indefinability
is a defect in our understanding of the functional nature of lambda terms. Below, for a different notion of lambda definition,
we shall prove a representation theorem (completeness theorem)
for morphisms. This theorem is based on a construction which
realizes certain partitions as collections of fibers of morphisms

defined by lambda terms in the classical sense of definition.

Our notion of definition begins with a Curried context F and ends with another G. We think of a combinator M printed in the righthand part of our screen and F printed in the lefthand part

FM

We convert this screen to others by beta conversion and we ask which screens of the form

GN

will appear. We also ask whether this relationship between M and N is functional (modulo conversion). When this relationship is functional we obviously have an Ershov morphism. We shall show that every morphism can be represented in this way.

We shall also present a number of related results about morphisms and partitions.

Preliminaries

For notation and terminology concerning lambda calculus we refer the reader to Barendregt [1]. For terminology and notation conerning recursion theory we refer the reader to Rogers [2] especially for the notions of e-reducibility and e-degree. In addition, for a discussion of DRE sets (Difference of RE sets sets) the reader should see Soare [3].

A combinator is a closed lambda term. A V-set is a set of
combinators which is both recursively enumerable and closed
under beta conversion. We have elsewhere called these sets "Visseral" ([5]) and "Varieties"([4]) hence the name "V-set".
is the Godel numbering of combinators from Barendregt [1]
page 143. If X is a V-set then $\#X = \{ \#M : M \in X \}$.
M = $(\lambda,\mathbb{M})$ is the Ershov numbering introduced by Albert Visser in [7] 1.6. Here $\mathbb{M}$ is the set of equivalence classes of
combinators under the equivalence relation of beta conversion
and λ: **N** -> $\mathbb{M}$ satisfying $\lambda(\#M) = (M/{=}\beta)$. A partial function μ:
$\mathbb{M}$ -> $\mathbb{M}$ is said to be a partial morphism if there is a partial recursive function ϕ with $\mathrm{dom}(\phi) = \{ \#M : M/{=}\beta \in \mathrm{dom}(\mu) \}$ such that the diagram

$$\mathbb{M} \xleftarrow{\mu} \mathbb{M} \xleftarrow{\lambda} \mathbf{N} \xrightarrow{\phi} \mathbf{N} \xrightarrow{\lambda} \mathbb{M}$$

commutes. The relation ~**M** is defined by

$$\#M \sim_{\mathbf{M}} \#N \iff M =_\beta N.$$

Given a V-set X a partition P of X is said to be V if the blocks of P are V-sets. Given a V-partition P of X, a set of natural
numbers is said to be a (multi-) representation of P if it contains

(at least) exactly one RE index for each block of P and nothing
else. P is said to be RE if it has an RE multi-representation and
P is said to be recursive if it has a recursive representation.
The V-partitions of X form a lower semi-lattice whose meet
operation is the usual meet in the lattice of partitions. The largest
element in the semi-lattice is $\{X\}$ here denoted '**1**' and the smallest element is $\{ M/{=}\beta : M \varepsilon X \}$ here denoted '**0**'. When **0**
is the subject of concern it is convenient to talk about systems
of distinct representatives (SDR's) instead of representations.
Indeed, for each representation R of **0** there is an SDR of $\leq$ e-degree obtained by taking the first member in the domain of
{e} for each e in R . Similarly for each SDR S of **0** there is a
representation of $\leq$ e-degree obtained by taking an index for the
function which converges on precisely those N such that N $=\beta$ M, for each M ε S . We define the degree of X, deg(X),
to be the infinum of the e-degrees of the SDR's of **0**, if this exists.

Morphisms

If you read Visser's paper [7] carefully you learn enough to
prove the following.

Theorem 1 (Visser fixed point theorem): Suppose that S is an
RE set, X is a V-set not including all combinators, and G is any combinator. Then there exists a combinator F such that

$$F\underline{n} =_\beta \begin{cases} G\underline{\#F}\ \underline{n} & \text{if } n \in S \\ \text{does not belong to } X & \text{otherwise.} \end{cases}$$

Proof: Let E be the Kleene enumerator constructed in [1] page
167. Let Φ be the usual partial recursive enumeration of the
partial recursive functions of one variable so that if ϕ has index e then $\Phi(e,x) = \phi(x)$. By Visser's theorem [7] 1.6.1.1 there exists a total recursive function ψ such that whenever
$\Phi(x,x)$ converges we have $\Phi(x,x)$ ~**M** $\psi(x)$. Now let ϕ be partial
recursive with index e. Kleene proved ([1]) that there is a combinator F such that

$$F\underline{n} =_\beta \begin{cases} \underline{\phi(n)} & \text{if } \phi(n) \text{ converges} \\ \text{a term with no normal form else} \end{cases}$$

Now the map e |-> #F is total recursive and thus has an index

i. This i is also mapped to a combinator F here denoted 'J'. Thus
for any e and F with e |-> #F we have E(J<u>e</u>) $=_\beta$ F. We shall also
use the following notation 'M' = <u>#M</u> a few times. Now define the
total recursive α by α(x,y) = #G ' \z. E(E(J<u>x</u>) z) ' <u>y</u> and define the
partial recursive ϕ by

$$\phi(x,y,z) = \begin{cases} \alpha(x,y) & \text{if } y \varepsilon S \text{ before } \psi(z) \varepsilon \#X \text{ in some given enumerations of } S \text{ and } X \\ k & \text{if } \psi(z) \varepsilon X \text{ before } y \varepsilon S \end{cases}$$

for k any natural number not in #X . By the S(m,n) theorem ([2])
there exitsts a total recursive ξ such that ϕ(x,y,z) = Φ(ξ(x,y),z).
Let ζ(x,y) = ψ(ξ(x,y)). We observe that
yεS => ϕ(x,y,ξ(x,y)) converges => ζ(x,y) ~**M** ϕ(x,y,ξ(x,y,)) =
α(x,y)
~(yεS) => ϕ(x,y,ξ(x,y)) diverges => ζ(x,y) does not belong to #X.
Now by the recursion theorem there exists an e such that
ζ(e,y) = Φ(e,y). We now set F $\equiv$ \z. E(E(J<u>e</u>)z) and we claim
that this is the desired combinator. Suppose first that yεS .
We have

$\Phi(e,y) = \zeta(e,y)$ ~**M** $\alpha(e,y)$ so
E$\underline{\Phi(e,y)}$ $=\beta$ E$\underline{\alpha(e,y)}$ and
E(E(J$\underline{e}$)$\underline{y}$) $=\beta$ G ' \z.E(E(J$\underline{e}$) z) ' $\underline{y}$ thus
F$\underline{y}$ $=\beta$ G$\underline{\#F}$ $\underline{y}$.
On the other hand if ~(yε*S*) then
$\Phi(e,y) = \zeta(e,y)$ which does not belong to #*X* , so
F$\underline{y}$ $=\beta$ E(E(J$\underline{e}$)$\underline{y}$) $=\beta$ E$\underline{\Phi(e,y)}$ which does not belong to *X* .
This completes the proof.
Theorem 2 (morphism extension): Suppose that μ is a partial
morphism and *X* is a V-set. Then there exists a total
morphism ν extending μ such that if M/$=\beta$ does not
belong to dom(μ) then ν(M/$=\beta$) is disjoint from *X* .
Proof: Suppose that μ and *X* are given together with the partial
recursive function ϕ which makes the morphism diagram commute. Now we can construct a combinator H such that

Ex$\underline{\#N}$ if N $=\beta$ M & #N < #M & N is the first such found in some enumeration of the beta converts of M

Hx$\underline{\#M}$ $=\beta$ {

E$\underline{\phi(\#M)}$ if ϕ(#M) converges before such N is found.

Where E is the Kleene enumerator constructed in [1] page 167.

Thus by the Visser fixed point theorem there exists a combinator
F such that

H#F #M   if there exists N =β M with #N < #M

F#M =β {   or φ(#M) converges

does not belong to *X*   otherwise.

Finally let ψ be defined by ψ(#M) = #F#M. Clearly ψ is a total
recursive function. Moreover it is easy to see that

(1) if M =β N then ψ(#M) ~**M** ψ(#N)

(2) if φ(#M) converges then φ(#M) ~**M** ψ(#M).

(3) if φ(#M) diverges then λ(ψ(#M)) does not belong to *X* .

Thus there is a total morphism ν with the desired properties
which makes the above diagram commute for ψ.

End of proof.

Each combinator M induces a morphism via the recursive
map #N |-> #MN. However, there are morphisms not induced
by such combinator application. For example, take a left invertible
combinator such as C* ≡ \xy.yx with left inverse C*I. Given X
define X+ by

x+ ≡ x

(XY)+ ≡ C*I(X+)(Y+)

(\x.X)+ ≡ C*(\x.(X+)).

Then the map M |-> M+ induces a morphism which is not induced by combinator application (just assume that $F\Omega =_\beta \Omega+$).

Theorem 3 (morphism representation): Suppose that μ is a morphism. Then there exist combinators F and G such that for any M and N we have

$$\mu(M/{=_\beta}) = (N/{=_\beta}) \iff FM =_\beta GN.$$

Proof: We shall actually prove a stronger result. Suppose that *P* is an RE V-partition of the V-set *X* . We shall construct a combinator H such that $HM =_\beta HN \iff M =_\beta N \vee$ M and N belong to the same block of *P* (and thus belong to *X*). Given this we get the theorem as follows. Suppose that μ is given. Let $X = \{ \langle \underline{0},M\rangle :$ all combinators $M \} \cup \{ \langle \underline{1},N\rangle : N/{=_\beta} \in rng(\mu) \}$ and let $P = \{ \{ \langle \underline{0},M\rangle : \mu(M/{=_\beta}) = (N/{=_\beta}) \} \cup \{\langle \underline{1},N\rangle \} :$ for all $N/{=_\beta} \in rng(\mu) \}$ where $\langle , \rangle$ is the pairing function in [1] pg 133. Then *P* is RE.

Given H as above we set $F \equiv \backslash x. H\langle \underline{0},x\rangle$ and $G \equiv \backslash x.H\langle \underline{1},x\rangle$. Now $FM =_\beta GN \iff H\langle \underline{0},M\rangle =_\beta H\langle \underline{1},N\rangle \iff \langle \underline{0},M\rangle$ and $\langle \underline{1},N\rangle$ belong to the same block of *P* $\iff \mu(M/{=_\beta}) = (N/{=_\beta})$.

Suppose now that *X* and *P* are given. Fix an RE represent-

ation of P . By Kleene's enumerator construction ([1] pg 167)
there exists a combinator E such that
E $\underline{(2r+1)*(2\wedge s)}$ $=_\beta$ the s-th member of the r-th block of P
where of course the same block of P may occur many times
in the enumeration because P is only assumed to be RE. We make
the following definitions; let a be a new free variable

$\underline{S} \equiv \backslash xyz.\ y(xyz)$
$\underline{T} \equiv \backslash xyz.\ xy(xyz)$
$Y \equiv (\backslash xy.\ y(xxy))(\backslash xy.\ y(xxy))$
$A \equiv \backslash fgxyz.\ fx(a(Ex))[f(\underline{S}x)y(g(\underline{S}x))z]$
$B \equiv \backslash fgx.\ f(\underline{S}x)(a(E(\underline{T}x))(g(\underline{S}))(gx)$
$G \equiv Y(\backslash u.\ B(Y(\backslash v.Auv))u)$
$F \equiv Y(\backslash u.\ AuG)$
$H \equiv \backslash xa.\ F\underline{1}(ax)(G\underline{1})$.

We now calculate for k>0

$H(E\underline{k}) \to \backslash a.\ F\underline{1}(a(E\underline{k}))(G\underline{1}) \twoheadrightarrow \backslash a.$
$F\underline{1}(a(E\underline{1}))[F\underline{2}(a(E\underline{k}))(G\underline{2})(G\underline{1})]$
$\twoheadrightarrow$
$\backslash a.F\underline{1}(a(E\underline{1}))(F\underline{2}(a(E\underline{2}))(\ldots F\underline{k-1}(a(E\underline{k-1}))[F\underline{k}(a(E\underline{k}))(G\underline{k})(G\underline{k-1})]$
$\ldots)(G\underline{2}))(G\underline{1}) \twoheadrightarrow \backslash a.$
$F\underline{1}(a(E\underline{1}))(F\underline{2}(a(E\underline{2}))(\ldots F\underline{k-1}(a(E\underline{k-1}))[F\underline{k}$
$(a(E\underline{k}))[F\underline{k+1}(a(E(\underline{T}\ \underline{k}))(G\underline{k+1})(G\underline{k})](G\underline{k-1})]\ldots)(G\underline{2}))(G\underline{1}) \twoheadrightarrow$
$\backslash a.\ F\underline{1}(a(E\underline{1}))(F\underline{2}(a(E\underline{2}))(\ldots F\underline{k-1}(a(E\underline{k-1}))[F\underline{k}(a(E\underline{k}))[F\underline{k+1}$
$(a(E(\underline{2k}))(G\underline{k+1})(G\underline{k})](G\underline{k-1})]\ldots)(G\underline{2}))(G\underline{1}) \twoheadleftarrow H(E\underline{2k})$.

Thus we conclude that if M and N belong to the same block of

P
then HM =β HN. Next we prove that if HM =β HN then either
M =β
N or M and N belong to the same block of *P* . Suppose that
F<u>k</u>(aM)(G<u>k</u>) =β F<u>k</u>(aN)(G<u>k</u>). Then by the Church-Rosser and standardization theorems there exists a common reduct Z
and
standard reductions *M* : F<u>k</u>(aM)(G<u>k</u>) ->> Z and *N* :
F<u>k</u>(aN)(G<u>k</u>)
->> Z. The proof is by induction on the sum of the lengths of these standard reduction sequences. The following facts
about
the definitions are easy to verify and will be used below.
(1) Fxyz is an oreder zero unsolvable
(2) \xy. y(xxy) =/=β A
(3) Y =/=β \u. AuG
(4) Y =/=β AF
(5) \u. AuG =/=β AF.
Basis; the sum of the lengths of the reductions is 0.
In this case it is clear that M =β N.
Induction step; the sum of the lengths is say r > 0.
For any term X the head reduction of F<u>k</u>(aX)(G<u>k</u>) cycles
through
segments which are 8 terms long viz;
F<u>k</u>(aX)Y ≡ Y(\u.AuG)<u>k</u>(aX)Y ->
(\x.x(Yx))(\u.AuG)<u>k</u>(aX)Y ->
(\u.AuG)F<u>k</u>(aX)Y ->
AFG<u>k</u>(aX)Y ->
(\gxyx.Fx(a(Ex))[F(<u>S</u>x)y(g(<u>S</u>))z])G<u>k</u>(aX)Y ->

$(\backslash xyz.Fx(a(Ex))[F(\underline{S}x)y(G(\underline{S}x))z])\underline{k}(aX)Y$ ->
$(\backslash yz.F\underline{k}(a(E\underline{k}))[F(\underline{S}\ \underline{k})y(G(\underline{S}\ \underline{k}))z])(aX)Y$ ->
$(\backslash z.F\underline{k}(a(E\underline{k}))[F(\underline{S}\ \underline{k})(aX)(G(\underline{S}\ \underline{k}))z])Y$ ->
$F\underline{k}(a(E\underline{k}))[F(\underline{S}\ \underline{k})(aX)(G(\underline{S}\ \underline{k}))Y]$

where $E\underline{k}$ becomes the new X and $F(\underline{S}\ \underline{k})(aX)(G(\underline{S}\ \underline{k}))Y$ becomes the new Y. First observe that the head reduction parts of *M* and *N* cannot terminate at different spots in the 8 term cycle. This can be seen by tracing the position of aX, which is the only component with a head normal form beginning with a. The positions of aX are resp 5,4,4,5,3,2, and finally not a component at all. Only like positions can match up by internal reductions by facts 2,3,4, and 5. If neither *M* nor *N* completes the first full 8 term cycle then clearly $M =_\beta N$ except possibly in the 8th case. In the 8th case we have standard reductions from $F\underline{k}(a(E\underline{k}))[F(\underline{S}\ \underline{k})(aM)(G(\underline{S}\ \underline{k}))z]$ and $F\underline{k}(a(E\underline{k}))[F(\underline{S}\ \underline{k})(aN)(G(\underline{S}\ \underline{k}))z]$ to a common reduct. The sum of the lengths of these reductions is < r. By the above analysis of the 8 term cycle and fact (1) the induction hypothesis applies to $F(\underline{S}\ \underline{k})(aM)(G(\underline{S}\ \underline{k}))$ and $F(\underline{S}\ \underline{k})(aN)(G(\underline{S}\ \underline{k}))$. Thus either $M =_\beta N$ or M and N belong to the same block of *P* . Thus we may assume that *M* completes the full 8 term cycle at least once and its head reduction part ends in a term

$U((F(\underline{S}\ \underline{k})(a(E\underline{k}))(G(\underline{S}\ \underline{k}))(\ldots(F(\underline{S}\ \underline{k})(a(E\underline{k}))(G(\underline{S}\ \underline{k})))[F(\underline{S}\ \underline{k})(aM)$
$(G(\underline{S}\ \underline{k}))(G\underline{k})]\ldots)$

where $F(\underline{S}\ \underline{k})(a(E\underline{k}))(G(\underline{S}\ \underline{k}))$ appears $t \geq 0$ times and $F\underline{k}(a(E\underline{k}))$ head reduces to U in ≤ 7 steps. We distinguish two caes.

Case 1; *N* does not complete the full cycle of the first 8 head reductions. In this case the head reduction part of *N* ends in $V(G\underline{k})$ where $F\underline{k}(aN)$ head reduces to V in the same ≤ 7 steps.

If the number of steps is indeed < 7 then clearly $N =_\beta E\underline{k}$. Suppose then that

$U \equiv \backslash z.F\underline{k}(a(E\underline{k}))[F(\underline{S}\ \underline{k})(a(E\underline{k}))(G(\underline{S}\ \underline{k}))z]$ and

$V \equiv \backslash z.F\underline{k}(a(E\underline{k}))[F(\underline{S}\ \underline{k})(aN)(G(\underline{S}\ \underline{k}))z]$.

Then $Z \equiv Z'Z''$ with standard reductions of U and V to Z' of total length < r. By our previous analysis of the 8 term cycle and fact (1) above the induction hypothesis applies to the pair $F(\underline{S}\ \underline{k})(a(E\underline{k}))(G(\underline{S}\ \underline{k}))$ and $F(\underline{S}\ \underline{k})(aN)(G(\underline{S}\ \underline{k}))$. Thus in any case $N =_\beta E\underline{k}$. In addition the argument of U and $G\underline{k}$ have a

common
reduct by standard reductions whose lengths are less than resp. M and N .Since the t+1 components of the argument of U are of order 0, we have Z ≡ Z0(Z1(...(Zt(Z'Z"))...) and for i= 1,...,t

F(S k)(a(Ek))(G(S k)) ->> Zi and

F(S k)(aM)(G(S k)) ->> Z'

all by standard reductions of length less than M . In addition, the head reduction of Gk begins

Gk ≡ Y(\u.B(Y(\v.Avu))u) k ->
(\x.x(Yx))(\u.B(Y(\v.Avu))u) k ->
(\u.B(Y(\v.Avu))u) G k ->
BFG k ->
(\gx.F(S x)(a(E(Tx)))(g(Sx))(gx)) G k ->
(\x.F(S x)(a(E(Tx)))(G(Sx))(Gx)) k ->
F(S k)(a(E(T k)))(G(S k))(Gk)

and none of the heads of any of these terms except the last is of order o. Thus the head reduction part of the standard reduction of Gk goes at least this far. Indeed, we may assume that t = 0 and the induction hypothesis applies to the pair F(S k)(aM)(G(S k)) and F(S k)(a(E(T k)))(G(S k)). This completes the proof for case 1.

Case 2; both M and N complete the full cycle of the first 8 head reductions. W.l.o.g. we may assume that the head

reduc-
tion part of *N* ends in
V((F(S̲ k̲)(a(Ek̲))(G(S̲ k̲))(...(F(S̲ k̲)(a(Ek̲))(G(S̲ k̲)))[F(S̲ k̲)(aN)
(G(S̲ k̲))(Gk̲)]...)
where (F(S̲ k̲)(a(Ek̲))(G(S̲ k̲)) appears s $\leq$ t times,and F(S̲ k̲)(a(Ek̲)) head reduces to V in $\leq$ 7 steps. We distinguish two subcases.

Subcase 1; s=t. In this case, by fact (1), the induction hypothesis
applies to the pair F(S̲ k̲)(aM)(G(S̲ k̲)) and F(S̲ k̲)(aN)(G(S̲ k̲)).

Subcase 2; s < t. In this case, by fact (1), the induction hypo-
thesis applies to the pair F(S̲ k̲)(aN)(G(S̲ k̲)) and F(S̲ k̲)(a(Ek̲))(G(S̲ k̲)). In addition, Gk̲ and F(S̲ k̲)(a(Ek̲))(G(S̲ k̲))(...(F(S̲ k̲)(a(Ek̲))(G(S̲ k̲)))[(F(S̲ k̲)(aM)(G(S̲ k̲))(Gk̲)]...) ,with t-s occurrences of F(S̲ k̲)(a(Ek̲))(G(S̲ k̲)),
have a common reduct by standard redution whose total length
is < r. Here the argument of case1 applies so that we may conclude that there exists an m whose odd part is the same as the odd part of k such that M =β Ek̲. This completes the proof
of case 2.

End of proof.

It is interesting to note here that uniformization of the relation defined by the equation FM =β GN cannot always be

carried out by a morphism, For example, if F $\equiv$ I and G $\equiv$ E then
for each M there exists N such that FM =β GN namely N $\equiv$ #M
but there is no morphism which takes M/=β to the equivalence
class of one particular #N for N =β M. For, such a morphism would solve the beta conversion problem effectively.

Recursive partitions

When it comes to the subsemilattice of recursive V-partitions of *X* it can be very small including only **1** or very large including also **0**. Our study actually began with trying to understand the following theorem. This lead to theorem 6 and then theorem 3.

Theorem 4 (Grzegorczyk-Scott) : The only recursive V-partition
of the set of all combinators is **1**.

We know 4 constructions which yield V-sets whose only recursive
V-partition is **1**. Ofcourse if *X* is such a V-set then the image of
X under any morphism also has this property.

Construction 1 (direct construction by the recursion theorem):

It is convenient to formulate this construction in terms of the
precompleteness of the Ershov numbering. Given partial recursive
functions ϕ and ψ define another partial recursive function

ξ which
matches the members of the domain of ϕ to the members of the
domain of ψ in the order in which they converge. By precomplete-
ness ξ "extends" to a total recursive ζ. Moreover the index of ζ
is a total recursive function f(x,y) of the indicies of ϕ and ψ. By
the Ershov fixed point theorem there exists a z = g(x,y) which
is a fixed point of ζ i.e. z ~**M** ζ(z). Now if we let X be the smallest
V-set such that #M ε dom(ϕ) => M εX , Y the smallest V-
set such that #M ε dom(ψ) => M εY , and #M = g(x,y), then
M ε Union(X ,Y) => M ε Intersection(X ,Y). In other words g
is a productive function for V-sets. In addition there is a well
known total recursive function h(x,y) which gives the index of
a partial recursive function whose domain is the intersection
of the domains of {x} and {y}. Finally let j(x) be the index of the
RE set { g(h(x,y),h(x,z)) : all natural numbers y,z }. By the
recursion theorem there exists an index e such that W(e) =
W(j(e)). It is easy to verify that the beta conversion closure of

{ M : #M ε W(e) } cannot be partitioned into two RE blocks which respect beta conversion. It is not clear to us exactly when this construction yields the set of all combinators. We leave this question open.

Construction 2 (the Grzegorczyk-Scott construction):

This construction uses the fixed point theorem. We suppose that we have a partition { X , Y } with M ε X and N ε Y. We define a total recursive ϕ by L ε X => ϕ(#L) = #N and L ε Y => ϕ(#L) = #M.

Here we can use the Ershov fixed point theorem ,where Scott and Grzegorczyk use Kleene's representation of the recursive functions and the traditional fixed point theorem. There exists L such that ϕ(#L) ~**M** #L. This contradicts the choice of the definition of ϕ.

This type of arguement applies to fragnments of lambda calculus suitable for representing all recursive functions such as the lambda I calculus or the combinators hereditarily of order 2 (HOT[6]).

Construction 3 (Visser's construction):

Visser's construction begins like construction (2) except we do not assume that X and Y are disjoint; we assume only

that M ε X - Y and N ε Y - X . Let A and B be a pair of recursively
inseparable sets and let A and B be the beta conversion closure of the set Church numerals representing the members
of A and ,resp., of B.By the precompleteness of the Ershov numbering the partial recursive function ϕ defined by L ε A =>
ϕ(#L) = #M and L ε B => ϕ(#L) = #N "extends" to a total recursive
function ψ.It is easy to obtain a recursive separation of A and B
using ψ, X , and Y . Indeed Visser's arguement reachees an apparently stronger conclusion; viz, the set of all combinators
cannot be covered by two incomparable V-sets.
Construction 4 (a simple set construction):

Let W(e) be a simple set and let ϕ be the partial morphism
defined by ϕ(#M) = #I if there exists a member of W(e) ~**M** #M.
By theorem 2 there exists a total morphism ξ "extending" ϕ, and there exists a combinator F representing ξ. Consider the V-set
Z which is the beta conversion closure of { E(F#M) : all combin-
ators M }. Now suppose that { X , Y } is a partition of Z into V-sets. First note that if E(F#M) εX then there are infinitely

many N beta convertible to M such that E(F#N) also belongs to X and similarly for Y . Now I belongs to one of these sets say X and let B be the set of all k such that E(Fk) ε Y. Since B
is infinite and RE it must intersect W(e) in, say, the natural number k. But since k ε W(e) we have E(Fk) =β I ε X . This is a contradiction.

0 can be recursive or not recursive. An example of the first is X = the beta conversion closure of the set of normal
forms. An example of the second is X = the set of all combinators.
Indeed, an SDR for this X solves the beta conversion problem.

If **0** is recursive then there is a clear sense in which every
recursive partition of natural numbers yields a recursive partition of X . The cannonical example is the beta closure of
the Church numerals. In [3] Visser observed the following.
Theorem 5 (Visser) : If **0** is recursive then any morphism into X
is constant.

The following theorem explains where all the RE V-partitions
of a V-set come from. It is a corollary to the proof of theorem 3.
Theorem 6 (partition representation theorem):
Suppose that P is an RE V-partition of the V-set

X.

Then there exists a combinator H such that HM =β HN

<=> M =β N v M and N belong to the same block of P

(and thus belong to X).

Partitions of higher complexity

It turns out that the most natural notion of the complexity of an SDR for **0** is the notion of e-degree (partial or enumeration degree [2]). The following theorem characterizes those e-degrees.

Theorem 7 (existence of deg(X)) : deg(X) exists and is the e-degree of a DRE SDR for **0**. Moreover each DRE e-degree is deg(X) for some V-set X.

Proof: Given a V-set X define X^* = {<M,N> : M =/= β N for M,N εX}.

Now let S be any SDR for **0**. Define the enumeration operator Φ by {<M,N,<P,Q>> : P =β M & Q =β N & M ≡/≡ N }. Then Φ(S) = X^*.

Now fix an enumeration of X. Define a sequence M(1), M(2),..., by M(m) is the first element in the enumeration of X not =β to one of the M(1),...,M(m-1). This sequence forms and SDR S for

0. Now define the enumeration operator Ψ by { <<N(1),N(2)>,
<N(1),N(3)>,...,<N(n),N(n-1)>, N(n) >: N(1),...,N(n) appear in order in an initial segment of the fixed enumeration of *X* and the other members of the initial segment beta convert to one or more of the N(i) }. Then $\Psi(X^*) = S$. Thus the e-degree of *S* = the e-degree of
X * which is less than or equal to the e-degree of any SDR for **0**. In addition, *S* is the difference of RE sets since it is the difference of the members of the enumeration and the members of the
enumeration that beta convert to earlier members. This proves the first part of the theorem. To prove the second part let Δ = A-B be
the difference of the RE sets A and B. By the Visser fixed point theorem there exists a combinator F such that

$$F\underline{n} = \begin{cases} KI & \text{if } n \; \varepsilon \; B \\ \text{a term of order o} & \text{otherwise.} \end{cases}$$

Let *X* be the beta conversion closure of the set { F$\underline{n}$ $\underline{n}$: n a natural
number }. Now for this *X* clearly **0** has an SDR consising of {I}
union with { F$\underline{n}$ $\underline{n}$: n ε A & ~(n ε B) } since by the construction of
F no two of the latter set can beta convert. Thus deg(*X*) is e-reducible to Δ. On the other hand, it is clear that Δ can be enumerated from *X* *and thus deg(*X*) = the e-degree of Δ. This completes the proof.

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