Finite models, stability and Ramsey's theorem

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Abstract

We prove some results on the border of Ramsey theory (finite partition calculus) and model theory. Also a beginning of classification theory for classes of finite models is undertaken.

1 Introduction

Frank Ramsey in his fundamental paper (see [21] and pages 18-27 of [10]) was interested in "a problem of formal logic." He proved the result now known as "(finite) Ramsey's theorem" which essentially states

> For all $k, r, c < \omega$, there is an $n < \omega$ such that however the r – subsets of $\{1, 2, \ldots, n\}$ are c – colored, there will exist a k – element subset of $\{1,2,\ldots,n\}$ which has all its r – subsets the same color.

(We will let $n(k, r, c)$ denote the smallest such n.) Ramsey proved this theorem in order to construct a finite model for a given finite universal theory so that the universe of the model is canonical with respect to the relations in the language. (For model theorists "canonical" means Δ – indiscernible as in Definition 2.1).

Much is known about the order of magnitude of the function $n(k,r,c)$ and some of its generalizations (see [8], for example). An upper bound on $n(k,r,c)$ is an $(r-1)$ – times iterated exponential of a polynomial in k and c. Many feel that the upper bound is tight. However especially for $r \geq 3$ the gap between the best known lower and upper bounds is huge.

In 1956 A. Ehrenfeucht and A. Mostowski [7] rediscovered the usefulness of Ramsey's theorem in logic and introduced the notion we now call indiscernibility. Several people continued exploiting the connections between partition theorems and logic (i.e. model theory), among them M. Morley (see [18] and [19]) and S. Shelah who has published a virtually uncountable number of papers related to indiscernibles (see [27]). Morley [19] used indiscernibles to construct models of very large cardinality (relative to the cardinality of the reals) — specifically, he proved that the Hanf number of $L_{\omega_1,\omega}$ is $\beth_{\omega_1}$.

One of the most important developments in mathematical logic — certainly the most important in model theory — in the last 30 years is what is known as "classification theory" or "stability theory". There are several books dedicated entirely to some aspects of the subject, including books by J. Baldwin [2], D. Lascar [16], S. Shelah [27], and A. Pillay [20]. Lately Shelah and others have done extensive work in extending classification theory from the context of first order logic, to the classification of arbitrary classes of models, usually for infinitary logics extending first order logic (for example see [3], [4], [5], [12], [14], [17], [23], [28], [24]). [24] contains several philosophical and personal comments about why this research is interesting, and [25] is the video tape of Shelah's plenary talk at the International Congress of Mathematics at Berkeley in 1986.

This raises a question of fundamental importance: *Is there a classification theory for finite structures?* In a more philosophical context: *Is the beautiful classification theory of Shelah completely detached from finite mathematics?* One of the fundamental difficulties to developing model theory for finite structures is the choice of an appropriate "submodel" relation — in category-theoretic terminology, the choice of a natural morphism. In classification theory for elementary classes (models of a first order, usually complete theory) the right notion of morphism is "elementary embedding" defined using the relation $M \prec N$, which is a strengthening of the notion of submodel (denoted by $M \subseteq N$). Unfortunately for finite structures, $M \prec N$ always implies $M = N$. Moreover, in many cases even $M \subseteq N$ implies $M = N$ (e.g., when N is a group of prime order). We need a substitute.

One of the basic observations to make is that when we limit our attention to structures in a relational language only (i.e., no function symbols), then $M \subseteq N$ does not imply $M = N$. In general this seems to be insufficient to force the substructure to inherit some of the properties of the bigger structure. It was observed already by Ramsey (in [21]) that if $M \subseteq N$, then for every universal sentence ϕ, $N \models \phi$ implies $M \models \phi$. So when studying the class of models of a universal first order theory, the relation $M \subseteq N$ is reasonable, but it is not for more complicated theories (e.g. not every subfield of an algebraically closed field is algebraically closed). Such a concept for classes of finite structures is introduced in Definition 5.9.

This paper has several goals:

1. *Study Ramsey numbers for definable coloring inside models of a stable theory.*

   This can be viewed as a direct extension of Ramsey's work, namely by taking into account the first order properties of the structures. A typical example is the field of complex numbers $\langle \mathbf{C}, +, \cdot \rangle$. It is well known that its first order theory $Th(\mathbf{C})$ has many nice properties — it is $\aleph_1$ – categorical and thus is $\aleph_0$ – stable and has neither the order property nor the finite cover property. We will be interested in the following general situation.

   Given a first order (complete) theory T, and (an infinite) model $M \models T$. Let k, r, and c be natural numbers, and let F be a coloring of a set of r – tuples from M by c colors which is definable by a first order formula in the language $L(T)$ (maybe with parameters from M). Let $n \stackrel{\text{def}}{=} n_F(k, r, c)$ be the least natural number such that for every $S \subseteq |M|$ of cardinality n, if $F : [S]^r \to c$ then there exists $S^* \subseteq S$ of cardinality k such that F is constant on $[S^*]^r$. It turns out that for stable theories, (or even for theories without the independence property) we get better upper bounds than for the general Ramsey numbers. This indicates that one can not improve the lower bounds by looking at stable structures.

2. *Introduce stability-like properties (e.g. n-order property, k-independence property, d-cover property), as well as averages of finite sequences of indiscernibles.*

   Some of the interconnections and the effect on the existence of indiscernibles are presented.

3. *Develop classification theory for classes of finite structures. In particular introduce a notion that correspond to stable amalgamation, and show that it is symmetric for many models.*

   See Example 5.8.

4. *Bring down uncountable techniques to a finite context.*

   We believe that much of the machinery developed (mainly by Shelah) to deal with problems concerning categoricity of infinitary logics and the behavior of the spectrum function at cardinalities $\geq \beth_{\omega_1}$ depends on some very powerful combinatorial ideas. We try here to extract some of these ideas and present them in a finite context.

Shelah [27] proved that instability is equivalent to the presence of either the strict order property or the independence property. In a combinatorial setting, stability implies that for arbitrarily large sets, the number of types over a set is polynomial in the cardinality of the set. We address the finite case here in which we restrict our attention to when the number of ϕ-types over a finite set is bounded by a polynomial in the size of the set of parameters.

First we find precisely the degree of the polynomial bound on the number of these types given to us by the absence of the strict order or independence properties. This is an example of something relevant in the finite case which is of no concern in the usual classification theory framework.

Once we have these sharper bounds we can find sequences of indiscernibles in the spirit of [27]. It should be noted here that everything we do is "local", involving just a single formula (or equivalently a finite set of formulas). We then work through the calculations for uniform hypergraphs as a case study. This raises questions about "stable" graphs and hypergraphs which we begin to answer.

In the second half of the paper, we examine classes of finite structures in the framework of Shelah's classification for non-elementary classes (see [26]). In particular, we make an analogy to Shelah's "abstract elementary classes" and prove results similar to his.

Notation: Everything is standard. We will typically treat natural numbers as ordinals (i.e., $n = \{0, 1, \ldots, n-1\}$). Often x, y, and z will denote free variables, or finite sequences of variables — it should be clear from the context whether we are dealing with variables or with sequences of variables. When x is a sequence, we let $l(x)$ denote its length. L will denote a similarity type (a.k.a. language or signature), Δ will stand for a finite set of L formulas. M and N will stand for L - structures, $|M|$ the universe of the structure M, and $\|M\|$ the cardinality of the universe of M. Given a fixed structure M, subsets of its universe will be denoted by A, B, C, and D. So when we write $A \subseteq M$ we really mean that $A \subseteq |M|$, while $N \subseteq M$ stands for "N is a submodel of M". When M is a structure then by $a \in M$ we mean $a \in |M|$, and when a is a finite sequence of elements, then $a \in M$ stands for "all the elements of the sequence a are elements of $|M|$".

Since all of our work will be inside a given structure M (with the exception of Section 4), all the notions are relative to it. For example for $a \in M$ and $A \subseteq M$ we denote by $tp_\Delta(a, A)$ the type $tp_\Delta(a, A, M)$ which is $\{\phi(x;b) : M \models \phi[a;b], b \in A, \phi(x;y) \in \Delta\}$ and if $A \subseteq M$ then $S_\Delta(A, M) \stackrel{\text{def}}{=} \{tp_\Delta(a, A) : a \in M\}$. Note that in [27] $S_\Delta(A, M)$ denotes the set of all complete Δ -types with parameters from A that are consistent with $Th(\langle M, c_a\rangle_{a\in A}$. It is important for us to limit attention to the types realized in M in order to avoid dependence on the compactness theorem. It is usually important that Δ is closed under negation, so when $\Delta = \{\phi, \neg\phi\}$, instead of writing $tp_\Delta(\cdots)$ and $S_\Delta(\cdots)$ we will write $tp_\phi(\cdots)$ and $S_\phi(\cdots)$, respectively.

2 The effect of the order and independence properties on the number of local types

In this section, we fix some notation and terms and then define the first important concepts. In the following definition, the first three parts are from [27], (4) is a generalization of a definition of Shelah, and (5) is from Grossberg and Shelah [13].

Definition 2.1 1. *For a set Δ of L – formulas and a natural number n, a (Δ, n) – type over a set A is a set of formulas of the form $\phi(x;a)$ where $\phi(x;y) \in \Delta$ and $a \in A$ with $l(x) = n$. If $\Delta = L$, we omit it, and we just say "ϕ – type" for a $(\{\phi(x;y), \neg\phi(x;y)\}, l(x))$ – type.*

2. *Given a (Δ, n) – type p over A, define $dom(p) = \{a \in A$: for some $\phi \in \Delta$, $\phi(x;a) \in p\}$.*

3. *A type p $\underline{(\Delta_0, \Delta_1)\text{ - splits over}}$ $B \subseteq dom(p)$ if there is a $\phi(x; y) \in \Delta_0$ and $b, c \in dom(p)$ such that $tp_{\Delta_1}(b, B) = tp_{\Delta_1}(c, B)$ and $\phi(x; b), \neg\phi(x; c) \in p$. If p is a Δ - type and $\Delta_0 = \Delta_1 = \Delta$, then we just say p splits over B.*

4. *We say that $(M, \phi(x; y))$ has the k - independence property if there are $\{a_i : i < k\} \subseteq M$, and $\{b_w : w \subseteq k\} \subseteq M$, such that $M \models \phi[a_i; b_w]$ if and only if $i \in w$. We will say that M has the k - independence property when there is a formula ϕ such that (M, ϕ) does.*

5. *$(M, \phi(x; y))$ has the n - order property (where $l(x) = l(y) = k$) if there exists a set of k - tuples $\{a_i : i < n\} \subseteq M$ such that $i < j$ if and only if $M \models \phi[a_i, a_j]$ for all $i, j < n$. We will say that M has the n - order property if there is a formula ϕ so that (M, ϕ) has the n - order property.*

WARNING: This use of "order property" corresponds to neither the order property nor the strict order property in [27]. The definition comes rather from [11].

The following monotonicity property is immediate from the definitions.

Proposition 2.2 *Let sets $B \subseteq C \subseteq A$ and a complete (Δ, n) - type p be given with $Dom(p) \subseteq A$. If p does not split over B, then p does not split over C.*

Fact 2.3 *(Shelah see [27]) Let T be a complete first order theory. The following conditions are equivalent:*

1. *T is unstable.*
2. *There are $\phi(x; y) \in L(M)$, $M \models T$, and $\{a_n : n < \omega\} \subseteq M$ such that $l(x) = l(y) = l(a_n)$, and for every $n, k < \omega$ we have $n < k \Leftrightarrow M \models \phi[a_n; a_k]$.*

The compactness theorem gives us the following

Corollary 2.4 *Let T be a stable theory, and suppose that $M \models T$ is an infinite model.*

1. *For every $\phi(x, y) \in L(M)$ there exists a natural number n_ϕ such that (M, ϕ) does not have the n_ϕ – order property.*

2. *For every $\phi(x, y) \in L(M)$ there exists a natural number k_ϕ such that (M, ϕ) does not have the k_ϕ – independence property.*

3. *If T is categorical in some cardinality greater than $|T|$, then for every $\phi(x, y) \in L(M)$ there exists a natural number d_ϕ such that (M, ϕ) does not have the d_ϕ – cover property (see Definition 2.14).*

We first establish that the failure of either the independence property or the order property for ϕ implies that there is a polynomial bound on the number of ϕ – types. The more complicated of these to deal with is the failure of the order property. At the same time this is perhaps the more natural property to look for in a given structure. The bounds in this case are given in Theorem 2.9. The failure of the independence property gives us a far better bound (i.e., smaller degree polynomial) with less work. Theorem 2.13 reproduces this result of Shelah paying attention to the specific connection between the bound and where the independence property fails.

This first lemma is a finite version of Lemma 5 from [11].

Lemma 2.5 *Let $\phi(x; y)$ be a formula in L, n a positive integer, $s = l(y)$, $r = l(x)$, $\psi(y; x) = \phi(x; y)$. Suppose that $\{A_i \subseteq M \ : \ i \leq 2n\}$ is an increasing chain of sets such that for every $B \subseteq A_i$ with $|B| \leq 3sn$, every type in $S_\phi(B, M)$ is realized in A_{i+1}. Then if there is a type $p \in S_\phi(A_{2n}, M)$ such that for all $i < 2n$, $p|A_{i+1}$ (ψ, ϕ) – splits over every subset of A_i of size at most $3sn$, then (M, ρ) has the $n-$ order property, where*

$$\rho(x_0, x_1, x_2; y_0, y_1, y_2) \stackrel{\text{def}}{=} [\phi(x_0; y_1) \leftrightarrow \phi(x_0; y_2)]$$

Proof: Choose $d \in M$ realizing p. Define $\{a_i, b_i, c_i \in A_{2i+2} \ : \ i < n\}$ by induction on i. Assume for $j < n$ that we have defined these for all $i < j$. Let $B_j = \bigcup\{a_i, b_i, c_i \ : \ i < j\}$. Notice that $|B_j| \leq 3sj < 3sn$, so by the assumption, $p|A_{2j+1}$ (ψ, ϕ) – splits over B_j. That is, there are a_j, $b_j \in A_{2j+1}$ such that

$$tp_\psi(a_j, B_j, M) = tp_\psi(b_j, B_j, M),$$

and

$$M \models \phi[d; a_j] \wedge \neg\phi[d; b_j].$$

Now choose $c_j \in A_{2j+1}$ realizing $tp(d, B_j \cup \{a_j, b_j\}, M)$ (which can be done since $|B_j \cup \{a_j, b_j\}| \leq 3sj + 2s < 3s(j+1) \leq 3sn$). This completes the inductive definition.

For each i, let $d_i = c_i a_i b_i$. We will check that the sequence of d_i and the formula

$$\rho(x_0, x_1, x_2; y_0, y_1, y_2) \stackrel{\text{def}}{=} [\phi(x_0; y_1) \leftrightarrow \phi(x_0; y_2)]$$

witness the n – order property for M.

If $i < j < n$, then $c_i \in B_j$. By choice of a_j and b_j, $tp_\psi(a_j, B_j, M) = tp_\psi(b_j, B_j, M)$, so in particular,

$$M \models \phi[c_i; a_j] \leftrightarrow \phi[c_i; b_j]$$

That is, $M \models \rho[d_i; d_j]$.

On the other hand, if $i \leq j < n$, then $\phi(x; a_i) \in tp_\phi(d, B_j \cup \{a_j, b_j\}, M)$ and $\phi(x; b_i) \notin tp_\phi(d, B_j \cup \{a_j, b_j\}, M)$, and so, by the choice of c_j, we have that

$$M \models \phi[c_j; a_i] \wedge \neg\phi[c_j; b_i].$$

That is, $M \models \neg\rho[d_j; d_i]$ in this case. $\square$

In order to see the relationship between this definition of the order property and Shelah's, we mention Corollary 2.8 below. Note that it is the formula ϕ, not the ρ of Lemma 2.5, which has the weak order property in the Corollary.

Definition 2.6 *(M, ϕ) has the weak m – order property if there exist $\{d_i : i < m\} \subseteq M$ such that for each $j < m$,*

$$M \models \exists x \bigwedge_{i<m} \phi(x; d_i)^{if(i \geq j)}$$

REMARK: This is what Shelah [27] calls the m – order property.

Definition 2.7 *We write $x \to (y)^a_b$ if for every partition Π of the a - element subsets of $\{1, \ldots, x\}$ with b parts, there is a y - element subset of $\{1, \ldots, x\}$ with all of its a – element subsets in the same part of Π.*

Corollary 2.8 *1. If in addition to the hypotheses of Lemma 2.5 we have that* $(2n) \longrightarrow (m+1)^2_2$*, then* ϕ *has the weak* m *– order property in* M.

2. If in addition to the hypotheses of Lemma 2.5 we have that $n \geq \frac{2^{2m-1}}{\pi m}$*, then* ϕ *has the weak* m *– order property in* M.

Proof: (This is essentially [27] I.2.10(2))

1. Let a_i, b_i, c_i for $i < n$ be as in the proof of Lemma 2.5. For each pair $i < j \leq n$, define

$$\chi(i,j) := \begin{cases} 1 & \text{if } M \models \phi[c_i; a_j] \\ 0 & \text{otherwise.} \end{cases}$$

Since $(2n) \longrightarrow (m+1)^2_2$, we can find a subset I of $2n$ of cardinality $m+1$ on which χ is constant and which we can enumerate as $I = \{i_0, \ldots, i_m\}$.

If χ is 1 on I, then for every k with $1 \leq k \leq m+1$

$$\{\neg\phi(x; b_{i_j})^{if(j>k)} \; : \; 1 \leq j < m\}$$

is realized by $c_{i_{k-1}}$. Therefore, the sequence $\{b_{i_0}, \ldots, b_{i_m}\}$ witnesses the weak m – order property of ϕ in M.

On the other hand, if χ is 0 on I, then for every k with $1 \leq k \leq m+1$

$$\{\neg\phi(x; a_{i_j})^{if(j>k)} \; : \; 1 \leq j < m\}$$

is realized by $c_{i_{k-1}}$. Therefore, the sequence $\{a_{i_0}, \ldots, a_{i_m}\}$ witnesses the weak m – order property of $\neg\phi$ in M. Of course, it is equivalent for ϕ and $\neg\phi$ to have the weak m – order property in M.

2. By Stirling's formula, $n \geq \frac{2^{2m-1}}{\pi m}$ implies that $n \geq \frac{1}{2}\binom{2m}{m}$, and from [10], $n \geq \frac{1}{2}\binom{2m}{m}$ implies that $(2n) \longrightarrow (m+1)^2_2$.

□

We can now establish the relationship between the number of types and the order property.

Theorem 2.9 *If $\phi(x;y) \in L(M)$ is such that*

$$\rho(x_0,x_1,x_2;y_0,y_1,y_2) \stackrel{\text{def}}{=} \phi[(x_0;y_1) \leftrightarrow \phi(x_0;y_2)]$$

does not have the n – order property in M, then for every set $A \subseteq M$ with $|A| \geq 2$, we have that $|S_\phi(A,M)| \leq 2n|A|^k$, where $k = 2^{(3ns)^{t+1}}$ for $r = l(x)$, and $s = l(y)$, and $t = \max\{r,s\}$.

Proof: Suppose that there is some $A \subseteq M$ with $|A| \geq 2$ so that $|S_\phi(A,M)| > (2n)|A|^k$. Let $\psi(y;x) = \phi(x;y)$, $m = |A|$, and let $\{a_i : i \leq (2n)m^k\} \subseteq M$ be witnesses to the fact that $|S_\phi(A,M)| > (2n)m^k$. (That is, each of these tuples realizes a different ϕ – type over A.) Define $\{A_i : i < 2n\}$, satisfying

1. $A \subseteq A_i \subseteq A_{i+1} \subseteq M$,
2. $|A_i| \leq c^{e(i)}m^{(3ns)^i}$,where $c := 2^{2+(3sn)^t}$ and $e(i) := \frac{(3ns)^i-1}{3ns-1}$, and
3. for every $B \subseteq A_i$ with $|B| \leq 3sn$, every $p \in S_\phi(B,M) \bigcup S_\psi(B,M)$ is realized in A_{i+1}.

To see that this can be done, we need only check the cardinality constraints. There are at most $|A_i|^{3sn}$ subsets of A_i with cardinality at most $3sn$, and over each such subset B, there are at most $2^{(3sn)^r}$ and $2^{(3sn)^s}$ types in $S_\psi(B,M)$ and $S_\phi(B,M)$, respectively, so there are at most $2^{(3sn)^r}+2^{(3sn)^s} \leq 2^{1+(3sn)^t}$ types in $S_\psi(B,M) \bigcup S_\phi(B,M)$ for each such B. Therefore, A_{i+1} can be defined so that

$$\begin{aligned}
|A_{i+1}| &\leq |A_i| + (2^{1+(3sn)^t})|A_i|^{3sn} \\
&\leq c|A_i|^{3sn} \\
&\leq c(c^{e(i)}m^{(3ns)^i})^{3sn} \\
&= c^{1+e(i)(3sn)}m^{(3sn)^{i+1}} \\
&= c^{e(i+1)}m^{(3sn)^{i+1}}
\end{aligned}$$

Claim 2.10 *There is a $j < (2n)m^k$ such that for every $i < 2n$ and every $B \subseteq A_i$ with $|B| \leq 3sn$, $tp(a_j, A_{i+1})$ (ψ,ϕ) – splits over B.*

Proof: (Of Claim 2.10) Suppose not. That is, for every $j \leq (2n)m^k$, there is an $i(j) < 2n$ and a $B \subseteq A_{i(j)}$ with $|B| \leq 3sn$, so that $tp(a_j, A_{i(j)+1})$ does not (ψ, ϕ) – split over B. Since i is a function from $1 + (2n)m^k$ to $2n$, there must be a subset S of $1 + (2n)m^k$ with $|S| > m^k$, and an integer $i_0 < 2n$ such that for all $j \in S$, $i(j) = i_0$. Now similarly, there are less than $|A_{i_0}|^{3sn}$ subsets of A_{i_0}, with cardinality at most $3sn$, so there is a $T \subseteq S$ with

$$|T| > \frac{m^k}{|A_{i_0}|^{3sn}}$$

and a $B_0 \subseteq A_{i_0}$, with $|B_0| \leq 3sn$ such that for all $j \in T$, $tp(a_j, A_{i_0+1})$ does not (ψ, ϕ) – split over B_0. Since $|A_{i_0}| \leq c^{e(i_0)} m^{(3ns)^{i_0}} \leq (cm)^{(3sn)^{2n}}$, then

$$|T| \geq \frac{m^k}{(cm)^{(3sn)^{2n}}}$$

Let $C \subseteq A_{i_0+1}$ be obtained by adding to B_0, realizations of every type in $S_\phi(B_0, M) \bigcup S_\psi(B_0, M)$. This can clearly be done so that $|C| \leq 3ns + 2^{(3ns)^r} + 2^{(3ns)^s}$. The maximum number of ϕ – types over C is at most $2^{|C|^s} \leq 2^{c^s}$.

Claim 2.11 $m^{k-(3ns)^{2n}} > (2^{c^s})(c^{(3ns)^{2n}})$

Proof: (Of Claim 2.11) Since $c = 2^{2+(3ns)^t}$, we have $c^s + (3ns)^{2n}(2 + (3ns)^t)$ as the exponent on the right-hand side above. Since $m \geq 2$, it is enough to show that

$$\begin{aligned} k &> (c^s + (3ns)^{2n}(2 + (3ns)^t) + (3ns)^{2n} \\ &= 2^{s(2+(3ns)^t)} + (3ns)^{2n}(3 + (3ns)^t) \end{aligned}$$

This follows from the definition of k (recall that $k = 2^{(3ns)^{t+1}}$), so we have established Claim 2.11. $\square_{2.11}$

Therefore, $|T|$ is greater than the number of ϕ – types over C, so there must be $i \neq j \in T$ such that $tp_\phi(a_i, C) = tp_\phi(a_j, C)$. Since $tp_\phi(a_i, A) \neq tp_\phi(a_j, A)$, we may choose $a \in A$ so that $M \models \phi[a_i, a] \wedge \neg\phi[a_j, a]$. Now choose $a' \in C$ so that $tp_\psi(a, B_0) = tp_\psi(a', B_0)$ (this is how C is defined after all). Since $tp_\phi(a_i, A_{i_0+1})$ does not (ψ, ϕ) – split over B_0, we have that

$$\phi(x; a) \in tp_\phi(a_i, A_{i_0+1}) \text{ if and only if } \phi(x; a') \in tp_\phi(a_i, A_{i_0+1})$$

so $M \models \phi[a_i, a'] \wedge \neg\phi[a_j, a']$, contradicting the fact that $tp_\phi(a_i, C) = tp_\phi(a_j, C)$ and thus completing the proof of Claim 2.10. Now letting j be as in Claim 2.10 above and applying Lemma 2.5 completes the proof of Theorem 2.9. $\square$

Theorem 2.13 below gives a better result under different assumptions. The next lemma is II, 4.10, (4) in [27]. It comes from a question due to Erdős about the so-called "trace" of a set system which was answered by Shelah and Perles [22] in 1972. Purely combinatorial proofs (i.e., proofs in the language of combinatorics) can also be found in most books on extremal set systems (e.g., Bollobas [6]).

Lemma 2.12 *If S is any family of subsets of the finite set I with*

$$|S| > \sum_{i<k} \binom{|I|}{i}$$

then there exist $\alpha_i \in I$ for $i < k$ such that for every $w \subseteq k$ there is an $A_w \in S$ so that $i \in w \Leftrightarrow \alpha_i \in A_w$. (The conclusion here is equivalent to trace(I) $\geq k$ in the language of [6].)

Proof: See Theorem 1 in Section 17 of [6] or Ap.1.7(2) in [27]. $\square$

Theorem 2.13 *If $\phi(x; y) \in L(M)$ ($r = l(x)$, $s = l(y)$) does not have the k – independence property in M, then for every set $A \subseteq M$, if $|A| \geq 2$, then $|S_\phi(A, M)| \leq |A|^{s(k-1)}$.*

Proof: (Essentially [27], II.4.10(4)) Let F be the set of ϕ – formulas over A. Then

$$|F| < |A|^s.$$

So if $|S_\phi(A, M)| > |A|^{s(k-1)}$, then certainly

$$|S_\phi(A, M)| > \sum_{i<k} \binom{|F|}{i},$$

in which case Lemma 2.12 can be applied to F and $S_\phi(A, M)$ to get witnesses to the k – independence property in M, a contradiction. $\square$

The "moral" of Theorem 2.9 and Theorem 2.13 is that when ϕ has some nice properties, there is a bound on the number of ϕ – types over A which is polynomial in $|A|$. Note that the difference between the two properties is that the degree of the polynomial in the absence of the k – independence property is linear in k while in the absence of the n – order property the degree is exponential in n. Also the bounds on ϕ – types in the latter case hold when a formula ρ related to ϕ (as opposed to ϕ itself) is without the n – order property.

Another property discovered by Keisler (in order to study saturation of ultrapowers, see [15]), and studied extensively by Shelah is the "finite cover property" (see [27]) whose failure essentially provides us with a strengthening of the compactness theorem.

Definition 2.14 *We say that (M, ϕ) does not have the d - cover property if for every $n \geq d$ and $\{b_i : i < n\} \subseteq M$,* **if**

$$\left(\forall w \subseteq n \left[|w| < d \Rightarrow M \models \exists x \bigwedge_{i \in w} \phi(x; b_i)\right]\right)$$

then

$$M \models \exists x \bigwedge_{i<n} \phi(x; b_i).$$

EXAMPLE **2.15** *If $M = (M, R)$ is the countable random graph, then (M, R) fails to have the 2 – cover property. If M is the countable universal homogeneous triangle-free graph, then (M, R) fails to have the 3 – cover property.*

3 Indiscernible sequences in large finite sets

NOTE: The next definition is an interpolant of Shelah's [27], I.2.3, and Ramsey's notion of canonical sequence.

Definition 3.1 *1. A sequence $I = \langle a_i : i < n \rangle \subseteq M$ is called a (Δ, m) – indiscernible sequence over $A \subseteq M$ (where Δ is a set of $L(M)$ – formulas) if for every $i_0 < \ldots < i_{m-1} \in I$, $j_0 < \ldots < j_{m-1} \in I$ we have that $tp_\Delta(a_{i_0} \cdots a_{i_{m-1}}, A, M) = tp_\Delta(a_{j_0} \cdots a_{j_{m-1}}, A, M)$*

2. A set $I = \{a_i : i < n\} \subseteq M$ is called a (Δ, m) – indiscernible set over $A \subseteq M$ if and only if for every $\{i_0, \ldots, i_{m-1}\}$, $\{j_0, \ldots, j_{m-1}\} \subseteq I$ we have

$$tp_\Delta(a_{i_0}\cdots a_{i_{m-1}}, A, M) = tp_\Delta(a_{j_0}\cdots a_{j_{m-1}}, A, M).$$

Note that if $\phi(x;b) \in tp(a_0 \ldots a_{m-1}, B, M)$, then necessarily $l(x) = m \cdot l(a_0)$.

EXAMPLE **3.2** *1. In the model $M_n^m = \langle m, 0, 1, \chi\rangle$ ($n \leq m < \omega$) where χ is function from the increasing n - tuples of m to $\{0,1\}$, any increasing enumeration of a monochromatic set is an example of a $(\Delta, 1)$ - indiscernible sequence over $\emptyset$ where $\Delta = \{\chi(x) = 0, \chi(x) = 1\}$.*

2. In a graph (G, R), cliques and independent sets are examples of $(R, 2)$ - indiscernible sets over $\emptyset$.

Recall that in a stable first order theory, every sequence of indiscernibles is a set of indiscernibles. In our finite setting this is also true if the formula fails to have the n - order property. The argument below follows closely that of Shelah [27].

Theorem 3.3 *If M does not have the n - order property, then any sequence $I = \langle a_i : i < n+m-1\rangle \subseteq M$ which is (ϕ, m) - indiscernible over $B \subseteq M$ is a set of (ϕ, m) - indiscernibles over B.*

Proof: Since any permutation of $\{1, \ldots, n\}$ is a product of transpositions $(k, k+1)$, and since I is a ϕ - indiscernible sequence over B, it is enough to show that for each $b \in B$ and $k < m$,

$$M \models \phi[a_0 \cdots a_{k-1}a_{k+1}a_k \cdots a_{m-1}; b] \leftrightarrow \phi[a_0 \cdots a_{k-1}a_k a_{k+1} \cdots a_{m-1}; b].$$

Suppose this is not the case. Then we may choose $b \in B$ and $k < m$ so that

$$M \models \neg\phi[a_0 \cdots a_{k-1}a_{k+1}a_k \cdots a_{m-1}; b] \wedge \phi[a_0 \cdots a_{k-1}a_k a_{k+1} \cdots a_{m-1}; b].$$

Let $c = a_0 \cdots a_{k-1}$ and $d = a_{n+k+1} \cdots a_{n+m-2}$ making $l(c) = k$ and $l(d) = m - k - 2$). By the indiscernibility of I,

$$M \models \neg\phi[ca_{k+1}a_k d; b] \wedge \phi[ca_k a_{k+1} d; b].$$

For each i and j with $k \leq i < j < n+k$, we have (again by the indiscernibility of the sequence I) that

$$M \models \neg\phi[ca_j a_i d; b] \wedge \phi[ca_i a_j d; b].$$

Thus the formula $\psi(x, y; cdb) \stackrel{\text{def}}{=} \phi(c, x, y, d; b)$ defines an order on $\langle a_i : k \leq i < n + k\rangle$ in M, a contradiction. $\square$

The following definition is a generalization of the notion of end-homogenous sets in combinatorics (see section 15 of [8]) to the context of Δ – indiscernible sequences.

Definition 3.4 *A sequence $I = \langle a_i : i < n\rangle \subseteq M$ is called a (Δ, m) – end-indiscernible sequence over $A \subseteq M$ (where Δ is a set of $L(M)$ – formulas) if for every $\{i_0, \ldots, i_{m-2}\} \subseteq n$ and $j_0, j_1 < n$ both larger than $\max\{i_0, \ldots, i_{m-2}\}$, we have*

$$tp_\Delta(a_{i_0} \cdots a_{i_{m-2}} a_{j_0}, A, M) = tp_\Delta(a_{i_0} \cdots a_{i_{m-2}} a_{j_1}, A, M)$$

Definition 3.5 *For the following lemma, let $F : \omega \to \omega$ be given, and fix the parameters, α, r, and m. We define the function F^* for each $k \geq m$ as follows:*

- $F^*(0) = 1$,
- $F^*(j + 1) = 1 + F^*(j) \cdot F(\alpha + m \cdot r \cdot j)$ *for* $j < k - 2 - m$, *and*
- $F^*(j + 1) = 1 + F^*(j)$ *for* $k - 2 - m \leq j < k - 2$.

We will not need $j \geq k - 2$.

Lemma 3.6 *Let $\psi(x; y) = \phi(x_1, \ldots, x_{m-1}, x_0; y)$. If for every $B \subseteq M$, $|S_\psi(B, M)| < F(|B|)$, and $I = \{c_i : i \leq F^*(k - 2)\} \subseteq M$ (where $l(c_i) = l(x_i) = r$, $\alpha = |A|$), then there is a $J \subseteq I$ such that $|J| \geq k$ and J is a (ϕ, m) – end-indiscernible sequence over A.*

Proof: (For notational convenience when we have a subset $S \subseteq I$, we will write $\min S$ instead of the clumsier $c_{\min\{i\,:\,c_i \in S\}}$.) We now construct $A_j = \{a_i : i \leq j\} \subseteq I$ and $S_j \subseteq I$ by induction on $j < k - 1$ so that

1. $a_j = \min S_j$,
2. $S_{j+1} \subseteq S_j$,

3. $|S_j| > F^*(k-2-j)$, and
4. whenever $\{i_0, \ldots, i_{m-1}\} \subseteq j$ and $b \in S_j$,

$$tp_\phi(a_{i_0} \cdots a_{i_{m-2}} a_j, A, M) = tp_\phi(a_{i_0} \cdots a_{i_{m-2}} b, A, M).$$

The construction is completed by taking an arbitrary $a_{k-1} \in S_{k-2} - \{a_{k-2}\}$. (which is possible by (3) since $F^*(0) = 1$), and letting $J = \langle a_i : i < k \rangle$. We claim that J will be the desired (ϕ, m) – end-indiscernible sequence over A.

To see this, let $\{i_0, \ldots, i_{m-2}, j_0, j_1\} \subseteq k$ with $\max\{i_0, \ldots, i_{m-2}\} < j_0 < j_1 < k$ be given. Certainly then $\{i_0, \ldots, i_{m-2}\} \subseteq j_0$ and $a_{j_1} \in S_{j_0}$, so by (4) we have that

$$tp_\phi(a_{i_0} \cdots a_{i_{m-2}} a_{j_0}, A, M) = tp_\phi(a_{i_0} \cdots a_{i_{m-2}} a_{j_1}, A, M).$$

To carry out the construction, first set $a_j = c_j$ and $S_j = \{c_i : j \le i \le F^*(k-2)\}$ for $0 \le j \le m-1$. Clearly, we have satisfied all conditions in this. Now assume for some $j \ge m$ that A_{j-1} and S_{j-1} have been defined satisfying the conditions.

Define the equivalence relation $\sim$ on $S_{j-1} - \{a_{j-1}\}$ by $c \sim d$ if and only if for all $\{i_0, \ldots, i_{r-1}\}$,

$$tp_\phi(a_{i_1} \cdots a_{i_{m-1}} c, A, M) = tp_\phi(a_{i_1} \cdots a_{i_{m-1}} d, A, M)$$

The number of $\sim$ – classes then is at most $|S_\psi(A \cup A_j)| < F(\alpha + m \cdot r \cdot j)$. Therefore, at least one class $S_m cdotrcdotj$ has cardinality at least $\frac{|S_{j-1}|-1}{F(\alpha+m \cdot r \cdot j)}$. Let $a_j = \min S_j$. By definition of F^*, $\frac{F^*(k-2-j+1)}{F(\alpha+m r \cdot j)} > F^*(k-2-j)$, so we have that $|S_j| > F^*(k-2-j)$. It is easy to see that condition (4) is satisfied. □

For the following lemma, we once again need a function defined in terms of the parameters of the problem. We will need the parameter r and the function F^* defined for Lemma 3.6 (which depends on r, α, and m). Let f_i be the F^* that we get when $m = i$ (and α and r are fixed) in Lemma 3.6.

For the following lemma define

$$g_i := \begin{cases} id & \text{if } i = 0 \\ f_{i-1} \circ (g_{i-1} - 2) & \text{otherwise} \end{cases}$$

Lemma 3.7 *If $J = \{a_i \ : \ i \leq g_{m-1}(k-1)\} \subseteq M$ is a (ϕ,m) – end-indiscernible sequence over $A \subseteq M$, then there is a $J' \subseteq J$ such that $|J'| \geq k$ and J' is a (ϕ, m) – indiscernible sequence over A.*

Proof: (By induction on m) Note that if $m = 1$, there is nothing to do since end - indiscernible *is* indiscernible in this case. Now let $m \geq 1$ be given, and assume that the result is true of all (θ, m) – end-indiscernible sequences.

Let a formula ϕ and a sequence J of $(\phi, m+1)$ – end-indiscernible sequences over $\emptyset$. Let c be the last element in J. Define ψ so that

$$M \models \psi[a_0, \ldots, a_{m-1}; b] \text{ if and only if } M \models \phi[a_0 \cdots a_{m-1}c; b]$$

for all $a_0, \ldots, a_{m-1} \in J$, $b \in M$. Note then that $|S_\psi(B, M) \leq |S_\phi(B, M)|$ for all $B \subseteq M$, so we can use the same F^* for ψ as for ϕ. (This result can be improved by using a sharper bound on the number of ψ – types.) By the definition of g_m and Lemma 3.6, there must be a subset J'' of J with cardinality at least $g_{m-1}(k-2)$ which is ψ – end-indiscernible over A. By the inductive hypothesis, there is a subsequence of J'' with cardinality at least $k-1$ which is (ψ, m) – indiscernible over A. Form J' by adding c to the end of this sequence. It follows from the $(\phi, m+1)$ – end-indiscernibility of J and the (ψ, m) – indiscernibility of J'' that J' is $(\phi, m+1)$ – indiscernible over A. □

Theorem 3.8 *For any $A \subseteq M$ and any sequence I from M with $|I| \geq g_m(k-1)$, there is a subsequence J of I with cardinality at least k which is (ϕ, m) – indiscernible over A.*

Proof: By Lemmas 3.6 and 3.7. □

Our goal now is to apply this to theories with different properties to see how these properties affect the size of a sequence one must look in to be assured of finding an indiscernible sequence. First we will do a basic comparison between the cases when we do and do not have a polynomial bound on the number of types over a set. In each of these cases, we will give the bound to find a sequence indiscernible over $\emptyset$. We will use the notation $\log^{(i)}$ for

$$\underbrace{\log_2 \circ \log_2 \circ \cdots \circ \log_2}_{i \text{ times}}$$

Corollary 3.9 *1. If $F(i) = 2^{i^m}$ (which is the worst possible case), then $\log^{(m)} g_m(k-1) \le 4k$.*

2. If $F(i) = i^p$, then $\log^{(m)} g_m(k-1) \le 2mk + \log_2 k + \log_2 p$.

We now combine part (2) above with the results from the previous section to see what happens in the specific cases of structures without the n – order property and structures without the n – independence property. We define by induction on i the function

$$\beth(i, x) = \begin{cases} x & \text{if } i = 0 \\ 2^{\beth(i-1,x)} & \text{if } i > 0 \end{cases}$$

Recall that for the formula $\phi(x; y)$ we have defined the parameters $r = l(x)$, $s = l(y)$, and $t = \max\{r, s\}$.

Corollary 3.9 *3. If (M,ϕ) fails to have the n – independence property and $I = \{a_i : i < \beth(m, 2k + \log_2 k + \log_2 n + \log_2 m)\} \subseteq M$, then there is a $J \subseteq I$ so that $|J| \ge k$ and J is a (ϕ, m) – indiscernible sequence over $\emptyset$.*

4. If (M, ϕ) fails to have the n – order property and $I = \{a_i : i < \beth(m, 2k + \log_2 k + (3ns)^{t+1})\} \subseteq M$, then there is a $J \subseteq I$ so that $|J| \ge k$ and J is a (ϕ, m) – indiscernible sequence over $\emptyset$.

Finally, note that with the additional assumption of failure of the d – cover property, if d is smaller than n, then from the assumptions in (3) and (4) above, we could infer a failure of the d – independence property or the d – order property improving the bounds even further.

4 Applications to Graph Theory

In this section we look to graph theory to illustrate some applications. The reader should be warned that the word "independent" has a graph - theoretic meaning, so care must be taken when reading "independent set" versus "independence property".

4.1 The independence property in random graphs

A first question is "How much independence can one expect a random graph to have?" We will approach the answer to this question along the lines of Albert & Frieze [1]. There an analogy is made to the Coupon Collector Problem, and we will continue this here.

The Coupon Collector Problem (see Feller [9]) is essentially that if n distinct balls are independently and randomly distributed among m labeled boxes (so each distribution has the same probability m^{-n} of occurring), then what is the probability that no box is empty? Letting $q(n,m)$ be this probability, it is easy to compute that

$$q(n,m) = \sum_{i=0}^{m}(-1)^i \binom{m}{i} \left(1 - \frac{i}{m}\right)^n = \frac{m!\, S_{m,n}}{m^n}$$

where $S_{n,m}$ is the Stirling number of the second kind.

It is well - known that, for $\lambda = me^{-n/m}$, $q(n,m) - e^{-\lambda}$ tends to 0 as n and m get large with λ bounded.

The way that this will be applied in our context is as follows. We will say that a certain set $\{v_1, \ldots, v_k\}$ of vertices *witnesses the k - independence property in G* if (G,R) has the k – independence property with $a_i = v_i$ (see Definition 4). Notice that any k vertices $\{v_1, \ldots, v_k\}$ determine 2^k "boxes" defined by all possible Boolean combinations of formulas $\{R(x,v_1), \ldots, R(x,v_k)\}$ (a vertex being "in a box" meaning it witnesses the corresponding formula in G). The remaining $n-k$ vertices are then equally likely to fill each of the 2^k boxes, so the probability that these k vertices witness the k – independence property in G is just $q(n-k, 2^k)$. So for $\lambda = \lambda(n,k) = 2^k \exp(-(n-k)/2^k)$ bounded (as $n, k \to \infty$) we will have the probability that k particular vertices witness the k - independence property in a graph on n vertices tends to $e^{-\lambda}$, and the probability that a graph on n vertices has the k – independence property is at most

$$\binom{r}{k} e^{-\lambda} \le n^k e^{-\lambda} \text{ as } n, k \to \infty$$

If $n = k + k2^k$, then $q(n-k, 2^k) \to 1$, so the particular vertices $\{1, \ldots, k\}$ witness k – independence in a graph on $k + k2^k$ vertices almost surely. On the other hand,

Theorem 4.1 *A random graph on $n = k + 2^k(\log k)$ vertices has the failure of the k – independence property almost surely.*

Proof: For $n = k + 2^k \log k$, the λ from above is $\frac{2^k}{k}$, and $\log(n^k e^{-\lambda}) = k\log(k + 2^k \log k) - 2^k/k$ which clearly goes to $-\infty$ as $k \to \infty$, so the probability that a graph on n vertices has the k – independence property goes to 0. □

4.2 Ramsey's theorem for finite hypergraphs

We can improve (for the case of hypergraphs without n – independence) the best known upper bounds for the Ramsey number $R_r(a, b)$. First we should say what this means.

Definition 4.2

1. *An r –* graph *is a set of* vertices V *along with a set of r – element subsets of V called* edges. *The edge set will be identified in the language by the r – ary predicate R.*
2. *A* complete r – graph *is one in which all r – element subsets of the vertices are edges. An* empty r – graph *is one in which none of the r – element subsets of the vertices are edges.*
3. *$R_r(a, b)$ denotes the smallest positive integer N so that in any r – hypergraph on N vertices there will be an induced subgraph which is either a complete r – graph on a vertices or an empty r – graph on b vertices.*
4. *We say that an r – graph G* has the n – independence property *if $(G, R(x))$ does (where $l(x) = r$).*

Note that the first suggested improvement of Lemma 3.6 applies in this situation – namely, the edge relation is symmetric. We can immediately make the following computations.

Lemma 4.3

1. *In an r – graph G, F is given by $F(i) = 2^q$ where $q = \begin{pmatrix} i \\ r-1 \end{pmatrix}$. Consequently, $F^*(k) \leq 2^{k^r}$ in this case.*

2. *In an r – graph G which does not have the n – independence property, F is defined by*

$$F(i) := \begin{cases} 1 & \textit{for } i < r \\ i^{(r-1)(n-1)} & \textit{otherwise} \end{cases}$$

Consequently $F^*(k) \leq k^{(r-1)(n-1)k}$ in this case.

For a fixed natural number p, define the functions $E_p^{(j)}$ by

- $E^{(1)} = E = (\alpha \mapsto (\alpha + 1)^{p(\alpha+1)})$, and
- $E^{(i+1)} = E \circ E^{(i)}$ for $i \geq 1$.

Theorem 4.4 *Let $n \geq 2$ and $k \geq 3$ be given, and let $p = (r-1)(n-1)$. If an r – graph G on at least $E_p^{(r-1)}(k-1)$ vertices does not have the n – independence property, then G has an induced subgraph on k vertices which is either complete or empty.*

Proof: (By induction on r)

For $r = 2$, the graph has at least $E_{n-1}^{(1)}(k-1) = k^{(n-1)k} \geq 2^{2k}$ vertices, and it is well-known (see e.g. [10]) that $2^{2k} \to (k)_2^2$.

Let $r \geq 3$ be given, and let $G = (V, R)$ be an r – graph as described and set $C = E_p^{(r-2)}(k)$, where $p = (r-1)(n-1)$. Using $F(i) = i^p$ for $i \geq 2$, $(F(0) = F(1) = 1)$ and computing F^* in Lemma 4.3, we first see from Lemma 3.6 that any r – graph on at least $(C+1)^{p(C+1)}$ vertices will have an $(R, 1)$ – end-indiscernible sequence J over $\emptyset$ of cardinality C. Let v be the last vertex in J and define the relation R' on the $(r-1)$ – sets from (the range of) J by

$$R'(X) \text{ if and only if } R(X \cup \{v\}).$$

Now (J, R) is an $(r-1)$ – graph of cardinality C, so by the inductive hypothesis there is an R' – indiscernible subsequence J_0 of J with cardinality k. Clearly $I = \{A \cup \{v\} : A \in J_0\}$ is an R – indiscernible sequence over $\emptyset$ of cardinality k. □

REMARK: Another way to say this is that in the class of r – graphs without the independence property $R_r(k, k) \leq E_p^{(r-1)}(k-1)$.

Comparing upper bounds for $r = 3$

Note that for $r = 3$ in Theorem 4.4, we have $p = 2(n-1)$, and so we get $E_p^{(2)}(k-1) = (2^{2k}+1)^{p(2^{2k}+1)}$ which is roughly $2^{nk(2^{2k}+2)}$. The upper bound for $R_3(k,k)$ in [8] is roughly $2^{2^{4k}}$. So $\log_2 \log_2(\text{their bound}) = 4k$ and

$$\log_2 \log_2(\text{our bound}) = \log_2(nk(2^{2k+2})) = \log_2 p + \log_2 k + (2k+2)$$

which is smaller than $4k$ as long as $2k - 2 - \log_2 k > \log_2 n$. This is true as long as $n < 2^{2k-2}/k$.

For example, for $k = 10$ our bound is about $2^{c(n-1)}$ where c is roughly 4×10^7 and theirs is about $2^{2^{40}}$. Since 2^{40} is roughly 10^{12}, this is a significant improvement in the exponent for 3 – graphs without the n – independence property.

Comparing upper bounds in general

Let a_r be the upper bound for $R_r(k,k)$ given in [8] and b_r be the upper bound as computed for the class of r – graphs without the n – independence property in Theorem 4.4 (both as a function of k, the size of the desired indiscernible set). Since we have $b_{r+1} \leq b_r^{(p)(b_r)}$, we get the relationship

$$\begin{aligned} \log^{(r)} b_{r+1} &\leq \log^{(r-1)}[p\, b_r (\log b_r)] \\ &= \log^{(r-2)}(\log(r-1) + \log(n-1) + \log b_r + \log\log b_r) \end{aligned}$$

for $r \geq 3$, $\log\log b_3 = 2k + \log_2 k + \log_2 n + \log_2 r$, and $\log b_2 = 2k$. It follows that $\log^{(r)} b_{r+1}$ is less than (roughly) $2k + \log_2 k + \log_2 n$ for every r.

In [8], the bounds a_r satisfy $\log a_2 = 2k$, $\log\log a_3 = 4k$, and for $r \geq 3$,

$$\begin{aligned} \log^{(r)} a_{r+1} &= \log^{(r-1)}(a_r^r) = \\ \log^{(r-2)}(r \log a_r) &= \log^{(r-3)}[\log r + \log\log a_r]. \end{aligned}$$

We can then show that $\log^{(r-1)} a_r < 4k + 2$ for all r.

Clearly for each $r \geq 3$, $\frac{b_r}{a_r} \to 0$ as m gets large.

FINAL REMARK: On a final note, the above comparison is only given for r – graphs with $r \geq 3$ because the technique enlisted does not give an improvement in the case of graphs. This has not been pursued in this paper because it seems to be of no interest in the general study. However, the techniques may be of interest to the specialist.

5 Toward a classification theory

5.1 Introduction

One of the most powerful concepts of model theory (discovered by Shelah) is the notion of forking, which from a certain point of view can be considered as an instrument to discover the structure of combinatorial geometry in certain definable subsets of models.

We were unable to capture the notion of forking (or a forking - like concept) for finite structures. What we *can* do is to present an alternative, more global property called stable – amalgamation (which is the main innovation in [26] in dealing with non-elementary classes). We do this by imitating [23]. We have reasonable substitutes for $\kappa(T)$ and $Av(I, A, M)$. The most important property we manage to prove is the symmetry property for stable amalgamation. This is the corresponding property to the exchange principle in combinatorial geometry.

The ultimate goal of the project started here is to have a decomposition theorem not unlike the theorem for finite abelian groups. We hope to identify some properties $P_1, \dots, P_n$ of a class of finite models K in such a way that the following conjecture will hold:

Conjecture 5.1 *If $\langle K, \prec_K\rangle$ satisfies $P_1, \dots, P_n$ then for every $M \in K$ large enough there exists a finite tree $T \subseteq {}^{\omega<}\omega$ and $\{M_\eta \prec_K M \ : \ \eta \in T\}$ such that*

1. *$\{M_\eta \ : \ \eta \in T\}$ is a "stable" tree* [1]
2. *For every $\eta \in T$ we have that $\|M_\eta\| \leq n(K)$* [2]
3. *M is uniquely determined by $\bigcup_{\eta\in T} |M_\eta|$.*

Ideally $P_1, \dots, P_n$ is a minimal list of properties sufficient to derive the above decomposition. We hope to be able to eventually emulate Theorem XI.2.17 in [27].

[1]Defined using the notion of stable amalgamation introduced below.

[2]We acknowledge that cardinality of the universe may not be an appropriate measure of "smallness" for a substructure in this context. The reader should also consider a restriction on the cardinality of a set of generators for M_η as another possibility.

Considering our present state of knowledge it seems that our conjecture is closer to a fantasy than to a mathematical statement. However we seem to have a start. Much of this section together with some of the earlier results can be viewed as a search for candidates for the above mentioned list of properties $P_1, \ldots, P_n$. We hope that this section might form an infrastructure for the classification project.

5.2 Abstract properties

We now begin to look at some of the abstract properties of a class K of finite L – structures with an appropriate partial ordering denoted by $\prec_K$. These properties come from Shelah's list of axioms in §1 of [26].

Definition 5.2 *Let L be a given similarity type, let Δ be a set of L – formulas, and let $n < \omega$, by Δ^*_n we denote the minimal set of L – formulas containing the following set and all its subformulas:*

$$\{\exists x[\bigwedge_{i\in w} \phi(x;y_i) \wedge \bigwedge_{i\in k\setminus w} \neg\phi(x;y_i)] : \phi(x;y) \in \Delta, k \leq n, w \subseteq k, l(y_i) = l(y)\}.$$

We will now look into natural values of k from the previous section.

Theorem 5.3 *Suppose the formula ϕ fails to have the weak n – order property (and hence fails to have the n – independence property in K), and let $\Delta \supseteq \{\phi\}^*_n$ be given. Then for every (Δ, n) – indiscernible sequence I over $\emptyset$ and every $c \in M$, either $|\{a \in I \ : \ M \models \phi[c; a]\}| < n$ or $|\{a \in I \ : \ M \models \neg\phi[c; a]\}| < n$.*

Proof: We may assume the length of I is at least $2n$ since otherwise the result is trivial. We proceed by contradiction. Suppose the result is not true. Then there is a (Δ, n) – indiscernible sequence I from M of length at least $2n$ and a $c \in M$ such that both $|\{\phi[c; a] \ : \ a \in I\}| \geq n$ and $|\{\neg\phi[c; a] \ : \ a \in I\}| \geq n$. Let $\{a_0, \ldots, a_{2n-1}\} \subseteq I$ be such that

$$M \models \bigwedge_{i<n} \phi[c, a_i] \wedge \bigwedge_{n\leq i<2n} \neg\phi[c, a_i] \tag{1}$$

We complete the proof by showing that $\{a_0, \dots, a_{n-1}\}$ exemplifies the n – independence property. Let $w \subseteq n$ be given. Consider the formula

$$\psi_w(y_0, \dots, y_{n-1}) \stackrel{\text{def}}{=} (\exists x) \left[\bigwedge_{i \in w} \phi(x; y_i) \wedge \bigwedge_{i \in n \setminus w} \neg\phi(x; y_i) \right]$$

Let $\{i_0, \dots, i_{k-1}\}$ be an increasing enumeration of w. By (1) the following holds

$$M \models \psi_w[a_{i_0}, \dots, a_{i_{k-1}}, a_n, \dots, a_{2n-1-k}]$$

Since $\psi_w \in \{\Delta\}^*_n$, by the indiscernibility of I we have that also $M \models \psi_w[a_0, \dots, a_{n-1}]$. So for every $w \subseteq n$, we may choose $b_w \in M$ so that

$$M \models \bigwedge_{i \in w} \phi[b_w, a_i] \wedge \bigwedge_{i \in n - w} \neg\phi[b_w, a_i].$$

We are done since $\{a_0, \dots, a_{n-1}\}$ and $\{b_w : w \subseteq n\}$ witness the fact that (M, ϕ) has the n – independence property. □

The following definition is inspired by $\kappa(T)$ in Chapter III of [27].

Definition 5.4 *Let $n < \omega$ be given, and let Δ be a finite set of formulas. $\kappa_{\Delta,n}(K)$ is the least positive integer so that for every $M \in K$, every sequence $I = \langle a_i : i < \beta < \omega \rangle \in M$ which is Δ^*_n – indiscernible over $\emptyset$ has either $M \models \phi[c; a_i]$ or $M \models \neg\phi[c; a_i]$ for less than $\kappa_{\Delta,n}(K)$ elements of I for each $\phi \in \Delta$ and $c \in M$. Recalling that Δ is to be closed under negation, we will write $\kappa_{\phi,n}$ instead of $\kappa_{\{\phi,\neg\phi\},n}$.*

So the previous theorem states that if the formula ϕ fails to have the n – independence property in M, then $\kappa_{\phi,n}(M) \leq n$. When ϕ is understood to not have the n – independence property κ_ϕ will stand for $\kappa_{\phi,n}$. In this case, the following definition makes sense.

Definition 5.5 *Let $n < \omega$ be given, Δ be a finite set of formulas and n be as above. Suppose I is a sequence of (Δ^*_n, n)-indiscernibles over $\emptyset$. Define*

$$\begin{aligned} Av_\Delta(I, A, M) &= \{\phi(x; a) : a \in A, \phi(x; y) \in \Delta \\ &\qquad \text{and } |\{c \in I : M \models \phi[c; a]\}| \geq \kappa_{\Delta,n}(K)\}. \end{aligned}$$

If M is understood, it will often be omitted.

Theorem 5.6 *Let $\psi(y;x) = \phi(x;y)$. If ϕ has neither the n – independence property nor the d – cover property in M, $\Delta \supseteq \{\phi\}^{\star}_{n}$ is finite, and I is a set of (Δ, n) – indiscernibles over $\emptyset$ of length greater than $\max\{d \cdot \kappa_{\psi}(M), 2n\}$, then $Av_{\phi}(I, A, M)$ is a complete ϕ – type over A.*

Proof: That $Av_{\psi}(I, A, M)$ is complete follows from the previous theorem. To see that it is consistent, we need only establish that every d formulas from it are consistent (by failure of the d – cover property), and this follows from the size of I and the pigeonhole principle. □

So we will use the following term to denote when we are in a model in which the notion of average type is well defined.

Definition 5.7 *Let $\psi(y;x) = \phi(x;y)$, and $\Delta = \{\phi, \psi, \neg\phi, \neg\psi\}$. We will say that M is $\underline{(\phi, n, d) \text{ – good}}$ if (M, Δ) has neither the n – independence property nor the d – cover property. In this case, we will define $\lambda_{\phi}(M) = \max\{d \cdot \kappa_{\Delta,n}(M), 2n\}$. We will sometimes refer to this same situation by saying (M, ϕ) is (n, d) – good.*

If K is a class of (ϕ, n, d) – good structures which all include a common set A, then we will say that K is (ϕ, n, d) – good, and define $\lambda(K) = \kappa_{\Delta,n}(K) \cdot |A|^{s}$, where $s = \max\{l(y) : \phi(x; y) \in \Delta\}$.

EXAMPLE **5.8** *Let T be an $\aleph_1$ – categorical theory in a relational language (no function symbols), and let $M \models T$ be an uncountable model (e.g., an uncountable algebraically closed field of positive characteristic). Let $K := \{N \subseteq M : \|N\| < \aleph_0\}$, and let $\phi \in L(T)$. By $\aleph_1$ – categoricity there exist integers n and d such that K is (ϕ, n, d) – good (see Corollary 2.4).*

Definition 5.9 *For a fixed (finite) relational language L and an L – formula ϕ (and $\psi(y; x) = \phi(x; y)$), let K be a class of finite (ϕ, n, d) – good L – structures all of which include a common set A. Fix a $k < \omega$ and and define $\prec_K$, as follows: $N \prec_K M$*

1. *$N \subseteq M$, and for all $a \in A$ and $b \in N$, $M \models \phi[b; a]$ if and only if $N \models \phi[b; a]$.*
2. *For every $a_0, \ldots, a_{k-1} \in A$, if $M \models \exists x \bigwedge_{i<k} \phi(x; a_i)$, then $M \models \bigwedge_{i<k} \phi[b; a_i]$ for some $b \in N$.*

3. *For every $a \in M$, there is a sequence $I \subseteq N$ which is $(\{\psi\}^*_n, n)$ - indiscernible over A with length at least $\lambda(K)$ so that $tp_\phi(a, A, M) = Av_\phi(I, A, M)$.*

We define the same relation for a set Δ of formulas simply by requiring that the above holds for each $\phi \in \Delta$ in the case that K is a class of finite (ϕ, n, d) - good structures.

REMARKS ON DEFINITION 5.9:

- Condition (1) ensures that the fact for elementary classes that forms the basis of the Tarski - Vaught (namely, $N \subseteq M \Rightarrow N \prec_{qf} M$ holds for ϕ – formulas even if ϕ is not quantifier free.
- Condition (2) is like k – saturation relative to ϕ – formulas with parameters from A (i.e. every ϕ – type with at most k parameters from A which is realized in M is also realized in N). It can be thought of as a generalization of the Tarski - Vaught test relativized to formulas from $\{\phi\}^*_k$.
- Condition (3) is a property like the one that guarantees in the first order case that types over a model are stationary. Here we are requiring a strong closure condition on N — namely, if $a \in M \setminus N$, then there is a strong reason why a does not belong to N : there is a long sequence of indiscernibles in N averaging the same ϕ – type over A.
- It should be emphasized that K (and hence $\prec_K$) has parameters A, k, and ϕ which are suppressed for notational convenience.

5.3 Properties of $\prec_K$

We prove the following facts about the relation $\prec_K$. The Roman numerals in parentheses indicate the corresponding Axioms in [26]. Δ is closed under negation and fixed throughout, and K is a class of (Δ, n, d) – good structures which all include a common set A.

Lemma 5.10 *1. (I) If $N \prec_K M$, then $N \subseteq M$.*

2. *(II)* $M_0 \prec_K M_1 \prec_K M_2$ *implies* $M_0 \prec_K M_2$. *Also* $M \prec_K M$ *for all* $M \in K$

3. *(V) If* $N_0 \subseteq N_1 \prec_K M$ *and* $N_0 \prec_K M$, *then* $N_0 \prec_K N_1$.

Proof: The first part of (I) is trivial, and the second part of (II) only requires that one chooses a constant sequence for I in Condition (3). Note that for all three statements, checking Condition (1) is routine.

For the first part of (II), assume the hypothesis is true and first look at Condition (2). Let $\phi \in \Delta$ and $a_i \in A$ for $i < k$ be given, and assume that $M_2 \models \exists x \bigwedge_{i<k} \phi(x; a_i)$ Since $M_1 \prec_K M_2$, we may choose $b \in M_1$ so that $M_2 \models \bigwedge_{i<k} \phi[b; a_i]$. Of course, $b, a_0, \dots, a_{k-1}$ are all from M_1, so we can conclude that $M_1 \models \bigwedge_{i<k} \phi[b; a_i]$, or less specifically that $M_1 \models \exists x \bigwedge_{i<k} \phi(x; a_i)$. Since $M_0 \prec_K M_1$, this in turn allows us to choose $b' \in M_0$ so that $M_1 \models \bigwedge_{i<k} \phi[b'; a_i]$, which means necessarily that $M_2 \models \bigwedge_{i<k} \phi[b'; a_i]$.

For condition (3), let $a \in M_2$ be given. Since $M_1 \prec_K M_2$, we may choose I from M_1 of length $\lambda(K)$ which is $(\{\psi\}^*_n, n)$ – indiscernible over A so that $tp_\phi(a, A, M_2) = Av_\phi(I, A, M_2)$. Because the length of I exceeds $|A|^{l(y)} \cdot \kappa_\phi(M)$, we may choose one element b_{i_0} in I so that $tp_\phi(b_{i_0}, A, M_2) = Av_\phi(I, A, M_2)$. (This can be accomplished by throwing out $< \kappa_\phi(M)$ elements of I for each instance $\phi(x; b)$ with $b \in A$ so that the elements of I that are left all realize the same instances of ϕ over A.) Since $M_0 \prec_K M_1$, we may choose a sequence J in M_0 which is $(\{\psi\}^*_n, n)$ – indiscernible over $\emptyset$ so that $Av_\phi(J, A, M_1) = tp_\phi(b_0, A, M_1)$. But then we have $Av_\phi(J, A, M_1) = Av_\phi(I, A, M_1)$, and consequently $Av_\phi(J, A, M_2) = Av_\phi(I, A, M_2) = tp_\phi(a, A, M_2)$, as desired.

For (V), consider first Condition (2). Assuming the hypotheses in (V) are true, we let $\phi \in \Delta$ and $a_i \in A$ for $i < k$ be given, and assume that $N_1 \models \exists x \bigwedge_{i<k} \phi(x; a_i)$. It then follows from $N_1 \prec_K M$ (Condition (2)) that $M \models \exists x \bigwedge_{i<k} \phi(x; a_i)$, and since $N_0 \prec_K M$, we may choose $b \in N_0$ so that $M \models \bigwedge_{i<k} \phi[b; a_i]$. Of course, $b, a_0, \dots, a_{k-1}$ are all from N_1, so from Condition (1) of $N_1 \prec_K M$, we can conclude that $N_1 \models \bigwedge_{i<k} \phi[b; a_i]$.

For condition (3), let $a \in N_1$ be given. Since $a \in M$ and $N_0 \prec_K M$, we may choose I from N_0 of length $\lambda(K)$ which is $(\{\psi\}^*_n, n)$ – indiscernible over $\emptyset$ so that $tp_\phi(a, A, M) = Av_\phi(I, A, M)$. Since $N_1 \subseteq M$, and a, I, and A are all included in N_1, it follows that $tp_\phi(a, A, N_1) = Av_\phi(I, A, N_1)$. $\square$

Definition 5.11 *Here again $\psi(y;x) = \phi(x;y)$. Given (ϕ, n, d) – good structures M, M_0, M_1, and M_2 with $M_l \prec_K M$, $M_0 \prec_K M_1$, and $M_0 \prec_K M_2$, we say that (M_0, M_1, M_2) is in ϕ – stable amalgamation inside M if for every $c \in M_2$ with $l(c) = l(x)$ there is a $(\{\psi\}_n^*, n)$ – indiscernible sequence $I \subseteq M_0$ over $\emptyset$, of length at least $\lambda(K)$ such that $Av_\phi(I, M_1, M) = tp_\phi(c, M_1, M)$.*

To prove symmetry of stable amalgamation (with the assumption of non-order), we must first establish the following lemma (corresponding to I.3.1 in [23]).

Lemma 5.12 *Let $\psi(y;x) = \phi(x;y)$ and $\Delta = \{\phi, \psi, \neg\phi, \neg\psi\}$, and let $\lambda = \max\{\lambda_\Delta(M), \kappa_\phi(M) + \kappa_\psi(M) + \kappa_\phi(M) \cdot \kappa_\psi(M)\}$. Assume M is (Δ, n, d) – good, $I_0 = \langle a_k^0 : k < m_0 \rangle$ is a $\{\psi\}_n^*$ – indiscernible sequence (over $\emptyset$) in M of length greater than λ, and $I_1 = \langle a_k^1 : k < m_1 \rangle$ is a $\{\phi\}_n^*$ – indiscernible sequence (over $\emptyset$) in M of length greater than λ. The following are equivalent:*

(i) There exists $i_k < m_0$ for $k < m_0 - \kappa_\phi(M)$ such that for each k,

$$\phi(a_{i_k}^0, y) \in Av_\psi(I_1, |M|, M).$$

(ii) There exists $j_l < m_1$ for $l < m_1 - \kappa_\psi(M)$ such that for each l,

$$\phi(x, a_{j_l}^1) \in Av_\phi(I_0, |M|, M).$$

Proof: Assume (i) holds. Choose $i_k < m_0$ for $k < m_0 - \kappa_\phi(M)$ and $j_{k,l} < m_1$ for $l < m_1 - \kappa_\psi(M)$ witnessing (i). Since $m_0 > \kappa_\psi(M) + \kappa_\phi(M)\kappa_\psi(M)$, we can find $\kappa_\psi(M)$ of the $j_{k,l}$ each of which occurs for at least $\kappa_\phi(M)$ different i_k. Thus for each of these, $\phi(x; a_{j_{k,l}}^1) \in Av_\phi(I_0, |M|, M)$.

Now assume (ii) does not hold. That is, there are $j_l < m_1$ for each $l < m_1 - \kappa_\psi(M)$ such that $\neg\phi(x; a_{j_l}^1) \in Av_\phi(I_0, |M|, M)$. Clearly one of these j_l must correspond to one of the $j_{k,l}$ from before that occurs at least $\kappa_\psi(M)$ times. But as we noted above $\phi(x; a_{j_{k,l}}^1) \in Av_\phi(I_0, |M|, M)$, a contradiction.

Note that (ii) implies (i) by the symmetric argument. □

Theorem 5.13 (Symmetry) *Let $\psi(y;x) = \phi(x;y)$ and $\Delta = \{\phi, \psi, \neg\phi, \neg\psi\}$. Let K be a class of (Δ, n, d) – good structures which all include a common set A. Suppose M_0, M_1, $M_2 \prec_K M$, $M_0 \prec_K M_1$, and $M_0 \prec_K M_2$.*

Then (M_0, M_1, M_2) *is in* Δ *– stable amalgamation inside* M *if and only if* (M_0, M_2, M_1) *is in* Δ *– stable amalgamation inside* M.

Proof: We show that (M_0, M_1, M_2) in ϕ – stable amalgamation implies that (M_0, M_2, M_1) is in ψ – stable amalgamation. The result follows from this. Assume that (M_0, M_1, M_2) is in ϕ – stable amalgamation in M. Let $c \in M_1$ with $l(c) = l(x)$ be given. (We need to find a $(\{\phi\}^*_n, n)$ – indiscernible sequence $I \subseteq M$, with $Av_\psi(I, M_2, M) = tp_\psi(c, M_2, M)$.) By the definition of $M_0 \prec_K M_1$, we may choose a $(\{\phi\}^*_n, n)$ – indiscernible $I \subseteq M_0$ of length at least $\lambda(K)$ so that $tp_\psi(c, M_0, M_1) = Av_\psi(I, M_0, M_1)$ (and so $tp_\psi(c, M_0, M) = Av_\psi(I, M_0, M)$).

We claim that $Av_\psi(I, M_2, M) = tp_\psi(c, M_2, M)$. (Note that the first type is defined since I is long enough.) To see this, let $b \in M_2$ be given such that $M \models \psi[c; b]$, and we will show that $\psi(x; b) \in Av_\psi(I, M_2, M)$.

Since (M_0, M_1, M_2) is in ϕ – stable amalgamation in M, we may choose a $(\{\psi\}^*_n, n)$ – indiscernible set $J \subseteq M_0$ of length at least $\lambda(K)$ so that $tp_\phi(b, M_1, M) = Av_\phi(J, M_1, M)$. Since $M \models \phi[b; c]$, we have $\phi(x; c) \in Av_\phi(J, M_1, M)$, so a large number of b_i from J have $M \models \phi[b_i; c]$, or rather $\psi(y; b_i) \in tp_\psi(c, M_0, M) = Av_\psi(I, M_0, M)$ for each of these b_i. So $\psi(y; b_i) \in Av_\psi(I, M_0, M)$ for each of these b_i.

But then by the previous Lemma, we may choose a large number of c_j from I for which $\phi(x; c_j) \in Av_\phi(J, M_1, M) = tp_\phi(b, M_1, M)$. That is, $M \models \phi[b; c_j]$ for each of these c_j, and so $\psi(y; b) \in Av_\psi(I, M_2, M)$ as desired. $\square$

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