Categories of Relations and Functional Relations

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Abstract. We define relations and their composition in a category with $(\mathcal{E},\mathcal{M})$-factorization structure, with $\mathcal{M}$ consisting of monomorphisms, but $\mathcal{E}$ not restricted to epimorphisms. We obtain an associativity criterion for composition of relations, and we study functional and induced relations.

Introduction

This paper presents the main results of the first author's thesis [9], with some related new results.

A relation $\rho : A \to B$ in a category is usually defined as a subobject of a product $A \times B$. Relations in this sense were introduced by S. Mac Lane [12] and D. Puppe [15] for abelian categories, and by M. Barr [2] and P.A. Grillet [3] for regular categories. A. Klein discussed in [11] the general case of categories with an $(\mathcal{E},\mathcal{M})$-factorization structure in the sense of [1], with $\mathcal{E}$ consisting of epimorphisms and $\mathcal{M}$ consisting of monomorphisms. In this generality, composition of relations is not always associative; Klein obtained a necessary and sufficient condition for this associativity. Relations and functional relations in this generality were also discussed by G.M. Kelly [10].

Subobjects and relations in a category can be used to obtain an internal logic. For topological categories over sets, with subspaces of a space X as subobjects, given by subsets of the underlying set of X, topological structures are completely lost in this process, and the resulting internal logic is classical two-valued logic. This is particularly embarrassing for the categories of fuzzy sets and "crisp" maps considered by various authors. These categories are topological over sets, and with subspaces as subobjects, their internal logic is classical two-valued logic, with no fuzzyness left.

L. Stout [16] showed a way out of this dilemma, and it was shown in [18] that Stout's theory postulates an $(\mathcal{E},\mathcal{M})$-factorization structure with $\mathcal{E}$ not consisting of epimorphisms. The results of [18] are generalized in [9] and this paper, with strictly categorical proofs.

Throughout the paper, we work with a category $\mathbf{C}$, not necessarily locally small, with finite limits, and with subobjects given by an $(\mathcal{E}, \mathcal{M})$-factorization structure, with minimal added assumptions. We define subobjects in 1.2 as special morphisms in $\mathcal{M}$; this avoids using large equivalence classes of morphisms, and several related complications. We discuss relations in $\mathbf{C}$ and their composition in Sections 1 and 2. From Section 3 on, we assume that $\mathcal{M}$ consists of monomorphisms, and that $\mathbf{C}$ is leg-regular, *i.e.* $\mathcal{E}$ is stable for pullbacks by legs r and s of relations (r, s). Composition of relations is associative in this setting. We define functional and induced relations and obtain their basic properties in Sections 3 through 6. Section 7 discusses some useful examples.

The most important examples in Section 7 are categories of fuzzy sets, with an appropriate lattice of truth values, and "crisp" maps. Thus our paper can be regarded as a contribution to a satisfactory mathematical theory of fuzzy sets. In the theory of fuzzy sets, there is strong evidence that functional relations rather than "crisp" maps are the appropriate morphisms; see *e.g.* [14] and [8], and from a categorical viewpoint [7] and [13]. These functional relations are obtained from factorization structures defined in Section 7 of this paper.

Notations: Superfluous parentheses are sometimes omitted; we may write Fx instead of $F(x)$, especially if F is a functor. For a product $A \xleftarrow{p} A \times B \xrightarrow{q} B$ and morphisms $f : X \to A$ and $g : X \to B$, we denote by $\langle f, g \rangle : X \to A \times B$ the unique morphism $h : X \to A \times B$ such that $ph = f$ and $qh = g$.

1. Subobjects, Spans and Relations

1.1. Assumptions. Throughout this paper, we work with a category $\mathbf{C}$ with finite limits, and with an $(\mathcal{E}, \mathcal{M})$-factorization structure, in the sense of [1]. This means that $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms of $\mathbf{C}$ with the following properties.

(i) Every morphism f of $\mathbf{C}$ factors $f = me$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

(ii) Every commutative square $mf = ge$ in $\mathbf{C}$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ has a unique diagonal d with $f = de$ and $g = md$ in $\mathbf{C}$.

(iii) Compositions ue in $\mathbf{C}$ with $e \in \mathcal{E}$ and u isomorphic are in $\mathcal{E}$, and compositions mv with $m \in \mathcal{M}$ and v isomorphic are in $\mathcal{M}$.

It follows (see [1], Section 14) that $\mathcal{E}$ and $\mathcal{M}$ are closed under compositions, $\mathcal{E} \cap \mathcal{M}$ consists of all isomorphisms of $\mathbf{C}$, and $\mathcal{M}$ is closed under pullbacks and products.

We usually assume that $\mathcal{M}$ consists of monomorphisms of $\mathbf{C}$, and that $\mathbf{C}$ is leg-regular in the sense of 1.9 below, but we do not assume that $\mathcal{E}$ consists of epimorphisms. We note without proof the following variation on [1] 14.11.

1.1.1. Proposition. *The following are equivalent.*

(i) *All strong monomorphisms of* $\mathbf{C}$ *are in* $\mathcal{M}$.

(ii) *All equalizers in* $\mathbf{C}$ *are in* $\mathcal{M}$.

(iii) *For every pullback square* $fu = gv$ *in* $\mathbf{C}$, *the morphism* $\langle u, v\rangle$ *of* $\mathbf{C}$ *is in* $\mathcal{M}$.

(iv) *All morphisms* $\langle \mathrm{id}, f\rangle$ *of* $\mathbf{C}$ *are in* $\mathcal{M}$.

(v) *All morphisms* $\langle \mathrm{id}_A, \mathrm{id}_A\rangle$ *of* $\mathbf{C}$ *are in* $\mathcal{M}$.

(vi) $\mathcal{E}$ *consists of epimorphisms of* $\mathbf{C}$.

1.2. Subobjects. Subobjects of an object A of $\mathbf{C}$ are usually defined as equivalence classes of morphisms in $\mathcal{M}$ with codomain A. Subobjects in this sense are usually proper classes. We avoid this and related nuisances by postulating the existence of a subclass $\mathcal{M}_0$ of $\mathcal{M}$ with the following property.

(i) Every morphism f of $\mathbf{C}$ has a unique factorization $f = me$ with e in $\mathcal{E}$ and m in $\mathcal{M}_0$.

Morphisms in $\mathcal{M}_0$ with codomain A in $\mathbf{C}$ will then be called *subobjects* of A. We note that $f \in \mathcal{M}$ in (i) iff e in the $(\mathcal{E}, \mathcal{M}_0)$-factorization $f = me$ is an isomorphism.

Examples of classes $\mathcal{M}_0$ are subset insertions for sets and subspace inclusions for topological constructs. In these examples, $\mathcal{M}_0$ is closed under composition and contains all identity morphisms of $\mathbf{C}$, but we shall not need these properties. Other examples are given in [18] and [9]; some of these are discussed in Section 7 of this paper.

1.3. Spans. A *span* $(u, v) : A \rightarrow B$ in $\mathbf{C}$ is a pair of morphisms u, v of $\mathbf{C}$ with codomains A for u and B for v, and with the same domain. Spans $(u, v) : A \rightarrow B$ are the objects of a category $\mathrm{Span}(A, B)$, with $f : (u, v) \rightarrow (u', v')$ in $\mathrm{Span}(A, B)$ if $u = u'f$ and $v = v'f$ in $\mathbf{C}$. Composition in $\mathrm{Span}(A, B)$ is composition of the underlying morphisms of $\mathbf{C}$. We say that spans α and α' are *span-equivalent*, and write $\alpha \simeq \alpha'$, if α and α' are isomorphic in a category $\mathrm{Span}(A, B)$.

Categories $\mathrm{Span}(A, B)$ have finite limits. Pullbacks in $\mathrm{Span}(A, B)$ are lifted from $\mathbf{C}$, and projections $A \xleftarrow{p} A \times B \xrightarrow{q} B$ of a product $A \times B$ define a terminal span (p, q), with

$$\langle r, s\rangle : (r, s) \rightarrow (p, q)$$

for a span $(r,s) : A \rightarrow B$. Products in $\mathrm{Span}(A,B)$ are called *intersections*, with

$$(r,s) \cap (u,v) \simeq (rh, sh) = (uk, vk)$$

for a pullback square

$$\begin{array}{ccc} \cdot & \xrightarrow{k} & \cdot \\ \big\downarrow h & & \big\downarrow \langle u,v \rangle \\ \cdot & \xrightarrow{\langle r,s \rangle} & A \times B \end{array}$$

1.4. Relations. For objects A, B of $\mathbf{C}$, we define a *relation* $\rho : A \rightarrow B$ as a span $\rho = (r,s) : A \rightarrow B$ with $\langle r,s \rangle$ in $\mathcal{M}$, and we denote by $\mathrm{Rel}(A,B)$ the full subcategory of $\mathrm{Span}(A,B)$ with relations as its objects. It is often convenient to restrict relations to subobjects in the sense of 1.2, and we denote by $\mathrm{Rel}_0(A,B)$ the full subcategory of $\mathrm{Rel}(A,B)$ with objects (r,s) such that $\langle r,s \rangle \in \mathcal{M}_0$. This is a skeleton of $\mathrm{Rel}(A,B)$. If $\mathcal{M}$ consists of monomorphism, then $\mathrm{Rel}_0(A,B)$ is a meet semilattice, possibly large.

Theorem and Definition. *For objects A and B of* $\mathbf{C}$, $\mathrm{Rel}(A,B)$ *and* $\mathrm{Rel}_0(A,B)$ *are $\mathcal{E}$-reflective full subcategories of* $\mathrm{Span}(A,B)$, *with reflections*

$$e : (a,b) \rightarrow \mathbf{r}(a,b) \tag{1}$$

given by $\mathbf{r}(a,b) = (r,s)$ for the $(\mathcal{E},\mathcal{M}_0)$-factorization $\langle a,b \rangle = \langle r,s \rangle \cdot e$. Thus $\mathrm{Rel}(A,B)$ *is closed under limits in* $\mathrm{Span}(A,B)$.

This follows immediately from the definitions. We shall always use the notation $\mathbf{r}(a,b)$ for this reflection, and we note that e in (1) is an isomorphism iff (a,b) is a relation, and an identity morphism for $\langle a,b \rangle$ in $\mathcal{M}_0$.

1.5. Dual spans and relations. For a span $(r,s) : A \rightarrow B$, we put

$$(r,s)^{\mathrm{op}} = (s,r) : B \rightarrow A$$

and call $(r,s)^{\mathrm{op}}$ the *dual span* of (r,s). Then clearly

$$((r,s)^{\mathrm{op}})^{\mathrm{op}} = (r,s),$$

and the dual span of a relation is a relation.

Duality defines isomorphisms between the categories $\mathrm{Span}(A,B)$ and $\mathrm{Span}(B,A)$, and also between $\mathrm{Rel}(A,B)$ and $\mathrm{Rel}(B,A)$. We also note without proof that

$$(\mathbf{r}(a,b))^{\mathrm{op}} \simeq \mathbf{r}((a,b)^{\mathrm{op}})$$

for every span (a,b). If $(r,s) \in \mathrm{Rel}_0(A,B)$, then (s,r) need not be in $\mathrm{Rel}_0(B,A)$, but $(r,s) \mapsto \mathbf{r}(s,r)$ provides an isomorphism $\mathrm{Rel}_0(A,B) \to \mathrm{Rel}_0(B,A)$ with all properties of duality.

1.6. Images and inverse images of relations. For $f : A \to B$ and a relation $(r,s) : A \to C$, we put

$$f^{\to}(r,s) = \mathbf{r}(fr,s)\,.$$

This clearly defines an *image functor*

$$f^{\to} : \mathrm{Rel}(A,C) \to \mathrm{Rel}(B,C)$$

for relations, with all values in $\mathrm{Rel}_0(B,C)$.

Now consider commutative squares

$$(1) \qquad \begin{array}{ccc} \bullet & \xrightarrow{f'} & \bullet \\ \big\downarrow{\scriptstyle u'} & & \big\downarrow{\scriptstyle u} \\ A & \xrightarrow{f} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \bullet & \xrightarrow{\quad f' \quad} & \bullet \\ \big\downarrow{\scriptstyle \langle u', vf' \rangle} & & \big\downarrow{\scriptstyle \langle u, v \rangle} \\ A \times C & \xrightarrow{f \times \mathrm{id}_C} & B \times C \end{array}$$

in $\mathbf{C}$. Then the lefthand square is a pullback iff the righthand square is one. As $\mathcal{M}$ is pullback-stable, putting

$$f^{\leftarrow}(u,v) = (u', vf')$$

defines an *inverse image relation* $f^{\leftarrow}(u,v)$ up to equivalence. We make the inverse image unique by choosing, as we may, pullbacks (1) with $\langle u', vf' \rangle$ in $\mathcal{M}_0$. Now we have a functor

$$f^{\leftarrow} : \mathrm{Rel}(B,C) \to \mathrm{Rel}(A,C)$$

for relations, with all values in $\mathrm{Rel}_0(A,C)$.

If gf is defined in $\mathbf{C}$, then it is easily seen that

$$(gf)^{\to} = g^{\to} \circ f^{\to}, \qquad \text{and} \qquad (gf)^{\leftarrow} = f^{\leftarrow} \circ g^{\leftarrow}.$$

We also note the following result.

Proposition. *$f^{\to} \dashv f^{\leftarrow}$ for f in* $\mathbf{C}$.

PROOF. From (1), we have a natural bijection of span morphisms

$$h : (fr, s) \to (u, v) \qquad \text{and} \qquad k : (r, s) \to f^{\leftarrow}(u, v),$$

given by $h = f'k$, and from 1.4 a natural bijection of morphisms

$$h : (fr, s) \to (u, v) \qquad \text{and} \qquad j : f^{\to}(r, s) \to (u, v).$$

1.7. Direct and inverse images for subobjects. For a terminal object 1 and an object A of $\mathbf{C}$, the projection $A \times 1 \to A$ is an isomorphism; thus there is a natural bijection between subobjects m of A and relations $\mathbf{r}(m, t)$ in $\mathrm{Rel}_0(A, 1)$. Thus for $f : A \to B$ in $\mathbf{C}$ and subobjects m of A and m_1 of B, and for morphisms t, t', t'', t_1 with codomain 1, putting

$$(f^{\to} m, t') \simeq f^{\to}(m, t) \qquad \text{and} \qquad (f^{\leftarrow} m_1, t'') \simeq f^{\leftarrow}(m_1, t_1)$$

defines subobjecs $f^{\to} m$ of B and $f^{\leftarrow} m_1$ of A, with the usual properties.

1.8. Legs. For a relation (r, s), we call r and s the *legs* of (r, s). Legs are compositions pm of a projection p of a product and a morphism m in $\mathcal{M}$. We note some basic properties of legs.

Proposition. *Projections of products, and morphisms in $\mathcal{M}$, are legs. Legs form a pullback-stable subcategory of $\mathbf{C}$, and a product $r \times u$ of legs r and u is again a leg. If $\mathcal{E}$ consists of epimorphisms, then every morphism of $\mathbf{C}$ is a leg.*

PROOF. The projections of $A \times B$ are the legs of the full relation $(p, q) : A \to B$. A span $(m, t) : A \to 1$, for a terminal object 1 of $\mathbf{C}$, is a relation iff $m \in \mathcal{M}$, since the projection $A \times 1 \to A$ is an isomorphism.

The construction of $f^{\leftarrow}(u, v)$ in 1.6.(1) shows that every pullback u' of a leg u is again a leg.

If $(r, s) : A \to B$ and $(u, v) : X \to C$ are relations with ru defined in $\mathbf{C}$, then we have a morphism

$$(\langle r, s\rangle \times \mathrm{id}_C) \cdot \langle u, v\rangle \;=\; \langle ru, su, v\rangle : \bullet \to A \times B \times C$$

in $\mathcal{M}$, and hence a relation $(ru, \langle su, v\rangle) : A \to B \times C$. Thus ru is a leg.

If (r, s) and (u, v) are relations, then $(r \times u, s \times v)$ is a relation, since

$$\langle r, s\rangle \times \langle u, v\rangle \;=\; \pi \cdot \langle r \times u, s \times v\rangle$$

for a "middle interchange" isomorphism π in $\mathbf{C}$, with both sides in $\mathcal{M}$.

Finally, if $\mathcal{E}$ consists of epimorphisms, then every span (id, f) is a relation by 1.1.1, and thus every morphism of $\mathbf{C}$ is a leg.

1.9. Regular legs. We say that $\mathbf{C}$ has *regular legs*, or that $\mathbf{C}$ is *leg-regular*, if $\mathcal{E}$ is stable under pullbacks by legs of relations. Since legs are compositions pm with $m \in \mathcal{M}$ and p a projection of a product, $\mathbf{C}$ has regular legs if and only if the following two conditions are satisfied.

(i) Every product $e_1 \times e_2$ of morphisms in $\mathcal{E}$ is in $\mathcal{E}$.

(ii) $\mathcal{E}$ is closed under pullbacks by morphisms in $\mathcal{M}$.

2. Composition of Spans and Relations

2.1. Span composition. Categories $\mathrm{Span}(A, B)$ form a bicategory in the sense of [4]. We compose spans $(r, s) : A \to B$ and $(u, v) : B \to C$ in $\mathbf{C}$ by putting

$$(u, v) \diamond (r, s) \;=\; (ru', vs') : A \to C$$

for a pullback square

$$\begin{array}{ccc} \bullet & \xrightarrow{s'} & \bullet \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ \bullet & \xrightarrow{s} & B \end{array}$$

in $\mathbf{C}$. This composition is defined up to span-equivalence.

Composition of spans is clearly functorial and associative, up to equivalence of spans. For an object A of $\mathbf{C}$, we put

$$\delta_A \;=\; (\mathrm{id}_A, \mathrm{id}_A) \, .$$

Spans δ_A are identity spans, again up to span-equivalence.

It is easily seen that

$$\alpha^{\mathrm{op}} \diamond \beta^{\mathrm{op}} \;\simeq\; (\beta \diamond \alpha)^{\mathrm{op}} \, ,$$

for spans α, β, if either composition is defined.

2.2. Composition of relations. The *composition* $\sigma \circ \rho : A \to C$ of relations $\rho : A \to B$ and $\sigma : B \to C$ in $\mathbf{C}$ is defined by

$$\sigma \circ \rho = \mathbf{r}(\sigma \diamond \rho) \, .$$

This clearly defines a functor from $\mathrm{Rel}(B,C)\times\mathrm{Rel}(A,B)$ to $\mathrm{Rel}(A,C)$, with all values in $\mathrm{Rel}_0(A,C)$. Thus composition of relations is order-preserving if $\mathcal{M}$ consists of monomorphisms. We note that

$$\sigma \circ (r,s) \;=\; r^{\rightarrow} s^{\leftarrow} \sigma \tag{1}$$

for relations $(r,s) : A \rightarrow B$ and $\sigma : B \rightarrow C$ in $\mathbf{C}$, and that always

$$(\sigma \circ \rho)^{\mathrm{op}} \;\simeq\; \rho^{\mathrm{op}} \circ \sigma^{\mathrm{op}},$$

by 1.5, 2.1 and the definitions, if either composition is defined. We also note that $\sigma' \circ \rho' = \sigma \circ \rho$ if $\rho' \simeq \rho$ and $\sigma' \simeq \sigma$, and the compositions are defined.

2.3. Theorem. *For a category* $\mathbf{C}$ *with* $(\mathcal{E},\mathcal{M})$*-factorization structure and finite limits, the following are equivalent.*

(i) $\sigma \circ \mathbf{r}\alpha = \mathbf{r}(\sigma \diamond \alpha)$ *for all spans* α *and relations* σ *for which the compositions are defined.*

(ii) $\mathbf{r}\beta \circ \rho = \mathbf{r}(\beta \diamond \rho)$ *for all relations* ρ *and spans* β *for which the compositions are defined.*

(iii) $\sigma \circ \mathbf{r}(a,b) = a^{\rightarrow} b^{\leftarrow} \sigma$ *for all spans* (a,b) *and relations* σ *such that the composition is defined.*

(iv) $e^{\rightarrow} e^{\leftarrow} \rho \simeq \rho$ *for all relations* ρ *in* $\mathbf{C}$ *and morphisms* e *in* $\mathcal{E}$ *with* $e^{\leftarrow}\rho$ *defined.*

If these conditions are satisfied, then composition of relations in $\mathbf{C}$ *is associative, with identity relations* $\mathbf{r}\delta_A$.

PROOF. (ii) is clearly the same as (i) for the dual relations; thus (i) and (ii) are equivalent,

If $b^{\leftarrow}\sigma = (u,v)$, then

$$\sigma \diamond (a,b) \;\simeq\; (au, v) \qquad \text{and} \qquad \mathbf{r}(\sigma \diamond (a,b)) \;=\; a^{\rightarrow} b^{\leftarrow} \sigma;$$

thus (i) and (iii) are equivalent.

It follows from (i) and (ii) that

$$\rho \circ \mathbf{r}\delta_A \;\simeq\; \rho \;\simeq\; \mathbf{r}\delta_B \circ \rho \tag{1}$$

for a relation $\rho : A \rightarrow B$, with equality iff $\rho \in \mathrm{Rel}_0(A,B)$. Since $\mathbf{r}(e,e) = \mathbf{r}\delta_A$ for e in $\mathcal{E}$ with codomain A, (iv) follows from this and (iii). Conversely, if $\langle a,b\rangle = \langle r,s\rangle \cdot e$ is an $(\mathcal{E},\mathcal{M})$-factorization of $\langle a,b\rangle$, then

$$a^{\rightarrow} b^{\leftarrow} \sigma \;=\; r^{\rightarrow} e^{\rightarrow} e^{\leftarrow} s^{\leftarrow} \sigma \;=\; r^{\rightarrow} s^{\leftarrow} \sigma \;=\; \sigma \circ (r,s)$$

if $\sigma \circ (r,s)$ is defined and (iv) is valid; thus (iv)$\Longrightarrow$(iii).

If (i) and (ii) are valid, then

$$\begin{aligned}\tau \circ (\sigma \circ \rho) &= \tau \circ \mathbf{r}(\sigma \diamond \rho) = \mathbf{r}(\tau \diamond (\sigma \diamond \rho)) \\ &= \mathbf{r}((\tau \diamond \sigma) \diamond \rho) = \mathbf{r}(\tau \diamond \sigma) \circ \rho = (\tau \circ \sigma) \circ \rho\end{aligned}$$

for relations ρ, σ, τ with $\sigma \circ \rho$ and $\tau \circ \sigma$ defined, and composition of relations is associative.

2.4. Regularity of legs has been defined in 1.9; we note the following result.

Proposition. *If* $\mathbf{C}$ *has regular legs, then*

$$\mathbf{r}((b', c) \diamond (a, b)) = \mathbf{r}(b', c) \circ \mathbf{r}(a, b) \tag{1}$$

for spans (a, b) *and* (b', c) *with the compositions defined and with* b *a leg or* b' *a leg, and* 2.3.(i)–(iv) *are valid for* $\mathbf{C}$. *Conversely, if* 2.3.(i)–(iv) *are valid for* $\mathbf{C}$ *and* $\mathcal{M}$ *consists of monomorphisms, then* $\mathbf{C}$ *has regular legs.*

PROOF. 2.3.(i) and 2.3.(ii) clearly are special cases of (1).
For (1), let

$$\langle a, b\rangle = \langle r, s\rangle \cdot e_1 \qquad \text{and} \qquad \langle b', c\rangle = \langle u, v\rangle \cdot e_2$$

be $(\mathcal{E}, \mathcal{M})$-factorizations, and construct a diagram

$$\begin{array}{ccccc} \bullet & \xrightarrow{e_1''} & \bullet & \xrightarrow{s''} & \bullet \\ \downarrow{\scriptstyle e_2''} & & \downarrow{\scriptstyle e_2'} & & \downarrow{\scriptstyle e_2} \\ \bullet & \xrightarrow{e_1'} & \bullet & \xrightarrow{s'} & \bullet \\ \downarrow{\scriptstyle u''} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ \bullet & \xrightarrow{e_1} & \bullet & \xrightarrow{s} & \bullet \end{array} \tag{2}$$

of pullback squares. Then

$$(b', c) \diamond (a, b) \simeq (re_1u''e_2'', ve_2s''e_1'') = (ru'e_1'e_2'', vs'e_1'e_2'').$$

Now s' and u' are legs, and e_1' and e_2' are in $\mathcal{E}$ if $\mathbf{C}$ has regular legs. If $b = se_1$ is a leg, then so is $s'e_1'$, and thus e_2'' is in $\mathcal{E}$ if $\mathbf{C}$ has regular legs. But then

$$\mathbf{r}((b', c) \diamond (a, b)) = \mathbf{r}(ru', vs') = (u, v) \circ (r, s),$$

so that (1) is valid. The proof of (1) with b' a leg is exactly analogous.

For a relation (r, s) and a pullback square

$$\begin{array}{ccc} \bullet & \xrightarrow{e'} & \bullet \\ \Big\downarrow r' & & \Big\downarrow r \\ \bullet & \xrightarrow{e} & \bullet \end{array}$$

in $\mathbf{C}$, we have

$$e^{\rightarrow}e^{\leftarrow}(r,s) = \mathbf{r}(er', se') = \mathbf{r}(re', se') .$$

If 2.3.(iv) is valid, and the pullback shown above exists with e in $\mathcal{E}$, then we have an $(\mathcal{E}, \mathcal{M})$-factorization

$$\langle re', se' \rangle = \langle r, s \rangle \cdot e_1 .$$

If $\mathcal{M}$ consists of monomorphisms, then $e' = e_1$; thus $\mathbf{C}$ has regular legs.

2.5. Discussion and problems. The system with the objects of $\mathbf{C}$ as objects, $\mathrm{Rel}(A, B)$ as the class of morphisms $\rho : A \nrightarrow B$ for objects A and B, and with compositions defined by 2.2 is almost, but not quite, a category, because we have span-equivalence instead of equality in 2.3.(1). We obtain a *bona fide* category, which we denote by $\mathrm{Rel}\,\mathbf{C}$, if we restrict morphisms $\rho : A \nrightarrow B$ to $\mathrm{Rel}_0(A, B)$. Every relation ρ in $\mathbf{C}$ is then span-equivalent to a unique morphism $\mathbf{r}\rho$ of $\mathrm{Rel}\,\mathbf{C}$.

The results of this Section are due to A. Klein [11] for the case that $\mathcal{E}$ consists of epimorphisms of $\mathbf{C}$ and $\mathcal{M}$ of monomorphisms. In this situation, all morphisms of $\mathbf{C}$ are legs, and composition of relations in $\mathbf{C}$ is associative, up to span-equivalence, iff $\mathcal{E}$ is stable for all pullbacks in $\mathbf{C}$.

If $\mathcal{E}$ does not consist of epimorphisms, then $\mathcal{E}$ can be stable for pullbacks by legs, but not for all pullbacks in $\mathbf{C}$. [18] 2.16.1 is an example for this. This leaves us with two unsolved problems.

- Is it possible for composition of relations to be associative, with identity relations $\mathbf{r}\delta_A$, but with 2.3.(i)–(iv) not valid?

- Can 2.3.(i)–(iv) be valid if $\mathbf{C}$ does not have regular legs?

3. Induced and Functional Relations

3.1. Induced relations. We assume from now on that $\mathcal{M}$ consists of monomorphisms, and that $\mathbf{C}$ has regular legs. We assign to every morphism $f : A \to B$ of $\mathbf{C}$ an *induced relation*

$$[f] = \mathbf{r}(\mathrm{id}_A, f) : A \to B\,.$$

Note that $[f]$ is always in $\mathrm{Rel}_0(A, B)$ for $f : A \to B$ in $\mathbf{C}$, and that $[\mathrm{id}_A]$ is an identity relation in $\mathrm{Rel}\,\mathbf{C}$ for every object A of $\mathbf{C}$.

Relations $[f]$ were denoted by $\langle f \rangle$ in [18] and [9]; we changed the notation for better distinction from morphisms $\langle f, g \rangle$.

3.2. Properties of induced relations. Using 2.3, and 2.4 for spans (id, f) and (f, id), we have the following results for morphisms f, g and relations ρ, σ, valid whenever the compositions are defined.

3.2.1. $[gf] = [g] \circ [f]$.

Since the relations

$$[\mathrm{id}_A] = \mathbf{r}\delta_A$$

are identity relations, this says that the assignments $f \mapsto [f]$ define a functor, from $\mathbf{C}$ to $\mathrm{Rel}\,\mathbf{C}$.

3.2.2. $\mathbf{r}(f, g) = [g] \circ [f]^{\mathrm{op}}$.

3.2.3. $f^{\to}\rho = \rho \circ [f]^{\mathrm{op}}$ *and* $f^{\leftarrow}\sigma = \sigma \circ [f]$.

Using this for dual spans and relations, we get

3.2.4. $[f] \circ \rho \simeq (f^{\to}\rho^{\mathrm{op}})^{\mathrm{op}}$ *and* $[f]^{\mathrm{op}} \circ \sigma \simeq (f^{\leftarrow}\sigma^{\mathrm{op}})^{\mathrm{op}}$.

3.3. Special relations. With terms borrowed from set theory, we say that a relation $\rho : A \to B$ is
single-valued if $\rho \circ \rho^{\mathrm{op}} \leq [\mathrm{id}_B]$,
total if $\rho^{\mathrm{op}} \circ \rho \geq [\mathrm{id}_A]$,
injective if $\rho^{\mathrm{op}} \circ \rho \leq [\mathrm{id}_A]$,
surjective if $\rho \circ \rho^{\mathrm{op}} \geq [\mathrm{id}_B]$.
We also say that ρ is *functional* if ρ is single-valued and total, and *bijective* if ρ is injective and surjective. We shall see in 3.7 that functional relations are the same as the maps of G.M. Kelly [10].

Clearly ρ is single-valued if and only if ρ^{op} is injective, and ρ is total if and only if ρ^{op} is surjective.

An identity relation $[\mathrm{id}_A]$, span-equivalent to its dual, satisfies all four conditions. It is also easy to verify that each of the four classes of relations defined above is closed under span-equivalence, and under composition of relations.

3.4. Theorem. *Every relation* $[f]$ *induced by a morphism* f *of* $\mathbf{C}$ *is functional.*

PROOF. By 3.2 and 1.6, we have

$$\sigma \circ [f] \circ [f]^{\mathrm{op}} = f^{\rightarrow} f^{\leftarrow} \sigma \leq \sigma$$

for every relation σ with $f^{\leftarrow}\sigma$ defined; thus $[f]$ is single-valued. Totality of $[f]$ is obtained similarly.

3.5. Our next result shows that the pre-order of relations becomes span-equivalence when restricted to functional relations.

Proposition. *If* $\rho \leq \sigma$ *for a total relation* ρ *and a single-valued relation* σ*, then* $\rho \simeq \sigma$*.*

PROOF. We have

$$\sigma \leq \sigma \circ \rho^{\mathrm{op}} \circ \rho \leq \sigma \circ \sigma^{\mathrm{op}} \circ \rho \leq \rho$$

since $[\mathrm{id}] \leq \rho^{\mathrm{op}} \circ \rho$, $\rho^{\mathrm{op}} \leq \sigma^{\mathrm{op}}$, and $\sigma \circ \sigma^{\mathrm{op}} \leq [\mathrm{id}]$.

3.6. The following result does not use regularity of legs. Similar characterizations of single-valued, injective and functional relations will be obtained in 4.3.

Proposition. *For a relation* $\rho = (r, s) : A \to B$*, the following are equivalent.*

(i) ρ *is total.*

(ii) $\sigma \circ \rho \geq [\mathrm{id}_A]$ *for some relation* $\sigma : B \to A$*.*

(iii) $r \in \mathcal{E}$*.*

PROOF. (i) $\Longrightarrow$ (ii) trivially; use $\sigma = \rho^{\mathrm{op}}$.

If $(u, v) \circ (r, s) = (a, a')$ and $(a, a') \geq [\mathrm{id}_A]$, let $su' = us'$ be a pullback square. Then ru' factors ae with $e \in \mathcal{E}$, and there is t in $\mathbf{C}$ with $at = \mathrm{id}_A$. If $mf = gr$ in $\mathbf{C}$ with m in $\mathcal{M}$, then $mfu' = gae$, and thus $ga = md$ for a morphism d. Now $g = mdt$. Since m is monomorphic, this determines dt uniquely, and $f = dtr$. Thus r has the diagonal property of $(\mathcal{E}, \mathcal{M})$-factorizations, and $r \in \mathcal{E}$ follows.

Since $\rho^{\mathrm{op}} \diamond \rho \simeq (rs', rs'')$ for a pullback square $ss' = ss''$, there is a morphism $h : (r, r) \to \rho^{\mathrm{op}} \diamond \rho$ in $\mathrm{Span}(A, A)$. If r is in $\mathcal{E}$, then $\mathbf{r}(r, r) = [\mathrm{id}_A]$, and $[\mathrm{id}_A] \leq \rho^{\mathrm{op}} \circ \rho$ follows. This completes the proof.

3.7. Proposition. *A relation* $\rho : A \to B$ *is functional if and only if there is a relation* $\sigma : B \to A$ *such that* $\sigma \circ \rho \geq [\mathrm{id}_A]$ *and* $\rho \circ \sigma \leq [\mathrm{id}_B]$*, and then* $\sigma \simeq \rho^{\mathrm{op}}$*.*

PROOF. If ρ is functional, then $\sigma = \rho^{\mathrm{op}}$ satisfies the inequalities. Conversely, it follows from the first inequality, by 3.6 for ρ and σ^{op}, that ρ is total and σ surjective. But then

$$\sigma \leq \rho^{\mathrm{op}} \circ \rho \circ \sigma \leq \rho^{\mathrm{op}}$$

with the second inequality, and also

$$\rho \leq \rho \circ \sigma \circ \sigma^{\mathrm{op}} \leq \sigma^{\mathrm{op}} .$$

It follows that $\sigma \simeq \rho^{\mathrm{op}}$, and ρ is functional.

3.8. Isomorphisms. We say as usual that relations $\rho : A \to B$ and $\sigma : B \to A$ are *inverse isomorphisms of relations* if

$$\sigma \circ \rho = [\mathrm{id}_A] \qquad \text{and} \qquad \rho \circ \sigma = [\mathrm{id}_B] .$$

It follows that inverses of relations are only determined up to equivalence if they exist. Inverses in $\mathrm{Rel}\,\mathbf{C}$ are unique.

Theorem. *A relation ρ is an isomorphism of relations if and only if ρ is functional and bijective, and then the inverses of ρ are the relations span-equivalent to ρ^{op}.*

PROOF. If ρ is functional and bijective, then ρ^{op} is an inverse of ρ by the definitions. Conversely, if ρ and σ are inverse relations, then ρ and σ are functional, and $\sigma \simeq \rho^{\mathrm{op}}$, by 3.7. But then ρ and σ are also bijective.

3.9. Proposition. *The following two statements are logically equivalent.*

(i) *$\mathcal{E}$ consists of epimorphisms.*

(ii) *If $[f] = [g]$, for morphisms f and g of $\mathbf{C}$ with the same domain and the same codomain, then always $f = g$.*

PROOF. Consider $f, g : A \to B$. If (i) is valid, then $[f] \simeq (\mathrm{id}_A, f)$ and $[g] \simeq (\mathrm{id}_A, g)$. These clearly are span-equivalent only if $f = g$. Conversely, suppose $fe = ge$, with $e \in \mathcal{E}$. Then

$$e^{\leftarrow}[f] = [fe] = [ge] = e^{\leftarrow}[g] .$$

Applying $e^{\rightarrow}$ to both sides, we get $[f] = [g]$ by 2.3.(iv). If (ii) is valid, then $f = g$ follows, and e is epimorphic.

4. Surjective-Injective Factorizations

4.1. Proposition. *Every relation ρ factors $\rho \simeq [m] \circ \varepsilon$, with ε surjective and m in $\mathcal{M}_0$, hence $[m]$ injective, with ε total if and only if ρ is total, and ε single-valued if and only if ρ is single-valued.*

Proof. Put $\rho = (r, s)$ and factor $s = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$. Since $\langle r, s\rangle = (\mathrm{id} \times m)\langle r, e\rangle$ with $\langle r, s\rangle$ and $\mathrm{id} \times m$ in $\mathcal{M}$, this defines a relation $\varepsilon = (r, e)$, with $\rho \simeq [m] \circ \varepsilon$ by 3.2.4, and ε is surjective by 3.6 for $\varepsilon^{\mathrm{op}}$. By 3.6, ε is total iff ρ is total. We have $\varepsilon \simeq [m]^{\mathrm{op}} \circ \rho$, and $[m]$ and $[m]^{\mathrm{op}}$ are single-valued. Thus ε is single-valued iff ρ is single-valued.

4.2. Theorem. *For a commutative square $\mu \circ \rho = \sigma \circ \varepsilon$ of relations, with ε single-valued and surjective, and μ injective and total, the following four statements are logically equivalent, and each of them determines τ up to equivalence:*

$$\rho \simeq \tau \circ \varepsilon, \qquad \tau \simeq \rho \circ \varepsilon^{\mathrm{op}}, \qquad \sigma \simeq \mu \circ \tau, \qquad \tau \simeq \mu^{\mathrm{op}} \circ \sigma.$$

In this situation, ρ and τ are single-valued if σ is single-valued, and σ and τ are total if ρ is total.

Proof. If $\rho \simeq \tau \circ \varepsilon$, then $\tau \simeq \rho \circ \varepsilon^{\mathrm{op}}$ since $\varepsilon \circ \varepsilon^{\mathrm{op}} = [\mathrm{id}]$, and then

$$\mu \circ \tau = \mu \circ \rho \circ \varepsilon^{\mathrm{op}} = \sigma \circ \varepsilon \circ \varepsilon^{\mathrm{op}} \simeq \sigma.$$

Similarly, $\sigma \simeq \mu \circ \tau \Longrightarrow \tau \simeq \mu^{\mathrm{op}} \circ \sigma \Longrightarrow \rho \simeq \tau \circ \varepsilon$, using $\mu^{\mathrm{op}} \circ \mu = [\mathrm{id}]$, and the last part follows immediately from the assumptions and the displayed statements.

4.3. Theorem. *An induced relation $[f]$ is surjective if and only if f is in $\mathcal{E}$.*

A relation (r, s) in $\mathbf{C}$ is total if and only if $[r]$ is surjective, single-valued if and only if $[r]$ is injective, and functional if and only if $[r]$ is bijective. Dually, (r, s) is surjecive, injective or bijective if and only if $[s]$ has the same property.

If $fu = rg$ is a pullback square with r a leg, and if $[r]$ is injective or surjective or bijective, then so is $[u]$.

Proof. If we $(\mathcal{E}, \mathcal{M})$-factor $\langle \mathrm{id}, f\rangle = \langle a, b\rangle \cdot e$, then $f = be$ is in $\mathcal{E}$ iff $b \in \mathcal{E}$, and so the first part and the first claim of the second part follow immediately from 3.6 for $[f]^{\mathrm{op}}$ and (r, s). For the second claim, factor $s = me$ as in 4.1, with (r, s) single-valued iff (r, e) is a single-valued relation. We have

$$(r, e) \circ (r, e)^{\mathrm{op}} = [r]^{\mathrm{op}} \circ [e] \circ [e]^{\mathrm{op}} \circ [r] = [r]^{\mathrm{op}} \circ [r],$$

by 3.4 and 4.1. Thus (r, s) is single-valued iff $[r]$ is injective. The third claim and the dual statements are now obvious.

For the third part, let r be a leg of (r, s), with $(r, s) \circ [f] \simeq (u, sg)$ by 3.2.3 and 1.6. If $[r]$ is injective, then (r, s) is single-valued, and so are the composition $(r,s) \circ [f]$ and (u, sg). But then $[u]$ is injective. The proof for surjective or bijective $[r]$ is exactly analogous.

4.4. For injective functional relations μ_1 and μ_2, we put $\mu_1 \leq \mu_2$ if $\mu_1 \simeq \mu_2 \circ \tau$ for a relation τ. This determines $\tau \simeq {\mu_1}^{\mathrm{op}} \circ \mu_2$ up to span-equivalence, and τ is single-valued. From

$$\tau^{\mathrm{op}} \circ \tau = \tau^{\mathrm{op}} \circ {\mu_2}^{\mathrm{op}} \circ \mu_2 \circ \tau = {\mu_1}^{\mathrm{op}} \circ \mu_1 = [\mathrm{id}]$$

we see that τ is also total and injective.

Proposition. *For m_1 and m_2 in $\mathcal{M}$ with the same codomain, we have $m_1 \leq m_2$ if and only if $[m_1] \leq [m_2]$.*

PROOF. We have $[m_1] \simeq [m_2] \circ (a,b)$ for a relation (a,b) iff

$$\langle \mathrm{id}, m_1 \rangle = \langle a, m_2 b \rangle \cdot e$$

for a morphism e in $\mathcal{E}$, and then $m_1 = m_2 b e$. Conversely, if $m_1 = m_2 h$, then $[m_1] = [m_2] \circ [h]$.

4.5. Proposition. *If $\mu \simeq [m]$ and $\sigma \simeq [g]$ in 4.2, with $m \in \mathcal{M}$, then $\tau \simeq [t]$ for a morphism t of $\mathbf{C}$ with $mt = g$.*

PROOF. If we $(\mathcal{E}, \mathcal{M})$-factor $g = m_1 e_1$, then $[m] \circ \tau \simeq [m_1] \circ [e_1]$, and $[m_1] \leq [m]$ follows by 4.2. But then $m_1 = ms$ for a morphism s by 4.4, and $g = mt$ for $t = se_1$. Now $\tau \simeq [t]$ since $[m]$ is monomorphic in $\mathrm{Rel}\,\mathbf{C}$.

4.6. We shall need the following result in Section 5.

Proposition. *For a functional relation $\rho : A \to B$, compositions*

$$\tau = \rho \circ [m] \circ [m]^{\mathrm{op}},$$

with m in $\mathcal{M}_0$ with codomain A, define an order preserving bijection between morphisms m in $\mathcal{M}_0$ with codomain A, and relations $\tau \leq \rho$ in $\mathrm{Rel}_0(A, B)$.

PROOF. For $\rho = (r,s)$, a composition $\rho \circ [m] \simeq (e, sh)$ is obtained from a pullback square

$$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle r} \\ \cdot & \xrightarrow{m} & A \end{array},$$

with $h \in \mathcal{M}$, and $e \in \mathcal{E}$ since $r \in \mathcal{E}$ by 3.6 and m is a leg. From these data, we get a relation

$$\tau = m^{\rightarrow}(e, rh) \simeq (me, rh) = (rh, sh) \leq \rho.$$

We get m back from τ by $(\mathcal{E}, \mathcal{M}_0)$-factoring $rh = me$.

If $\tau \simeq (rh, sh)$ is given, with h in $\mathcal{M}$, then we obtain m from an $(\mathcal{E}, \mathcal{M}_0)$-factorization $rh = me$. In this situation,

$$\langle sh, rh \rangle = (m \times \mathrm{id})\langle e, sh \rangle$$

in $\mathbf{C}$. It follows that (e, sh) is a relation, total since $e \in \mathcal{E}$. From the construction of $\rho \circ [m]$ by a pullback, it follows that

$$(e, sh) \leq \rho \circ [m].$$

Since (e, sh) is total and $\rho \circ [m]$ single-valued, the two relations are equivalent by 3.5. But then τ is a composition $\rho \circ [m] \circ [m]^{\mathrm{op}}$ for this m.

If $m' \leq m$ in $\mathcal{M}_0$, with common codomain A, then $h' \leq h$ for the pullbacks h and h' of m and m' by r. Conversely, if $(rh', sh') \leq (rh, sh)$ in $\mathrm{Rel}(A, B)$, then $h' = hu$ for a morphism u, and $(\mathcal{E}, \mathcal{M})$-factoring $rh = me$ and $rh' = m'e'$ produces a commutative square $meu = m'e'$. With 1.1.(ii), $m' \leq m$ follows.

4.7. The following result complements 1.1.1, 1.8 and 3.9.

Proposition. *The following are equivalent.*

(i) *$\mathcal{E}$ consists of epimorphisms.*

(ii) *An induced relation $[f]$ is injective if and only if f is monomorphic in $\mathbf{C}$.*

(iii) *A relation (r, s) is single-valued if and only if r is monomorphic in $\mathbf{C}$.*

(iv) *A relation (r, s) is injective if and only if s is monomorphic in $\mathbf{C}$.*

(v) *An induced relation $[f]$ is bijective if and only if f is monomorphic and in $\mathcal{E}$.*

(vi) *A relation (r, s) is functional if and only if r is monomorphic and in $\mathcal{E}$.*

(vii) *A relation (r, s) is bijective if and only if s is monomorphic and in $\mathcal{E}$.*

Proof. We have (ii)$\Longrightarrow$(iii)$\Longrightarrow$(vi), and (ii)$\Longrightarrow$(v)$\Longrightarrow$(vi), by 4.3 and 3.6, and (iv) and (vii) are (iii) and (vi) for the dual relations.

If (i) is valid, then (id, f) is a relation. If $fu = fv$ is a pullback square, then (u, v) is a relation by 1.1.1. It follows that

$$[f]^{\mathrm{op}} \circ [f] = (f, \mathrm{id}) \circ (\mathrm{id}, f) \simeq (u, v).$$

Again by 1.1.1, this is an identity relation iff u is isomorphic and $u = v$, and thus iff f is monomorphic.

Now consider $\mathbf{r}\delta_A = (a, b)$, with $ae = be = \mathrm{id}_A$ for a morphism e in $\mathcal{E}$. If (vi) is valid, then a is monomorphic, and then a and e are inverse isomorphisms, and $b = a$. But then δ_A is a relation, and thus (vi) $\Longrightarrow$ (i) by 1.1.1.

5. Finite Limits for Functional Relations

5.1. Definitions. We denote by Fun $\mathbf{C}$ the subcategory of Rel $\mathbf{C}$ with functional relations as morphisms, and by Ind $\mathbf{C}$ the subcategory of Fun $\mathbf{C}$ with induced relations $[f]$ as morphisms. The aim of this Section is to construct finite limits in these categories.

5.2. Terminal objects. *A terminal object* 1 *of* $\mathbf{C}$ *is also a terminal object of* Fun $\mathbf{C}$ *and of* Ind $\mathbf{C}$.

PROOF. Since the projection $A \times 1 \to A$ is an isomorphism, a span $(m, t) : A \to 1$ is a relation iff $m \in \mathcal{M}$, and a functional relation iff also $m \in \mathcal{E}$, hence iff m is an isomorphism. Thus there is exactly one morphism $[t_A] = \mathbf{r}(\mathrm{id}_A, t_A) : A \to 1$ in Fun $\mathbf{C}$, for the morphism $t_A : A \to 1$.

5.3. Binary products. *If* $A \xleftarrow{p} A \times B \xrightarrow{q} B$ *are the projections of a product in* $\mathbf{C}$, *then* $A \xleftarrow{[p]} A \times B \xrightarrow{[q]} B$ *are the projections of a product in* Fun $\mathbf{C}$ *and in* Ind $\mathbf{C}$, *with*

$$(1) \qquad \langle \rho, \sigma \rangle = [p]^{\mathrm{op}} \circ \rho \cap [q]^{\mathrm{op}} \circ [\sigma] : C \to A \times B$$

in Fun $\mathbf{C}$, *for morphisms* $\rho : C \to A$ *and* $\sigma : C \to B$ *of* Fun $\mathbf{C}$.

PROOF. For a relation $\varphi : C \to A \times B$, we have

$$[p] \circ \varphi \leq \rho \quad \Longleftrightarrow \quad \varphi \leq [p]^{\mathrm{op}} \circ \rho,$$

by 3.2.4, and 1.6 for the dual relations. In the same way,

$$[q] \circ \varphi \leq \sigma \quad \Longleftrightarrow \quad \varphi \leq [q]^{\mathrm{op}} \circ \sigma.$$

It follows that

$$[p] \circ \varphi \leq \rho \ \text{ and } \ [q] \circ \varphi \leq \sigma \quad \text{iff} \quad \varphi \leq \langle \rho, \sigma \rangle,$$

with $\langle \rho, \sigma \rangle$ defined by (1). By 3.6, these inequalities become equations if φ is a morphism of $\mathrm{Fun}\,\mathbf{C}$. Thus it remains only to prove that (1) defines a functional relation $\langle \rho, \sigma \rangle$.

Let $\rho = (r, s) : C \to A$ and $\sigma = (u, v) : C \to B$. Then

$$[p]^{\mathrm{op}} \circ \rho \simeq (rp', s \times \mathrm{id}_B) \qquad \text{and} \qquad [q]^{\mathrm{op}} \circ \sigma \simeq (uq', \mathrm{id}_A \times v)$$

for pullback squares

$$\begin{array}{ccc} X \times B & \xrightarrow{p'} & X \\ \big\downarrow{\scriptstyle s \times \mathrm{id}_B} & & \big\downarrow{\scriptstyle s} \\ A \times B & \xrightarrow{p} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A \times Y & \xrightarrow{q'} & Y \\ \big\downarrow{\scriptstyle \mathrm{id}_A \times v} & & \big\downarrow{\scriptstyle v} \\ A \times B & \xrightarrow{q} & B \end{array} ,$$

with p' and q' projections of products. Thus a further pullback

$$\begin{array}{ccc} \bullet & \xrightarrow{\langle sx, y \rangle} & A \times Y \\ \big\downarrow{\scriptstyle \langle x, vy \rangle} & & \big\downarrow{\scriptstyle \langle uq', \mathrm{id}_A \times v \rangle} \\ X \times B & \xrightarrow{\langle rp', s \times \mathrm{id}_B \rangle} & C \times A \times B \end{array} ,$$

with x and y given by a pullback

$$\begin{array}{ccc} \bullet & \xrightarrow{y} & Y \\ \big\downarrow{\scriptstyle x} & & \big\downarrow{\scriptstyle u} \\ X & \xrightarrow{r} & A \end{array} ,$$

gives the desired intersection

$$\langle \rho, \sigma \rangle \simeq (rx, \langle sx, vy \rangle) .$$

By 4.3, $[r]$ and $[u]$ are bijective in the present situation, and so are $[x]$ and $[y]$ in the pullback diagram $rx = uy$. But then $[rx]$ is bijective, and $\langle \rho, \sigma \rangle$ functional, again by 4.3.

For morphisms $f : C \to A$ and $g : C \to B$ of $\mathbf{C}$, we have

$$[p] \circ [\langle f, g \rangle] = [f] \qquad \text{and} \qquad [q] \circ [\langle f, g \rangle] = [g] .$$

Thus $\langle [f], [g] \rangle = [\langle f, g \rangle]$, and $A \times B$ with projections $[p]$ and $[q]$ is also a product in $\mathrm{Ind}\,\mathbf{C}$.

5.4. Equalizers. *For morphisms* $\rho = (r,s) : A \to B$ *and* $\sigma = (u,v) : A \to B$ *of* Fun **C**, *and for a pullback square*

$$(1) \qquad \begin{array}{ccc} \cdot & \xrightarrow{k} & \cdot \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \langle u,v\rangle} \\ \cdot & \xrightarrow{\langle r,s\rangle} & A \times B \end{array}$$

and an $(\mathcal{E},\mathcal{M})$*-factorization* $rh = m_1 e = uk$, *the morphism* $[m_1]$ *is an equalizer of* ρ *and* σ *in* Fun **C**. *If* ρ *and* σ *are morphisms of* Ind **C**, *this is also an equalizer in* Ind **C**,

PROOF. For a functional relation $\xi = [m] \circ \varepsilon$ with codomain A, factored by 4.1, we have $\rho \circ \xi = \sigma \circ \xi$ iff $\rho \circ [m] = \sigma \circ [m]$, since ε is epimorphic. This is the case iff $\rho \circ [m] \circ [m]^{\mathrm{op}} = \sigma \circ [m] \circ [m]^{\mathrm{op}}$, and hence iff the relations (rh, sh) and (uk, vk) corresponding to $[m]$ by 4.6 are equal. Thus we obtain the desired equalizer $[m_1]$, with $m_1 \in \mathcal{M}$, by forming a pullback diagram (1) and factoring $rh = uk$ as shown.

If we factor $[g] = [m_1] \circ \tau$ in Fun **C**, then τ is a morphism of Ind **C** by 4.5. Thus $[m_1]$ is also an equalizer in Ind **C** if ρ and σ are induced relations.

5.5. Pullbacks. *For morphisms* $\rho = (r,s) : A \to C$ *and* $\sigma = (u,v) : B \to C$ *of* Fun **C**, *we obtain a pullback square*

$$(1) \qquad \begin{array}{ccc} P & \xrightarrow{[b]} & B \\ \downarrow{\scriptstyle [a]} & & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow{\rho} & C \end{array} ,$$

in Fun **C** *by*

$$\sigma^{\mathrm{op}} \circ \rho = (a,b)$$

in Rel **C**. *For morphisms* ρ *and* σ *of* Ind **C**, *this is a pullback square in* Ind **C**.

5.5.1. Corollary. *Every pullback square*

$$\begin{array}{ccc} P & \xrightarrow{\beta} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow{\rho} & C \end{array}$$

in Fun $\mathbf{C}$ *or in* Ind $\mathbf{C}$ *satisfies the Beck condition*

$$\sigma^{\mathrm{op}} \circ \rho = \beta \circ \alpha^{\mathrm{op}} .$$

PROOF. By the general theory, the components of the desired pullback are the components of an equalizer $[\langle a, b\rangle]$ of $\rho \circ [p]$, and $\sigma \circ [q]$, for the projections p and q of $A \times B$. For this, we need the intersection

$$\rho \circ [p] \cap \sigma \circ [q] : A \times B \to C .$$

This is dual to the intersection $\langle \rho^{\mathrm{op}}, \sigma^{\mathrm{op}} \rangle$ of 5.3, and thus $(\langle rh, uk\rangle, sh)$ for a pullback square $sh = vk$ in $\mathbf{C}$. Now $(a, b) = \mathbf{r}(rh, uk)$; this is $\sigma^{\mathrm{op}} \circ \rho$.

For ρ and σ in Ind $\mathbf{C}$, we have constructed an equalizer $[\langle a, b\rangle]$ in Ind $\mathbf{C}$, and thus a pullback in Ind $\mathbf{C}$.

The Corollary is valid for the pullback (1) since $(a, b) = [b] \circ [a]^{\mathrm{op}}$, and hence for every pullback square in Fun $\mathbf{C}$.

5.6. Theorem. *The following properties of* $\mathbf{C}$ *are equivalent.*

(i) $\mathcal{E}$ *is stable for all pullbacks in* $\mathbf{C}$.

(ii) *All span compositions* $\beta \diamond \alpha$ *in* $\mathbf{C}$ *satisfy*

$$\mathbf{r}(\beta \diamond \alpha) = \mathbf{r}\beta \circ \mathbf{r}\alpha \tag{1}$$

(iii) *The functor* $f \mapsto [f] : \mathbf{C} \to \mathrm{Fun}\,\mathbf{C}$ *preserves pullbacks.*

(iv) *The functor* $f \mapsto [f] : \mathbf{C} \to \mathrm{Fun}\,\mathbf{C}$ *preserves equalizers.*

(v) *The functor* $f \mapsto [f] : \mathbf{C} \to \mathrm{Fun}\,\mathbf{C}$ *preserves all finite limits.*

If these conditions are satisfied, then $[m]$ *is injective for every monomorphism* m *in* $\mathbf{C}$.

PROOF. If $\mathcal{E}$ is preserved by all pullbacks, then the proof of 2.4 works without any restrictions on b or b', and thus (1) is valid for all span compositions. If $m : A \to B$ is monomorphic, then

$$(\mathrm{id}_A, m)^{\mathrm{op}} \diamond (\mathrm{id}_A, m) \simeq \delta_A ,$$

and $[m]^{\mathrm{op}} \circ [m] = [\mathrm{id}_A]$ follows if (1) is valid.

Now assume (ii) and consider a pullback square

$$\begin{array}{ccc} \bullet & \xrightarrow{k} & B \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$$

in $\mathbf{C}$, with $\langle h,k\rangle \simeq (\mathrm{id}_B, g)^{\mathrm{op}} \diamond (\mathrm{id}_A, f)$. Thus

$$[g]^{\mathrm{op}} \circ [f] = \mathbf{r}(h,k)$$

by (1). If we $(\mathcal{E}, \mathcal{M}_0)$-factor $\langle h,k\rangle = \langle a,b\rangle \cdot e$, then $[f] \circ [a] = [g] \circ [b]$ is a pullback square in $\mathrm{Fun}\,\mathbf{C}$ by 5.5. Now e in $\mathcal{E}$ is monomorphic since $\langle h,k\rangle$ is, and thus $[e]$ is injective as well as surjective. But then $[f] \circ [h] = [g] \circ [k]$ is also a pullback square, and so (iii) is valid.

Since $f \mapsto [f]$ preserves finite products, (iii), (iv) and (v) are equivalent. If they are valid and $fh = gk$ is a pullback square in $\mathbf{C}$ with f in $\mathcal{E}$, then $[f] \circ [h] = [g] \circ [k]$ is a pullback square, with $[f]$ surjective by 4.3. But then $[k]$ is surjective by 6.2 below, and k is in $\mathcal{E}$.

5.7. Remarks. If $\mathcal{E}$ consists of epimorphisms, then all morphisms of $\mathbf{C}$ are legs. Thus 5.6.(i) is equivalent to leg regularity of $\mathbf{C}$, and the functor $f \mapsto [f] : \mathbf{C} \to \mathrm{Fun}\,\mathbf{C}$ preserves finite limits.

It does not follow from 5.6.(iii) that the functor $f \mapsto [f] : \mathbf{C} \to \mathrm{Ind}\,\mathbf{C}$ preserves pullbacks and equalizers. An isomorphism $[f]$ in $\mathrm{Fun}\,\mathbf{C}$ need not be an isomorphism in $\mathrm{Ind}\,\mathbf{C}$; there may be no morphism of $\mathbf{C}$ which induces the isomorphism $[f]^{-1}$ of $\mathrm{Fun}\,\mathbf{C}$.

6. Relations in Fun C and in Ind C

6.1. Discussion. We have seen in Section 5 that $\mathrm{Ind}\,\mathbf{C}$ and $\mathrm{Fun}\,\mathbf{C}$ have finite limits, and that the embedding $\mathrm{Ind}\,\mathbf{C} \to \mathrm{Fun}\,\mathbf{C}$ preserves them. The functor $f \mapsto [f] : \mathbf{C} \to \mathrm{Fun}\,\mathbf{C}$ preserves products, but in general not equalizers or pullbacks. An example for this is given in [18].

By 4.1, 4.2 and 4.5, the categories $\mathrm{Ind}\,\mathbf{C}$ and $\mathrm{Fun}\,\mathbf{C}$ have a (surjective,injective) factorization structure, preserved by the embedding $\mathrm{Ind}\,\mathbf{C} \to \mathrm{Fun}\,\mathbf{C}$. We assume from now on that $\mathrm{Fun}\,\mathbf{C}$ and $\mathrm{Ind}\,\mathbf{C}$ are provided with this structure.

Every injective functional relation is span-equivalent to a relation $[m]$ with m in $\mathcal{M}$, and by 4.4 to $[m]$ for exactly one m in $\mathcal{M}_0$. Thus we can, and shall, use the relations $[m]$ with m in $\mathcal{M}_0$ as subobjects for $\mathrm{Fun}\,\mathbf{C}$ and $\mathrm{Ind}\,\mathbf{C}$. With this convention, the functor $f \mapsto [f]$ also preserves and reflects subobjects.

Now only one thing is missing.

6.2. Proposition. $\mathrm{Fun}\,\mathbf{C}$ *and* $\mathrm{Ind}\,\mathbf{C}$ *have regular legs.*

PROOF. It is sufficient to prove this for the pullback square

$$\begin{array}{ccc} P & \xrightarrow{[b]} & B \\ \big\downarrow{\scriptstyle [a]} & & \big\downarrow{\scriptstyle \sigma} \\ A & \xrightarrow{\rho} & C \end{array},$$

constructed in 5.5, with $(a,b) = \sigma^{\mathrm{op}} \circ \rho$, and σ^{op} surjective since σ is total. If ρ is surjective, then (a,b) is surjective. But then $b \in \mathcal{E}$, and $[b]$ is surjective, by 3.6 and 4.3.

6.3. We can now construct categories of relations and of functional relations in $\mathrm{Fun}\,\mathbf{C}$ and in $\mathrm{Ind}\,\mathbf{C}$, with all the properties obtained in this paper. However, we get nothing new. Every injective morphism $\langle \rho, \sigma \rangle : \bullet \to A \times B$ in $\mathrm{Fun}\,\mathbf{C}$ is equivalent to exactly one morphism $[\langle r,s \rangle] = \langle [r],[s] \rangle$ with $\langle r,s \rangle : \bullet \to A \times B$ in $\mathcal{M}_0$. Thus the morphisms of $\mathrm{Rel}\,\mathrm{Fun}\,\mathbf{C}$ are just the pairs $([r],[s])$ with (r,s) a morphism of $\mathrm{Rel}\,\mathbf{C}$.
This leads up to the following result.

Theorem. *The bijection* $(r,s) \mapsto ([r],[s])$ *from* $\mathrm{Rel}\,\mathbf{C}$ *to* $\mathrm{Rel}\,\mathrm{Fun}\,\mathbf{C}$ *is an isomorphism of categories which preserves and reflects order and duality, and maps* $\mathrm{Fun}\,\mathbf{C}$ *to* $\mathrm{Fun}\,\mathrm{Fun}\,\mathbf{C} = \mathrm{Ind}\,\mathrm{Fun}\,\mathbf{C}$.

PROOF. For $(r,s) : A \to B$ and $(u,v) : B \to C$, consider pullbacks

$$\begin{array}{ccc} \bullet & \xrightarrow{k} & \bullet \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle u} \\ \bullet & \xrightarrow{s} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} \bullet & \xrightarrow{[b]} & \bullet \\ \big\downarrow{\scriptstyle [a]} & & \big\downarrow{\scriptstyle [u]} \\ \bullet & \xrightarrow{[s]} & B \end{array}$$

in $\mathbf{C}$ and $\mathrm{Fun}\,\mathbf{C}$, with $(a,b) = [u]^{\mathrm{op}} \circ [s]$ b 5.6. Then $(u,v) \circ (r,s) = \mathbf{r}(rh, vk)$, and $([u],[v]) \circ ([r],[s])$ is obtained fro an $(\mathcal{E}, \mathcal{M}_0)$-factorization of $\langle [ra],[vb] \rangle$, or equivalently of $\langle ra, vb \rangle$. Now

$$(h,k) \simeq (\mathrm{id}, u)^{\mathrm{op}} \diamond (\mathrm{id}, s).$$

Since s and u are legs, we can apply 2.4 to this composition, getting

$$\mathbf{r}(h,k) = [u]^{\mathrm{op}} \circ [s] = (a,b).$$

Thus $\langle rh, vk \rangle = \langle ra, vb \rangle \cdot e$ with $e \in \mathcal{E}$, and $\mathbf{r}(rh,vk) = \mathbf{r}(ra,vb)$ follows. This shows that composition of relations is preserved, and we have an isomorphism of categories.

Since the isomorphism preserves order and duality, it also preserves the four properties defined in 3.3, and thus maps functional relations to functional relations. If (r, s) is functional, then $[r]$ is an isomorphism of Fun **C** by 4.3, and

$$([r],[s]) \simeq ([\mathrm{id}],[s] \circ [r]^{-1}) \qquad \text{and} \qquad ([r],[s]) = [[s] \circ [r]^{-1}]$$

follow. Thus every functional relation in Fun **C** is induced by a morphism of Fun **C**,

6.4. Remarks. The functor $f \mapsto [f] : \mathbf{C} \to \mathrm{Ind}\,\mathbf{C}$ is full, and by 3.9 an isomorphism of categories if $\mathcal{E}$ consists of epimorphism. Subobjects in Fun **C** are the same as subobjects in Ind **C**, and thus Rel Ind **C** and Fun Ind **C** are the same as Rel Fun **C** and Fun Fun **C**.

We haver seen that every functional relation $([r],[s])$ in Fun **C** is an induced relation $[[s] \circ [r]^{-1}]$. However, $([r],[s])$ need not be induced by a morphism of Ind **C**. The relation $[r]^{-1} \simeq [r]^{\mathrm{op}}$ need not be induced by a morphism of **C**, even if $\mathcal{E}$ consists of epimorphisms.

All monomorphisms in Ind **C** and in Fun **C** are injective by 4.7; thus the surjective morphisms in these categories are the strong epimorphisms We note that not all epimorphisms in Ind **C** or in Fun **C** need be strong.

7. Examples

7.1. Final maps. If **C** is topological over a category **A** with finite limits, with forgetful functor $U : \mathbf{C} \to \mathbf{A}$, then a morphism f of **C** is called a *final map* if f by itself is a final sink. With final maps as the class $\mathcal{E}$, and $\mathcal{M}_0$ consisting of all maps of the form $\mathrm{id}_S : A \to B$, for objects of **C** with $UA = S = UB$, we get an $(\mathcal{E}, \mathcal{M}_0)$-factorization structure of **C**.

A trivial example is given by **C** topological over itself. In this situation, every object A has exactly one subobject id_A, and there is always exactly one morphism $A \to B$ in Rel **C**. Fuzzy sets, discussed below, provide a non-trivial example.

We have not investigated leg-regularity for topological categories with this factorization structure.

7.2. Fuzzy sets. Fuzzy sets were introduced by L. Zadeh [19], essentially as pairs $A = (|A|, \varepsilon_A)$, consisting of a set $|A|$ and a mapping $\varepsilon_A : |A| \to [0,1]$, which assigns to every $x \in |A|$ a membership degree $\varepsilon_A(x)$, a number in the real unit interval.

Zadeh noted that membership degrees $\varepsilon_A(x)$ can be viewed as truth-values, with 1 meaning "true" and 0 "false". J.A. GOGUEN [6] pointed out that truth-values could be taken in any lattice, preferably a complete Heyting algebra H, with mappings $\varepsilon_A : |A| \to H$ defining truth-values of statements $x \in A$. Goguen also pointed out that H-valued fuzzy sets, with morphisms $f : A \to B$ given by mappings $f : |A| \to |B|$ such that $\varepsilon_A(x) \le \varepsilon_B(fx)$ for $x \in |A|$, define a topological construct. These morphisms are called *crisp maps.*

If we define a *fuzzy subset* of an H-valued fuzzy set A as an H-valued fuzzy set B with $|B| = |A|$, and $\varepsilon_B(x) \le \varepsilon_A(x)$ for $x \in |A|$, then subset insertions $\mathrm{id}_{|A|} : B \to A$ provide the class $\mathcal{M}_0$ for an $(\mathcal{E}, \mathcal{M}_0)$-factorization structure of H-valued fuzzy sets, with $\mathcal{E}$ consisting of all final maps.

7.3. ***H*-sets.** For a complete Heyting algebra H, we define an H-set as a pair $A = (|A|, \delta_A)$ consisting of an ordinary set $|A|$ and a mapping $\delta_A : |A| \times |A| \to H$ with the following two properties.

(1) $\delta_A(x,y) = \delta_A(y,x)$ for all x, y in $|A|$.

(2) $\delta_A(x,y) \wedge \delta_A(y,z) \le \delta_A(x,z)$ for all x, y, z in $|A|$.

A *crisp map* $f : A \to B$ is defined as a mapping $f : |A| \to |B|$ of the underlying sets such that $\delta_A(x,y) \le \delta_B(fx, fy)$ for all x, y in $|A|$.

With composition of the underlying mappings as composition of crisp maps, H-sets and crisp maps form a topological construct.

For an H-set A, we interpret δ_A as H-valued equality. Putting $\varepsilon_A(x) = \delta_A(x,x)$ for $x \in |A|$ defines H-valued membership in an H-set, and thus an underlying H-valued fuzzy set.

For an H-set A, we define an *H-subset* of A as an H-set B with $|B| = |A|$, and the following properties.

(i) $\delta_B(x,y) \le \delta_A(x,y)$ for all x, y in $|A|$.

(ii) $\delta_A(x,y) \wedge \delta_B(y,z) \le \delta_B(x,z)$ for all x, y, z in $|A|$.

These properties are easily seen to be equivalent to:

(a) $\varepsilon_B(x) \le \varepsilon_A(x)$ for all x in $|A|$.

(b) $\varepsilon_B(y) \le \varepsilon_A(x) \wedge \delta_A(x,y)$ for all x and y in $|A|$.

(c) $\delta_B(x,y) = \varepsilon_B(x) \wedge \delta_A(x,y)$ for all x, y in $|A|$.

Thus an H-subset B of an H-set A is determined by its membership mapping ε_B which must satisfy (a) and (b).

With H-subset insertions $\mathrm{id}_{|A|} : B \to A$ as the class $\mathcal{M}_0$, we get an $(\mathcal{E}, \mathcal{M}_0)$-factorization structure for H-sets. Morphisms in $\mathcal{E}$ for this structure are called *covers* in [18] and [9]. It is easily seen that a crisp map $f : A \to B$ is a cover iff

$$\varepsilon_B(y) = \bigvee_{x \in |A|} (\varepsilon_A(x) \wedge \delta_B(fx, y))$$

for all y in $|B|$.

7.4. Discussion. Underlying fuzzy sets define a concrete functor from H-sets with crisp maps to fuzzy sets with crisp maps. This functor turns out to be topological. Its concrete left adjoint D assigns to a fuzzy set A a *discrete* H-set DA, with $|DA| = |A|$, $\varepsilon_{DA} = \varepsilon_A$, and $\delta_{DA}(x, y) = \bot$, the bottom element of H, for $x \neq y$ in A. Its concrete right adjoint assigns to a fuzzy set A an H-set CA with $|CA| = |A|$, and $\delta_{CA}(x, y) = \varepsilon_A(x) \wedge \varepsilon_A(y)$ for all x, y in A.

Covers for H-sets and crisp maps are stable for pullbacks by legs, and covers for H-valued fuzzy sets and crisp maps are stable for all pullbacks. Thus categories Fun **C** and Ind **C** are defined for both categories.

For H-sets and discrete maps, Fun **C** turns out to be the topos of H-sets and functional relations introduced by D. HIGGS in [7], and also studied in detail by M.P. FOURMAN and D.S. SCOTT [5]. For this example, Ind **C** is a quasitopos, introduced independently by G.P. MONRO [13] and D. PONASSE (see [14]). H-sets were called *totally fuzzy sets* by Ponasse. We note that $[f] = [g]$, for crisp maps $A \underset{g}{\overset{f}{\rightrightarrows}} B$ of H-sets, iff $\varepsilon_A(x) \leq \delta_B(fx, gx)$, for all x in $|A|$.

The categories Fun **C** and Ind **C** for H-valued fuzzy sets have also been studied, but they turn out to be much less interesting and well behaved that the corresponding categories for H-sets. This is not astonishing. For fuzzy sets, membership is fuzzy but equality crisp, with "true" and "false" as the only truth-values. For H-sets, membership and equality both have the same spectrum of truth-values. For the same reason, categories of functional relations, with fuzzy function values, are more interesting than categories of crisp maps, with crisp function values.

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