# An effective one-step Church-Rosser strategy for combinators

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## Introduction

In this note we shall construct an effective one-step Church-Rosser conversion strategy F. We wish to emphasize that our strategy is not a reduction strategy since it on occasion expands rather than contracts; however, F is indeed a Church-Rosser strategy since X = Y => there exist n and m such that X ->> F^n(X) ≡ F^m(Y) <<- Y. Our strategy only works for combinators , since it makes use of our effective one-step cofinal reduction strategy [ 3 ] which only works for combinators; however, it does yield an effective one-step conversion strategy for lambda terms which the reader will easily see.

In short F has the following properties;

(1) F is effective

(2) either X -> F(X) or F(X) -> X

(3) if X beta converts to Y then for some n and m
X ->> F^n(X) which is identical to F^m(Y) << - Y.

## Preliminaries

Below '=' denotes beta conversion and '≡' denotes syntactic identity.

A combinator is an applicative combination of S and K. D is a the digraph whose points consist of the combinators and whose lines are defined by the one-step reduction relation X -> Y. The depth d(X) of a combinator X is defined by
d(S) = d(K) = 1 and d(XY) = max {d(X), d(Y)} +1.

D(m) is the subgraph of D induced by { X : d(X) ≤ m } and D(X) is the weakly connected (i.e. connected in the undirected sense) component of D(d(X)) containing X. We assume that the combinators have been ordered by < so that d(X)<d(Y) => X<Y. Let t(X) be the < least element of D(X). In [ 3 ] we defined an effective one-step cofinal reduction strategy C. The nth iterate of C on X is denote C^n (X). Here we recall that either there are infinitely many n such that X,C(X),C^2(X),...,C^(n-1)(X) belong to D(C^n(X)) (these C^n(X) are called the principal reducts of X ) or there is some n such that ,for all m>n, C^m(X) belongs to D(C^n(X)) ( such a C^n(X) is called a sink for X ). If there is a sink in D(X) we let s(X) be the < least such sink. Given a reduction sequence R = X1 -> X2 -> ...-> Xn we define lh(R) = n, df(R) = $S_{i=1}^{n}$ max {d(X(i+1)) - d(Xi), 0 }, wk(R) = | { Xi : t(Xi) > t(X1) } |. Now we order the triples

trip(R) = (df(R),wk(R),lh(R))

lexicographically and observe that among all the reduction sequences from X1 to Xn there are only finitely many paths R with df(R) < m for any fixed m. This is because any term in such a path has depth at most d(X1) + m. We shall assume that all of these paths have been ordered by << so that trip(R1) < trip(R2) => R1 << R2. Now given X find p(X) the << least reduction path from X to a principal reduct of t(X) or a sink of t(X) which ever exists. Let q(X) be the << least reduction path from X to s(X) if this exists. p(X) and q(X) can be effectively constructed from X. Finally we set ord(X) = (t(X),trip(p(X))) and ord'(X) = (s(X), trip(q(X))) if the latter exists. These quadruples are ordered lexicaographically.

**The algorithm**

We now give the definition of F(X). We assume that we have a Godel numbering of combinators X such that d(X) < the Godel number of X.

First we determine whether D(X) contains a sink and if one exists we compute s(X) and q(X).In addition, we find a principal reduct or a sink of t(X), Y, such that X ->> Y. This can be done by iterating C on t(X) while simultaneously enumerating the reduction paths beginning with X. By the definition of C ([ 3 ]), a sink for t(X) can be effectively recognized. Thus either a sink will be found or a reduction to a principal reduct. Next we find p(X); this can be found by the above remark from any reduction to Y. Let

A1 = S(KK)(S(SKK)(SKK))(S(KK)(S(SKK)(SKK)))

(this is just a combinatory fixed point of K)

A2 = KK(S(KK)(S(SKK)(SKK)))A1

We distinguish several cases.

Case 1; s(X) exists.

In case either X ≡ Ks(X)(K^nA1) or X ≡ K(s(X))(K^nA2) we put resp. F(X) ≡ Ks(X)(K^nA2) and F(X) ≡ Ks(X)(K^(n+1)A1). Similarly if X ≡ s(X) we put F(X) ≡ KXA1. Otherwise let X+ be the next point on q(X). If s(X+) exists and s(X+) ≤ s(X) then put F(X) ≡ X+. Otherwise set F(X) ≡ KXN for N a combinatory integer representing the Godel number of X.

Case 2 s(X) does not exist.

In case X ≡ KC^n(Y)N where N is the combin- atory integer representing the Godel number of Y a principal reduct of t(X) and none of the C^j(Y) for j = 1,...,n are principal reducts of t(X) then we put F(X) ≡ KC^(n+1)(Y)N.Otherwise ,we distinguish several subcases.

Subcase 1. lh(p(X)) = 1.

If C(X) is a principal reduct of t(X) then we set F(X) ≡ C(X). Otherwise ,we set F(X) ≡ KXN for

N a combinatory integer representing the Godel number of X

Subcase 2. lh(p(X)) > 1.

Let X+ be the next point on p(X). If t(X+) ≤ t(X) then we put F(X) ≡ X+ unless X ≡ KX+N for N a combinatory integer representing the Godel number of X+. In the latter case we put F(X) ≡ KC(X+)N .Otherwise, we set F(X) ≡ KXN where N is a combinatory integer representing the Godel number of X.

A correctness proof

First consider the sequence of iterations of F X,F(X),F(F(X)),.....,F^n(X),......

We claim that this sequence is unbounded in depth. Indeed if s(Y) is defined for any Y ≡ F^n(X) then for a < smallest such s(Y) we observe that there are two cases. If Y is s(Y) , Ks(Y)(K^mA1), or Ks(Y)(K^mA2) then F(Y) is Ks(Y)A1, Ks(Y)(K^mA2), or Ks(Y)(K^(m+1) A1) and s(F(Y)) ≡ s(Y). Otherwise s(F(Y)) ≡ s(Y) and q(F(Y)) < q(Y) . Thus the first case eventually comes up and ,once it is established ,it persists. Otherwise s(Y) is not defined for any Y ≡ F^m(X). Let Y1 ≡ F^m(X) be such that t(Y1) is < smallest and among those such that p(Y1) is << least. We claim that there is some principal reduct of t(Y1) in the original iterative sequence. Write p(Y1) = Y1 -> Y2 -> ...->Yk. If for some i>1 t(Y(i+1)) > t(Y1) ,for smallest such i we have F(Y1)≡ Y2, F(Y2) ≡ Y3,..., F(Y(i-1)) ≡ Yi, F(Yi) ≡ KYiN, and KYiN -> KY(i+1)N -> ... -> KYkN -> Yk. Thus p(F(Yi)) << p(Y1) contradicting the choice of Y1. Thus for j = 1,...,k-2 F(Yj) ≡ Y(j+1). If FY(k-1) ≡ Yk then the principal reduct Yk of t(Y1) is in the original iterative sequence. It is possible that F(Y(k-1)) =/≡ Yk. In this case , F(Y(k-1)) ≡ KY(k-1)N for N the Godel number of Y(k-1) and the next principal reduct of t(Y1) is a member of the original iterative sequence. For, if the next principal reduct of t(Y1) is C^r(Yk) we have

F(Y(k-1)) ≡ KY(k-1)N, F^2(Y(k-1)) ≡ KYkN,
F^3(Y(k-1) ≡ KC(Yk)N,...,F^(r+2)(Y(k-1)) ≡ C^r(Yk).
Once a principal reduct of t(Y1) is found in the original iterative sequence ,the sequence of iterates alternates between the first part of case 2 and subcase 1 of case 2 forever. Since s(Y) never exists the sequence must grow unbounded in depth.

Since the sequence of iterates is unbounded in depth there exists an infinite sequence of points F^n(X) such that X,F(X),...,F^(n-1)(X) belong to D(F^n(X)). Let all of these points in order be Y1, Y2,...,Yn,..... Now if there exists a sink Z = X then there exists some Yn such that Z belongs to D(Yn). Thus s(Yn) exists and for all but finitely many n s(F^n(X)) is the < smallest sink beta convertible to X. Similarly, for all but finitely many n ,t(F^n(X)) is the < smallest combinator beta convertible to X.

Finally ,if there is some sink = X then ,for all but finitely many n , F^n(X) alternates between KZ(K^mA1) and KZ(K^mA1) where Z is the < smallest sink = X. And, if there is no such sink then ,for all but finitely many n ,F^n(X) alternates between the principal reducts Y of the < smallest combinator = X and the terms K(C^m(Y))N for N the combinatory integer representing the Godel number of Y (and all but finitely many such Y are included). It follows that F is a Church -Rosser strategy.

References

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