# An Effective One-Step Hyperperpetual Reduction Strategy for Combinators

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**Abstract:**

We construct an effective one-step reduction strategy ϕ for combinators such that ϕ is normalizing and if M has no normal form then the iterations M,ϕ(M),...,ϕ^m(M),... are all distinct.

**The Algorithm:**

We adopt the terminology and notations of [ 1 ].

Barendregt in [ 1 ] defines a perpetual reduction strategy to be a function ϕ : CL -> CL such that for any combinator M we have M ->> ϕ(M) and if M is the sourse of non-terminating reduction then the reduction M ->> ϕ(M) ->> ...->> ϕ^m(M) ->>... is such a non-terminating reduction. In addition, a function ϕ : CL -> CL is said to be a normalizing reduction strategy if for any combinator M we have M ->> ϕ(M) and if M has a normal form then, for some m, ϕ^m(M) is the normal form of M. In applications, one usually wants a strategy which, in the absence of a normal form, produces, after iteration, arbitrarily long combinators. This is not done by conventional perpetual or normalizing strategies.

In the case of CL we have the following consequence of a theorem of Jan Willem Klop ([ 2 ] 1.13.(iv)); namely, if a combinator has only finitely many reducts then it has a normal form. This theorem is simply false for lambda calculus. Thus we can ask for a bit more out of a perpetual reduction strategy. We can ask that all the iterates M, ϕ(M),...,ϕ^m(M),... be distinct when M has no normal form. If we combine this idea with the idea of a normalizing reduction strategy we obtain the notion of a hyperpetual reduction strategy.

Definition: A function ϕ : CL -> CL is a hyperpetual reduction strategy if for any M

(1) M ->> ϕ(M) and M ≡ ϕ(M) iff M is normal.

(2) if M has a normal form then, for some m, ϕ^m(M) is the normal form of M.

(3) if M has no normal form then all the iterates M,ϕ(M),...,ϕ^m(M),... are distinct.

We shall prove the following theorem.

Theorem : There is an effective one-step hyperperpetual reduction strategy for CL.

Proof: We shall construct an algorithm ℙ by recursion for this purpose.

The algorithm $\mathbb{P}$

Input: a combinator M

Output: a combinator $\mathbb{P}$(M) such that

if M is normal then $\mathbb{P}$(M) $\equiv$ M

else M -> $\mathbb{P}$(M)

Begin:

(1) If M is head normal then

case 1; M is normal. Set $\mathbb{P}$(M) $\equiv$ M.

case 2; M $\equiv$ SLN and L is not normal. Set $\mathbb{P}$(M) $\equiv$ S($\mathbb{P}$(L))N.

case 3; M $\equiv$ SLN and N is not normal. Set $\mathbb{P}$(M) $\equiv$ SL($\mathbb{P}$(N)).

case 4; M $\equiv$ KN. Set $\mathbb{P}$(M) $\equiv$ K($\mathbb{P}$(N))

else

(2) If M $\equiv$ SPQRM(1)...M(m) then $\mathbb{P}$(M) $\equiv$ PR(QR)M(0)...M(m)

else

(3) If M $\equiv$ KPQM(1)...M(m) then

case 1; P is head normal. Set $\mathbb{P}$(M) $\equiv$ PM(1)...M(m).

case 2; P is not head normal. Set $\mathbb{P}$(M) $\equiv$

K($\mathbb{P}$(P))QM(1)...M(m).

End.

We need three lemmas to prove the theorem.

Lemma : Suppose that M has no normal form. Then the reduction sequence

M -> $\mathbb{P}$(M) -> ... -> $\mathbb{P}$^m(M) -> ... does not cycle.

Proof: Suppose that the lemma is false. Of all the counter-examples to the lemma let M be the shortest in length. Let M $\equiv$ AN(1)...N(n) for A an atom i.e. M $\varepsilon$ {S.K}. In the cycle M -> $\mathbb{P}$(M) -> ...-> $\mathbb{P}$^m(M) $\equiv$ M some trace of N(n) must be the last arguement of a redex. For otherwise we can shorten M to AN(1)...N(n-1). Moreover, after zero or more S reductions there must be at least one K reduction whose last arguement is the last component of the term, for otherwise the last components have the form N(n), L(1)N(n), L(2)(L(1)N(n)),..., L(k)(...L(1)N(n)....) which can never repeat N(n). Thus there is some $\mathbb{P}$^k(M) $\equiv$ KPQ and $\mathbb{P}$^(k+1)(M) $\equiv$ P. Hence P is head normal and this contradicts the choice of M. End of proof.

Definition; If M has a head normal form then the principal head normal form of M is the one obtained by performing all possible head reductions of M.

Indeed if P is the principal head normal form of M and H is any other head normal form then P ->>(internal) H.

Lemma : If M has a head normal form then, for some m, $\mathbb{P}$^m(M) is the principal head normal form of M.

Proof ; by induction on the number of head reductions from M to its principal head normal form. The basis case is when M is already head normal and this case is trivial.

Next suppose that M has a head redex. When this head redex is an S redex then the result follows directly from the induction hypothesis applied to $\mathbb{P}(M)$. Now suppose that M begins with a K redex $M \equiv KPQM(1)...M(m)$. Then P has a head normal form and if $P \to P(1) \to ... \to P(p)$ the head reduction of P to principal head normal form then the head reduction of M to principal head normal from begins $M \to PM(1)...M(m) \to P(1)M(1)...M(m) \to ... \to P(p)M(1)...M(m)$. By induction hypothesis there is an n such that $\mathbb{P}^n(P) \equiv P(p)$ and there is an m such that $\mathbb{P}^m(P(p)M(1)...M(m))$ is the principal head normal form of M. But then $\mathbb{P}^n(M) \equiv KP(p)QM(1)...M(m)$, $\mathbb{P}^{(n+1)}(M) \equiv P(p)M(1)...M(m)$,
and $\mathbb{P}^{(n+m+1)}(M)$ is the principal head normal form of M as desired. End of proof.

Lemma : If M has a normal form then, for some m, $\mathbb{P}^m(M)$ is the normal form of M.

Proof : By the standardization theorem if M has a normal form then there is a standard reduction to the normal form of M. Let #M be the length of such a standard reduction. The proof is by induction on #M. The basis case is when M is already normal and this is trivial. For the induction step we distinguish several cases.

Case 1; M is head normal.

When #M is fixed we prove this by induction on the length of M. We have either $M \equiv SPQ$ or $M \equiv KP$ with one or the other induction hypothesis applying to P and Q. Here the result follows from (1) in the definition of $\mathbb{P}$.

Case 2; M begins with a head redex.

By the previous lemma there exists an n such that $\mathbb{P}^n(M)$ is the principal head normal form of M and moreover $\#\mathbb{P}^n(M) < \#M$. Thus the induction hypothesis applies to $\mathbb{P}^n(M)$ and there is an m such that $\mathbb{P}^m(\mathbb{P}^n(M)) \equiv \mathbb{P}^{(n+m)}(M)$ is the normal form $\mathbb{P}^n(M)$ i.e. the normal form of M.This completes the proof of the lemma and the theorem.

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