Ranks and Pregeometries in Finite Diagrams

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ABSTRACT. The study of classes of models of a finite diagram was initiated by S. Shelah in 1969. A *diagram* D is a set of types over the empty set, and the class of models of the diagram D consists of the models of T which omit all the types not in D. In this work, we introduce a natural dependence relation on the subsets of the models for the $\aleph_0$-stable case which share many of the formal properties of forking. This is achieved by considering a rank for this framework which is bounded when the diagram D is $\aleph_0$-stable. We can also obtain pregeometries with respect to this dependence relation. The dependence relation is the natural one induced by the rank, and the pregeometries exist on the set of realizations of types of minimal rank. Finally, these concepts are used to generalize many of the classical results for models of a totally transcendental first-order theory. In fact, strong analogies arise: models are determined by their pregeometries or their relationship with their pregeometries; however the proofs are different, as we do not have compactness. This is illustrated with positive results (categoricity) as well as negative results (construction of nonisomorphic models).

0. INTRODUCTION

The problem of categoricity has been a driving force in model theory since its early development in the late 1950's. For the countable first-order case, M. Morley in 1965 ([Mo]) introduced a rank which captures $\aleph_0$-stability, and used it to construct prime models and give a proof of Łoś conjecture. In 1971, J. Baldwin and A. Lachlan [BlLa] gave an alternative proof using the fact that algebraic closure induces a pregeometry on strongly minimal sets. Their proof generalizes ideas from Steinitz's famous 1910 theorem of categoricity for algebraically closed fields. Łoś conjecture for uncountable languages was solved in 1970 by S. Shelah [Sh 70] introducing a rank which corresponds to the superstable case. Later, Shelah discovered a dependence relation called forking and more general pregeometries, and since then, these ideas have been extended to more and more general first-order contexts, each of them corresponding to a specific rank: $\aleph_0$-stable, superstable, stable and simple.

The problem of categoricity for non-elementary classes is quite considerably more involved. In 1971, H. J. Keisler (see [Ke]) proved a categoricity

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theorem for Scott sentences $\psi \in L_{\omega_1\omega}$, which in a sense generalizes Morley's Theorem. To achieve this, Keisler made the additional assumption that ψ admits $\aleph_1$-homogeneous models. Later, L. Marcus, with the assistance of Shelah (see [MaSh]), produced an example of a categorical $\psi \in L_{\omega_1\omega}$ that does not have any $\aleph_1$-homogeneous model, so this is not the most general case. Since then, many of Shelah's hardest papers in model theory have been dedicated the categoricity problem and to the development of general classification theory for non-elementary classes. Among the landmarks, one should mention [Sh 4] about sentences in $L_{\omega_1\omega}(Q)$ which answers a question of Harvey Friedman's list (see [Fr]). In [Sh 87a] and [Sh 87b] a version of Morley's Theorem is proved for a special kind of formulas $\psi \in L_{\omega_1\omega}$ which are called excellent. It is noteworthy that to deal with these non-elementary classes, these papers introduced several crucial ideas, among them stable amalgamation, 2-goodness and others, which are now essential parts of the proof of the "Main Gap" for first-order, countable theories. Later, R. Grossberg and B. Hart completed the classification of excellent classes and gave a proof of the Main Gap for those classes ([GrHa]). H. Kierstead also continued the study of sentences in $L_{\omega_1\omega}(Q)$ (see [Ki]). He introduced a generalization of strongly minimal formulas by replacing "non-algebraic" by "there exists uncountably many" and obtained results about countable models of these classes using [Sh 4]. In [Sh 300], Shelah began the classification theory for universal classes (see also ICM 1986/videotape) and is currently working on a book entirely dedicated to them. He also started the classification of classes in a context somewhat more general than $PC(T_1, T, \Gamma)$, see [Sh 88], [Sh 576] and [Sh 600]. In a related work, Grossberg started studying the classification of $\text{Mod}(\psi)$ for $\psi \in L_{\lambda^+\omega}$ under the assumption that there exists a "Universal Model" for ψ and studied relatively saturated substructures (see [Gr 1] and [Gr 2]). This seems to be a natural hypothesis which others have made as well (for example [Sh 88], [KlSh] and [BlSh 3]). As a matter of fact, it is conjectured that if an abstract class of models $\mathcal{K}$ is categorical above the Hanf number, then $\mathcal{K}$ has the μ-amalgamation property for every μ (this implies the existence of μ^+-universal models, under the General Continuum Hypothesis).

There are several striking differences between the problem of categoricity for first-order and the non-elementary case. First, it appears that classification for non-elementary classes is sensitive to the axioms of set theory. Second, the methods used are heavily combinatorial: there is no "forking" (though splitting and strong splitting are sometimes well-behaved), and the presence of pregeometries to understand systematically models of a given class is scarce. (A nice example of pregeometries is hidden in the last section of [Sh 4] and only [Ki] has used them to study countable models.) However, stability was not developed originally for first-order. In 1970, Shelah published [Sh 1], where he introduced some of the most fundamental ideas of classification theory (stability, splitting of types, existence of indiscernibles, several notions of prime models etc.). In this paper, Shelah considered classes of models which omit all types in $D(T) - D$, for a fixed *diagram* $D \subseteq D(T)$. This class is usually denoted $EC(T, \Gamma)$, where Γ stands for $D(T) - D$.

He made assumptions of two kinds (explicitly in his definition of stability): (1) restriction on the cardinality of the space of types realizable by the models, and (2) existence of models realizing many types. In fact, the context studied by Keisler in his categoricity result for $L_{\omega_1\omega}$, turns out to be the $\aleph_0$-stable case in the above sense. This is made precise by the following results. (C.-C. Chang:) The class of models of a sentence $\psi \in L_{\omega_1\omega}$ is equal to the class $PC(T_1, T, \Gamma)$, which is the class of reducts to $L(T)$ of models of a first-order countable theory T_1 containing T, and omitting a set of types $\Gamma \subseteq D(T_1)$. (Shelah:) The number of models of a Scott sentence $\psi \in L_{\omega_1\omega}$ is equal to the number of models of $EC(T, \Gamma)$, for some countable T, where Γ the set of isolated types of T.

In retrospective, it seems that what prevented the emergence of a smooth theory for $\aleph_0$-stable diagrams is the absence of a rank like Morley's rank. Considering the success of the use of pregeometries to understand models in the first-order $\aleph_0$-stable case, if one hopes to lift these ideas to more general contexts, it appears that $\aleph_0$-stable diagrams constitute a natural test case. This is the main goal of this paper. We try to develop what Shelah calls the structure part of the theory for the class $EC(T, \Gamma)$, under the assumption that it is $\aleph_0$-stable (in the sense of [Sh 1]). In fact, as in [Sh 2], we assume that $EC(T, \Gamma)$ contains a large homogeneous model (which follows from Shelah's original definition of stability for $EC(T, \Gamma)$, see Theorem 3.4. in [Sh 1]), so that the stability assumptions only deal with the cardinality of the spaces of types. This hypothesis allows us to do all the work in ZFC, in contrast to [Sh 4], [Sh 87a], [Sh 87b] or [Ki] for example.

The paper is organized as follows.

Section 1: We describe the general context.

Section 2: We introduce a rank for this framework which captures $\aleph_0$-stability (it does not generalize Morley rank, but rather generalizes what Shelah calls $R[p, L, 2]$). This rank differs from previously studied ranks in two ways: (1) it allows us to deal with general diagrams (as opposed to the atomic case or the first-order case) and (2) the definition is relativized to a given set (which allows us to construct prime models). By analogy with the first-order case, we call D totally transcendental when the rank is bounded. For the rest of the paper, we only consider totally transcendental D, and we make no assumption on the cardinality of T. We study basic properties of this rank, and examine the natural dependence relation that it induces on the subsets of the models. We are then able to obtain many of the classical properties of forking, which we summarize in Theorem 2.21. We also obtain stationary types with respect to this dependence relation, and they turn out to behave well: they satisfy in addition the symmetry property, and can be represented by averages.

Section 3: We focus on pregeometries. Regular types are defined in the usual manner (but with this dependence relation instead of forking, of course), and the dependence relation on the set of realizations of a regular type yields a

pregeometry. We can show that stationary types of minimal rank are regular, and this is used to show that they exist very often. We also consider a more concrete kind of regular types, which are called minimal. They could be defined independently by replacing "non-algebraic" by "realized outside any model which contains the set of parameters" in the usual definition of strongly minimal formulas. (This can be done for any suitable class of models, as in the last section of [Sh 4].) We could show directly that the natural closure operator induces a pregeometry on the set of realizations in any $(D, \aleph_0)$-homogeneous model. We choose not to do this, and instead we consider minimal types only when the natural dependence relation coincides with the one given by the rank. This allows us to use the results we have already obtained and have a picture which is conceptually similar to the first-order totally transcendental case (where strongly minimal types are stationary and regular, and the unique nonforking extension is also the unique non-algebraic one). Another reason is that the proofs are identical to those which use the rank, and this presentation permits us to skip them.

Section 4: Here, we give various applications of both the rank and the pregeometries to the class $\mathcal{K}$ of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram. We introduce unidimensionality for diagrams. We are able to adapt techniques of Baldwin-Lachlan (see [BlLa]) to our context for the categoricity proof. In fact, we obtain a picture strikingly similar to the first-order totally transcendental case. (1) If D is totally transcendental, then over any D-set there is a prime model for $\mathcal{K}$ (this improves parts of Theorems 5.3 and 5.10 of [Sh 1]). (2) If D is totally transcendental, then $\mathcal{K}$ is categorical in some $\lambda > |T| + |D|$ if and only if $\mathcal{K}$ is categorical in every $\lambda > |T| + |D|$ if and only if every model of $\mathcal{K}$ is prime and minimal over the set of realizations of a minimal type if and only if every model of $\mathcal{K}$ of cardinality $> |T| + |D|$ is D-homogeneous. (3) If D is totally transcendental and if there is a model of $\mathcal{K}$ of cardinality above $|T| + |D|$ which is not D-homogeneous, then for any $|T| + |D| \leq \mu \leq \lambda$, there exists maximally (D, μ)-homogeneous models in $\mathcal{K}$ of cardinality λ (see the definition below). If T is countable this implies, in particular, that for each ordinal α the class $\mathcal{K}$ has at least $|\alpha|$ models of cardinality $\aleph_\alpha$. When $|T| < 2^{\aleph_0}$, the categoricity assumption on $\mathcal{K}$ implies that D is totally transcendental, if D is the set of isolated types of T. As a byproduct, this gives an alternative proof to Keisler's theorem which works so long as $|T| < 2^{\aleph_0}$ (whereas Keisler's soft $L_{\omega_1 \omega}$ methods do not generalize).

Using regular types and prime models, we could also give a decomposition theorem, but we do not include it here since it is a particular case of a more general abstract decomposition theorem, part of a joint work with R. Grossberg.

1. The Context

Let T be a first-order theory in the language $L(T)$. Let $\overline{M}$ be a very large saturated model of T. All sets are assumed to be subsets of $\overline{M}$. As usual,

$$\text{tp}(\bar{c}, A) = \{ \phi(\bar{x}, \bar{a}) \mid \overline{M} \models \phi[\bar{c}, \bar{a}], \ell(\bar{c}) = \ell(\bar{x}), \phi \in L(T) \}.$$

We say that $p(\bar{x})$ is a *complete type over* A in n variables if $\ell(\bar{x}) = n$ and there is $\bar{c}$ in $\overline{M}$ such that $p(\bar{x}) = \text{tp}(\bar{c}, A)$. The *diagram of* T, denoted by $D(T)$, is the set of complete types over the empty set. $S^n(A)$ is the set of all complete types over A in n variables. $S^1(A)$ is written $S(A)$. Given a set of formulas p, we let $\text{dom}(p)$ be the set of parameters appearing in the formulas of p. We say that p is over A if $\text{dom}(p)$ is contained in A. Finally, given a type p and a model M, we denote by $p(M)$ the set of realizations of p in M.

The following notions of *diagram D* were defined by Shelah in [Sh 1].

Definition 1.1. (1) For any set A, let $D(A) = \{ \text{tp}(\bar{c}, \emptyset) \mid \bar{c} \in A \} \subseteq D(T)$;
(2) For a model M of T, let $D(M) = D(|M|)$.

Definition 1.2. Let $D \subseteq D(T)$.

(1) A is called a *D-set* if $D(A) \subseteq D$;
(2) A model M of T is called a *D-model* if $D(M) \subseteq D$;
(3) Define $S_D(A) = \{ p \in S(A) \mid \text{ if } \bar{c} \models p \text{ then } A \cup \bar{c} \text{ is a } D\text{-set} \}$.

Remark 1.3. $|S_D(A)| = |S_D^n(A)|$ provided both are infinite, so we will usually not write the superscript.

Here, we follow [Sh 2].

Definition 1.4. Let $D \subseteq D(T)$.

(1) The diagram D is called *stable in* λ if for any D-set A of cardinality at most λ, we have $|S_D(A)| \leq \lambda$;
(2) The diagram D is called *stable* if there is λ such that D is stable in λ, and we say that D is *unstable* if D is not stable;
(3) A D-model M is called (D, λ)-*homogeneous* if M realizes every type $p \in S_D(A)$ over subsets A of $|M|$ of cardinality less than λ;
(4) A D-model M is D-*homogeneous* if M is $(D, \|M\|)$-homogeneous.

The following definition is due to Grossberg and Shelah in [GrSh 2].

Definition 1.5. We say that D has the ∞-*order property* if for every λ, there is a formula $\phi(\bar{x}, \bar{y}, \bar{z})$, a sequence $\bar{c}$ and a set of sequences $I = \{ \bar{a}_i \mid i < \lambda \}$, such that the following two conditions hold:

(1) $I \cup \bar{c}$ is a D-set;
(2) $\models \phi[\bar{a}_i, \bar{a}_j, \bar{c}]$ if and only if $i < j < \lambda$.

Theorem 1.6. [GrSh 2] *D has the ∞-order property if and only if there is a formula* $\phi(\bar{x}, \bar{y}, \bar{z})$, *a sequence* $\bar{c}$ *and a set of sequences* $I = \{\, \bar{a}_i \mid i < \beth_{(2^{|T|})^+} \,\}$, *such that the following two conditions hold:*

(1) $I \cup \bar{c}$ *is a D-set;*

(2) $\models \phi[\bar{a}_i, \bar{a}_j, \bar{c}]$ *if and only if* $i < j < \beth_{(2^{|T|})^+}$.

Definition 1.7. Let $D \subseteq D(T)$ and let $\Gamma = D(T) - D$. Define

$$\mathrm{EC}(T, \Gamma) = \{\, M \models T \mid M \text{ omits every type in } \Gamma \,\}.$$

Equivalently,

$$\mathrm{EC}(T, \Gamma) = \{\, M \models T \mid M \text{ is a } D\text{-model } \}.$$

For the rest of the paper, we will study the class $\mathrm{EC}(T, \Gamma)$, where $\Gamma = D(T) - D$ for a fixed diagram $D \subseteq D(T)$, under the following hypothesis.

Hypothesis 1.8. There exists a (D, χ)-homogeneous model $\mathfrak{C} \in \mathrm{EC}(T, \Gamma)$ for some χ larger than any cardinality mentioned in this paper.

This implies that all D-models can be assumed to sit inside $\mathfrak{C}$, and that model satisfaction is with respect to $\mathfrak{C}$. In this context, Shelah proved the following results.

Theorem 1.9 (The Stability Spectrum). [Sh 1] *One of the following conditions must hold:*

(1) *D is unstable;*

(2) *There are* $\kappa(D) \leq \lambda(D) < \beth_{(2^{|T|})^+}$ *such that for every* μ, *D is stable in* μ *if and only if* $\mu \geq \lambda(D)$ *and* $\mu^{<\kappa(D)} = \mu$.

Theorem 1.10 (The Homogeneity Spectrum). [Sh 2]
There is a D-homogeneous model of cardinality λ *if and only if* $\lambda \geq |D|$ *and D is stable in* λ *or* $\lambda^{<\lambda} = \lambda$.

For an alternative and self-contained exposition of above two theorems, see [GrLe].

In the same paper, Shelah proved the following theorem. We will make use of a particular case which we will prove using the rank.

Theorem 1.11. [Sh 2] *Let D be stable. If* $\langle M_i \mid i < \alpha \rangle$ *is an increasing sequence of* (D, μ)-*homogeneous models and the cofinality of* α *is at least* $\kappa(D)$, *then* $\bigcup_{i < \alpha} M_i$ *is* (D, μ)-*homogeneous.*

The next theorem will be used to show the symmetry property of the rank.

Theorem 1.12. [Sh 5] *D is unstable if and only if D has the ∞-order property.*

Using [Sh 1] together with the method of [Sh a] Theorem 2.12 and Theorem 1.12, one can easily show:

Theorem 1.13. *If D is stable in λ, A is a D-set of cardinality at most λ, and I is a D-set of finite sequences of cardinality at least λ^+, then there is $J \subseteq I$ of cardinality λ^+, such that J is an indiscernible set over A.*

We will use the following properties of $\kappa(D)$ in the case when $\kappa(D) = \aleph_0$, and we will actually provide alternative proofs to these facts using the rank.

Definition 1.14. Suppose D is stable, I is a D-set, which is a set of indiscernibles and A is a D-set. Define

$$\mathrm{Av_D}(I, A) = \{\, \phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, \phi(\bar{x}, \bar{y}) \in L(T) \text{ and } |\phi(I, \bar{a})| \geq \kappa(D) \,\}.$$

Lemma 1.15. [Sh 2] *Suppose D is stable, I is a D-set, which is a set of indiscernibles and A is a D-set. Then*

(1) $\mathrm{Av_D}(I, A) \in S_D(A)$;

(2) *There exists J a subset of I with $|J| < |A|^+ + \kappa(D)$ such that $I - J$ is indiscernible over $A \cup J$;*

(3) *If $|I| \geq |A|^+ + \kappa(D)$, then there is $\bar{a}$ in I realizing $\mathrm{Av_D}(I, A)$.*

2. RANK, STATIONARY TYPES AND DEPENDENCE RELATION

We first introduce a rank for the class of D-models (see Definition 1.2) which generalizes the rank from [Sh 87a]. We then prove basic properties of it which show that it is well-behaved and is natural for this class.

Definition 2.1. For any set of formulas $p(\bar{x}, \bar{b})$ with parameters in $\bar{b}$, and A a subset of $\mathfrak{C}$ containing $\bar{b}$, we define the *rank* $R_A[p]$. The rank $R_A[p]$ will be an ordinal, -1, or ∞ and we have the usual ordering $-1 < \alpha < \infty$ for any ordinal α. We define the relation $R_A[p] \geq \alpha$ by induction on α.

(1) $R_A[p] \geq 0$ if $p(\bar{x}, \bar{b})$ is realized in $\mathfrak{C}$;

(2) $R_A[p] \geq \delta$, when δ is a limit ordinal, if $R_A[p] \geq \alpha$ for every $\alpha < \delta$;

(3) $R_A[p] \geq \alpha + 1$ if the following two conditions hold:

    (a) There is $\bar{a} \in A$ and a formula $\phi(\bar{x}, \bar{y})$ such that

$$R_A[p \cup \phi(\bar{x}, \bar{a})] \geq \alpha \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{a})] \geq \alpha;$$

    (b) For every $\bar{a} \in A$ there is $q(\bar{x}, \bar{y}) \in D$ such that

$$R_A[p \cup q(\bar{x}, \bar{a})] \geq \alpha.$$

We write:

$R_A[p] = -1$ if p is not realized in $\mathfrak{C}$;

$R_A[p] = \alpha$ if $R_A[p] \geq \alpha$ but it is not the case that $R_A[p] \geq \alpha + 1$;

$R_A[p] = \infty$ if $R_A[p] \geq \alpha$ for every ordinal α.

For any set of formulas $p(\bar{x})$ over $A \subseteq \mathfrak{C}$, we let

$$R_A[p] = \min\{ R_A[q] \mid q \subseteq p \upharpoonright B, B \subseteq \mathrm{dom}(p), \ B \text{ finite } \}.$$

We omit the subscript A when $A = \mathfrak{C}$.

We need several basic properties of this rank. Some of them are purely technical and are stated here for future reference. Most of them are analogs of the usual properties for ranks in the first-order case, with the exception of (2) and (3). The proofs vary from the first-order context because of the second clause at successor stage, but they are all routine inductions.

Lemma 2.2. *Let A be a subset of $\mathfrak{C}$.*

(1) $R_A[\{ \bar{x} = \bar{c} \}] = 0$.

(2) *If p is over a finite set or p is complete, then $R_A[p] \geq 0$ if and only if there is $B \subseteq A$ and $q \in S_D(B)$ such that $p \subseteq q$.*

(3) *If A is $(D, \aleph_0)$-homogeneous and $\mathrm{tp}(\bar{a}, \emptyset) = \mathrm{tp}(\bar{b}, \emptyset)$ (for $\bar{a}, \bar{b} \in A$), then $R_A[p(\bar{x}, \bar{b})] = R_A[p(\bar{x}, \bar{a})]$.*

(4) (Monotonicity) *If $p \vdash q$ and p is over a finite set, then $R_A[p] \leq R_A[q]$.*

(5) *If p is over $B \subseteq A$ and $f \in \mathrm{Aut}(\mathfrak{C})$ then $R_A[p] = R_{f(A)}[f(p)]$.*

(6) (Monotonicity) *If $p \subseteq q$ then $R_A[p] \geq R_A[q]$.*

(7) (Finite Character) *There is a finite $B \subseteq \mathrm{dom}(p)$ such that*

$$R_A[p] = R_A[p \upharpoonright B].$$

(8) *If $R_A[p] = \alpha$ and $\beta < \alpha$, then there is q over A such that $R_A[q] = \beta$.*

(9) *If $R_A[p] \geq (|A| + 2^{|T|})^+$, then $R_A[p] = \infty$.*
Moreover, when A is $(D, \aleph_0)$-homogeneous, the bound is $(2^{|T|})^+$.

Proof. (1) Trivial

(2) Suppose $p \subseteq q \in S_D(B)$, and $B \subseteq A$. Since $\mathfrak{C}$ is (D, χ)-homogeneous, and $q \in S_D(B)$, then q is realized in $\mathfrak{C}$. Hence p is realized in $\mathfrak{C}$ and $R_A[p] \geq 0$.

For the converse, if p is over a finite set, and $R_A[p] \geq 0$, then there is $\bar{c} \in \mathfrak{C}$ realizing p. Thus $\mathrm{tp}(\bar{c}, \mathrm{dom}(p))$ extends p and $\mathrm{tp}(\bar{c}, \mathrm{dom}(p)) \in S_D(\mathrm{dom}(p))$.

If p is complete, then there is $B \subseteq A$ such that $p \in S(B)$. Now let $\bar{c}$ (not necessarily in $\mathfrak{C}$) realize p. For every $\bar{b} \in B$, $R_A[p \upharpoonright \bar{b}] \geq 0$, and so there is $\bar{c}' \in \mathfrak{C}$ realizing $p \upharpoonright \bar{b}$. But $\mathrm{tp}(\bar{c}, \bar{b}) = p \upharpoonright \bar{b} = \mathrm{tp}(\bar{c}', \bar{b})$ since p is complete. Thus $\mathrm{tp}(\bar{c}\bar{b}, \emptyset) \in D$, so $p \in S_D(B)$.

(3) By symmetry, it is enough to show that for every ordinal α,

$$R_A[p(\bar{x}, \bar{b})] \geq \alpha \qquad \text{implies} \qquad R_A[p(\bar{x}, \bar{a})] \geq \alpha.$$

We prove that this is true for all types by induction on α.

- When $\alpha = 0$, we know that there is $\bar{c} \in \mathfrak{C}$ realizing $p(\bar{x}, \bar{a})$. Then, since $\text{tp}(\bar{a}, \emptyset) = \text{tp}(\bar{b}, \emptyset)$ and A is $(D, \aleph_0)$-homogeneous, there is $\bar{d} \in A$ such that $\text{tp}(\bar{c}\bar{a}, \emptyset) = \text{tp}(\bar{d}\bar{b}, \emptyset)$. But then $p(\bar{x}, \bar{b}) \subseteq \text{tp}(\bar{d}, \bar{b})$. Hence $p(\bar{x}, \bar{b})$ is realized in $\mathfrak{C}$, so $R_A[p(\bar{x}, \bar{b})] \geq 0$.
- When α is a limit ordinal, this is true by induction.
- Suppose $R_A[p(\bar{x}, \bar{a})] \geq \alpha + 1$. First, there is $\bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in \text{Fml}(T)$ such that both

$$R_A[p(\bar{x}, \bar{a}) \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[p(\bar{x}, \bar{a}) \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$

Since A is $(D, \aleph_0)$-homogeneous, there is $\bar{d} \in A$ such that $\text{tp}(\bar{c}\bar{a}, \emptyset) = \text{tp}(\bar{d}\bar{b}, \emptyset)$. Therefore by induction hypothesis, both

$$R_A[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{d})] \geq \alpha \quad \text{and} \quad R_A[p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{d})] \geq \alpha.$$

Second, for every $\bar{d} \in A$, there is $\bar{c} \in A$ such that $\text{tp}(\bar{c}\bar{a}, \emptyset) = \text{tp}(\bar{d}\bar{b}, \emptyset)$. Thus, since $R_A[p(\bar{x}, \bar{a})] \geq \alpha+1$, there is $q(\bar{x}, \bar{y}) \in D$, such that $R_A[p(\bar{x}, \bar{a}) \cup q(\bar{x}, \bar{c})] \geq \alpha$. Therefore, by induction hypothesis, $R_A[p(\bar{x}, \bar{b}) \cup q(\bar{x}, \bar{d})] \geq \alpha$. This shows that $R_A[p(\bar{x}, \bar{b})] \geq \alpha + 1$.

(4) Suppose $p \vdash q$. By definition of the rank, we may choose $q_0 \subseteq q$ over a finite set, such that $R_A[q_0] = R_A[q]$. Hence, since $p \vdash q_0$, it is enough to show the lemma when q is over a finite set also. Write $p = p(\bar{x}, \bar{b}) \vdash q = q(\bar{x}, \bar{a})$. We show by induction on α that for every such pair of types over finite sets, we have

$$R_A[p(\bar{x}, \bar{b})] \geq \alpha \quad \text{implies} \quad R_A[q(\bar{x}, \bar{b})] \geq \alpha.$$

- For $\alpha = 0$, this is true by definition.
- For α a limit ordinal, this is true by induction.
- Suppose $R_A[p(\bar{x}, \bar{b})] \geq \alpha+1$. On the one hand, there is $\bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in \text{Fml}(T)$ such that both

$$R_A[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$

But

$$p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c}) \vdash q(\bar{x}, \bar{a}) \cup \phi(\bar{x}, \bar{c})$$

and similarly

$$p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{c}) \vdash q(\bar{x}, \bar{a}) \cup \neg\phi(\bar{x}, \bar{c}),$$

so by induction hypothesis, both

$$R_A[q(\bar{x}, \bar{a}) \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[q(\bar{x}, \bar{a}) \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$

On the other hand, given any $\bar{c} \in A$, there is $r(\bar{x}, \bar{y}) \in D$, such that $R_A[p(\bar{x}, \bar{b}) \cup r(\bar{x}, \bar{c})] \geq \alpha$. But

$$p(\bar{x}, \bar{b}) \cup r(\bar{x}, \bar{c}) \vdash q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{c}),$$

so by induction hypothesis, $R_A[q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{c})] \geq \alpha$. Hence $R_A[q(\bar{x}, \bar{a})] \geq \alpha + 1$.

(5) First, choose $q(\bar{x}, \bar{a}) \subseteq p$, such that $R_A[q] = R_A[p]$ (this is possible by definition of the rank). Similarly, since $f(q) \subseteq f(p)$, we could have chosen q so that in addition $R_{f(A)}[f(q)] = R_{f(A)}[f(p)]$. Now, by symmetry, it is enough to show that if $R_A[q] \geq \alpha$ then $R_{f(A)}[f(q)] \geq \alpha$.

- For $\alpha = 0$ or α a limit ordinal, it is obvious by definition.
- Suppose $\alpha = \beta + 1$. First, there exists $\phi(\bar{x}, \bar{b})$ such that

$$R_A[q \cup \phi(\bar{x}, \bar{b})] \geq \beta \quad \text{and} \quad R_A[q \cup \neg\phi(\bar{x}, \bar{b})] \geq \beta.$$

Thus, by induction hypothesis, we have

$$R_{f(A)}[f(q) \cup \phi(\bar{x}, f(\bar{b}))] \geq \beta \quad \text{and} \quad R_{f(A)}[f(q) \cup \neg\phi(\bar{x}, f(\bar{b}))] \geq \beta.$$

Second, notice that for every $\bar{b} \in f(A)$, there is $\bar{c} \in A$, such that $f(\bar{c}) = \bar{b}$. Since $R_A[q] \geq \beta + 1$, there exists $r(\bar{x}, \bar{y}) \in D$, such that $R_A[q \cup r(\bar{x}, \bar{c})] \geq \beta$. Hence, by induction hypothesis, $R_{f(A)}[f(q) \cup r(\bar{x}, \bar{b})] \geq \beta$. This shows that $R_{f(A)}[f(q)] \geq \beta + 1$.

(6) This is immediate by definition of the rank.

(7) By definition of the rank, let $B \in \text{dom}(p)$ and $q \subseteq p \restriction B$ be such that $R_A[q] = R_A[p]$. Now, clearly $q \subseteq p \restriction B \subseteq p$, so $R_A[q] \geq R_A[p \restriction B] \geq R_A[p]$ by Lemma 6. So $R_A[p \restriction B] = R[p]$.

(8) Suppose there is α_0 such that $R_A[p] \neq \alpha_0$ for every p. We prove by induction on $\alpha \geq \alpha_0$, that for no type p do we have $R_A[p] = \alpha$.

- For $\alpha = \alpha_0$, this is the definition of α_0.
- Now suppose that there is p such that $R_A[p] = \alpha + 1$. By 7, we may assume that p is over a finite set. Then there is $\bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in \text{Fml}(T)$ such that both

$$R_A[p \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$

But by induction hypothesis, neither can be equal to α, so we must have both

$$R_A[p \cup \phi(\bar{x}, \bar{c})] \geq \alpha + 1 \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha + 1.$$

Similarly, given any $\bar{c} \in A$, there is $q(\bar{x}, \bar{y}) \in D$, such that $R_A[p \cup q(\bar{x}, \bar{c})] \geq \alpha$. But, by induction hypothesis, we cannot have $R_A[p \cup q(\bar{x}, \bar{c})] = \alpha$, so $R_A[p \cup q(\bar{x}, \bar{c})] \geq \alpha + 1$. But this shows that $R_A[p] \geq \alpha + 2$, a contradiction.

- Suppose $\alpha > \alpha_0$ is a limit ordinal. Then $\alpha \geq \alpha_0 + 1$, so as in the previous case, there is $\bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in \text{Fml}(T)$ such that both

$$R_A[p \cup \phi(\bar{x}, \bar{c})] \geq \alpha_0 \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha_0.$$

But by induction hypothesis, for no β such that $\alpha > \beta \geq \alpha_0$ can we have $R_A[p \cup \phi(\bar{x}, \bar{c})] = \beta$ or $R_A[p \cup \neg\phi(\bar{x}, \bar{c})] = \beta$, so necessarily since α is a limit ordinal, we have

$$R_A[p \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$

Similarly, for any $\bar{c} \in A$, there is $q(\bar{x}, \bar{y}) \in D$, such that $R_A[p \cup q(\bar{x}, \bar{c})] \geq \alpha_0$ and hence by induction hypothesis $R_A[p \cup q(\bar{x}, \bar{c})] > \beta$ for any $\alpha_0 \leq \beta < \alpha$ so since α is a limit ordinal, we have $R_A[p \cup q(\bar{x}, \bar{c})] \geq \alpha$. But this shows that $R_A[p] \geq \alpha + 1$, a contradiction.

(9) By the previous lemma, it is enough to find $\alpha_0 < (|A| + 2^{|T|})^+$, (respectively $< (2^{|T|})^+$ if A is a $(D, \aleph_0)$-homogeneous model) such that

(*) $R_A[p] \neq \alpha_0$ for every type over A.

We do this by counting the number of possible values for the rank. By 7 it is enough to count the values achieved by types over finite subsets of A. But there are at most $|A|^{<\aleph_0} \leq |A| + \aleph_0$ finite subsets of A, and given any finite subset, there are only $2^{|T|}$ distinct types over it. Hence there are at most $|A| + 2^{|T|}$ many different ranks, and so by the pigeonhole principle (*) holds for some $\alpha_0 < (|A| + 2^{|T|})^+$.

When A is a $(D, \aleph_0)$-homogeneous model, the bound can be further reduced by a use of 3, since only the type of each of those finite subset of A is relevant.   □

The next lemma shows that the rank is especially well-behaved when the parameter A is the universe of a $(D, \aleph_0)$-homogeneous model. This is used in particular to study $(D, \aleph_0)$-homogeneous models in the last two sections.

Lemma 2.3.   (1) *If p is over a subset of a $(D, \aleph_0)$-homogeneous model M, then $R_M[p] = R[p]$.*
    (2) *If p is over $M_1 \cap M_2$, with M_l $(D, \aleph_0)$-homogeneous, for $l = 1, 2$, we have $R_{M_1}[p] = R_{M_2}[p]$.*
    (3) *If $q(\bar{x}, \bar{a}_l)$ are sets of formulas, with $a_l \in M_l$ for $l = 1, 2$ satisfying $\mathrm{tp}(\bar{a}_1, \emptyset) = \mathrm{tp}(\bar{a}_2, \emptyset)$, then $R_{M_1}[q(\bar{x}, \bar{a}_1)] = R_{M_2}[q(\bar{x}, \bar{a}_2)]$.*

Proof. (1) First, by Finite Character, we may assume that p is over a finite set. Now we show by induction on α that

$$R_M[p] \geq \alpha \qquad \text{implies} \qquad R[p] \geq \alpha.$$

When $\alpha = 0$ or α is a limit, it is clear. Suppose $R_M[p] \geq \alpha + 1$. Then there is $\bar{b} \in M$ and $\phi(\bar{x}, \bar{y})$ such that both

$$R_M[p \cup \phi(\bar{x}, \bar{b})] \geq \alpha \quad \text{and} \quad R_M[p \cup \neg\phi(\bar{x}, \bar{b})] \geq \alpha.$$

By induction hypothesis, we have

$$R[p \cup \phi(\bar{x}, \bar{b})] \geq \alpha \quad \text{and} \quad R[p \cup \neg\phi(\bar{x}, \bar{b})] \geq \alpha.$$

Further, if $\bar{b} \in \mathfrak{C}$, choose $\bar{b}' \in M$, such that $\mathrm{tp}(\bar{b}, \bar{a}) = \mathrm{tp}(\bar{b}', \bar{a})$. Since $R_M[p] \geq \alpha + 1$, there is $q(\bar{x}, \bar{y}) \in D$ such that $R_M[p \cup q(\bar{x}, \bar{b}')] \geq \alpha$. Thus, since $\mathfrak{C}$ is $(D, \aleph_0)$-homogeneous, by induction hypothesis we have $R[p \cup q(\bar{x}, \bar{b}')] \geq \alpha$, and so by Lemma 2.2 3 $R[p \cup q(\bar{x}, \bar{b})] \geq \alpha$. Hence $R[p] \geq \alpha + 1$.

For the converse, similarly by induction on α we show that

$$R[p] \geq \alpha \qquad \text{implies} \qquad R_M[p] \geq \alpha.$$

Again, for $\alpha = 0$ or α a limit, it is easy. Suppose $R[p] \geq \alpha + 1$. Then there is $\bar{b} \in \mathcal{C}$ and $\phi(\bar{x}, \bar{y})$ such that both

$$R[p \cup \phi(\bar{x}, \bar{b})] \geq \alpha \quad \text{and} \quad R[p \cup \neg\phi(\bar{x}, \bar{b})] \geq \alpha.$$

Since M is $(D, \aleph_0)$-homogeneous, there exists $\bar{b}' \in M$, such that $\mathrm{tp}(\bar{b}, \bar{a}) = \mathrm{tp}(\bar{b}', \bar{a})$. By Lemma 2.2 3, we have

$$R[p \cup \phi(\bar{x}, \bar{b}')] \geq \alpha \quad \text{and} \quad R[p \cup \neg\phi(\bar{x}, \bar{b}')] \geq \alpha.$$

Hence, by induction hypothesis, we have (since $\bar{b}' \in M$)

$$R_M[p \cup \phi(\bar{x}, \bar{b}')] \geq \alpha \quad \text{and} \quad R_M[p \cup \neg\phi(\bar{x}, \bar{b}')] \geq \alpha.$$

Also, for any $\bar{b} \in M$, since $\bar{b} \in \mathcal{C}$ there is $q(\bar{x}, \bar{y}) \in D$ such that $R[p \cup q(\bar{x}, \bar{b})] \geq \alpha$. By induction hypothesis, we have $R_M[p \cup q(\bar{x}, \bar{b})] \geq \alpha$, which finishes to show that $R_M[p] \geq \alpha + 1$ and completes the proof.

(2) By (1) applied twice, $R_{M_1}[p] = R[p] = R_{M_2}[p]$.

(3) Since $R_{M_1}[q(\bar{x}, \bar{a}_1)] = R[q(\bar{x}, \bar{a}_1)] = R[q(\bar{x}, \bar{a}_2)] = R_{M_2}[q(\bar{x}, \bar{a}_2)]$. $\qquad\square$

We now show that the rank is bounded when D is $\aleph_0$-stable.

Theorem 2.4. *If D is stable in λ for some $\aleph_0 \leq \lambda < 2^{\aleph_0}$ then $R_A[p] < \infty$ for every type p and every subset A of $\mathcal{C}$.*

Proof. We prove the contrapositive. Suppose there is a subset A of $\mathcal{C}$ and a type p over A such that $R_A[p] = \infty$. We construct sets $A_\eta \subseteq A$ and types p_η, for $\eta \in {}^{<\omega}2$, such that:

(1) $p_\eta \in S_D(A_\eta)$;
(2) $p_\eta \subseteq p_\nu$ when $\eta < \nu$;
(3) A_η is finite;
(4) $p_{\eta\hat{~}0}$ and $p_{\eta\hat{~}1}$ are contradictory;
(5) $R_A[p_\eta] = \infty$;

This is possible: Let $\mu = (2^{|T|})^+$ if A is a $(D, \aleph_0)$-homogeneous model, and $\mu = (|A| + 2^{|T|})^+$ otherwise. The construction is by induction on $n = \ell(\eta)$.

- For $n = 0$, by Finite Character we choose first $\bar{b} \in A$, such that $R_A[p] = R_A[p \restriction \bar{b}] = \infty$. Since $R_A[p \restriction \bar{b}] = \infty$, in particular $R_A[p \restriction \bar{b}] \geq \mu + 1$ so there exists $q(\bar{x}, \bar{y}) \in D$, such that $R_A[(p \restriction \bar{b}) \cup q(\bar{x}, \bar{b})] \geq \mu$. But then $p \restriction \bar{b} \subseteq q(\bar{x}, \bar{b})$, $q(\bar{x}, \bar{b}) \in S_D(\bar{b})$ and $R_A[q(\bar{x}, \bar{b})] \geq \mu$, so $R_A[q(\bar{x}, \bar{b})] = \infty$ by Lemma 2.2 9. Therefore, we let $A_{<>} = \bar{b}$ and $p_{<>} = q(\bar{x}, \bar{b})$ and the conditions are satisfied.

- Assume $n \geq 0$ and that we have constructed $p_\eta \in S_D(A_\eta)$ with $\ell(\eta) = n$. Since $R_A[p_\eta] = \infty$, in particular $R_A[p_\eta] \geq (\mu + 1) + 1$. Hence, there is $\bar{a}_\eta \in A$ and $\phi(\bar{x}, \bar{y})$ such that

(*) $\quad R_A[p_\eta \cup \phi(\bar{x}, \bar{a}_\eta)] \geq \mu + 1 \quad$ and $\quad R_A[p_\eta \cup \neg\phi(\bar{x}, \bar{a}_\eta)] \geq \mu + 1$.

Let $A_{\eta^\frown 0} = A_{\eta^\frown 1} = A_\eta \cup \bar{a}_\eta \subseteq A$. Both $A_{\eta^\frown 0}$ and $A_{\eta^\frown 1}$ are finite, so (*) and the definition of the rank imply that there are $q_l(\bar{x}, \bar{y}) \in D$ for $l = 0, 1$, such that

$$R_A[p_\eta \cup \phi(\bar{x}, \bar{a}_\eta) \cup q_0(\bar{x}, A_{\eta^\frown 0})] \geq \mu$$

and

$$\dot{R}_A[p_\eta \cup \neg\phi(\bar{x}, \bar{a}_\eta) \cup q_1(\bar{x}, A_{\eta^\frown 1})] \geq \mu.$$

Define $p_{\eta^\frown 0} := p_\eta \cup \phi(\bar{x}, \bar{a}_\eta) \cup q_0(\bar{x}, A_{\eta^\frown 0})$ and $p_{\eta^\frown 1} := p_\eta \cup \neg\phi(\bar{x}, \bar{a}_\eta) \cup q_1(\bar{x}, A_{\eta^\frown 1})$. Then $p_{\eta^\frown l} \in S_D(A_{\eta^\frown l})$ since $q_l(\bar{x}, A_{\eta^\frown l}) \in S_D(A_{\eta^\frown l})$ and $A_{\eta^\frown l}$ is finite for $l = 0, 1$. Moreover, $p_{\eta^\frown 0}$ and $p_{\eta^\frown 1}$ are contradictory by construction. Finally $R_A[p_{\eta^\frown l}] = \infty$, since $R_A[p_{\eta^\frown l}] \geq \mu$. Hence all the requirements are met.

This is enough: For each $\eta \in {}^\omega 2$, define $A_\eta := \bigcup_{n \in \omega} A_{\eta \restriction n}$ and $p_\eta := \bigcup_{n \in \omega} p_{\eta \restriction n}$. We claim that $p_\eta \in S_D(A_\eta)$. Certainly $p_\eta \in S(A_\eta)$, so we only need to show that if $\bar{c} \models p_\eta$, then $A_\eta \cup \bar{c}$ is a D-set ($\bar{c}$ is not assumed to be in $\mathfrak{C}$). It is enough to show that $\text{tp}(\bar{c}\bar{d}, \emptyset) \in D$ for every finite $\bar{d} \in A_\eta$. But, if $\bar{d} \in A_\eta$, then there is $n \in \omega$ such that $\bar{d} \in A_{\eta \restriction n}$. Since $\bar{c} \models p_{\eta \restriction n}$ and $p_{\eta \restriction n} \in S_D(A_{\eta \restriction n})$, then $\bar{c} \cup A_{\eta \restriction n}$ is a D-set, and therefore $\text{tp}(\bar{c}\bar{d}, \emptyset) \in D$, which is what we wanted. Now that we have established that $p_\eta \in S_D(A_\eta)$, since $\mathfrak{C}$ is (D, χ)-homogeneous, there is $\bar{c}_\eta \in \mathfrak{C}$ such that $\bar{c}_\eta \models p_\eta$. Now let $C = \bigcup_{\eta \in {}^{\omega \geq} 2} A_\eta$. Then $|C| = \aleph_0$ and if $\eta \neq \nu \in {}^\omega 2$, then $\text{tp}(\bar{c}_\eta, C) \neq \text{tp}(\bar{c}_\nu, C)$, since p_η and p_ν are contradictory. Therefore $|S_D(C)| \geq 2^{\aleph_0}$, which shows that D is not stable in λ for any $\aleph_0 \leq \lambda < 2^{\aleph_0}$. $\qquad \square$

Remark 2.5. Recall that in [Sh 1], D is stable in λ if and only if there is a (D, λ^+)-homogeneous model and $|S_D(A)| \leq \lambda$ for all D sets A of cardinality at most λ (this is Definition 2.1 of [Sh 1]). The proof of the previous theorem shows that if D is stable in λ for some $\aleph_0 \leq \lambda < 2^{\aleph_0}$ in the sense of [Sh 1] then $R_A[p] < \infty$ for all D-set A and D-type p. In other words, we do not really need $\mathfrak{C}$ for this proof.

By analogy with the first-order case (see [Sh a] definition 3.1), we introduce the following definition:

Definition 2.6. We say that D is *totally transcendental* if $R_A[p] < \infty$ for every subset A of $\mathfrak{C}$ and every type p over A.

For the rest of the paper, we will make the following hypothesis. We will occasionally repeat that D is totally transcendental for emphasis.

Hypothesis 2.7. D is totally transcendental.

In what follows, we shall show that when D is totally transcendental, the rank affords a well-behaved dependence relation on the subsets of $\mathfrak{C}$. We first focus on a special kind of types.

Definition 2.8. A type p is called *stationary* if for every B containing $\text{dom}(p)$ there is a unique type $p_B \in S_D(B)$, such that p_B extends p and $R[p] = R[p_B]$.

Note that since our rank is not an extension of Morley's rank, one does not necessarily get the usual stationary types when the class is first-order. The argument in the next lemma is a generalization of Theorem 1.4.(1)(b) in [Sh 87a]. Recall that $p \in S_D(A)$ *splits over* $B \subseteq A$ if there exists $\phi(\bar{x}, \bar{y})$ and $\bar{a}, \bar{c} \in A$ with $\text{tp}(\bar{a}, B) = \text{tp}(\bar{c}, B)$, such that $\phi(\bar{x}, \bar{a}) \in p$ and $\neg\phi(\bar{x}, \bar{c}) \in p$.

Lemma 2.9. *Suppose there is $\bar{d} \in \mathfrak{C}$ realizing $p(\bar{x}, \bar{b})$ and a $(D, \aleph_0)$-homogeneous model M such that*

(*) $$R[\text{tp}(\bar{d}, M)] = R[p(\bar{x}, \bar{b})] = \alpha.$$

Then, for any $A \subseteq \mathfrak{C}$ containing $\bar{b}$ there is a unique $p_A \in S_D(A)$ extending $p(\bar{x}, \bar{b})$, such that

$$R[p_A] = R[p(\bar{x}, \bar{b})] = \alpha.$$

Moreover, p_A does not split over $\bar{b}$.

Proof. We first prove uniqueness. Suppose two different types p_A and $q_A \in S_D(A)$ extend $p(\bar{x}, \bar{b})$ and

$$R[p_A] = R[p(\bar{x}, \bar{b})] = R[q_A] = \alpha.$$

Then there is $\phi(\bar{x}, \bar{c}) \in p_A$ such that $\neg\phi(\bar{x}, \bar{c}) \in q_A$. Thus, by Monotonicity,

$$R[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c})] \geq R_A[p] = \alpha \quad \text{and} \quad R[p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{c})] \geq R_A[p] = \alpha.$$

Further, for every $\bar{c} \in \mathfrak{C}$, there is $\bar{c}' \in M$ such that $\text{tp}(\bar{c}, \bar{b}) = \text{tp}(\bar{c}', \bar{b})$ since M is $(D, \aleph_0)$-homogeneous. Now write $q(\bar{x}, \bar{c}') = \text{tp}(\bar{d}, \bar{c}')$, and notice that

$$R[p(\bar{x}, \bar{b}) \cup q(\bar{x}, \bar{c}')] \geq R[\text{tp}(\bar{d}, \bar{b} \cup \bar{c}')] \geq R[\text{tp}(\bar{d}, M)] = \alpha.$$

But $q(\bar{x}, \bar{y}) \in D$ by definition and so by Lemma 2.2 (2) $R[p(\bar{x}, \bar{b}) \cup q(\bar{x}, \bar{c})] \geq \alpha$ since $\text{tp}(\bar{c}\bar{b}, \emptyset) = \text{tp}(\bar{c}'\bar{b}, \emptyset)$. But this shows that $R[p(\bar{x}, \bar{b})] \geq \alpha + 1$, which contradicts (*).

We now argue that p_A does not split over $\bar{b}$. Suppose it does, and choose a formula $\phi(\bar{x}, \bar{y}) \in \text{Fml}(T)$ and sequences $\bar{c}_0, \bar{c}_1 \in A$ with $\text{tp}(\bar{c}_0, \bar{b}) = \text{tp}(\bar{c}_1, \bar{b})$ such that $\phi(\bar{x}, \bar{c}_0)$ and $\neg\phi(\bar{x}, \bar{c}_1)$ both belong to p_A. Then by Monotonicity,

$$R[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c}_0)] \geq R_A[p] = \alpha \quad \text{and} \quad R[p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{c}_1)] \geq R_A[p] = \alpha.$$

But $\text{tp}(\bar{c}_0, \bar{b}) = \text{tp}(\bar{c}_1, \bar{b})$ so by Lemma 2.2(3) we have

$$R[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c}_1)] \geq \alpha.$$

An argument similar to the uniqueness argument in the first paragraph finishes to show that $R[p(\bar{x}, \bar{b})] \geq \alpha + 1$, which is again a contradiction to (*).

For the existence, let p_A be the following set of formulas with parameters in A:

$$\{ \phi(\bar{x}, \bar{c}) \mid \text{There exists } \bar{c}' \in M \text{ such that } \mathrm{tp}(\bar{c}, \bar{b}) = \mathrm{tp}(\bar{c}', \bar{b}) \text{ and } \models \phi[\bar{d}, \bar{c}'] \}.$$

By the non-splitting part, using the fact that M is $(D, \aleph_0)$-homogeneous, we have that $\mathrm{tp}(\bar{d}, M)$ does not split over $\bar{b}$. Hence $p_A \in S_D(A)$ and does not split over $\bar{b}$. We show that this implies that $R[p_A] = R[\mathrm{tp}(\bar{d}, M)] = \alpha$. Otherwise, since p_A extends $p(\bar{x}, \bar{b})$, by Monotonicity we must have $R[p_A] \leq \alpha$, and therefore $R[p_A] < \alpha$. Let us choose $\bar{b}' \in A$ such that $\bar{b} \subseteq \bar{b}'$ and $R[p_A] = R[p_A \upharpoonright \bar{b}']$. For convenience, we write $q(\bar{x}, \bar{b}') := p_A \upharpoonright \bar{b}'$, and so $R[q(\bar{x}, \bar{b}')] < \alpha$. Now since M is $(D, \aleph_0)$-homogeneous, we can choose $\bar{b}'' \in M$ such that $\mathrm{tp}(\bar{b}'', \bar{b}) = \mathrm{tp}(\bar{b}', \bar{b})$. Hence

(**) $\qquad\qquad R[q(\bar{x}, \bar{b}'')] = R[q(\bar{x}, \bar{b}')] < \alpha.$

But by definition of p_A, we must have $q(\bar{x}, \bar{b}'') \subseteq \mathrm{tp}(\bar{d}, M)$, so by Monotonicity we have $R[q(\bar{x}, \bar{b}'')] \geq R[\mathrm{tp}(\bar{d}, M)] = \alpha$, which contradicts (**). $\qquad\square$

Corollary 2.10. *The following conditions are equivalent:*

(1) *$p \in S_D(A)$ is stationary.*

(2) *There is a $(D, \aleph_0)$-homogeneous model M containing A and $\bar{d} \in \mathfrak{C}$ realizing p such that $R[\mathrm{tp}(\bar{d}, M)] = R[p]$.*

Definition 2.11. A stationary type $p \in S_D(A)$ is *based on B* if $R[p] = R[p \upharpoonright B]$.

Remark 2.12. (1) If p is stationary, there is a finite $B \subseteq \mathrm{dom}(p)$ such that p is based on B.

(2) If p is based on B, then $p \upharpoonright B$ is also stationary and p is the only extension of $p \upharpoonright B$ such that $R[p] = R[p \upharpoonright B]$.

(3) If p is stationary and $\mathrm{dom}(p) \subseteq A \subseteq B$, then $p_A = p_B \upharpoonright A$.

(4) Suppose $\mathrm{tp}(\bar{a}, \emptyset) = \mathrm{tp}(\bar{a}', \emptyset)$. Then $p(\bar{x}, \bar{a}')$ is stationary if and only if $p(\bar{x}, \bar{a})$ is stationary. (Use an automorphism of $\mathfrak{C}$ sending $\bar{a}$ to $\bar{a}'$.)

Stationary types allow us to prove a converse of Theorem 2.4.

Theorem 2.13. *If D is totally transcendental then D is stable in every $\lambda \geq |D| + |T|$. In particular $\kappa(D) = \aleph_0$.*

Proof. Let $\lambda \geq |D| + |T|$, and let A be a subset of $\mathfrak{C}$ of cardinality at most λ. Since $\lambda \geq |D| + |T|$, by using a countable, increasing chain of models we can find a $(D, \aleph_0)$-homogeneous model M containing A of cardinality λ. Since $|S_D(A)| \leq |S_D(M)|$, it is enough to show that $|S_D(M)| \leq \lambda$. Suppose that $|S_D(M)| \geq \lambda^+$. Since M is $(D, \aleph_0)$-homogeneous, each $p \in S_D(M)$ is stationary. Hence, for each $p \in S_D(M)$, we can choose a finite $B_p \subseteq M$ such that p is based on B_p. Since there are only λ many finite subsets of M, by the pigeonhole principle there is a fixed finite subset B of M such that λ^+ many types $p \in S_D(M)$ are based on B. Since $\lambda^+ > |S_D(B)| = |D|$, another application of the pigeonhole principle

shows that there a single stationary type $q \in S_D(B)$ with λ^+ many extensions in $S_D(M)$ of the same rank. This contradicts the stationarity of q. Hence D is stable in λ.

For the last sentence, let $\lambda = \beth_\omega(|D| + |T|)$. By Zermelo-König, $\lambda^{\aleph_0} > \lambda$, hence by Theorem 1.9 $\kappa(D) = \aleph_0$. $\quad\square$

The following results show that stationary types behave nicely. Not only do they have the uniqueness and the extension properties, but they can be represented by averages. Surprisingly, it turns out that every type is reasonably close to a stationary type (this is.made precise in Lemma 4.8).

Definition 2.14. Let $p \in S_D(A)$ be stationary and let α be an infinite ordinal. The sequence $I = \{\, c_i \mid i < \alpha \,\}$ is called a *Morley sequence based on* p if for each $i < \alpha$ we have c_i realizes p_{A_i}, where $A_i = A \cup \{c_j \mid j < i\}$.

Lemma 2.15. *Let* $p \in S_D(A)$ *be stationary. If I is a Morley sequence based on p, then I is indiscernible over A.*

Proof. By stationarity $p_{A_i} \subseteq p_{A_j}$ when $i < j$, and by the previous lemma each p_{A_i} does not split over A. Hence, a standard result (see for example [Sh a] Lemma I.2.5) implies that I is an indiscernible sequence over A. $\quad\square$

Definition 2.16. ($\kappa(D) = \aleph_0$) For I an infinite set of indiscernibles and A a set (with $I \cup A \subseteq \mathfrak{C}$), recall that

$$\mathrm{Av}_D(I, A) = \{\, \phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, \phi(\bar{x}, \bar{y}) \in L(T) \text{ and } |\phi(I, \bar{a})| \geq \aleph_0 \,\}.$$

Lemma 2.17. *Suppose $p \in S_D(A)$ is stationary and I is a Morley sequence based on p. Then for any B containing A we have that $p_B = \mathrm{Av}_D(I, B)$.*

Proof. Let $B \subseteq \mathfrak{C}$ and write $I = \{c_i \mid i < \alpha\}$. Choose $c_i \in \mathfrak{C}$ for $\alpha \leq i < \alpha + \omega$ realizing p_{B_i}, where $B_i = B \cup \bigcup\{a_j \mid j < i\}$. Since $\mathrm{Av}_D(I, B) \in S_D(B)$ extends p, it is enough to show that $R[\mathrm{Av}_D(I, B)] = R[p]$. Suppose $R[\mathrm{Av}_D(I, B)] \neq R[p]$. Then, by Monotonicity, we must have $R[\mathrm{Av}_D(I, B)] < R[p]$. We can find a finite $C \subseteq B$ such that p is based on C and by Finite Character, we may assume in addition that

(*) $R[\mathrm{Av}_D(I, B)] = R[\mathrm{Av}_D(I, C)] < R[p].$

But, since C is finite and $\kappa(D) = \aleph_0$, by Lemma 1.15 there is $c_i \in I$ for $\alpha \leq i < \alpha + \omega$ realizing $\mathrm{Av}_D(I, C)$, and since $C \subseteq B$, we must have $\mathrm{tp}(c_i, C) = \mathrm{Av}_D(I, C) = p_C$ (since c_i realizes p_{B_i}). But then, by choice of C we have $R[\mathrm{Av}_D(I, C)] = R[p_C] = R[p]$ which contradicts (*). $\quad\square$

Lemma 2.18. *Let I be an infinite indiscernible set, A be finite and $p = \mathrm{Av}_D(I, A)$ be stationary. Then for any $C \supseteq A$ we have $p_C = \mathrm{Av}_D(I, C)$.*

Proof. Write $I = \{c_i \mid i < \alpha\}$, for $\alpha \geq \omega$ and let C be given. Choose $c_i \in \mathfrak{C}$ for $\alpha \leq i < \alpha + \omega$ realizing p_{C_i}, where $C_i = C \cup \bigcup\{c_j \mid j < i\}$. Let $I' = \{c_i \mid i < \alpha + \omega\}$ and notice that necessarily $\mathrm{Av_D}(I, B) = \mathrm{Av_D}(I', B)$ for any B. Suppose $p_C \neq \mathrm{Av_D}(I, C)$, then since $\mathrm{Av_D}(I, A) \subseteq \mathrm{Av_D}(I, C)$, we must have $R[\mathrm{Av_D}(I, C)] < R[p]$, so $R[\mathrm{Av_D}(I', C)] < R[p_C]$. Choose C' finite, with $A \subseteq C' \subseteq C$, such that $R[\mathrm{Av_D}(I', C)] = R[\mathrm{Av_D}(I', C')]$. Now there is $J \subseteq I'$ finite such that $I' - J$ is indiscernible over C'. Choose $c_i \in I' - J$ with $i > \alpha$. Then c_i realizes $\mathrm{Av_D}(I', C')$, so $\mathrm{Av_D}(I', C') = \mathrm{tp}(c_i, C') \subseteq p_{C_i}$ by choice of c_i. But then

$$R[\mathrm{Av_D}(I', C')] \geq R[p_{C_i}] = R[p] > R[\mathrm{Av_D}(I, C)] = R[\mathrm{Av_D}(I', C')],$$

a contradiction. $\qquad\square$

It is natural at this point to introduce the forking symbol, by analogy with the first-order case (see for example [Bl] or [Ma]). We do not claim that the two notions coincide even when both are defined.

Definition 2.19. Suppose $A, B, C \subseteq \mathfrak{C}$, with $B \subseteq A$. We say that

$$A \underset{B}{\downarrow} C \quad \text{if} \quad R[\mathrm{tp}(\bar{a}, B)] = R[\mathrm{tp}(\bar{a}, B \cup C)], \qquad \text{for every } \bar{a} \in A.$$

As in many other contexts, the symmetry property can be obtained from the failure of the order property.

Theorem 2.20 (Symmetry). *If* $\mathrm{tp}(\bar{a}, B)$ *and* $\mathrm{tp}(\bar{c}, B)$ *are stationary, then*

$$\bar{a} \underset{B}{\downarrow} \bar{c} \qquad \text{if and only if} \qquad \bar{c} \underset{B}{\downarrow} \bar{a}.$$

Proof. First, D is stable by Theorem 2.13, and therefore does not have the ∞-order property by Theorem 1.12. Suppose, for a contradiction, that

$$R[\mathrm{tp}(\bar{c}, B \cup \bar{a})] < R[\mathrm{tp}(\bar{c}, B)] \quad \text{and} \quad R[\mathrm{tp}(\bar{a}, B \cup \bar{c})] = R[\mathrm{tp}(\bar{a}, B)].$$

Let $\lambda = \beth_{(2^{|T|})^+}$ and let $\mu = (2^\lambda)^+$. We use Theorem 1.6 to show that D has the ∞-order property, by constructing an order of length λ. Choose $p(\bar{x}, \bar{y}, \bar{b}) \in S_D(\bar{b})$ with $\bar{b} \in B$, such that

$$R[\mathrm{tp}(\bar{a}, B \cup \bar{c})] = R[p(\bar{x}, \bar{c}, \bar{b})] = R[\mathrm{tp}(\bar{a}, B)]$$

and

$$R[\mathrm{tp}(\bar{c}, B \cup \bar{a})] = R[p(\bar{c}, \bar{y}, \bar{b})] < R[\mathrm{tp}(\bar{c}, B)].$$

Let $\bar{a}_\alpha, \bar{c}_\alpha \in \mathfrak{C}$ for $\alpha < \mu$ and $B_\alpha = \bigcup\{\bar{a}_\beta, \bar{c}_\beta \mid \beta < \alpha\}$ be such that:

(1) $B_0 = B$;
(2) $\bar{a}_\alpha$ realizes $\mathrm{tp}(\bar{a}, B)$ and $R[\mathrm{tp}(\bar{a}_\alpha, B_\alpha)] = R[\mathrm{tp}(\bar{a}, B)]$;
(3) $\bar{c}_\alpha$ realizes $\mathrm{tp}(\bar{c}, B)$ and $R[\mathrm{tp}(\bar{c}_\alpha, B_\alpha \cup \bar{a}_\alpha)] = R[\mathrm{tp}(\bar{c}, B)]$.

This is achieved by induction on $\alpha < \mu$. Let $B_0 := B$, $\bar{a}_0 := \bar{a}$ and $\bar{c}_0 := \bar{c}$. At stage α, we let first $B_\alpha := \bigcup\{\bar{a}_\beta, \bar{c}_\beta \mid \beta < \alpha\}$ which is well-defined by induction hypothesis. We then satisfy in this order (2) by stationarity of $\mathrm{tp}(\bar{a}, B)$, and (3) by stationarity of $\mathrm{tp}(\bar{c}, B)$.

This is enough: First, notice that $\bar{c}_\alpha$ does not realize $p(\bar{a}, \bar{y}, \bar{b})$, otherwise

$$R[\mathrm{tp}(\bar{c}_\alpha, B_\alpha \cup \bar{a}_\alpha)] \le R[p(\bar{a}, \bar{y}, \bar{b})] < R[\mathrm{tp}(\bar{c}, B)],$$

contrary to the choice of $\bar{c}_\alpha$. Similarly, since $\mathrm{tp}(\bar{a}_\alpha, B) = \mathrm{tp}(\bar{a}, B)$ and $\bar{b} \in B$, then

$$R[p(\bar{a}_\beta, \bar{y}, \bar{b})] < R[\mathrm{tp}(\bar{c}, B)],$$

so $\bar{c}_\alpha$ does not realize $p(\bar{a}_\beta, \bar{y}, \bar{b})$ when $\alpha \ge \beta$.

Now suppose $\alpha < \beta$. Then $\bar{a}_\beta$ realizes $p(\bar{x}, \bar{c}, \bar{b})$ since by stationarity, we must have $\mathrm{tp}(\bar{a}_\beta, A \cup \bar{c}) = \mathrm{tp}(\bar{a}, B \cup \bar{c})$. Further, since $\mathrm{tp}(\bar{a}_\alpha, B_\alpha)$ does not split over B and $\mathrm{tp}(\bar{c}_\alpha, B) = \mathrm{tp}(\bar{c}, B)$ we must have $p(\bar{x}, \bar{c}_\alpha, \bar{b}) \subseteq \mathrm{tp}(\bar{a}_\alpha, B_\alpha)$. So $\bar{a}_\beta$ realizes $p(\bar{x}, \bar{c}_\alpha, \bar{b})$.

Let $\bar{d}_\alpha = \bar{c}_\alpha \bar{a}_\alpha$ and let $q(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \bar{b}) := p(\bar{x}_1, \bar{y}_2, \bar{b})$ (we may assume that q is closed under finite conjunction). Then, above construction shows that

(*) $\bar{d}_\alpha \bar{d}_\beta \models q(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \bar{b})$ if and only if $\alpha < \beta < \mu$,

i.e. we we have an order of length μ witnessed by the type q.

We use (*) to obtain an order of length λ witnessed by a formula as follows. On the one hand, (*) implies that for any $\phi(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{c}) \in q$, the following holds:

(**) $\models \phi[\bar{d}_\alpha, \bar{d}_\beta, \bar{b}]$ whenever $\alpha < \beta$.

On the other hand, if $\alpha \ge \beta$, by (*) again, there is $\phi_{\alpha,\beta}(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{b}) \in q$, such that $\models \neg\phi_{\alpha,\beta}[\bar{d}_\alpha, \bar{d}_\beta, \bar{b}]$. Hence, by the Erdös-Rado Theorem, since $|q| \le |T|$, we can find $S \subseteq \mu$ of cardinality λ and $\phi(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{b}) \in q$, such that

(***) $\models \neg\phi[\bar{d}_\alpha, \bar{d}_\beta, \bar{b}]$ whenever $\alpha \ge \beta$, $\alpha, \beta \in S$.

Therefore, (**) and (***) together show that we can find an order of length λ, which is the desired contradiction. $\square$

We close this section by gathering together the properties of the forking symbol. They are stated with the names of the first-order forking properties to which they correspond.

Theorem 2.21.   (1) (Definition) $A \underset{B}{\downarrow} C$ if and only if $A \underset{B}{\downarrow} B \cup C$.

(2) (Existence) $A \underset{B}{\downarrow} B$

(3) ($\kappa(D) = \aleph_0$) For all $\bar{a}$ and C, there is a finite $B \subseteq C$ such that $\bar{a} \underset{B}{\downarrow} C$.

(4) (Invariance under automorphisms) *Let* $f \in \mathrm{Aut}(\mathfrak{C})$.

$$A \underset{B}{\downarrow} C \qquad \textit{if and only if} \qquad f(A) \underset{f(B)}{\downarrow} f(C).$$

(5) (Finite Character)

$$A \underset{B}{\downarrow} C \qquad \textit{if and only if} \qquad A' \underset{B}{\downarrow} C',$$

for every finite $A' \subseteq A$, *and finite* $C' \subseteq C$.

(6) (Monotonicity) *Suppose* A' *and* C' *contain* A *and* C *respectively and that* B' *is a subset of* B. *Then*

$$A \underset{B}{\downarrow} C \qquad \textit{implies} \qquad A' \underset{B'}{\downarrow} C'.$$

(7) (Transitivity) *If* $B \subseteq C \subseteq D$, *then*

$$A \underset{B}{\downarrow} C \quad \textit{and} \quad A \underset{C}{\downarrow} D \qquad \textit{if and only if} \qquad A \underset{B}{\downarrow} D.$$

(8) (Symmetry) *Let* M *is a* $(D, \aleph_0)$-*homogeneous model.*

$$A \underset{M}{\downarrow} C \qquad \textit{if and only if} \qquad C \underset{M}{\downarrow} A.$$

(9) (Extension) *Let* M *be a* $(D, \aleph_0)$-*homogeneous model. For every* A, C *there exists* A' *such that*

$$\mathrm{tp}(A, M) = \mathrm{tp}(A', M) \quad \textit{and} \quad A' \underset{M}{\downarrow} C.$$

(10) (Uniqueness) *Let* M *be a* $(D, \aleph_0)$-*homogeneous model. If* A, A' *satisfy*

$$\mathrm{tp}(A, M) = \mathrm{tp}(A', M) \qquad \textit{and both} \qquad A \underset{M}{\downarrow} C \quad \textit{and} \quad A' \underset{M}{\downarrow} C$$

then $\mathrm{tp}(A, MC) = \mathrm{tp}(A', MC)$.

Proof. (1) This is just by Definition 2.19.

(2) Immediate from Definition 2.19.

(3) By Finite Character of the rank and Definition 2.19.

(4) Follows from Lemma 2.2 5.

(5) Immediate by finite definition and finite character of the rank.

(6) Assume $C \underset{M}{\not\downarrow} A$. Then, by Finite Character, there is $\bar{c} \in C$, such that $R[\mathrm{tp}(\bar{c}, M)] < R[\mathrm{tp}(\bar{c}, M)]$. Also by Finite Character , there exists $\bar{a} \in A$ such that $R[\mathrm{tp}(\bar{c}, M \cup \bar{a})] = R[\mathrm{tp}(\bar{c}, M)]$. Hence $\bar{c} \underset{M}{\not\downarrow} \bar{a}$. But, by Corollary 2.10, both $\mathrm{tp}(\bar{a}, M)$ and $\mathrm{tp}(\bar{c}, M)$ are stationary, so by Theorem 2.20 we must have $\bar{a} \underset{M}{\not\downarrow} \bar{c}$. By Finite Character, this shows that $A \underset{M}{\not\downarrow} C$.

(7) Let $\bar{a} \in A$. Then, by Finite Character, $\bar{a} \underset{B}{\downarrow} C$, and $\bar{a} \underset{C}{\downarrow} D$, so by Definition 2.19 $R[\mathrm{tp}(\bar{a}, C)] = R[\mathrm{tp}(\bar{a}, B)]$ and $R[\mathrm{tp}(\bar{a}, D)] = R[\mathrm{tp}(\bar{a}, C)]$. Thus

$R[\mathrm{tp}(\bar{a}, B)] = R[\mathrm{tp}(\bar{a}, D)]$, so $\bar{a} \underset{B}{\downarrow} D$. Hence, by Finite Character, we must have $A \underset{B}{\downarrow} D$. The converse is just by Monotonicity.

(8) Immediate by Theorem 2.20 and Corollary 2.10.

(9) Follows from Corollary 2.10 and Definition 2.19.

(10) Follows from Corollary 2.10 and Definition 2.19.

□

3. REGULAR AND MINIMAL TYPES

In this section, we prove the existence of various pregeometries for totally transcendental diagrams. First, we make the following definition (a similar definition appears in [Sh 4]).

Definition 3.1.   (1) Let $\bar{a}$ be in M and $q(\bar{x}, \bar{a})$ be a type. We say that $q(\bar{x}, \bar{a})$ is *big for M* if $q(\bar{x}, \bar{a})$ is realized outside M;

(2) We say that $q(\bar{x}, \bar{a})$ is *big* if $q(\bar{x}, \bar{a})$ is big for any M containing $\bar{a}$;

(3) A type $q \in S_D(A)$ is *big (for M)* if $q \upharpoonright \bar{a}$ is big (for M) for every $\bar{a} \in A$.

In presence of the compactness theorem, big types are the same as non-algebraic types. Even in the general case, we have a nice characterization of bigness when the types are stationary.

Lemma 3.2. *Let $q \in S_D(A)$ be stationary. The following conditions are equivalent:*

(1) *q is big for some $(D, \aleph_0)$-homogeneous M containing A;*

(2) *$R[q] \geq 1$;*

(3) *q is big.*

Proof. (1) $\Rightarrow$ (2): Since M is $(D, \aleph_0)$-homogeneous, by Lemma 2.3, $R[q] = R_M[q]$, so it is enough to show $R_M[q] \geq 1$. Let $\bar{a} \in A$ be such that $R_M[q] = R_M[q \upharpoonright \bar{a}]$. Since $q \upharpoonright \bar{a}$ is big for M, there exists $\bar{c} \notin M$ realizing $q \upharpoonright \bar{a}$. Also, since M is $(D, \aleph_0)$-homogeneous, there is $\bar{c}' \in M$ realizing $q \upharpoonright \bar{a}$. Hence

$$R_M[(q \upharpoonright \bar{a}) \cup \{\bar{x} = \bar{c}'\}] \geq 0 \quad \text{and} \quad R_M[(q \upharpoonright \bar{a}) \cup \{\bar{x} \neq \bar{c}'\}] \geq 0.$$

Moreover, for every $\bar{b} \in M$, $(q \upharpoonright \bar{a}) \cup \mathrm{tp}(\bar{c}, \bar{b})$ is realized by $\bar{c}$, and so

$$R_M[(q \upharpoonright \bar{a}) \cup \mathrm{tp}(\bar{c}, \bar{b})] \geq 0,$$

and $\mathrm{tp}(\bar{c}, \bar{b}) \in S_D(\bar{b})$. This shows that $R_M[q \upharpoonright \bar{a}] \geq 1$.

(2) $\Rightarrow$ (3): Suppose q is stationary, $R[q] \geq 1$ and M containing $\bar{a}$ are given. By taking a larger M if necessary, we may assume that M is $(D, \aleph_0)$-homogeneous. Since q is stationary, there exists $q_M \in S_D(M)$, such that $R[q_M] =$

$R[q] \geq 1$. Let $\bar{c}$ realize q_M. If $\bar{c} \in M$, then $\{x = \bar{c}\} \in q_M$, so

$$0 = R[\bar{x} = \bar{c}] \geq R[q_M] \geq 1,$$

which is a contradiction. Hence $\bar{c} \notin M$, so q is big for M.

(3) $\Rightarrow$ (1): Clear by definition. $\qquad\square$

Definition 3.3. Let $p \in S_D(A)$ be a big, stationary type.

(1) We say that p is *regular for M* if $A \subseteq M$ and for every $B \subseteq M$ we have

$$\bar{a} \underset{A}{\downarrow} B \text{ and } \bar{b} \underset{A}{\not\downarrow} B \quad \text{imply} \quad \bar{a} \underset{A}{\downarrow} B \cup \bar{b}, \qquad \text{for all } \bar{a}, \bar{b} \in p(M).$$

(2) We say that p is *regular* if p is regular for $\mathfrak{C}$.

Lemma 3.4. *Let $p \in S_D(A)$ be a big, stationary type based on $\bar{c} \in A$. If $p \upharpoonright \bar{c}$ is regular, then p is regular.*

Proof. First notice that stationarity and bigness are preserved (bigness is the content of Lemma 3.2). Suppose p is not regular. We will show that $p \upharpoonright \bar{c}$ is not regular. Let $\bar{a}, \bar{b} \models p$ and B be such that

$$\bar{a} \underset{A}{\downarrow} B, \quad \bar{b} \underset{A}{\not\downarrow} B \quad \text{and yet} \quad \bar{a} \underset{A}{\not\downarrow} B \cup \bar{b}.$$

Therefore $\mathrm{tp}(\bar{a}, A \cup B) = p_{A \cup B}$ and so by choice of $\bar{c}$ we have $\mathrm{tp}(\bar{a}, A \cup B) = (p \upharpoonright c)_{A \cup B}$, i.e. $\bar{a} \underset{\bar{c}}{\downarrow} A \cup B$. Now since $R[p] = R[p \upharpoonright \bar{c}]$,

$$R[\mathrm{tp}(\bar{b}, A \cup B)] < R[\mathrm{tp}(\bar{b}, A)] \quad \text{implies} \quad R[\mathrm{tp}(\bar{b}, A \cup B)] < R[p \upharpoonright \bar{c}],$$

i.e. $\bar{b} \underset{\bar{c}}{\not\downarrow} A \cup B$. We show similarly that $\bar{a} \underset{\bar{c}}{\not\downarrow} A \cup B \cup \bar{b}$, which shows that $p \upharpoonright \bar{c}$ is not regular. $\qquad\square$

Remark 3.5. If $p(\bar{x}, \bar{a})$ is regular and $\bar{a}' \in M$ is such that $\mathrm{tp}(\bar{a}, \emptyset) = \mathrm{tp}(\bar{a}', \emptyset)$, then $p(\bar{x}, \bar{a}')$ is regular.

Definition 3.6. Let $p \in S_D(B)$, $B \subseteq M$ and $W = p(M) - B \neq \emptyset$. Define

$$a \in cl(C) \quad \text{if} \quad a \underset{B}{\not\downarrow} C, \qquad \text{for } a \in W \text{ and } C \subseteq W.$$

Theorem 3.7. *Let M be $(D, \aleph_0)$-homogeneous containing B and $p \in S_D(B)$ be realized in M. If p is regular then (W, cl) is a pregeometry.*

Proof. We need to show that the four axioms of pregeometry hold (notice that $W \neq \emptyset$).

(1) We show that for every $C \subseteq W$, $C \subseteq cl(C)$.
Let $c \in C$, then $\{x = c\} \in \mathrm{tp}(c, A \cup C)$, hence

$$R[\mathrm{tp}(c, B \cup C)] = 0 < R[p],$$

so $c \underset{B}{\not\downarrow} C$ and thus $c \in cl(C)$.

(2) We show that if $c \in cl(C)$, there is $C' \subseteq C$ finite, such that $c \in cl(C')$.

Let $c \in cl(C)$. By Definition 3.6 $c \underset{B}{\not\downarrow} C$ so by Theorem 2.21 5 there exists $C' \subseteq C$ finite, such that $c \underset{B}{\not\downarrow} C'$, hence $c \in cl(C')$.

(3) We show that if $a \in cl(C)$ and $C \subseteq cl(E)$, then $a \in cl(E)$.

Write $C = \{c_i \mid i < \alpha\}$. Then $a \underset{B}{\not\downarrow} \{c_i \mid i < \alpha\}$. Suppose $a \underset{B}{\downarrow} E$. We show by induction on $i < \alpha$ that $a \underset{B}{\downarrow} E \cup \{c_j \mid j < i\}$.

- For $i = 0$ this is the assumption and for i a limit ordinal, this is true by Theorem 2.21 5.
- For the successor case, suppose it is true for i. Then $a \underset{B}{\downarrow} E \cup \{c_l \mid l < i\}$. Since $C \subseteq cl(E)$, we have $c_i \underset{B}{\not\downarrow} E$, so by Theorem 2.21 6 $c_i \underset{B}{\not\downarrow} E \cup \{c_l \mid l < i\}$. Hence, since p is regular, we must have $a \underset{B}{\downarrow} E \cup \{c_l \mid l < i\} \cup c_i$.

Thus $a \underset{B}{\downarrow} E \cup C$, and since $C \subseteq C \cup E$, we must have $a \underset{B}{\downarrow} C$. Hence $a \notin cl(C)$, which contradicts our assumption.

(4) We show that if $c \in cl(Ca) - cl(C)$, then $a \in cl(Cc)$.

Since symmetry has been shown only for stationary types, this statement is not immediate from Theorem 2.20.

Suppose that $c \underset{B}{\not\downarrow} Ca$ and $c \underset{B}{\downarrow} C$. Then $c \underset{C}{\not\downarrow} a$, since

$$R[\mathrm{tp}(c, B \cup Ca)] < R[\mathrm{tp}(c, B)] = R[\mathrm{tp}(c, B \cup C)].$$

Therefore c realizes $p_{B \cup C}$, so $\mathrm{tp}(c, B \cup C)$ is stationary. If $a \underset{B}{\not\downarrow} C$, then by Theorem 2.21 6 we must have $a \underset{B}{\not\downarrow} Cc$, and we are done.

Otherwise, $a \underset{B}{\downarrow} C$. Hence a realizes $p_{B \cup C}$ and so $\mathrm{tp}(a, B \cup C)$ is stationary. Therefore by Theorem 2.20 we must have $a \underset{C}{\not\downarrow} c$, a contradiction. Hence by Theorem 2.21 6, we have $a \underset{B}{\not\downarrow} Cc$, i.e. $a \in cl(Cc)$.

$\square$

We now show the connection between independent sets in the pregeometries, averages and stationarity.

Lemma 3.8. *Let $p(\bar{x}, \bar{c})$ be regular. Suppose I is infinite and independent in $p(\mathfrak{C}, \bar{c})$. Then I is indiscernible and for every B containing $\bar{c}$ we have $p_B = \mathrm{Av}_{\mathrm{D}}(I, B)$.*

Proof. Write $I = \{\bar{a}_i \mid i < \alpha\}$. Then since I is independent, $\bar{a}_{i+1} \models p_{A_i}$, where $A_i = \bar{c} \cup \{\bar{a}_j \mid j < i\}$. Thus I is a Morley sequence based on p, so the result follows from Lemmas 2.15 and 2.17. $\square$

Now we turn to existence. In order to do this, we need a lemma.

Lemma 3.9. *Let M be $(D, \aleph_0)$-homogeneous, and $p(\bar{x}, \bar{c})$ over M be big and stationary. Then $p(\bar{x}, \bar{c})$ is regular if and only if $p(\bar{x}, \bar{c})$ is regular for M.*

Proof. If $p(\bar{x}, \bar{c})$ is regular, then $p(\bar{x}, \bar{c})$ is clearly regular for M. Suppose $p(\bar{x}, \bar{c})$ is not regular. Then there are $B \subseteq \mathfrak{C}$, and $\bar{a}, \bar{b}$ realizing $p(\bar{x}, \bar{c})$, such that

$$\bar{a} \underset{\bar{c}}{\downarrow} B, \quad \bar{b} \underset{\bar{c}}{\not\downarrow} B, \quad \text{and} \quad \bar{a} \underset{\bar{c}}{\not\downarrow} B\bar{b}.$$

First, we may assume that B is finite: choose $B' \subseteq B$ such that

$$R[\mathrm{tp}(\bar{a}, B' \cup \bar{c}\bar{b})] = R[\mathrm{tp}(\bar{a}, B \cup \bar{c}\bar{b})]$$

and then choose $B'' \subseteq B$ finite, such that $\bar{b} \not\models p_B \restriction B''$. Hence, for $B_0 = B' \cup B'' \subseteq B$, we have

$$\bar{a} \underset{\bar{c}}{\downarrow} B_0, \quad \bar{b} \underset{\bar{c}}{\not\downarrow} B_0, \quad \text{and} \quad \bar{a} \underset{\bar{c}}{\not\downarrow} B_0\bar{b}.$$

Now, since M is $(D, \aleph_0)$-homogeneous and $\bar{c} \in M$, we can find $B_1, \bar{a}_1$ and $\bar{b}_1$ inside M such that $\mathrm{tp}(B_0\bar{a}\bar{b}, \bar{c}) = \mathrm{tp}(B_1\bar{a}_1\bar{b}_1, \bar{c})$. Therefore, by invariance we have:

$$\bar{a} \underset{\bar{c}}{\downarrow} B_1, \quad \bar{b} \underset{\bar{c}}{\not\downarrow} B_1, \quad \text{and} \quad \bar{a} \underset{\bar{c}}{\not\downarrow} B_1\bar{b}.$$

This shows that p is not regular for M. $\square$

The following argument for the existence of regular types is similar to Claim V.3.5. of [Sh a]. However, since our basic definitions are different, we provide a proof.

Theorem 3.10 (Existence of regular types). *Let $M \subseteq N$ be $(D, \aleph_0)$-homogeneous. If $M \neq N$, then there exists $p(x, \bar{a})$ regular, realized in $N - M$. In fact, if $p(x, \bar{a})$ is big and stationary, and has minimal rank among all big, stationary types over M realized in $N - M$, then $p(x, \bar{a})$ is regular.*

Proof. The first statement follows from the second. To prove the second statement, we first choose $c' \in N - M$, be such that $\mathrm{tp}(c', M)$ has minimal rank among all types over M realized in $N - M$, say $R[\mathrm{tp}(c', M)] = \alpha$. We then choose $\bar{a} \in M$ such that $R[\mathrm{tp}(c', M)] = R[\mathrm{tp}(c', \bar{a})] = \alpha$. Write $\mathrm{tp}(c', \bar{a}) = p(x, \bar{a})$ and notice that p is stationary and big for M, hence big, by Lemma 3.2.

By the previous lemma, to show that $p(x, \bar{a})$ is regular, it is equivalent to show that $p(x, \bar{a})$ is regular for M. For this, let $a, b \in p(M)$ and $B \subseteq M$ such that

$$a \underset{\bar{a}}{\downarrow} B \quad \text{and} \quad b \underset{\bar{a}}{\not\downarrow} B.$$

We must show that $a \underset{\bar{a}}{\downarrow} Bb$. Suppose, by way of contradiction that this is not the case. Then, by definition, we have $R[\mathrm{tp}(a, B\bar{a}b)] < \alpha$. We now choose $\bar{c}, \bar{d} \in B$ such that

$$R[\mathrm{tp}(a, B\bar{a}b)] = R[\mathrm{tp}(a, \bar{c}\bar{a}b)] < \alpha \quad \text{and} \quad R[\mathrm{tp}(b, B\bar{a})] = R[\mathrm{tp}(b, \bar{d}\bar{a})] < \alpha.$$

Since N is $(D, \aleph_0)$-homogeneous and $c', a, b, \bar{a}, \bar{c}, \bar{d} \in N$, there is $b' \in N$ such that $\mathrm{tp}(ab, \bar{a}\bar{c}\bar{d}) = \mathrm{tp}(a'b', \bar{a}\bar{c}\bar{d})$. Now, $\mathrm{tp}(b', \bar{a}\bar{d}) = \mathrm{tp}(b', \bar{a}\bar{d})$, so

$$R[\mathrm{tp}(b', M)] \leq R[\mathrm{tp}(b', \bar{a}\bar{d})] = R[\mathrm{tp}(b, \bar{a}\bar{d})] < \alpha.$$

By minimality of α, we must have $b' \in M$. This implies that $R[\mathrm{tp}(a', M)] \leq R[\mathrm{tp}(a', \bar{c}\bar{a}b')]$, so $R[\mathrm{tp}(a', \bar{c}\bar{a}b')] = \alpha$. Now there is $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(a') = a$, $f(b') = b$ and $f \upharpoonright \bar{c}\bar{a} = id_{\bar{c}\bar{a}}$, by choice of b'. Hence, by property of the rank

$$\alpha = R[\mathrm{tp}(a', \bar{c}\bar{a}b')] = R[f(\mathrm{tp}(a', \bar{c}\bar{a}b'))] = R[\mathrm{tp}(a, \bar{c}\bar{a}b)] < \alpha,$$

which is a contradiction. Hence $a \underset{\bar{a}}{\downarrow} Bb$, so that $p(x, \bar{a})$ is regular. $\qquad\square$

By observing what happens when $N = \mathfrak{C}$ in above theorem, one discovers more concrete regular types. For this, we make the following definition. A similar definition in the context of $L_{\omega_1\omega}(Q)$ appears in the last section of [Sh 4]. An illustration of why this definition is natural can be found in the proof of Lemma 4.20. In presence of the compactness theorem, S-minimal is the same as strongly minimal.

Definition 3.11.   (1) A big, stationary type $q(\bar{x}, \bar{a})$ over M is said to be *S-minimal for M* if for any $\theta(\bar{x}, \bar{b})$ over M not both $q(\bar{x}, \bar{a}) \cup \theta(\bar{x}, \bar{b})$ and $q(\bar{x}, \bar{a}) \cup \neg\theta(\bar{x}, \bar{b})$ are big for M.

(2) A big, stationary type $q(\bar{x}, \bar{a})$ is said to be *S-minimal* if $q(\bar{x}, \bar{a})$ is S-minimal for for every M containing $\bar{a}$.

(3) If $q \in S_D(A)$ is big and stationary, we say that q is *S-minimal* if $q \upharpoonright \bar{a}$ is S-minimal for some $\bar{a}$.

Remark 3.12.   (1) Let $q(\bar{x}, \bar{c})$ be S-minimal for the $(D, \aleph_0)$-homogeneous model M. Let $W = q(M, \bar{c})$ and for $a \in W$ and $B \subseteq W$ define

$$a \in cl(B) \quad \text{if} \quad \mathrm{tp}(a, B \cup \bar{c}) \text{ is not big (for } M).$$

Then it can be shown directly from the assumption that D is totally transcendental, that (W, cl) is a pregeometry.

(2) If M is $(D, \aleph_0)$-homogeneous and $q(x, \bar{c})$ has minimal rank among all big, stationary $q(x, \bar{c})$ over M, then the previous theorem shows that q is regular. But q is also S-minimal for M. As a matter of fact, if $a \underset{\bar{c}}{\downarrow} B$, then

$R[\text{tp}(a, B \cup \bar{c})] = R[q(\bar{x}, \bar{c})] \geq 1$ and $\text{tp}(a, B \cup \bar{c})$ is stationary, so $\text{tp}(a, b \cup \bar{c})$ is big, so $a \notin cl(B)$. Conversely, if $a \underset{\bar{c}}{\not\downarrow} B$, then $R[\text{tp}(a, B\bar{c})] < R[q(x, \bar{c})]$.

But if $\text{tp}(a, B \cup \bar{c})$ was big, then we could find $a' \notin M$ such that $\text{tp}(a', B \cup \bar{c}) = \text{tp}(a, B \cup \bar{c})$, so

$$R[\text{tp}(a', M)] \leq R[\text{tp}(a', B \cup \bar{c})] = R[\text{tp}(a, B \cup \bar{c})] < R[q(x, \bar{c})],$$

contradicting the minimality of $R[q(x, \bar{c})]$. Hence $\text{tp}(a, B \cup \bar{c})$ is not big, and so $a \in cl(B)$. In other words, both pregeometries coincide.

(3) Using the results that we have proven so far, it is not difficult to show that if M, N are $(D, \aleph_0)$-homogeneous, and $q(x, \bar{c})$ has minimal rank among all big, stationary types over M and $\bar{c}' \in N$ such that $\text{tp}(\bar{c}, \emptyset) = \text{tp}(\bar{c}', \emptyset)$, then $q(x, \bar{c}')$ has minimal rank among all big, stationary types over N, hence if $q(x, \bar{c}')$ is S-minimal for N.

In the light of these remarks, we will make the following definition.

Definition 3.13. Let M be $(D, \aleph_0)$-homogeneous. A big, stationary type $q(\bar{x}, \bar{c})$ with $\bar{c} \in M$ is called *minimal* if $q(\bar{x}, \bar{c})$ has minimal rank among all big, stationary types over M.

We close this section by summarizing above remark in the following theorem.

Theorem 3.14. (1) *For any $(D, \aleph_0)$-homogeneous model, there exists a minimal $q(x, \bar{c})$ with $\bar{c} \in M$.*

(2) *Minimal types are regular and moreover for every A containing $\bar{c}$, every set B and $a \models q_A$ we have*

$$\text{tp}(a, A \cup B) \quad \text{is big if and only if} \quad a \underset{A}{\downarrow} B.$$

Proof. The first item is clear by definition. The second follows by Theorem 3.10, and Remark 3.12 2 and 3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. APPLICATIONS

In this section, we give a few applications of our concepts. The rank is especially useful to study the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental D. In the first subsection, we start with the existence of prime models.

4.1. Prime models. We give definitions from [Sh 1] in more modern terminology.

Definition 4.1. (1) We say that $p \in S_D(A)$ is D^s_λ-isolated over $B \subseteq A$, $|B| < \lambda$, if for any $q \in S_D(A)$ extending $p \restriction B$, we have $q = p$.

(2) We say that $p \in S_D(A)$ is D_λ^s-*isolated* if there is $B \subseteq A$, $|B| < \lambda$, such that p is D_λ^s-isolated over B.

The following are verifications of Axioms X.1 and XI.1 from Chapter IV of [Sh a].

Theorem 4.2 (X.1). *Let $A \subseteq \mathfrak{C}$ and $\mu \geq \aleph_0$. Every $\phi(\bar{x}, \bar{a})$ over A realized in $\mathfrak{C}$ can be extended to a D_μ^s-isolated type $p \in S_D(A)$.*

Proof. It is enough to show the result for $\mu = \aleph_0$.

Since $\mathfrak{C} \models \exists \bar{x} \phi[\bar{x}, \bar{a}]$, there exists $\bar{c} \in \mathfrak{C}$ such that $\mathfrak{C} \models \phi[\bar{c}, \bar{a}]$. Thus there exists is $p \in S_D(A)$, namely $\mathrm{tp}(\bar{c}, A)$, containing $\phi(\bar{x}, \bar{a})$. Since D is totally transcendental and $A \subseteq \mathfrak{C}$ we must have $R_A[p] < \infty$. Among all those $p \in S_D(A)$ containing $\phi(\bar{x}, \bar{a})$ choose one with minimal rank. Say $R_A[p] = \alpha \geq 0$.

We claim that p is $D_{\aleph_0}^s$-isolated. First, there is $\bar{b} \in A$ such that $R_A[p] = R_A[p \upharpoonright \bar{b}]$. We may assume that $p \upharpoonright \bar{b}$ contains $\phi(\bar{x}, \bar{a})$ by Lemma 2.2 6. Suppose that there is $q \in S_D(A)$, $q \neq p$, such that q extends $p \upharpoonright \bar{b}$. Then $R_A[q] \geq \alpha$ by choice of p (since q contains $\phi(\bar{x}, \bar{a})$). Now, choose $\psi(\bar{x}, \bar{c})$ with $\bar{c} \in A$ such that $\psi(\bar{x}, \bar{c}) \in p$ and $\neg \psi(\bar{x}, \bar{c}) \in q$. Then since $(p \upharpoonright \bar{b}) \cup \psi(\bar{x}, \bar{c}) \subseteq p$, by Lemma 2.2 6 we have

$$R_A[(p \upharpoonright \bar{b}) \cup \psi(\bar{x}, \bar{c})] \geq R_A[p] \geq \alpha.$$

Similarly

$$R_A[(p \upharpoonright \bar{b}) \cup \neg \psi(\bar{x}, \bar{c})] \geq R_A[q] \geq \alpha.$$

Now, given any $\bar{d} \in A$, $R_A[p \upharpoonright \bar{b} \cup \bar{d}] \geq \alpha$ (again by Lemma 2.2 6). Since $p \in S_D(A)$, necessarily if we write $p \upharpoonright \bar{d} = p(\bar{x}, \bar{d})$, then we have $p(\bar{x}, \bar{y}) \in D$ (since $p(\bar{x}, \bar{d}) \in S_D(\bar{d})$). Hence since $p \upharpoonright \bar{b} \cup \bar{d} \vdash p \upharpoonright \bar{b} \cup p(\bar{x}, \bar{d})$ we have

$$R_A[(p \upharpoonright \bar{b}) \cup p(\bar{x}, \bar{d})] \geq R_A[p \upharpoonright \bar{b} \cup \bar{d}] \geq \alpha.$$

But this shows that $R_A[p \upharpoonright \bar{b}] \geq \alpha + 1$, a contradiction.

Hence p is the only extension of $p \upharpoonright b$, so p is $D_{\aleph_0}^s$-isolated. $\square$

Theorem 4.3 (XI.1). *Let μ be infinite and $B \subseteq A$. Every D_μ^s-isolated $r \in S_D(B)$ can be extended to a D_μ^s-isolated type $p \in S_D(A)$.*

Proof. Since $\mathfrak{C}$ is (D, χ)-homogeneous, there exists $\bar{c} \in \mathfrak{C}$ realizing r. Hence there is $p \in S_D(A)$ extending r, namely $\mathrm{tp}(\bar{c}, A)$. Since D is totally transcendental and $A \subseteq \mathfrak{C}$ we must have $R_A[p] < \infty$. Among all those $p \in S_D(A)$ extending r choose one with minimal rank. Say $R_A[p] = \alpha \geq 0$.

We claim that p is D_μ^s-isolated. First, there is $\bar{b} \in A$ such that $R_A[p] = R_A[p \upharpoonright \bar{b}]$. Also, since r is D_μ^s-isolated, there is $C \subseteq B$, $|C| < \mu$ such that $r \upharpoonright C$ isolates r. We may assume that $R_A[r] = R_A[r \upharpoonright C]$, by Lemma 2.2 7. We claim

that $(r \upharpoonright C) \cup (p \upharpoonright \bar{b})$ isolates p. By contradiction, suppose that there is $q \in S_D(A)$ extending $(r \upharpoonright C) \cup (p \upharpoonright \bar{b})$ such that $q \neq p$. Notice that $r \subseteq q$, since r was isolated by $r \upharpoonright C$, and hence $R_A[q] \geq R_A[p] = \alpha$ by choice of p. Now, choose $\psi(\bar{x}, \bar{a})$ with $\bar{a} \in A$ such that $\psi(\bar{x}, \bar{a}) \in p$ and $\neg\psi(\bar{x}, \bar{a}) \in q$. By Lemma 2.2 6 (since $(p \upharpoonright \bar{b}) \cup \psi(\bar{x}, \bar{c}) \subseteq p$), we must have

$$R_A[(p \upharpoonright \bar{b}) \cup \psi(\bar{x}, \bar{c})] \geq R_A[p] = \alpha.$$

Similarly

$$R_A[(p \upharpoonright \bar{b}) \cup \neg\psi(\bar{x}, \bar{c})] \geq R_A[q] \geq \alpha.$$

Now, given any $\bar{d} \in A$ we have that $R_A[p \upharpoonright \bar{b} \cup \bar{d}] \geq \alpha$ (again by Lemma 2.2 6). Since $p \in S_D(A)$, necessarily if we write $p \upharpoonright \bar{d} = p(\bar{x}, \bar{d})$, then we have $p(\bar{x}, \bar{y}) \in D$ (since $p(\bar{x}, \bar{d}) \in S_D(\bar{d})$). Hence

$$R_A[(p \upharpoonright b) \cup p(\bar{x}, \bar{d})] \geq R_A[p \upharpoonright \bar{b} \cup \bar{d}] \geq \alpha,$$

since $p \upharpoonright \bar{b} \cup \bar{d} \vdash (p \upharpoonright b) \cup p(\bar{x}, \bar{d})$. But this shows that $R_A[p \upharpoonright \bar{b}] \geq \alpha + 1$, a contradiction.

Hence p is the only extension of $(r \upharpoonright C) \cup (p \upharpoonright b)$, so p is D_μ^s-isolated. $\square$

Following Chapter IV of [Sh a], we set:

Definition 4.4. (1) We say that $C = \{\langle a_i, A_i, B_i \rangle \mid i < \alpha\}$ is a (D, λ)-*construction of C over A* if

(a) $C = A \cup \bigcup\{a_i \mid i < \alpha\}$;

(b) $B_i \subseteq A_i$, $|B_i| < \lambda$, where $A_i = A \cup \bigcup\{a_j \mid j < i\}$;

(c) $\text{tp}(a_i, A_i) \in S_D(A_i)$ is D_λ^s-isolated over B_i.

(2) We say that M is D_λ^s-*constructible over A* if there is a (D, λ)-construction for M over A.

(3) We say that M is D_λ^s-*primary over A*, if M is D_λ^s-constructible over A and M is (D, λ)-homogeneous.

(4) We say that M is D_λ^s-*prime over A* if

(a) M is (D, λ)-homogeneous and

(b) if N is (D, λ)-homogeneous and $A \subseteq N$, then there is $f : N \to M$ elementary such that $f \upharpoonright A = id_A$.

(5) We say that M is D_λ^s-*minimal over A*, if M is D_λ^s-prime over A and for every (D, λ)-homogeneous model N, if $A \subseteq N \subseteq M$, then $M = N$.

Remark 4.5. We use the same notation as in [Sh a], except that we replace $\mathbb{F}$ by D to make it explicit that we deal exclusively with D-types (or equivalently, types realized in $\mathfrak{C}$). In particular, for example if M is $D_{\aleph_0}^s$-primary over A, then M is $D_{\aleph_0}^s$-prime over A.

Theorem 4.6 (Existence of prime models). *Let D be totally transcendental. Then for all $A \subseteq \mathfrak{C}$ and infinite μ there is a D_μ^s-primary model M over A of cardinality $|A| + |T| + |D| + \mu$. Moreover, M is D_μ^s-prime over A.*

Proof. See page 175 of [Sh a] and notice that we just established $X.1$ and $XI.1$. Observe that in the construction, each new element realizes a D-type, so that the resulting model is indeed a D-model. The optimal bound on the cardinality follows from Theorem 2.13. The second sentence follows automatically.   □

Remark 4.7. A similar theorem, with a stronger assumption (D is $\aleph_0$-stable) and without the bound on the cardinality appears in [Sh 1]. Note that D_μ^s-primary, is called $(D, \mu, 1)$-prime there.

Notice that this allows us to show how any type can be decomposed into stationary and isolated types. A similar result appears in [Sh 87a].

Lemma 4.8. *Let* $p \in S_D(A)$ *and suppose* $\bar{a}$ *realizes* p. *Then there is* $\bar{b} \in \mathfrak{C}$ *such that*

    (1) $\text{tp}(\bar{b}, A)$ *is* $D_{\aleph_0}^s$*-isolated;*

    (2) $\text{tp}(\bar{a}, A\bar{b})$ *is stationary;*

    (3) $R[\text{tp}(\bar{a}, A\bar{b})] = R[\text{tp}(\bar{a}, \bar{b})]$.

Furthermore, p does not split over a finite set.

Proof. Let $\bar{a} \models p$. Let M be $D_{\aleph_0}^s$-primary model over A. Then $\text{tp}(\bar{a}, M)$ is stationary since M is $(D, \aleph_0)$-homogeneous, and there is $\bar{b} \in M$ finite, such that $R[\text{tp}(\bar{a}, M)] = R[\text{tp}(\bar{a}, \bar{b})]$. Hence $R[\text{tp}(\bar{a}, A\bar{b})] = R[\text{tp}(\bar{a}, \bar{b})]$ by Lemma 2.2 6, and so $\text{tp}(\bar{a}, A\bar{b})$ is stationary. Also, $\text{tp}(\bar{b}, A)$ is $D_{\aleph_0}^s$-isolated, since M is $D_{\aleph_0}^s$-primary over A.

Finally, to see that p does not split over a finite set, assume $\bar{a} \models p$, $\text{tp}(\bar{b}, A)$ is $D_{\aleph_0}^s$-isolated, $\text{tp}(\bar{a}, A\bar{b})$ is stationary, and $R[\text{tp}(\bar{a}, A\bar{b})] = R[\text{tp}(\bar{a}, \bar{b})]$. Then there is $C \subseteq A$ finite, such that $\text{tp}(\bar{b}, A)$ is $D_{\aleph_0}^s$-isolated over C. Also, since $\text{tp}(\bar{a}, A\bar{b})$ is stationary, it does not split over $\bar{b}$. Now it is easy to see that p does not split over C: otherwise there are $\bar{c}_l \in A$, and $\phi(\bar{x}, \bar{y})$ such that $\text{tp}(\bar{c}_1, C) = \text{tp}(\bar{c}_2, C)$, $\bar{c}_l \in A$ for $l = 1, 2$, and $\models \phi[\bar{a}, \bar{c}_1]$ and $\models \neg\phi[\bar{a}, \bar{c}_2]$. But $\text{tp}(\bar{b}, A)$ does not split over C, and so $\text{tp}(\bar{c}_1, \bar{b}) = \text{tp}(\bar{c}_2, \bar{b})$. However, this contradicts the fact that $\text{tp}(\bar{a}, A\bar{b})$ does not split over $\bar{b}$. All the conditions are satisfied.   □

This gives us an alternative and short proof that averages are well-defined, and in fact, allows us to give short proofs of all the facts in Lemma 1.15.

Lemma 4.9. *Let I be infinite and $A \subseteq \mathfrak{C}$. Then* $\text{Av}_D(I, A) \in S_D(A)$

Proof. Completeness is clear. To see that $\text{Av}_D(I, A)$ is consistent, suppose that both $\phi(x, \bar{a})$ and $\neg\phi(x, \bar{a})$ are realized by infinitely many elements of I. But $\text{tp}(\bar{a}, I)$ does not split over a finite set $B \subseteq I$ by the previous lemma. Hence, by choice of $\phi(x, \bar{a})$, we can find $b, c \in I - B$ such that $\models \phi[b, \bar{a}]$ and $\models \neg\phi[c, \bar{a}]$. This

however, shows that $\mathrm{tp}(\bar{a}, I)$ splits over B, since $\mathrm{tp}(b, B) = \mathrm{tp}(c, B)$ by indiscernibility of I and both $\phi(b, \bar{y})$, $\neg\phi(c, \bar{y}) \in \mathrm{tp}(\bar{a}, I)$. Now $\mathrm{Av_D}(I, A) \in S_D(A)$ since we can extend I to a D-set of indiscernible J of cardinality $|A|^+$, and then some element of J realizes $\mathrm{Av_D}(I, A)$. $\square$

The following is a particular case of Theorem 1.11. We include it here not just for completeness, but because the proof is different from the proof of 1.11 and very similar in the conceptual framework to the first-order case.

Theorem 4.10. *Let D be totally transcendental. If $\langle M_i \mid i < \alpha \rangle$ is an increasing chain of (D, μ)-homogeneous models, then $\bigcup_{i<\alpha} M_i$ is (D, μ)-homogeneous (μ infinite).*

Proof. Let $M = \bigcup_{i<\alpha} M_i$ and notice that M is $(D, \aleph_0)$-homogeneous. Let $p \in S_D(A)$, $A \subseteq M$, $|A| < \mu$ and choose $q \in S_D(M)$ extending p. Then, by Corollary 2.10, q is stationary and there is $B \subseteq M$, finite such that q is based on B. Let $i < \alpha$, be such that $B \subseteq M_i$. Since M_i is (D, μ)-homogeneous, there is $I = \{a_j \mid j < \mu\} \subseteq M_i$ a Morley sequence for q_B. Then, by Lemma 2.17, $q_{AB} = \mathrm{Av_D}(I, A \cup B)$. But $|I| > |A \cup B|$, so by Lemma 1.15 there is $a_j \in I$ realizing $\mathrm{Av_D}(I, A \cup B)$. But $q_{AB} \supseteq p$, so p is realized in M. This shows that M is (D, μ)-homogeneous. $\square$

4.2. Categoricity. We now focus on the structure of $(D, \aleph_0)$-homogeneous models. Notice that when D is the set of isolated types over the empty set or when D comes from a Scott sentence of $L_{\omega_1\omega}$, this class coincides with the class of D-models. When $D = D(T)$, then $\mathcal{K}$ is the class of $\aleph_0$-saturated models (of a totally transcendental theory, in our case).

Definition 4.11. Define

$$\mathcal{K} = \{ M \mid M \text{ is } (D, \aleph_0) - \text{homogeneous} \}.$$

Remark 4.12. We will say that $M \in \mathcal{K}$ is *prime over A* or *minimal over A*, when M is $D^s_{\aleph_0}$-prime over A or $D^s_{\aleph_0}$-minimal over A respectively.

By analogy with the first-order case, we set the following definition.

Definition 4.13. Let D be totally transcendental. We say that D is *unidimensional* if for every pair of models $M \subseteq N$ in $\mathcal{K}$ and minimal type $q(x, \bar{a})$ minimal over M,

$$q(M, \bar{a}) = q(N, \bar{a}) \quad \text{implies} \quad M = N.$$

Unidimensionality for a totally transcendental diagram D turns out to be a weak dividing line. When it fails, we can construct non-isomorphic models, like in the next theorem (this justifies the name), and when it holds we get a strong structural theorem (see Theorem 4.19, which implies categoricity). In fact, the conclusion of our next theorem is similar to (but stronger than) the conclusion of Theorem 6.9 of [Sh 1] (we prove it for every μ, not just regular μ, and can obtain

these models of cardinality exactly λ, not arbitrarily large). The assumptions of Theorem 6.9 of [Sh 1] are weaker and the proof considerably longer. Actually, Corollary 4.25 makes the connection with Theorem 6.9 of [Sh 1] clearer.

We first prove two technical lemmas which are similar to Lemma 3.4 and fact 3.2.1 from [GrHa] respectively. The proofs are straightforward generalizations and are presented here for the sake of completeness.

Lemma 4.14. *Let* $p, q \in S_D(M)$ *and* $M \subseteq N$ *be in* $\mathcal{K}$. *If* $a \underset{M}{\downarrow} b$ *for every* $a \models q$ *and* $b \models p$, *then* $a \underset{N}{\downarrow} b$ *for every* $a \models q_N$ *and* $b \models p_N$.

Proof. Suppose not. Then there are $a \models p_N$ and $b \models q_N$ such that $a \underset{N}{\not\downarrow} b$. Choose $E \subseteq N$ finite such that $a \underset{ME}{\not\downarrow} b$ and $\mathrm{tp}(ab, N)$ is based on E. This is possible by Theorem 2.21 5 and by the fact that $\mathrm{tp}(ab, N)$ is stationary. Similarly, we can find $C \subseteq M$ finite, such that p_M and q_M are based on C and $a \underset{CE}{\not\downarrow} b$. Since $C \subseteq M$ finite and $M \in \mathcal{K}$, there exists $a^*, b^*, E^* \subseteq M$, such that $\mathrm{tp}(abE, C) = \mathrm{tp}(a^*b^*E^*, C)$, and so $a^* \underset{CE^*}{\not\downarrow} b^*$. Since $\mathrm{tp}(ab, N)$ is based on E, then $\mathrm{tp}(ab, CE)$ is stationary based on E, so $\mathrm{tp}(a^*b^*, CE^*)$ is stationary based on E^*. Therefore, we can choose $a'b' \models \mathrm{tp}(a^*b^*, CE^*)_M$, and by choice of C, necessarily $a' \models p_M$ and $b' \models q_M$.

Hence, by assumption on p_M, q_M, we have $a' \underset{M}{\downarrow} b'$, so also $a' \underset{CE^*}{\downarrow} b'$. But this implies $a^* \underset{CE^*}{\downarrow} b^*$, by choice of $a'b'$, a contradiction. $\square$

Lemma 4.15. *Let* N *be* (D, μ)-*homogeneous. If* $a \underset{N}{\downarrow} b$ *and* $\mathrm{tp}(a, Nb)$ *is* D^s_μ-*isolated, then* $a \in N$.

Proof. Since $p = \mathrm{tp}(a, Nb)$ is D^s_μ-isolated, there is $C \subseteq N$, $|C| < \mu$ such that $\mathrm{tp}(a, Cb)$ isolates p. Since $\mathrm{tp}(b, N)$ is stationary, we may assume that $\mathrm{tp}(b, N)$ does not split over C. Since, by Theorem 2.21 8 also $b \underset{N}{\downarrow} a$, so we may assume that $\mathrm{tp}(b, Na)$ does not split over C.

Now, since N is (D, μ)-homogeneous, there is $a' \in N$, such that $\mathrm{tp}(a, C) = \mathrm{tp}(a', C)$. But since $\mathrm{tp}(b, Na)$ does not split over C, then $\mathrm{tp}(ab, C) = \mathrm{tp}(a'b, C)$. Hence $\mathrm{tp}(a, N) = \mathrm{tp}(a', N)$, so that $a \in N$. $\square$

We recall a definition from [Sh 1].

Definition 4.16. A D-model M is *maximally* (D, μ)-*homogeneous* if M is (D, μ)-homogeneous, but not (D, μ^+)-homogeneous.

Theorem 4.17. *Suppose D is not unidimensional. Then there is a maximally* (D, μ)-*homogeneous model M of cardinality λ, for every $\lambda \geq \mu \geq |T| + |D|$.*

Proof. Suppose D is totally transcendental and not unidimensional. Then there exists M, N in $\mathcal{K}$ and a minimal type $q(x, \bar{a})$ over M with the property that

(*) $\qquad q(M, \bar{a}) = q(N, \bar{a})$ and $M \subseteq N$, $M \neq N$.

Using the Downward Löwenheim Skolem Theorem and prime models, we may assume that $|q(M, \bar{a})| \leq |T| + |D|$. Let $\lambda \geq \mu \geq |T| + |D|$ be given. We first show that we can find $M, N \in \mathcal{K}$ satisfying (*) such that in addition $\|M\| = |q(M, \bar{a})| = \mu$.

Since $M \neq N \in \mathcal{K}$, there is $b \in N - M$, so $p = \mathrm{tp}(b, M) \in S_D(M)$ is big and stationary. This implies that $a' \underset{M}{\downarrow} b'$ for any $a' \models q_M$ and $b' \models p$ (by an automorphism sending b' to b, it is enough to see $a' \underset{M}{\downarrow} b$, but this is obvious, otherwise $\mathrm{tp}(a', Mb)$ is not big, thus cannot be big for N by Lemma 3.2, hence it has to be realized in $N - M$, which implies that $a' \in N - M$, contradicting $q(M, \bar{a}) = q(N, \bar{a})$).

Construct $\langle M_i \mid i \leq \mu \rangle$ increasing and $I = \{a_i \mid i < \mu\}$, $a_i \notin M_i$ realizing q_{M_i}, such that:

(1) $M_{i+1} \in \mathcal{K}$ is $D^s_{\aleph_0}$-primary over $M_i \cup a_i$;
(2) $M_0 = M$;
(3) $M_i = \bigcup_{j < i} M_j$ when j is a limit ordinal;
(4) If b' realizes p_{M_i}, and N^* is $D^s_{\aleph_0}$-primary over $M_i \cup b'$, then $q(M_i, \bar{a}) = q(N^*, \bar{a})$.

This is enough: Consider N $D^s_{\aleph_0}$-primary over $M_\mu \cup b'$, where $b' \models p_{M_\mu}$. Then $b' \in N - M_\mu$ and yet $q(M_\mu, \bar{a}) = q(N, \bar{a})$, so (*) holds. Furthermore, $\|M_\mu\| = |q(M_\mu, \bar{a})| = \mu$.

This is possible:

- For $i = 0$, this follows from the definition of q (send b' to b by an automorphism, fixing M, to obtain a realization of q_M in $N - M$).
- If i is a limit ordinal, and $b' \models p_{M_i}$, then this implies that $b' \models p_{M_j}$, for any $j < i$. Also, if N^* is prime over $M_i \cup b'$, and $c \in N^* - M_i$ realizes $q(x, \bar{a})$, then $\mathrm{tp}(c, M_i b')$ is $D^s_{\aleph_0}$-isolated over some $\bar{m}b$, and $\bar{m}b \in M_j$ for some $j < i$, hence $c \in M_j$ by induction hypothesis, a contradiction.
- For $i = j + 1$. Let $b' \models p_{M_j}$ and N^* be prime over $M_j \cup b'$. Suppose that $c \in N^* - M_j$ realizes $q(x, \bar{a})$. Then, since $c \notin M_j$, we must have $\mathrm{tp}(c, M_j)$ is big, so $c \models q_{M_j}$. Hence, by Lemma 4.14 we have $c \underset{M_j}{\downarrow} b'$. But

$\text{tp}(c, M_j b')$ is $D^s_{\aleph_0}$-isolated, so by Lemma 4.15, we must have $c \in M_j$, a contradiction. Hence $q(M_i) = q(N^*)$ and we are done.

Let $M^* = M_\mu$, and fix $b \models p_{M^*}$. We now show that we can find a (D, μ)-homogeneous model $N \in \mathcal{K}$ of cardinality λ such that M^* and N satisfy (*). This implies the conclusion of the theorem: N is (D, μ)-homogeneous of cardinality λ; N is *not* (D, μ^+)-homogeneous, since N omits $q_{M^*} \in S_D(M^*)$, and $\|M^*\| = \mu$.

We construct $\langle N_i \mid i \leq \lambda \rangle$ increasing, and $b_i \notin N_i$ realizing p_{N_i} such that:

(1) $b_0 = b$ and N_0 is D^s_μ-primary over $M^* \cup b$;
(2) N_{i+1} is D^s_μ-primary over $N_i \cup b_i$;
(3) $N_i = \bigcup_{j<i} N_i$, when i is a limit ordinal;
(4) $\|N_i\| \leq \lambda$;
(5) N_i is (D, μ)-homogeneous;
(6) $q(N_i, \bar{a}) = q(M^*, \bar{a})$.

This is clearly enough: N_λ is as required.

This is possible: We construct N_i by induction on $i \leq \lambda$.

- For $i = 0$, let $N^* \subseteq N_0$ be $D^s_{\aleph_0}$-primary over $M^* \cup b$. We have $q(N^*, \bar{a}) = q(M^*, \bar{a})$ by construction of M^*, so it is enough to show that $q(N^*, \bar{a}) = q(N_0, \bar{a})$. Suppose not and let $c \in N_0 - N^*$ realize $q(x, \bar{a})$. Then, c realizes q_{N^*} since $\text{tp}(c, N^*)$ is big, and further there is $A \subseteq M^*$, $|A| < \mu$ such that $\text{tp}(c, Ab)$ isolates $\text{tp}(c, M^*b)$. By Lemma 2.17 since I is based on q, we have $\text{Av}_D(I, N^*) = q_{N^*}$, where $I = \{a_i \mid i < \mu\} \subseteq M^*$ defined above. But since both $\text{tp}(c, Ab)$ and $\text{tp}(c, M^*)$ are big, we must have $\text{tp}(c, Ab) = \text{Av}_D(I, Ab)$ and $\text{tp}(c, M^*) = \text{Av}_D(I, M^*)$. Hence $\text{Av}_D(I, Ab) \vdash \text{Av}_D(I, M^*)$. Now, by Lemma 1.15, we can find $I' \subseteq I$, $|I'| < \mu$ such that $I - I'$ is indiscernible over Ab. Since $|I| = \mu$, then $I - I' \neq \emptyset$ and all elements of $I - I'$ realize $\text{Av}_D(I, Ab)$, hence also $\text{Av}_D(I, M^*) = q_{M^*}$. But this is impossible since $I \subseteq M^*$. Therefore $q(N_0, \bar{a}) = q(N^*, \bar{a}) = q(M^*, \bar{a})$.
- For i a limit ordinal, the only condition to check is that N_i is (D, μ)-homogeneous, but this follows from Theorem 4.10.
- For $i = j + 1$, by induction hypothesis, we have $q(N_j, \bar{a}) = q(M^*, \bar{a})$, so it is enough to show that $q(N_{j+1}, \bar{a}) = q(N_j, \bar{a})$. Suppose $c \in N_{j+1}$ realizes q. Since N_{j+1} is D^s_μ-primary over $N_j \cup b_j$, we have $\text{tp}(c, N_j \cup b_j)$ is D^s_μ-isolated. But $c \underset{N_j}{\downarrow} b_j$, by Lemma 4.14. Therefore, by Lemma 4.15, we have that $c \in N_j$. This shows that $q(N_{j+1}, \bar{a}) = q(M^*, \bar{a})$.

This completes the proof. $\square$

Corollary 4.18. *Let D be totally transcendental. If $\mathcal{K}$ is categorical in some $\lambda > |T| + |D|$ then D is unidimensional.*

Proof. Otherwise, there is a D-homogeneous model of cardinality λ and a maximally $(D, |T| + |D|)$-homogeneous model of cardinality λ. Hence $\mathcal{K}$ is not categorical in λ, since these models cannot be isomorphic. $\qquad\square$

We now obtain strong structural results when D is unidimensional.

Theorem 4.19. *Let D be unidimensional. Then every $M \in \mathcal{K}$ is prime and minimal over $q(M, \bar{a})$, for any minimal type $q(x, \bar{a})$ over M.*

Proof. Let $M \in \mathcal{K}$ be given. Since D is totally transcendental, there exists a minimal type $q(x, \bar{a})$ over M. Consider $A = q(M, \bar{a})$. To check minimality, suppose there was $N \in \mathcal{K}$, such that $A \subseteq N \subseteq M$. Since $q(N, \bar{a}) = A = q(M, \bar{a})$, we must have $N = M$, by unidimensionality of D. We now show that M is prime over A. Since D is totally transcendental, there is $M^* \in \mathcal{K}$ prime over A. Hence, we may assume that $A \subseteq M^* \subseteq M$. Now the minimality of M implies that $M = M^*$, so M is prime over A. Clearly, any other minimal type would have the same property. $\qquad\square$

We next establish two lemmas, which are key results to carry out the geometric argument for the categoricity theorem.

Lemma 4.20. *Let $M \in \mathcal{K}$ and suppose that $q(x, \bar{a})$ is minimal over M. If $W = q(M, \bar{a})$ has dimension λ infinite, then W realizes every extension $p \in S_D(A)$ of type q, provided A is a subset of W of cardinality less than the dimension λ.*

Proof. Let $p \in S_D(A)$ be given extending q. Let $c \in \mathfrak{C}$ realize p. If p is not big for M, then p is not realized outside M so $c \in M$. Hence $c \in W$ since p extends q. If however p is big for M, then p is big and then by Lemma 3.8 and Theorem 3.14 we have that $p = \mathrm{Av}_D(I, A)$, where I is any basis of W of cardinality λ. But $|I| = \lambda \geq |A|^+ + \aleph_0$, so by Lemma 1.15 and definition of averages, $\mathrm{Av}_D(I, A)$ is realized by some element of $I \subseteq W$. Hence p is realized in W. $\qquad\square$

Lemma 4.21. *Let D be unidimensional and let M be in $\mathcal{K}$ of cardinality $\lambda > |T| + |D|$. Suppose $q(x, \bar{a})$ is minimal over M. Then $q(M, \bar{a})$ has dimension λ.*

Proof. Let $M \in \mathcal{K}$ be given and $q(x, \bar{a})$ be minimal. Construct $\langle M_\alpha \mid \alpha < \lambda \rangle$ strictly increasing and continuous such that $\bar{a} \in M_0$, $M_\alpha \subseteq M$ and $\|M_\alpha\| = |\alpha| + |T| + |D|$.

This is possible by Theorem 4.6: For $\alpha = 0$, just choose $M_0 \subseteq M$ prime over $\bar{a}$. For α a limit ordinal, let $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$. At successor stage, since $\|M_\alpha\| \leq |\alpha| + |T| + |D| < \lambda$, there exists $a_\alpha \in M - M_\alpha$, so we can choose $M_{\alpha+1} \subseteq M$ prime over $M_\alpha \cup a_\alpha$.

This is enough: Since D is unidimensional, we can find $c_\alpha \in M_{\alpha+1} - M_\alpha$ realizing q. By definition, $\mathrm{tp}(c_\alpha, \bigcup\{c_\beta \mid \beta < \alpha\})$ is big, since $c_\alpha \notin M_\alpha$. Hence

$c_\alpha \notin cl(\bigcup\{c_\beta \mid \beta < \alpha\})$. Therefore $\{c_\alpha \mid \alpha < \lambda\}$ is independent and so $q(M, \bar{a})$ has dimension at least λ. Hence since $\|M\| = \lambda$, then $q(M, \bar{a})$ has dimension λ. $\square$

Theorem 4.22. *Let D be unidimensional. Then $\mathcal{K}$ is categorical in every $\lambda > |T| + |D|$.*

Proof. Let $M_l \in \mathcal{K}$ for $l = 1, 2$ be of cardinality $\lambda > |T| + |D|$. Since D is totally transcendental, we can choose, $q(x, \bar{a}_1)$ minimal, with $\bar{a}_1 \in M_1$. Now, since M_2 is $(D, \aleph_0)$-homogeneous, we can find $\bar{a}_2 \in M_2$ such that $\mathrm{tp}(\bar{a}_1, \emptyset) = \mathrm{tp}(\bar{a}_2, \emptyset)$. Then $q(x, \bar{a}_1)$ is minimal also. Let $W_l = q(M_l, \bar{a}_l)$ for $l = 1, 2$. Since D is unidimensional, by Lemma 4.21, we have $\dim(W_l) = \lambda > |T| + |D|$. Hence, by Lemma 4.20 every type extending $q(x, \bar{a}_l)$ over a subset of W_l of cardinality less than λ is realized in W_l, for $l = 1, 2$. This allows us to construct by induction an elementary mapping g from W_1 onto W_2 extending $\langle \bar{a}_1, \bar{a}_2 \rangle$. By Theorem 4.19, M_l is prime and minimal over W_l, for $l = 1, 2$. Hence, in particular M_1 is prime over W_1, so there is $f : M_1 \to M_2$ elementary extending g. But now $\mathrm{rang}(f)$ is a $(D, \aleph_0)$-homogeneous model containing W_2, so by minimality of M_2 over W_2 we have $\mathrm{rang}(f) = M_2$. Hence f is also onto, and so M_1 and M_2 are isomorphic. $\square$

We can now summarize our results.

Corollary 4.23. *Let D be totally transcendental. The following conditions are equivalent:*

(1) *$\mathcal{K}$ is categorical in every $\lambda > |T| + |D|$;*
(2) *$\mathcal{K}$ is categorical in some $\lambda > |T| + |D|$;*
(3) *D is unidimensional;*
(4) *Every $M \in \mathcal{K}$ is prime and minimal over $q(M, \bar{a})$, where $q(x, \bar{a})$ is any minimal type over M;*
(5) *Every model $M \in \mathcal{K}$ of cardinality $\lambda > |T| + |D|$ is D-homogeneous.*

Proof.   (1) implies (2) is trivial.
    (2) implies (3) is Theorem 4.18.
    (3) implies (1) is Theorem 4.22.
    (3) implies (4) is Theorem 4.19.
    (4) implies (3) is clear since prime models exist by Theorem 4.6.
    (5) implies (1) is by back and forth construction, similarly to the corresponding proof with saturated models.
    (1) implies (5) since for each $\lambda > |D| + |T|$ there exist a (D, λ)-homogeneous model of cardinality λ (e.g. by Theorem 4.6).

$\square$

Corollary 4.24. *Let D be totally transcendental. If $\mathcal{K}$ is not categorical in some $\lambda_1 > |T| + |D|$, then*

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(1) *If T is countable, then there are at least $|\alpha|$ models of cardinality $\aleph_\alpha$ in $\mathcal{K}$;*

(2) *For every $\lambda \geq \mu \geq |T| + |D|$ there is a maximally (D,μ)-homogeneous of cardinality λ.*

Proof. (1) follows from (2). For (2), notice that D is not unidimensional by above Corollary, so the result follows from Theorem 4.17.   □

Corollary 4.25. *Let D be totally transcendental. Suppose there is a maximally (D,μ)-homogeneous model of cardinality $\lambda > |T| + |D|$ for some $\lambda > \mu \geq \aleph_0$. Then for every $\lambda \geq \mu \geq |T| + |D|$ there is a maximally (D,μ)-homogeneous of cardinality λ.*

Proof. Notice that $M \in \mathcal{K}$, and so $\mathcal{K}$ is not categorical in λ. Hence, by the previous corollary, D is not unidimensional, so the result follows from Theorem 4.17.   □

As a last Corollary, we obtain a generalization of Keisler's Theorem. We *do not* assume that D is totally transcendental.

Corollary 4.26. *Let $|T| < 2^{\aleph_0}$, and suppose D is the set of isolated types of T. The following conditions are equivalent.*

(1) *$\mathcal{K}$ is categorical in every $\lambda > |T|$;*

(2) *$\mathcal{K}$ is categorical in some $\lambda > |T|$;*

(3) *D is totally transcendental and unidimensional;*

(4) *D is totally transcendental and every model of $\mathcal{K}$ is prime and minimal over over $q(M, \bar{a})$, where $q(x, \bar{a})$ is any minimal type over M;*

(5) *Every model $M \in \mathcal{K}$ of cardinality strictly above $|T|$ is D-homogeneous.*

Proof. Using Ehrenfeucht-Mostowski models, since $\mathcal{K}$ and the class of D-models coincide, it is easy to show that D must be stable in $|T|$. Hence D is totally transcendental by Theorem 2.13, and therefore the result follows from Corollary 4.23.   □

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