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CONFIGURATIONAL FORCES
AND THE BASIC LAWS FOR CRACK PROPAGATION

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Dedicated to our friend Gianfranco Capriz on the occasion of his seventieth birthday.

ABSTRACT This paper develops a framework for dynamical fracture, concentrating on the derivation of basic field equations that describe the motion of the crack tip in two space-dimensions. The theory is based on the notion of configurational forces in conjunction with a mechanical version of the second law.

1. INTRODUCTION

In this paper we develop a (two-dimensional) framework for dynamical fracture, concentrating on the derivation of general field equations that govern the motion of the crack tip regardless of constitutive assumptions. We work within the nonlinear theory because the basic ideas are most easily explained within a framework that distinguishes between reference and deformed configurations;[1] moreover, instead of laying down specific assumptions regarding the strength of the crack-tip singularities, we consider hypotheses motivated by the requirement that the underlying physical laws make sense.

We base the theory on the notion of *configurational forces*. In classical continuum mechanics the response of a body to deformation is described by standard *deformational forces* consistent with balance laws for linear and angular momentum. *Configurational forces* are less intuitive: they are related to the intrinsic coherency of a body's material structure and perform work in the addition and removal of material and in the evolution of structural defects. Following GURTIN and STRUTHERS (1990) and GURTIN (1995), we view configurational forces as basic primitive objects consistent with their own force balance. Configurational forces *defined* via the calculus of variations as derivatives of an energy have been introduced earlier, *e.g.*, in the classic work of ESHELBY (1951) on lattice defects.[2] The role of configurational forces, however, seems more pervasive and fundamental than problems susceptible to a variational formulation can indicate, a view we hope to demonstrate within the context of fracture dynamics.

The configurational force system we envisage in our discussion of crack-tip motions has three components: a stress $\mathbf{C}$; a force $\mathbf{f}$ distributed continuously over the body; and a force $\mathbf{g}$ concentrated at the tip, where

$$\mathbf{g} = \mathbf{g}_b + \mathbf{g}_d \tag{1.1}$$

with $\mathbf{g}_d$ inertial and $\mathbf{g}_b$ an internal force that maintains the integrity of the tip when the crack is stationary and acts in response to the breaking of bonds during propagation. What is most important, we postulate a configurational force balance that has the following form when applied at the crack tip:

$$\lim_{\delta\to 0} \int_{\partial D_\delta} \mathbf{C}\mathbf{n} + \mathbf{g} = \mathbf{0}, \tag{1.2}$$

[1]However, our analysis is applicable to small strains (*cf.* the remark containing (3.6)).

[2]The recent monograph by MAUGIN (1993) features a comprehensive treatment, inspired by Eshelby's work, of configurational forces (there called *material forces*) in nonlinear elasticity and other branches of continuum physics.

where $D_\delta = D_\delta(t)$ is a tip disk, that is, a referential control volume having the form of small disk of radius δ centered at the tip and moving with it.[3] (We omit the area measure in integrals over regions in $\mathbb{R}^2$ and the arc-length measure in integrals over the boundaries of such regions.)

A basic ingredient of our theory is a mechanical version of the second law of thermodynamics (GURTIN, 1995) which asserts that, for each control volume R(t),

$$(d/dt)\{\text{free energy of } R(t)\} + \mathcal{K}(R(t)) \le \{\text{rate at which work is performed on } R(t)\}, \tag{1.3}$$

where $\mathcal{K}(R(t))$ represents the temporal change in the kinetic energy of R(t) plus the outflow of kinetic energy due to the motion of $\partial R(t)$. We point out three features of this formulation of the second law:

(1) As with a tip disk, R(t) evolves through the reference configuration; we view the dependence of R(t) on t as representing the addition of material to — or the removal of material from — the boundary $\partial R(t)$.

(2) We do not include a term representing the outflow of free energy across $\partial R(t)$, as we view all noninertial interactions with material exterior to R(t) in terms of *working*, with an accounting that includes *work performed in the addition and removal of material at the boundary.*

(3) The configurational forces **f** and **g** do not enter (1.3), as their noninertial components represent forces internal to the portion of the body represented by R(t), while their inertial components are presumed to be accounted for by $\mathcal{K}(R(t))$ (*vid.* relation (1.6) just below).

A consequence of the second law formulated in this manner is the Eshelby relation

$$\mathbf{C} = \Psi\mathbf{1} - \mathbf{F}^{\mathsf{T}}\mathbf{S} \tag{1.4}$$

for the configurational stress, where Ψ is the free energy density, **S** the deformational stress, $\mathbf{F}^{\mathsf{T}}$ the transpose of the deformation gradient **F**, and **1** the unit tensor. This result is a consequence of the requirement that the second law be invariant under changes in parametrization of the boundary $\partial R(t)$ (GURTIN, 1995); it is independent of the particular constitutive equations satisfied by Ψ and **S**.

[3] We choose a disk for convenience only; we could equally well choose $D_\delta(t)$ to be a family of "nice regions that tends to the tip" as $\delta \to 0$, uniformly in t. The balance (1.2) does not involve **f** since $\lim_{\delta\to 0} \int_{D_\delta} \mathbf{f} = \mathbf{0}$.

To characterize the inertial component $\mathbf{g}_d$ of the configurational force $\mathbf{g}$, we first define the deformational inertial-force $\mathbf{b}_d$ at the tip through the balance

$$\lim_{\delta\to 0} \int_{\partial D_\delta} \mathbf{S}\mathbf{n} + \mathbf{b}_d = \mathbf{0}, \tag{1.5}$$

which identifies $\mathbf{b}_d$ as the limiting value of the inflow of momentum across ∂D_δ. We then require[4] that the working of $\mathbf{g}_d$ and $\mathbf{b}_d$ be equal to $-\lim_{\delta\to 0}\mathcal{K}(D_\delta)$ using as the kinematical quantities relevant to this working the velocities $\mathbf{v}$ and $\bar{\mathbf{v}}$ of the tip in the reference and deformed configurations:

$$\mathbf{g}_d\cdot\mathbf{v} + \mathbf{b}_d\cdot\bar{\mathbf{v}} = -\lim_{\delta\to 0}\mathcal{K}(D_\delta). \tag{1.6}$$

As a consequence of these assumptions we have for the configurational inertial-force $\mathbf{g}_d$ at the tip the relation

$$\mathbf{g}_d = \lim_{\delta\to 0} \int_{\partial D_\delta} (k_{rel})\mathbf{n}, \tag{1.7}$$

with k_{rel} the kinetic energy density measured relative to the crack tip; this allows us to write the configurational balance (1.2) in the form

$$\lim_{\delta\to 0} \int_{\partial D_\delta} \{(\Psi + k_{rel})\mathbf{1} - \mathbf{F}^{\mathsf{T}}\mathbf{S}\}\mathbf{n} + \mathbf{g}_b = \mathbf{0}. \tag{1.8}$$

The second law (1.3) and our results concerning tip inertia yield the internal dissipation inequality

$$\mathbf{g}_b\cdot\mathbf{v} \le 0, \tag{1.9}$$

a central result of our theory. This inequality shows $\mathbf{g}_b$ to be a dissipative force that opposes motion of the tip, a result consistent with our association of $\mathbf{g}_b$ with the breaking of bonds.

We next establish the energy balance

$$\lim_{\delta\to 0} \int_{\partial D_\delta} \{\mathbf{S}\mathbf{n}\cdot\mathbf{y}^{\cdot} + (\Psi + k)(\mathbf{v}\cdot\mathbf{n})\} = -\mathbf{g}_b\cdot\mathbf{v}, \tag{1.10}$$

[4] The use of a requirement of this type to relate the expressions for "inertial force" and "kinetic energy" in general theories of continua has been proposed by PODIO-GUIDUGLI (1995).

a result that allows us to relate $\mathbf{g}_b$ to more classical concepts. Assume that $V=|\mathbf{v}|\neq 0$ and write $\mathbf{v}=V\mathbf{e}$. Then the quantity $\mathcal{G}=\mathcal{G}(t)$ defined by setting $\mathcal{G}V$ equal to the left side of (1.10) is usually referred to as the (dynamic) energy release-rate;[5] by (1.10), $\mathcal{G}$ coincides with the component of the internal configurational force opposing crack propagation:

$$\mathcal{G} = -\mathbf{g}_b\cdot\mathbf{e}. \tag{1.11}$$

In accord with a practice now standard in continuum mechanics, we view the internal dissipation inequality (1.9) as indicating the need for constitutive relations involving $\mathbf{g}_b$ and $\mathbf{v}$ and as a means of suitably restricting such relations. A discussion of general constitutive assumptions is beyond the scope of this paper, but as an example we give an elementary discussion of possible constitutive equations for straight cracks.

Our goal is a clearer understanding of those basic concepts that underlie dynamical fracture. Our final results are not different in form from results well known by experts on fracture (FREUND, 1990); what is different is our derivation and our interpretation in terms of configurational forces, which we believe to most accurately describe the underlying physics.

To facilitate our discussion, we derive two general transport relations appropriate to evolving control volumes $R=R(t)$ that contain a portion $\mathcal{C}_R=\mathcal{C}_R(t)$ of the crack, including the tip. These relations, valid for $\Phi(\mathbf{X},t)$ a sufficiently regular field that is smooth away from the tip and up to the crack from either side, assert that

$$(d/dt)\{\int_R \Phi\} = \int_R \dot{\Phi} + \int_{\partial R} \Phi U_{\partial R} - \lim_{\delta\to 0} \int_{\partial D_\delta} \Phi(\mathbf{v}\cdot\mathbf{n}), \tag{1.12a}$$

$$(d/dt)\{\int_R \Phi\} = \int_R \Phi^\circ + \int_{\partial R} \Phi(U_{\partial R}-\mathbf{v}\cdot\mathbf{n}) + \int_{\mathcal{C}_R} [\Phi]\mathbf{m}\cdot\mathbf{v}, \tag{1.12b}$$

where Φ° is the time derivative of Φ following the tip, $U_{\partial R}$ is the (scalar) velocity of ∂R in the direction of its outward normal, $\mathbf{n}$ is the outward unit normal to $\partial D_\delta=\partial D_\delta(t)$, $\mathbf{m}$ is a unit normal to the crack, and, for $\mathbf{X}$ on the crack, $[\Phi](\mathbf{X},t)=\Phi(\mathbf{X}+0\mathbf{m},t)-\Phi(\mathbf{X}-0\mathbf{m},t)$.

A defect of the theory as presented here is that the presumed regularity of the crack rules out kinking; in particular, at initiation the crack has a prescribed

[5] *Cf.* ATKINSON and ESHELBY (1968), KOSTROV and NIKITIN (1970), FREUND (1972), WILLIS (1975), GURTIN and YATOMI (1980), and the comprehensive treatise of FREUND (1990).

direction. An extension of the framework developed here to irregularly propagating cracks is therefore warranted.

2. KINEMATICS OF CRACKED BODIES

a. Cracks. Time-dependent control volumes.

Let B denote a closed region of $\mathbb{R}^2$ with boundary ∂B and, for each t in some open time interval, let $\mathcal{C}(t)$ be a smooth, connected, oriented curve in B with one end fixed at the boundary ∂B, with the remainder of $\mathcal{C}(t)$ — including the other end point $\mathbf{Z}(t)$ — contained in the interior of B, and with

$$\mathcal{C}(\tau) \subset \mathcal{C}(t) \quad \text{for all} \quad t \geq \tau. \tag{2.1}$$

We view

$$\mathcal{B}(t) = B \setminus \mathcal{C}(t) \tag{2.2}$$

as a *referential neighborhood of a growing crack* $\mathcal{C}(t)$ with $\mathbf{Z}(t)$ the *crack tip* (Figure 1. Note that B contains the points of $\mathcal{C}(t)$ while $\mathcal{B}(t)$ does not; hence $\mathcal{B}(t)$ is cracked, while B is not. Note also that the assumed regularity of $\mathcal{C}(t)$ precludes singularities such as kinks and bifurcations.) We let $\mathbf{e}(t)$ denote the unit tangent to $\mathcal{C}(t)$ at $Z(t)$ in the direction of (possible) propagation. Then the *tip velocity*

$$\mathbf{v}(t) = dZ(t)/dt \tag{2.3}$$

may be written in the form

$$\mathbf{v}(t) = V(t)\mathbf{e}(t), \qquad V(t) \geq 0, \tag{2.4}$$

with V the *speed*. Finally, we choose a continuous unit normal field $\mathbf{m}(\mathbf{X},t)$ for $\mathcal{C}(t)$.

By a *control volume* we mean a closed subregion R(t) of B for which $\partial R(t)$ evolves smoothly with t, and for which

$$\mathcal{C}_R(t) = \mathcal{C}(t) \cap R(t), \tag{2.5}$$

the portion of the crack in R(t), does not intersect $\partial R(t)$ at more than two points

(Figure 2). For convenience, we limit our discussion to two classes of control volumes: those that do not intersect the tip and those that contain the tip in their interior. We view the dependence of R(t) on t as resulting from the addition and removal of material points. Our definition of a control volume does not preclude control volumes R that are independent of time.

For R(t) a control volume, $\mathbf{n}(\mathbf{X},t)$ designates the outward unit normal to $\partial R(t)$, and $U_{\partial R}$ the (scalar) *normal velocity* of the boundary curve in the direction $\mathbf{n}$. A useful example of a time-dependent control volume is the *tip disc*

$$D_\delta(t) = \{\mathbf{X}\in B:\ |\mathbf{X}-\mathbf{Z}(t)| \le \delta\}, \tag{2.6}$$

a disc of radius δ centered at the tip $\mathbf{Z}(t)$; here the normal velocity is

$$U_{\partial D_\delta} = \mathbf{v}\cdot\mathbf{n}. \tag{2.7}$$

For convenience, we write

$$\mathcal{C}_\delta(t) = \mathcal{C}_{D_\delta}(t) = \mathcal{C}(t)\cap D_\delta(t). \tag{2.8}$$

b. Derivatives following the crack tip. Tip integrals. Transport theorems. We refer to a field $\Phi(\mathbf{X},t)$ as *smooth away from the tip* if $\Phi(\mathbf{X},t)$ is defined for all $\mathbf{X}\in\mathcal{B}(t)$ and all t, and if, away from the tip, $\Phi(\mathbf{X},t)$ and its derivatives have limits up to the crack from either side; we then write, for $\mathbf{X}\in\mathcal{C}(t)$,

$$\Phi^\pm(\mathbf{X},t) = \lim_{\varepsilon\to 0}\Phi(\mathbf{X}\pm\varepsilon\mathbf{m}(\mathbf{X},t),t), \qquad [\Phi] = \Phi^+ - \Phi^-. \tag{2.9}$$

Given such a field $\Phi(\mathbf{X},t)$, consider the corresponding field $\hat{\Phi}(\mathbf{Y},t)$ in which $\mathbf{Y}$ represents the position of the material point $\mathbf{X}$ relative to the tip $\mathbf{Z}(t)$:

$$\hat{\Phi}(\mathbf{Y},t) = \Phi(\mathbf{X},t), \qquad \mathbf{Y} = \mathbf{X}-\mathbf{Z}(t). \tag{2.10}$$

The partial derivative

$$\Phi^\circ(\mathbf{X},t) = \partial\hat{\Phi}(\mathbf{Y},t)/\partial t$$

with respect to t holding $\mathbf{Y}$ fixed represents the time derivative of $\Phi(\mathbf{X},t)$ following the tip $\mathbf{Z}(t)$; by the chain rule,

$$\Phi^\circ = \Phi^\bullet + \nabla\Phi\cdot\mathbf{v} \tag{2.11}$$

away from the tip, where

$$\Phi^\bullet(\mathbf{X},t) = \partial\Phi(\mathbf{X},t)/\partial t.$$

We will repeatedly take limits, as $\delta\to 0$, of integrals of fields over $\partial D_\delta(t)$; we refer to such limits, when meaningful, as *tip integrals;* examples, for φ a scalar field, $\mathbf{w}$ a vector field, and $\mathbf{T}$ a tensor field, are:

$$\oint_{\text{tip}}\varphi\mathbf{n} = \lim_{\delta\to 0}\int_{\partial D_\delta(t)}\varphi\mathbf{n}, \tag{2.12a}$$

$$\oint_{\text{tip}}(\mathbf{w}\otimes\mathbf{n}) = \lim_{\delta\to 0}\int_{\partial D_\delta(t)}\mathbf{w}\otimes\mathbf{n}, \tag{2.12b}$$

$$\oint_{\text{tip}}\mathbf{T}\mathbf{n} = \lim_{\delta\to 0}\int_{\partial D_\delta(t)}\mathbf{T}\mathbf{n}. \tag{2.12c}$$

Let R(t) be a control volume that includes the tip and consider the region

$$R_\delta(t) = R(t)\setminus D_\delta(t), \tag{2.13}$$

with $\delta>0$ sufficiently small that $\partial R_\delta(t) = \partial R(t)\cup\partial D_\delta(t)$. Then, using the same letter $\mathbf{n}$ for the outward unit normal on both ∂R and ∂D_δ, and bearing in mind that the outward unit normal to ∂R_δ on ∂D_δ is $-\mathbf{n}$, we may use the gradient theorem in the usual manner — with $\mathfrak{C}_{R\delta}$ considered as a "slit in R_δ" giving rise to an additional pair of boundary segments (Figure 3), and with $\int_{R_\delta}\nabla\Phi$ interpreted accordingly — to conclude that, for Φ smooth away from the tip,

$$\int_{R_\delta}\nabla\Phi = \int_{\partial R}\Phi\mathbf{n} - \int_{\mathfrak{C}_{R\delta}}[\Phi]\mathbf{m} - \int_{\partial D_\delta}\Phi\mathbf{n}. \tag{2.14}$$

(Here, for convenience, we have suppressed the argument t.) Thus, if $\oint_{\text{tip}}\Phi\mathbf{n}$ exists, and if $[\Phi]\mathbf{m}$ is integrable on $\mathfrak{C}$, then $\int_R\nabla\Phi$ exists as the limit $\lim_{\delta\to 0}\int_{R_\delta}\nabla\Phi$ and we have the *generalized gradient theorem*

$$\int_R\nabla\Phi = \int_{\partial R}\Phi\mathbf{n} - \int_{\mathfrak{C}_R}[\Phi]\mathbf{m} - \oint_{\text{tip}}\Phi\mathbf{n}. \tag{2.15}$$

The next definition allows us to state succintly our hypotheses concerning momenta and energies. We will refer to Φ as *regular* if, in addition to being smooth away from the tip,

(R1) Φ is integrable on B; given any control volume R(t), the mapping $t \mapsto \int_{R(t)}\Phi$ is differentiable;

(R2) Φ° is integrable on B and $[\Phi]\mathbf{m}\cdot\mathbf{v}$ is integrable on $\mathcal{C}(t)$, both uniformly in t;

(R3) $\oint_{tip}\Phi\mathbf{n}$ exists.

(The phrase "uniformly in t" signifies "uniformly for t in any compact interval".) By (R2) and (2.8),

$$\int_{\mathcal{C}_\delta(t)}[\Phi]\mathbf{m}\cdot\mathbf{v} \quad \text{approaches zero as} \quad \delta\to 0. \tag{2.16}$$

Remark. In actual solutions of crack problems the underlying fields are generally singular in the distance $r = |\mathbf{X} - \mathbf{Z}(t)|$ from the tip, but are otherwise well-behaved; if a field Φ has a tip singularity and yet is integrable (*cf.* (R1)), neither $\dot{\Phi}$ nor $\nabla\Phi\cdot\mathbf{v}$ are generally integrable, but Φ° may well be, which is why Φ° rather than $\dot{\Phi}$ was used in (R2). For example, let Φ be a field that is smooth away from the tip and has the form

$$\Phi(\mathbf{X},t) = r^{-1}\varphi(\mathbf{X},t).$$

Conditions sufficient for the regularity of Φ are: (i) that $\varphi(\mathbf{X},t)$ have a limit as $\mathbf{X} \to \mathbf{Z}(t)$, uniformly in t; and (ii) that $\varphi^{\circ}(\mathbf{X},t)$ be uniformly bounded in $(\mathbf{X},t)$. Indeed, a direct consequence of these assumptions is that Φ and $\Phi^{\circ} = r^{-1}\varphi^{\circ}$ are integrable on B, uniformly in t, and that (R3) is satisfied. The differentiability of the mapping $t \mapsto \int_{R(t)}\Phi(\mathbf{X},t)\,d\mathbf{X}$ then follows upon changing the integration variable from $\mathbf{X}$ to $\mathbf{Y} = \mathbf{X} - \mathbf{Z}(t)$, while the integrability of $[\Phi]\mathbf{m}\cdot\mathbf{v}$ on $\mathcal{C}(t)$, uniformly in t, follows from the fact that $\Phi(\mathbf{X},t) = O(r^{-1})$ and $\mathbf{m}(\mathbf{X},t)\cdot\mathbf{v}(t) \to 0$ as $\mathbf{X} \to \mathbf{Z}(t)$, both uniformly in t. □

The following well known transport theorem is valid when $\Phi(\mathbf{X},t)$ is smooth away from the tip and R(t) does not contain the tip:

$$(d/dt)\Big\{\int_{R(t)}\Phi\Big\} = \int_{R(t)}\dot{\Phi} + \int_{\partial R(t)}\Phi U_{\partial R}. \tag{2.17}$$

We now give two generalizations of (2.17) that account for the crack tip.

Transport theorem. For R(t) a control volume that includes the tip, if $\Phi(\mathbf{X},t)$ is regular, then

$$(d/dt)\{\int_{R(t)}\Phi\} = \int_{R(t)}\dot{\Phi} + \int_{\partial R(t)}\Phi U_{\partial R} - \oint_{tip}\Phi(\mathbf{v}\cdot\mathbf{n}), \tag{2.18a}$$

$$(d/dt)\{\int_{R(t)}\Phi\} = \int_{R(t)}\Phi^{\circ} + \int_{\partial R(t)}\Phi(U_{\partial R} - \mathbf{v}\cdot\mathbf{n}) + \int_{\mathcal{C}_R(t)}[\Phi]\mathbf{m}\cdot\mathbf{v} \tag{2.18b}$$

(with $\int_{R(t)}\dot{\Phi}$ defined as $\lim_{\delta\to 0}\int_{R_\delta}\dot{\Phi}$, which exists).

We regard the transport theorem (2.18) as a central result of our theory. Relation (2.18b) expresses $(d/dt)\int_{R(t)}\Phi$ in terms of the derivative Φ° following the tip and the inflow $\Phi(U_{\partial R} - \mathbf{v}\cdot\mathbf{n})$ measured in a frame moving with the tip. By (2.7), it follows from (2.18b) that

$$(d/dt)\{\int_{D_\delta(t)}\Phi\} = \int_{D_\delta(t)}\Phi^{\circ} + \int_{\mathcal{C}_\delta(t)}[\Phi]\mathbf{m}\cdot\mathbf{v}, \tag{2.19}$$

so that, by (R2) and (2.16), as $\delta\to 0$,

$$(d/dt)\{\int_{D_\delta(t)}\Phi\} \to 0, \tag{2.20}$$

a result we will use often in what follows.

To verify (2.18), we consider the region $R_\delta(t)$ defined in (2.13) with δ sufficiently small. Then (2.17) holds with R(t) replaced by $R_\delta(t)$, so that,

$$(d/dt)\{\int_{R_\delta(t)}\Phi\} = \int_{R_\delta(t)}\dot{\Phi} + \int_{\partial R(t)}\Phi U_{\partial R} - \int_{\partial D_\delta(t)}\Phi(\mathbf{v}\cdot\mathbf{n}), \tag{2.21}$$

since $U_{\partial D_\delta} = \mathbf{v}\cdot\mathbf{n}$, and therefore, by (2.11),

$$(d/dt)\{\int_{R_\delta(t)}\Phi\} = \int_{R_\delta(t)}(\Phi^{\circ} - \nabla\Phi\cdot\mathbf{v}) + \int_{\partial R(t)}\Phi U_{\partial R} - \int_{\partial D_\delta(t)}\Phi(\mathbf{v}\cdot\mathbf{n}). \tag{2.22}$$

But, by (2.14),

$$\int_{R_\delta(t)} \nabla\Phi\cdot\mathbf{v} = \int_{\partial R(t)} \Phi(\mathbf{v}\cdot\mathbf{n}) - \int_{\partial D_\delta(t)} \Phi(\mathbf{v}\cdot\mathbf{n}) - \int_{\mathcal{C}_{R_\delta}(t)} [\Phi]\mathbf{m}\cdot\mathbf{v}; \tag{2.23}$$

thus

$$(d/dt)\{\int_{R_\delta(t)} \Phi\} = \int_{R_\delta(t)} \Phi^\circ + \int_{\partial R(t)} \Phi(U_{\partial R} - \mathbf{v}\cdot\mathbf{n}) + \int_{\mathcal{C}_{R_\delta}(t)} [\Phi]\mathbf{m}\cdot\mathbf{v}. \tag{2.24}$$

Let

$$\varphi(t) = \int_{R(t)} \Phi, \qquad \varphi_\delta(t) = \int_{R_\delta(t)} \Phi. \tag{2.25}$$

Then (2.24) and (R2) yield the conclusion that, as $\delta \to 0$, $d\varphi_\delta/dt$ tends to the right side of (2.18b) uniformly in t. Further, since, by (R1), $\varphi_\delta \to \varphi$, this uniformity implies that $d\varphi_\delta/dt \to d\varphi/dt$. Thus (2.18b) holds and, in addition, the left side and the last term of (2.21) each approach the corresponding terms in (2.18a) (*cf.* (R3)). Thus (2.18a) is valid *modulo* the asserted definition of $\int_R \Phi^\cdot$.

c. Motions of cracked bodies.

Let $\mathbf{y}(\mathbf{X},t)$ be a *motion* of $\mathcal{B}(t)$; that is, let $\mathbf{y}(\mathbf{X},t)$ be smooth away from the tip with $\mathbf{y}(\mathbf{X},t)$ one-to-one in $\mathbf{X}$ on $\mathcal{B}(t)$ for each t. The deformation gradient

$$\mathbf{F} = \nabla\mathbf{y} \tag{2.26}$$

and the material velocity $\mathbf{y}^\cdot$ are then smooth away from the tip.

Let R(t) be a control volume. The boundary curve $\partial R(t)$ may be parametrized in a sufficiently small time interval and in a neighborhood of any of its points by a function of the form $\mathbf{X} = \hat{\mathbf{X}}(\sigma,t)$ (σ a scalar variable); the field

$$\mathbf{u}(\mathbf{X},t) = \partial\hat{\mathbf{X}}(\sigma,t)/\partial t \tag{2.27}$$

then represents a velocity field for $\partial R(t)$ in that neighborhood. It is possible to use such parametrizations to construct a *velocity field* for $\partial R(t)$; that is, a smooth field $\mathbf{u}(\mathbf{X},t)$ defined for all $\mathbf{X}$ on $\partial R(t)$ and all t in any (sufficiently small) time interval. A field $\mathbf{u}$ so constructed depends on the choice of local parametrizations, but its normal component is intrinsic:

$$\mathbf{u}\cdot\mathbf{n} = U_{\partial R}. \tag{2.28}$$

Each local parametrization $\mathbf{X}=\hat{\mathbf{X}}(\sigma,t)$ induces a corresponding local parametrization $\mathbf{x}=\hat{\mathbf{x}}(\sigma,t)=\mathbf{y}(\hat{\mathbf{X}}(\sigma,t),t)$ for the deformed boundary curve $\mathbf{y}(\partial R(t),t)$; the corresponding *induced velocity field*

$$\bar{\mathbf{u}}(\mathbf{X},t) = \partial\hat{\mathbf{x}}(\sigma,t)/\partial t \tag{2.29}$$

for the deformed boundary $\mathbf{y}(\partial R(t),t)$ is related to $\mathbf{u}$ by the formula

$$\bar{\mathbf{u}} = \mathbf{y}^{\bullet} + \mathbf{F}\mathbf{u}. \tag{2.30}$$

The tip velocity $\mathbf{v}(t)$ may be considered as a velocity field for the boundary of the disc $D_\delta(t)$ using as a parametrization

$$\mathbf{X} = \hat{\mathbf{X}}_\delta(\sigma,t) = \mathbf{Z}(t) + \delta\boldsymbol{\nu}(\sigma), \tag{2.31}$$

with $\boldsymbol{\nu}(\sigma)$ a unit vector at an angle σ from a fixed axis. Then

$$\mathbf{y}^{\circ} = \mathbf{y}^{\bullet} + \mathbf{F}\mathbf{v}, \tag{2.32}$$

the time derivative following $\mathbf{Z}(t)$, represents the corresponding induced velocity field for $\mathbf{y}(\partial D_\delta(t),t)$. We assume that:

(A1) there is a function $\bar{\mathbf{v}}(t)$ such that

$$\mathbf{y}^{\circ}(\mathbf{X},t) \to \bar{\mathbf{v}}(t) \quad \text{as} \quad \mathbf{X} \to \mathbf{Z}(t), \quad \text{uniformly in } t. \tag{2.33}$$

One might expect that $\bar{\mathbf{v}}(t)$ represents the velocity of the deformed crack tip. Granted sufficient regularity this is indeed the case. Assume for the moment that $\mathbf{y}(\mathbf{X},t)$ has a limiting value $\mathbf{y}(\mathbf{Z}(t),t)$ as $\mathbf{X}\to\mathbf{Z}(t)$, so that the *deformed crack tip* is well defined. Then $\mathbf{y}(\mathbf{Z}(t),t)$ is differentiable in t and

$$\bar{\mathbf{v}}(t) = d\mathbf{y}(\mathbf{Z}(t),t)/dt. \tag{2.34}$$

To verify (2.34) consider (2.31) with σ fixed, and let $\mathbf{y}_\delta(t)=\mathbf{y}(\hat{\mathbf{X}}_\delta(\sigma,t),t)$. Then $d\mathbf{y}_\delta(t)/dt=\mathbf{y}^{\circ}(\hat{\mathbf{X}}_\delta(\sigma,t),t)$, so that, by (2.33), $d\mathbf{y}_\delta(t)/dt\to\bar{\mathbf{v}}(t)$ as $\delta\to 0$, uniformly in t. But, by hypothesis, $\mathbf{y}_\delta(t)\to\mathbf{y}(\mathbf{Z}(t),t)$; thus $\mathbf{y}(\mathbf{Z}(t),t)$ is differentiable in t and (2.32) holds at $\mathbf{Z}(t)$.

3. BASIC LAWS

a. Balance laws for deformational and configurational forces.
We let ρ denote the reference *mass density,* write

$$\mathbf{p} = \rho \mathbf{y}^{\cdot} \tag{3.1}$$

for the *momentum,* let $\mathbf{S}$ denote the *Piola-Kirchhoff stress* that arises in response to deformation, neglect external body forces, and assume that the crack faces are traction-free:

$$\mathbf{S}^{\pm}\mathbf{m} = \mathbf{0} \quad \text{on } \mathcal{C}(t). \tag{3.2}$$

The *balance laws for linear and angular momentum* then take the form

$$(d/dt)\{\int_{R(t)} \mathbf{p}\} = \int_{\partial R(t)} (\mathbf{p}U_{\partial R} + \mathbf{Sn}), \tag{3.3a}$$

$$(d/dt)\{\int_{R(t)} \mathbf{y}\times\mathbf{p}\} = \int_{\partial R(t)} \mathbf{y}\times(\mathbf{p}U_{\partial R} + \mathbf{Sn}), \tag{3.3b}$$

for each control volume $R(t)$.

We consider, in addition, a configurational stress $\mathbf{C}$, a configurational force $\mathbf{f}$ distributed over $\mathcal{B}(t)$, and a configurational force $\mathbf{g}$ concentrated at the tip; these are presumed consistent with the *configurational force balance:*

$$\int_{\partial R(t)} \mathbf{Cn} + \int_{\mathcal{C}_R(t)} [\mathbf{C}]\mathbf{m} + \int_{R(t)} \mathbf{f} = \mathbf{0}, \quad \text{if } R(t) \text{ does not contain the tip;} \tag{3.4a}$$

$$\int_{\partial R(t)} \mathbf{Cn} + \int_{\mathcal{C}_R(t)} [\mathbf{C}]\mathbf{m} + \int_{R(t)} \mathbf{f} + \mathbf{g}(t) = \mathbf{0}, \quad \text{if } R(t) \text{ contains the tip.} \tag{3.4b}$$

We assume that each of $\mathbf{f}$ and $\mathbf{g}$ consists of internal and inertial portions. While the decomposition of $\mathbf{f}$ is irrelevant to most of our discussion, determining the inertial portion of $\mathbf{g}$ will form a major part of our analysis.

To ensure that the balances (3.4) are well defined and that their localization to the crack tip (in Section 4) is meaningful, we assume that

(A2) ρ is continuous; $\mathbf{p}$ and $\mathbf{y}\times\mathbf{p}$ are regular; $\mathbf{S}$ and $\mathbf{C}$ are smooth away from the tip; $\mathbf{f}$ is integrable over B; $\int_{\partial D_\delta} |\mathbf{Sn}|$ remains bounded as $\delta \to 0$; $[\mathbf{C}]\mathbf{m}$ is integrable on $\mathcal{C}(t)$.

We then have the following local relations away from the crack:

$$\operatorname{div}\mathbf{S} = \mathbf{p}^{\cdot}, \qquad \mathbf{S}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{S}^{\mathsf{T}}, \tag{3.5a}$$

$$\operatorname{div}\mathbf{C} + \mathbf{f} = \mathbf{0}. \tag{3.5b}$$

Remark. The theory we here develop is valid for *infinitesimal* displacements $\mathbf{u}$ provided we redefine $\mathbf{F}$ to be $\nabla\mathbf{u}$ and replace $\mathbf{y}^{\cdot}$ and $\mathbf{y}^{\circ}$ with $\mathbf{u}^{\cdot}$ and $\mathbf{u}^{\circ}$. In particular, all definitions and results are unchanged except those related to angular momentum; there $\mathbf{y}$ in (3.3b), (3.7), and (6.1) should be replaced by $\mathbf{X}$, while the second of (3.5a) should be replaced by $\mathbf{S} = \mathbf{S}^{\mathsf{T}}$. Within this framework, classical additional estimates for a *linearly elastic material* are that, as $r = |\mathbf{X} - \mathbf{Z}(t)| \to 0$,

$$\mathbf{u} \sim r^{\frac{1}{2}}, \quad \mathbf{u}^{\cdot} \sim r^{-\frac{1}{2}}, \quad \nabla\mathbf{u} \sim r^{-\frac{1}{2}}, \quad \mathbf{S} \sim r^{-\frac{1}{2}} \tag{3.6}$$ [6]

(FREUND, 1990, p. 43), estimates which yield the additional results

$$\oint_{\text{tip}} \mathbf{Sn} = \mathbf{0}, \qquad \oint_{\text{tip}} (\mathbf{X} \times \mathbf{Sn}) = \mathbf{0}, \tag{3.7a}$$

$$\oint_{\text{tip}} (\mathbf{p} \otimes \mathbf{n}) = \mathbf{0}, \qquad \oint_{\text{tip}} ((\mathbf{X} \times \mathbf{p}) \otimes \mathbf{n}) = \mathbf{0}. \tag{3.7b}$$

On the other hand, the configurational balances suggest that the configurational stress satisfy $\mathbf{C} \sim r^{-1}$, an assumption consistent with the classical assumptions (3.6) in the light of the interpretation of $\mathbf{C}$ in terms of the Eshelby relation (1.4).[7] □

b. Mechanical version of the second law.

In the absence of defects (such as cracks), of external body forces, and of thermal and compositional effects, classical continuum mechanics may be based on a "second law" that utilizes *stationary* control volumes R and has the form

$$(d/dt)\{\int_R \Psi\} + \mathcal{K}(R) \le \mathcal{W}(R), \tag{3.8}$$

[6]Even within the infinitesimal theory, these estimates are generally not valid beyond linear elasticiticity (RICE and ROSENGREN, 1968; HUTCHINSON, 1968).

[7]Since $\mathbf{C} = \Psi\mathbf{1} - \mathbf{F}^{\mathsf{T}}\mathbf{S}$, and since $(3.6)_{3,4}$ imply that $\mathbf{F}^{\mathsf{T}}\mathbf{S} \sim r^{-1}$ as $r \to 0$. In fact, one expects $\Psi \sim r^{-1}$ as $r \to 0$ (*cf.* the Remark in Section 2b), hence the integrability of the free energy Ψ.

where Ψ is the *free energy* density,

$$\mathcal{K}(R) = (d/dt)\{\int_R k\} \tag{3.9}$$

with

$$k = \tfrac{1}{2}\rho|\mathbf{y}^{\cdot}|^2 \tag{3.10}$$

the *kinetic energy* density, and

$$\mathcal{W}(R) = \int_{\partial R} \mathbf{Sn}\cdot\mathbf{y}^{\cdot} \tag{3.11}$$

is the *boundary working* (GURTIN and STRUTHERS, 1990; GURTIN, 1995).

For an evolving control volume R(t) generalization of (3.8)-(3.11) is necessary, but by no means obvious. We consider the dependence of R(t) on t as representing the addition of material to — or the removal of material from — the boundary $\partial R(t)$, and we write the second law in a manner reflecting this view. To begin with we take

$$(d/dt)\{\int_{R(t)} \Psi\}$$

as the sole term involving free energy; we do not include the outflow term

$$\int_{\partial R(t)} \Psi U_{\partial R}$$

as we view noninertial interactions with the material exterior to R(t) in terms of *working*, rather than transport.

This leads to the main issue: generalization of the expression (3.11) to account for the work performed in the addition and removal of material at the boundary. We assume that $\mathbf{Cn}\cdot\mathbf{u}$ represents the boundary working of the configurational stress $\mathbf{C}$, where $\mathbf{u}$ is the velocity field computed *via* a particular choice of local parametrizations $\mathbf{X} = \hat{\mathbf{X}}(\sigma,t)$ for $\partial R(t)$. The working of the deformational stress $\mathbf{S}$ must also be taken into account. When the control volume depends on time there is no intrinsic material description of its deformed boundary $\mathbf{y}(\partial R(t),t)$, as material is continually being added and removed, and it would seem appropriate to use, as a velocity for $\dot{\mathbf{y}}(\partial R(t),t)$, the derivative $\bar{\mathbf{u}}(\mathbf{X},t)$ of $\mathbf{y}(\hat{\mathbf{X}}(\sigma,t),t)$ with respect to t holding the surface parameter σ fixed; we therefore write the boundary working of **S** in the form $\mathbf{Sn}\cdot\bar{\mathbf{u}}$.

Finally, as we view the kinetic energy as "independent" of the internal structure of the material, we generalize $\mathcal{K}(R)$ in the standard manner, *viz.,*

$$\mathcal{K}(R(t)) = (d/dt)\Big\{\int_{R(t)} k\Big\} - \int_{\partial R(t)} k U_{\partial R}. \tag{3.12}$$

In conclusion, we write the second law for an evolving control volume R(t) — that may or may not contain the crack tip — in the form

$$(d/dt)\Big\{\int_{R(t)} \Psi\Big\} + \mathcal{K}(R(t)) \le \int_{\partial R(t)} (\mathbf{Sn}\cdot\bar{\mathbf{u}} + \mathbf{Cn}\cdot\mathbf{u}), \tag{3.13}$$

with **u** a velocity field for $\partial R(t)$ and $\bar{\mathbf{u}}$ the corresponding induced velocity field for $\mathbf{y}(\partial R(t),t)$. (The configurational forces **f** and **g** perform no work, as their inertial components are accounted for by $\mathcal{K}(R(t))$, while their noninertial components are internal; moreover, there is no contribution from $\mathfrak{C}(t)$ because of (3.2) and since only the tip of $\mathfrak{C}(t)$ evolves.) Note that, by (2.30), the deformational working $\mathbf{Sn}\cdot\bar{\mathbf{u}}$ consists of a classical term $\mathbf{Sn}\cdot\mathbf{y}^{\cdot}$ plus a term $\mathbf{Sn}\cdot\mathbf{Fu}$ that accounts for the addition of strained material to ∂R. Note also that for R independent of time (3.13) reduces to the standard inequality (3.8), so there is no conflict with classical continuum mechanics.

To ensure that this version of the second law be meaningful, and to allow for its localization, we assume that:

(A3) Ψ and k are regular.

c. The Eshelby tensor as a consequence of invariance under reparametrization (GURTIN, 1995).

We require that our theory be independent of the choice of parametrization for $\partial R(t)$. This requirement of *invariance under reparametrization* has important consequences. In particular, the invariance of (3.13) is equivalent to invariance of the boundary working, which, by (2.30), can be given the form

$$\mathcal{W}(R(t)) = \int_{\partial R(t)} (\mathbf{Sn}\cdot\bar{\mathbf{u}} + \mathbf{Cn}\cdot\mathbf{u}) = \int_{\partial R(t)} \{\mathbf{Sn}\cdot\mathbf{y}^{\cdot} + (\mathbf{F}^{\mathsf{T}}\mathbf{Sn} + \mathbf{Cn})\cdot\mathbf{u}\}. \tag{3.14}$$

Changes in parametrization affect the tangential component of $\mathbf{u}$, but leave the normal component unaltered. In fact, invariance of (3.14) under reparametrization is equivalent to the requirement that $(\mathbf{F}^T\mathbf{S}\mathbf{n}+\mathbf{C}\mathbf{n})\cdot\mathbf{t}=0$ on $\partial R(t)$ for all tangential vector fields $\mathbf{t}$ on $\partial R(t)$; thus, since $R(t)$ is arbitrary, $(\mathbf{F}^T\mathbf{S}+\mathbf{C})\mathbf{n}$ must be parallel to $\mathbf{n}$ for all $\mathbf{n}$, so that

$$\mathbf{C}+\mathbf{F}^T\mathbf{S} = \pi\mathbf{1} \tag{3.15}$$

and, by (2.28), the working has the intrinsic form

$$\mathfrak{W}(R(t)) = \int_{\partial R(t)} \mathbf{S}\mathbf{n}\cdot\mathbf{y}^{\cdot} + \int_{\partial R(t)} \pi U_{\partial R}. \tag{3.16}$$

The scalar field π is a *configurational tension* that works to increase the volume of $R(t)$ through the addition of material at its boundary. Referring to the final term in (3.16) as the *configurational working*, (3.16) may be stated more suggestively as *boundary working equals deformational working plus configurational working*. Note that the configurational working $\pi U_{\partial R}$ is not due solely to the action of the configurational stress $\mathbf{C}$; the deformational stress contributes also through the term $(\mathbf{S}\mathbf{n}\cdot\mathbf{F}\mathbf{n})U_{\partial R}$.

Next, assuming that $R(t)$ does not contain the crack, and using (2.17) and (3.16), the inequality (3.13) may be rewritten as

$$\int_{R(t)} (\Psi + k)^{\cdot} \le \int_{\partial R(t)} \{\mathbf{S}\mathbf{n}\cdot\mathbf{y}^{\cdot} + (\pi - \Psi)U_{\partial R}\}. \tag{3.17}$$

Given a time τ, it is possible to find a second referential control volume $R'(t)$ with $R'(\tau)=R(\tau)$, but with $U_{\partial R'}(\mathbf{X},\tau)$, the normal velocity of $\partial R'(\tau)$, an *arbitrary* scalar field on $\partial R'(\tau)$; satisfaction of (3.17) for all such $U_{\partial R'}$ implies

$$\pi = \Psi. \tag{3.18}$$

Therefore, *configurational tension coincides with free energy*, a result analogous to the coincidence of surface tension and surface free-energy; what is more important, (3.16) and (3.18) yield the *Eshelby relation*

$$\mathbf{C} = \Psi\mathbf{1} - \mathbf{F}^T\mathbf{S} \tag{1.4}$$

for the configurational stress $\mathbf{C}$.

This derivation of the Eshelby relation was accomplished without recourse to constitutive equations or to a variational principle; the derivation was based on a version of the second law appropriate to referential control volumes whose boundaries evolve with time. The result (3.18) is a consequence of the invariance of $\mathfrak{W}(R)$ under reparametrization; it is independent of the particular form chosen for the second law and is hence more basic than (1.4).

Given the first of the balance equations (3.5a), the second law (3.17) has the classical local form:

$$\Psi^{\cdot} \le \mathbf{S}\cdot\mathbf{F}^{\cdot}. \tag{3.19}$$

The results (3.18)-(3.19) are valid away from the crack. Since the configurational force $\mathbf{f}$ does not appear in (3.19), we consider $\mathbf{f}$ to be *indeterminate*,[8] in fact, as *defined* by the balance (3.5b). The theory away from the tip is therefore equivalent to the classical theory: only Ψ and $\mathbf{S}$ need constitutive specification, while (1.4) and (3.5b) are regarded as *defining* relations for $\mathbf{f}$ and $\mathbf{C}$; configurational forces play no role. On the other hand, as we shall show in the next few sections, configurational forces play a pivotal role in the evolution of the crack tip, as it is there that the material structure undergoes change.

Finally, by (2.17), (3.17), and (3.18), we can write the second law in the form

$$(d/dt)\Big\{\int_{R(t)}(\Psi + k)\Big\} \le \int_{\partial R(t)}\mathbf{S}\mathbf{n}\cdot\mathbf{y}^{\cdot} + \int_{\partial R(t)}(\Psi + k)U_{\partial R}, \tag{3.20}$$

again showing consistency with classical ideas.

Remark. With (1.4), the first of (3.5a) and (5.5b) yield

$$(\mathbf{f} - \mathbf{F}^{T}\mathbf{p}^{\cdot})\cdot\mathbf{a} = -\nabla\Psi\cdot\mathbf{a} + \mathbf{S}\cdot\nabla(\mathbf{F}\mathbf{a}) \tag{3.21}$$

for all vectors $\mathbf{a}$. Suppose that the body $\mathcal{B}(t)$ is *elastic* and *homogeneous,* with constitutive equations giving the stress $\mathbf{S}$ as the derivative of the free energy Ψ:

$$\Psi = \hat{\Psi}(\mathbf{F}), \qquad \mathbf{S} = \partial_{\mathbf{F}}\hat{\Psi}(\mathbf{F}) \tag{3.22}$$

(so that, in particular, $\Psi^{\cdot} = \mathbf{S}\cdot\mathbf{F}^{\cdot}$, in accord with (3.19)). The right side of (3.21) then vanishes for all $\mathbf{a}$; hence $\mathbf{f} = \mathbf{F}^{T}\mathbf{p}^{\cdot}$ and the configurational force $\mathbf{f}$ has only an inertial part. This is a direct consequence of *homogeneity;* for an *inhomogeneous* material with energy $\hat{\Psi}(\mathbf{F},\mathbf{X})$ and stress $\mathbf{S} = \partial_{\mathbf{F}}\hat{\Psi}(\mathbf{F},\mathbf{X})$, (3.21) yields $\mathbf{f} = -\partial_{\mathbf{X}}\hat{\Psi} + \mathbf{F}^{T}\mathbf{p}^{\cdot}$,

[8] That is, not specified constitutively (TRUESDELL and NOLL, 1965, p. 70).

and both internal and inertial components are present (ESHELBY, 1975). □

4. BASIC LAWS AT THE CRACK TIP

a. Balance laws.

By (A2), (A3), (2.7), (2.12), and (2.20) applied to $\mathbf{p}$ and $\mathbf{y}\times\mathbf{p}$, the momentum balances have the limiting forms:

$$\oint_{\text{tip}}\mathbf{Sn} = -\oint_{\text{tip}}(\mathbf{p}(\mathbf{v}\cdot\mathbf{n})), \tag{4.1a}$$

$$\oint_{\text{tip}}(\mathbf{y}\times(\mathbf{Sn})) = -\oint_{\text{tip}}((\mathbf{y}\times\mathbf{p})(\mathbf{v}\cdot\mathbf{n})), \tag{4.1b}$$

while the limiting form of the balance for configurational force is

$$\oint_{\text{tip}}\mathbf{Cn} = -\mathbf{g}. \tag{4.2}$$

To see how, in particular, (4.1a) is arrived at, write (3.3a) for a tip disk, taking (2.7) into account, and then make use of (2.20) applied to $\mathbf{p}$; the result is

$$\int_{\partial D_\delta(t)}\mathbf{Sn} + \int_{\partial D_\delta(t)}\mathbf{p}(\mathbf{v}\cdot\mathbf{n}) = o(1), \tag{4.3}$$

where $o(1)$ represents terms that approach zero as $\delta\to 0$; the definition of the tip integrals (*cf.* (2.12b) and (2.12c)) then yields (4.1a).

b. Tip inertia. Resistance to the breaking of bonds.

We define the tip inertial force $\mathbf{b}_d$ in the deformational system through the balance (1.5); then, by (4.1a),

$$\mathbf{b}_d = \oint_{\text{tip}}\mathbf{p}(\mathbf{v}\cdot\mathbf{n}). \tag{4.4}$$

We then assume that the configurational force $\mathbf{g}$ concentrated at the tip admits the decomposition (1.1), with $\mathbf{g}_d$ the inertial force at the tip and $\mathbf{g}_b$ an internal force that resists the breaking of bonds during crack propagation.

The identification of $\mathbf{g}_d$ is not immediate, chiefly because of the nonintuitive nature of configurational forces. For that reason, following a procedure of PODIO-GUIDUGLI (1995), we characterize $\mathbf{g}_d$ through the *equivalence of inertial working and temporal changes in kinetic energy;* precisely, we require that the working of $\mathbf{g}_d$ *and* $\mathbf{b}_d$ be equal to $-\lim_{\delta\to 0}\mathcal{K}(D_\delta)$. Arguing as in our discussion in the paragraph following (3.11), the velocities relevant to the working of $\mathbf{g}_d$ and $\mathbf{b}_d$ are

$\mathbf{v}$ and $\bar{\mathbf{v}}$, respectively; therefore, appealing to (3.12),

$$\lim_{\delta\to 0} \Big(\int_{\partial D_\delta(t)} k\mathbf{v}\cdot\mathbf{n} - (d/dt)\{ \int_{D_\delta(t)} k \}\Big) = \mathbf{b}_d\cdot\bar{\mathbf{v}} + \mathbf{g}_d\cdot\mathbf{v}, \tag{4.5}$$

so that, by (2.20) applied to k,

$$\oint_{tip} k(\mathbf{v}\cdot\mathbf{n}) = \mathbf{b}_d\cdot\bar{\mathbf{v}} + \mathbf{g}_d\cdot\mathbf{v}. \tag{4.6}$$

With (4.4), (4.6) yields

$$\mathbf{g}_d\cdot\mathbf{v} = \mathbf{v}\cdot\oint_{tip}(k - \mathbf{p}\cdot\bar{\mathbf{v}})\mathbf{n}. \tag{4.7}$$

Further, as a consequence of (3.8),

$$k - \mathbf{p}\cdot\bar{\mathbf{v}} = k_{rel} - \tfrac{1}{2}\rho|\bar{\mathbf{v}}|^2 \tag{4.8}$$

with

$$k_{rel} = \tfrac{1}{2}\rho|\mathbf{y}^{\cdot} - \bar{\mathbf{v}}|^2, \tag{4.9}$$

the kinetic energy measured relative to the tip. Since ρ is continuous, the integral of $\rho|\bar{\mathbf{v}}|^2\mathbf{n}$ over ∂D_δ tends to zero, and

$$\oint_{tip}(k - (\mathbf{p}\cdot\bar{\mathbf{v}}))\mathbf{n} = \oint_{tip}(k_{rel}\,\mathbf{n}), \tag{4.10}$$

so that, by (4.7),

$$\mathbf{g}_d\cdot\mathbf{v} = \mathbf{v}\cdot\oint_{tip}(k_{rel}\,\mathbf{n}). \tag{4.11}$$

This should at least motivate our identification of $\mathbf{g}_d$ with the tip integral of the relative kinetic energy:

$$\mathbf{g}_d = \oint_{tip}(k_{rel}\,\mathbf{n}). \tag{4.12}$$

Finally, by (3.15), (4.3), and (4.12), we can write the configurational balance (4.1c) for the crack tip in the form

$$\oint_{tip}\{(\Psi + k_{rel})\mathbf{1} - \mathbf{F}^{\mathsf{T}}\mathbf{S}\}\mathbf{n} = -\mathbf{g}_b. \tag{4.13}$$

c. Internal dissipation inequality. Energy production. Energy release-rate.

We now localize the second law (3.13) to the crack tip. Because of (A3) and (2.20) applied to Ψ, (1.1) and the second law (3.13) with $R(t)=D_\delta(t)$, $\mathbf{u}=\mathbf{v}$, and $\bar{\mathbf{u}}=\mathbf{y}^\circ$ (*cf.* (2.32)) yield the inequality

$$0 \le \int_{\partial D_\delta(t)} (\mathbf{Sn}\cdot\mathbf{y}^\circ + \mathbf{Cn}\cdot\mathbf{v}) + \mathbf{b}_d\cdot\bar{\mathbf{v}} + \mathbf{g}_d\cdot\mathbf{v} + o(1). \tag{4.14}$$

The next two estimates use the spatial constancy of $\mathbf{v}$ and $\bar{\mathbf{v}}$. By (A1), the assertion of boundedness in (A2), and (1.5),

$$\int_{\partial D_\delta(t)} \mathbf{Sn}\cdot\mathbf{y}^\circ = \int_{\partial D_\delta(t)} \mathbf{Sn}\cdot\bar{\mathbf{v}} + o(1) = -\mathbf{b}_d\cdot\bar{\mathbf{v}} + o(1). \tag{4.15}$$

Thus, by (4.1c),

$$\int_{\partial D_\delta(t)} (\mathbf{Sn}\cdot\mathbf{y}^\circ + \mathbf{Cn}\cdot\mathbf{v}) = -\mathbf{b}_d\cdot\bar{\mathbf{v}} - \mathbf{g}\cdot\mathbf{v} + o(1); \tag{4.16}$$

this estimate, with (1.1), reduces (4.14) to

$$\mathbf{g}_b\cdot\mathbf{v} \le 0, \tag{1.9}$$

an inequality requiring that the internal configurational force oppose motion of the tip. The relation (1.9), the main result of this section, represents an *internal-dissipation inequality* for the crack tip.

The quantity

$$\mathcal{E}(R(t)) = (d/dt)\{\int_{R(t)} (\Psi + k)\} - \int_{\partial R(t)} \mathbf{Sn}\cdot\mathbf{y}^\cdot - \int_{\partial R(t)} (\Psi + k)U_{\partial R} \le 0. \tag{4.17}$$

is the *production of energy* in $R(t)$, with $-\mathcal{E}(R(t))$ the corresponding energy dissipated (*cf.* (3.21); equivalently, $\mathcal{E}(R(t))$ may be defined as the left side of (3.13) minus the right.) If we take $R(t)=D_\delta(t)=D_\delta$ in (4.17), we find, with the aid of (2.20), that

$$\oint_{tip}\{\mathbf{Sn}\cdot\mathbf{y}^\cdot + (\Psi + k)(\mathbf{v}\cdot\mathbf{n})\} = -\lim_{\delta\to 0} \mathcal{E}(D_\delta), \tag{4.18}$$

provided the tip integral on the left exists. By (A3), this integral, which represents the working on and energy flow out of an infinitesimal neighborhood of the tip, will be well defined provided $\oint_{tip}\mathbf{Sn}\cdot\mathbf{y}^{\cdot}$ exists; but (1.4) and (2.32) yield

$$\mathbf{Sn}\cdot\mathbf{y}^{\cdot} = \mathbf{Sn}\cdot\mathbf{y}^{\circ} + \mathbf{Cn}\cdot\mathbf{v} - \Psi\mathbf{n}\cdot\mathbf{v}, \tag{4.19}$$

so that, by (A3) and (4.14), $\oint_{tip}\mathbf{Sn}\cdot\mathbf{y}^{\cdot}$ exists. Further, appealing to (4.6), (4.14), and (4.16), we find that the left side of (4.18) equals $-\mathbf{g}_b\cdot\mathbf{v}$. Thus

$$\mathbf{g}_b\cdot\mathbf{v} = \lim_{\delta\to 0} \mathcal{E}(D_\delta), \tag{4.20}$$

establishing $-\mathbf{g}_b\cdot\mathbf{v}$, and hence the breaking of bonds, as the sole source of dissipation at the tip. An immediate consequence of (4.18) and (4.20) is the tip balance

$$\oint_{tip}\{\mathbf{Sn}\cdot\mathbf{y}^{\cdot} + (\Psi + k)(\mathbf{v}\cdot\mathbf{n})\} = -\mathbf{g}_b\cdot\mathbf{v}. \tag{4.21}$$

The quantity $\mathcal{G} = \mathcal{G}(t)$ defined for $V \neq 0$ by setting $\mathcal{G}V$ equal to the left side of (4.21) is usually referred to as the (dynamic) *energy release-rate*. A consequence of (2.4) is then the identity

$$\mathcal{G} = -\mathbf{g}_b\cdot\mathbf{e}, \tag{1.11}$$

and the energy release-rate coincides with the component of the internal configurational force opposing crack propagation. Finally, by (4.13),

$$\mathcal{G} = \mathbf{e}\cdot\oint_{tip}\{(\Psi + k_{rel})\mathbf{1} - \mathbf{F}^{\mathsf{T}}\mathbf{S}\}\mathbf{n}. \tag{4.22}$$

Remark. For a straight crack ($\mathbf{e}$ = constant) in a homogeneous elastic material (*cf.* the Remark in Section 3c), neglecting inertia, the energy release-rate may be computed *via* an integration along a path away from the tip. Let $\Gamma = \Gamma(t)$ denote any smooth, closed, nonintersecting path that begins and ends on the crack and surrounds the tip, let $\mathbf{n}$ denote the outward unit normal to Γ, let

$$J(\Gamma) = \mathbf{e}\cdot\int_{\Gamma}(\Psi\mathbf{1} - \mathbf{F}^{\mathsf{T}}\mathbf{S})\mathbf{n}, \tag{4.23}$$

and keep in mind that $\operatorname{div}\mathbf{C} = \mathbf{0}$, since $\mathbf{f} = \mathbf{0}$ (*cf.* again the Remark in Section 3c), and that $\mathbf{e}\cdot[\mathbf{C}]\mathbf{m} = \Psi\mathbf{e}\cdot\mathbf{m} = 0$ (*cf.* (1.4) and (3.2)). Then, applying the (tensorial version of) the generalized gradient theorem (2.15) to $\operatorname{div}\mathbf{C}$, with R the region bounded by Γ, we may conclude that (ESHELBY, 1956; RICE, 1968):

$$\mathcal{G} = J(\Gamma) \qquad \text{for any choice of the path } \Gamma. \quad \square \tag{4.24}$$

5. Second law revisited.

At this point we have identified the forms of energy dissipation, with $-\mathbf{g}_b\cdot\mathbf{v}$ the dissipation at the tip and $(\mathbf{S}\cdot\mathbf{F}^\cdot - \Psi^\cdot)$ the dissipation in the bulk material away from the tip (*cf.* (3.19)). We now show that, granted an accounting of these forms of dissipation, the second law takes the form of an energy balance.

We begin by writing

$$\mathcal{D}(R) = \int_R (\mathbf{S}\cdot\mathbf{F}^\cdot - \Psi^\cdot) \geq 0 \tag{5.1}$$

for the *bulk dissipation* in any control volume $R = R(t)$, where here and throughout this section $\int_R$ is defined, when meaningful, as $\lim_{\delta\to 0}\int_{R_\delta}$, with $R_\delta = R_\delta(t)$ given by (2.13). In fact, in each subsequent appearance $\int_R$ will be well-defined, thereby establishing the existence of (5.1).

Next, the (vectorial version of the) generalized gradient theorem (2.15) applied to $\mathbf{S}^T\mathbf{y}^\cdot$ yields

$$\int_{\partial R} \mathbf{S}\mathbf{n}\cdot\mathbf{y}^\cdot = \int_R (\mathbf{S}\cdot\mathbf{F}^\cdot + \mathbf{y}^\cdot\cdot\operatorname{div}\mathbf{S}) + \oint_{\text{tip}} \mathbf{S}\mathbf{n}\cdot\mathbf{y}^\cdot \tag{5.2}$$

(*cf.* (3.2)), and, by (3.5a) and (3.8), we have the power identity

$$\int_{\partial R} \mathbf{S}\mathbf{n}\cdot\mathbf{y}^\cdot - \oint_{\text{tip}} \mathbf{S}\mathbf{n}\cdot\mathbf{y}^\cdot = \int_R (\mathbf{S}\cdot\mathbf{F}^\cdot + \mathbf{k}^\cdot). \tag{5.3}$$

On the other hand, (4.17) and the transport identity (2.18a), with $\Phi = \Psi + k$, yield

$$\mathcal{E}(R) = \int_R (\Psi + k)^\cdot - \oint_{\text{tip}} (\Psi + k)(\mathbf{v}\cdot\mathbf{n}) - \int_{\partial R} \mathbf{S}\mathbf{n}\cdot\mathbf{y}^\cdot. \tag{5.4}$$

The last two relations and (5.1) imply that

$$\mathcal{E}(R) = -\mathcal{D}(R) + \mathbf{g}_b\cdot\mathbf{v}, \tag{5.5}$$

showing that the production of energy in a control volume R is balanced by the energy dissipated in bulk plus the energy dissipated by the moving tip. More specifically, (5.5) yields the second law as a balance:

$$(d/dt)\{\int_{R(t)}(\Psi + k)\} = \int_{\partial R(t)} \mathbf{Sn}\cdot\mathbf{y}^{\cdot} + \int_{\partial R(t)}(\Psi + k)U_{\partial R} - \mathcal{D}(R(t)) + \mathbf{g}_b\cdot\mathbf{v}. \tag{5.6}$$

For an elastic material $\mathcal{D} \equiv 0$ and (5.6) takes the simple form

$$(d/dt)\{\int_{R(t)}(\Psi + k)\} = \int_{\partial R(t)} \mathbf{Sn}\cdot\mathbf{y}^{\cdot} + \int_{\partial R(t)}(\Psi + k)U_{\partial R} + \mathbf{g}_b\cdot\mathbf{v}. \tag{5.7}$$

6. SUMMARY OF BASIC RESULTS FOR THE CRACK TIP

The basic equations for the crack tip consist of the *momentum balances*

$$\oint_{tip}\mathbf{Sn} = -\oint_{tip}(\mathbf{p}(\mathbf{v}\cdot\mathbf{n})), \tag{6.1a}$$

$$\oint_{tip}(\mathbf{y}\times\mathbf{Sn}) = -\oint_{tip}((\mathbf{y}\times\mathbf{p})(\mathbf{v}\cdot\mathbf{n})), \tag{6.1b}$$

and the *configurational balance*

$$\oint_{tip}\{(\Psi + k_{rel})\mathbf{1} - \mathbf{F}^{T}\mathbf{S}\}\mathbf{n} = -\mathbf{g}_b, \qquad k_{rel} = \tfrac{1}{2}\rho|\mathbf{y}^{\cdot} - \bar{\mathbf{v}}|^2. \tag{6.2}$$

These balance laws are supplemented by the *internal dissipation inequality*

$$\mathbf{g}_b\cdot\mathbf{v} \le 0, \tag{6.3}$$

which represents the second law localized to the crack tip.

The fields $\mathbf{S}$ and Ψ are generally given by constitutive equations defining the material properties away from the crack. On the other hand, the quantities $\mathbf{g}_b$ and $\mathbf{v}$, which characterize the mechanics and kinematics of the crack tip, require constitutive specification, as without further restriction the internal dissipation inequality (6.3) may be violated. The basic theory for the crack tip is therefore closed by relating $\mathbf{g}_b$ and $\mathbf{v}$ constitutively in a manner consistent with (6.3). In the next section we will discuss a possible constitutive specification for the kinetics of a straight crack.

Remarks.

1. It is important to differentiate between the roles played by the energy release-rate $\mathcal{G}$ and the internal configurational force $\mathbf{g}_b$. Throughout the literature one finds constitutive prescriptions for $\mathcal{G}$ (or equivalently for the stress intensity factor). We believe this to be conceptually incorrect, as it is $\mathbf{g}_b$ that is constitutive, with $\mathcal{G}$ a defined quantity related to $\mathbf{g}_b$ through a reduced version (1.11) of the configurational balance. In this regard, note that, with the exception of the inertial term k_{rel}, $\mathcal{G}$ is represented by quantities that already have constitutive prescriptions.[9] More pragmatically, $\mathbf{g}_b$, being a vector, accounts for directional resistance to motion of the tip, and might therefore be useful in modeling the curving and kinking of cracks.

2. Granted the classical estimates leading to (3.7), the momentum balance laws (6.1) are satisfied automatically and k_{rel} in (6.2) may be replaced by k (*cf.* (3.8)). □

[9] *E.g.*, for an elastic material in a quasi-static process, $\mathcal{G}$, as defined in (4.22), depends on Ψ, $\mathbf{S}$, and $\mathbf{F}$; but for such a material Ψ and $\mathbf{S}$ are already prescribed constitutively as functions of $\mathbf{F}$.

7. CONSTITUTIVE EQUATIONS FOR STRAIGHT CRACKS

We assume that the crack is straight. The direction of propagation $\mathbf{e}$ (*cf.* (2.4)) is then a prescribed constant, and it is convenient to write the internal dissipation inequality (5.3) in a form that singles out the component of $\mathbf{g}_b$ in the direction $\mathbf{e}$:

$$V\mathbf{e}\cdot\mathbf{g}_b \le 0. \tag{7.1}$$

The component of $\mathbf{g}_b$ normal to $\mathbf{e}$ does not enter the local form (7.1) of the second law, and for that reason we shall regard this component of $\mathbf{g}_b$ as indeterminate, an assumption consistent with the requirement that $\mathbf{v}$ be constrained to the direction $\mathbf{e}$.

We therefore let

$$g_b = \mathbf{e}\cdot\mathbf{g}_b \tag{7.2}$$

and rewrite (7.1) as

$$g_b V \le 0, \tag{7.3}$$

so that $-g_b$ is conjugate to the speed $V \ge 0$ of the tip; a necessary condition for crack propagation is then that g_b be resistive:

$$g_b \le 0. \tag{7.4}$$

The cohesive force $\mathbf{g}_b$ is related to the breaking of bonds at the crack tip, and it seems reasonable to suppose that crack propagation is accompanied by a resistive force dependent on the velocity V, with propagation possible only when $|g_b|$ is sufficiently large. We therefore introduce a *limiting value* L and a *tip viscosity* A, with

$$L = \hat{L}(\ldots) > 0, \qquad A = \hat{A}(\ldots) > 0, \tag{7.5}$$

such that

$$g_b = -L - AV \quad \text{for} \quad V > 0, \qquad -L \le g_b < \infty \quad \text{for} \quad V = 0 \tag{7.6}$$

(Figure 4.[10] The notation $L = \hat{L}(\ldots)$ and $A = \hat{A}(\ldots)$ signifies that L and A are

[10] The experimental results of ROSAKIS *et al* (1984) and ZENDER and ROSAKIS (1990), as

material moduli whose values may depend, constitutively, on quantities associated with the crack tip.) These constitutive relations are consistent with the internal dissipation inequality (7.3). Note that, by (7.9), the configurational balance (5.2) may be written in the form

$$\oint_{tip}\{(\Psi + k_{rel})\mathbf{1} - \mathbf{F}^T\mathbf{S}\}\mathbf{n} \;=\; L + AV \tag{7.7}$$

for $V > 0$.

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displayed in Figure 11 of the latter, at least indicate behavior of this form, as does the micromechanical model of LAM and FREUND (1985) (FREUND, 1990, §8.3).

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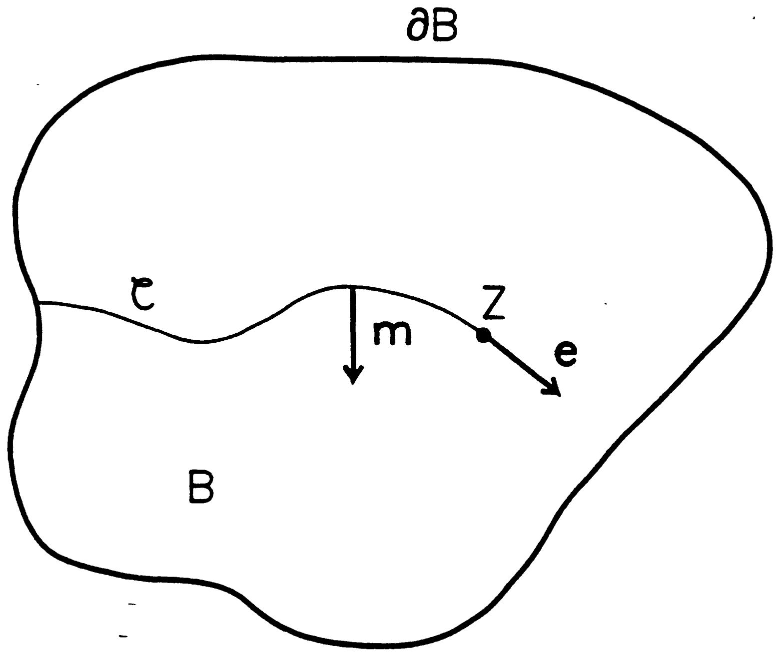

Figure 1
Referential neighborhood of a crack.

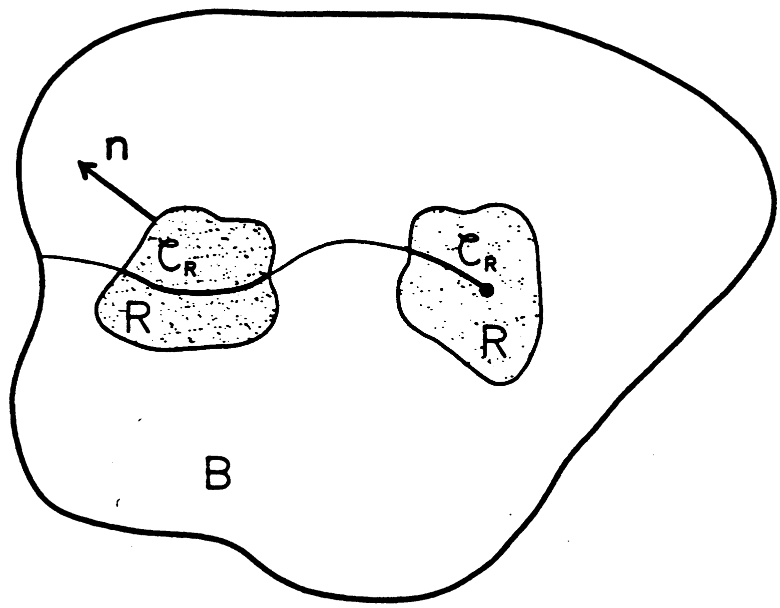

Figure 2
Types of control volumes.

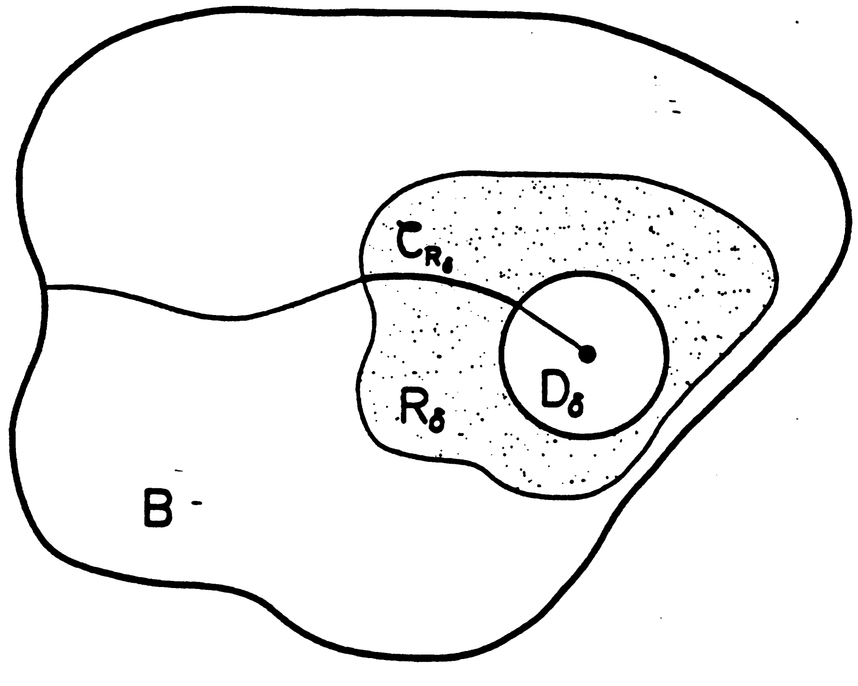

Figure 3

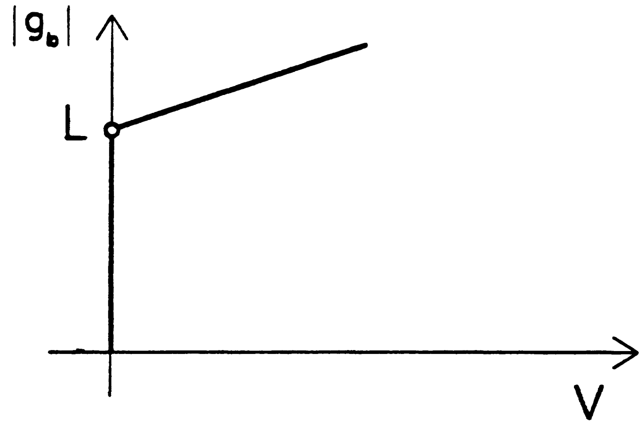

Figure 4
Tip force *vs.* tip velocity.