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Fang-Hua Lin
Courant Institute

Chun Liu
Carnegie Mellon University

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Partial Regularity Of The Dynamic System Modeling The Flow Of Liquid Crystals

Fang-Hua Lin * and Chun Liu*

Courant Institute

Abstract

Here we established the partial regularity of suitable weak solutions to the dynamical systems modelling the flow of liquid crystals. It is a natural generalization of an earlier work of Caffarelli-Kohn-Nirenberg on the Navier-Stokes system with some simplications due to better estimates on the presure term.

§1. Introduction.

In [LL] we studied the following dissipative system which comes from the modeling of the flow of liquid crystals:

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla\mathbf{P} = -\lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) \tag{1.1}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{1.2}$$

$$\mathbf{d}_t + (\mathbf{v}\cdot\nabla)\mathbf{d} = \gamma(\Delta\mathbf{d} - f(\mathbf{d})) \tag{1.3}$$

where Ω is a smooth, bounded domain in $\mathbf{R}^3$, $\mathbf{v}(x,t)$ represents the velocity of the flow, and $\mathbf{d}(x,t)$ is the optical molecule direction. Here we take $f(\mathbf{d})$ to be the gradiant of a scale function $F(\mathbf{d})$,

$$f(\mathbf{d}) = \nabla F(\mathbf{d}) \tag{1.4}$$

A typical example of $F(\mathbf{d})$ is given by $F(\mathbf{d}) = \frac{1}{\epsilon^2}(|\mathbf{d}|^2-1)^2$. As ϵ goes to zero, we see that $\mathbf{d}$ becomes an unit vector field.

We also have the following initial and boundary conditions:

$$\mathbf{v}(x,0) = \mathbf{v}_0(x) \quad \text{with } \nabla\cdot\mathbf{v}_0 = 0, \mathbf{d}(x,0) = \mathbf{d}_0(x), \text{for } x\in\Omega, \tag{1.5}$$

$$\mathbf{v}(x,t) = 0, \mathbf{d}(x,t) = \mathbf{d}_0(x), \text{for } (x,t)\in\partial\Omega\times(0,\infty). \tag{1.6}$$

In that paper, we have already shown the existence of the global weak solutions and the global classical solutions under certain conditions. The local existence of classical solutions

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was established as well. The most important property of the solutions of (1.1)—(1.6) is the following "basic energy inequality":

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(|\mathbf{v}|^2+\lambda|\nabla\mathbf{d}|^2+2\lambda F(\mathbf{d}))dx \le -\int_{\Omega}(\nu|\nabla\mathbf{v}|^2+\lambda\gamma|\Delta\mathbf{d}-f(\mathbf{d})|^2)dx,$$

for almost all $t \in (0,T]$.

However, there is still a gap between the case of existence and the case the solutions being regular. Because of that, in this paper, we want to give a partial regularity result for suitable weak solutions of the system (1.1)—(1.6). The existence of such solutions can be shown easily, see the discussions in Section 2.

The situation here is very much like that for the Navier-Stokes equations. There have been a lot of works concerning the regularity properties of the Navier-Stokes equation:

$$\mathbf{v}_t - \Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathbf{P} = f \tag{1.7}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{1.8}$$

in $\Omega \times [0.T]$. with the initial-boundary condition:

$$\mathbf{v}(\cdot,0) = \mathbf{v}_0, \quad \mathbf{v}|_{\partial\Omega} = 0 \tag{1.9}$$

Serrin[Se] has shown that a weak solution of (1.7)—(1.9) will be locally bounded (hence locally regular in spatial direction) under the following assumption:

$$\mathbf{v} \in \mathbf{L}^s(\mathbf{L}^r(\Omega)) \quad \text{with } \frac{n}{r}+\frac{2}{s}<1$$

Fabes, Jones, Riviere[FJR] and Sohr, von Wahl[W] extended this result to the equal case.

We note that, in Serrin's result, the regularity with respect to the time, t, variable is much weaker. The so called classical solution usually means the one with infinitely smoothness with respect to the spatial variables. Actually the following example constructed be Serrin prevent one from further expectation of the regularity in the time direction. Let $a(t)$ be any measurable function and $f(x)$ to be a harmonic function, then $\mathbf{v}(t) = a(t)\nabla f(x)$ is a weak solution of equations (1.7)—(1.9). Of course, for the boundary value problems, the situation may be better, see [La].

The gap between the case of existence and the case of regularity still exists for the Navier-Stokes equations. Schefer ([S1]—[S4]) first proved that under some conditions, the Hausdorff dimension of the singular set for the weak solutions of Navier-Stokes equation is $\frac{5}{3}$. The best known result in this direction is probably that of Caffarelli, Kohn Nirenberg

[CKN] which shows that the singular set of a "suitable weak solution" of the system (1.7)—(1.9) will satisfies:

$$\mathbf{P}^1(\mathbf{S}) = 0$$

where $\mathbf{P}^1$ is the one-dimension Hausdorff measure with respect to the parabolic matric in $\mathbf{R}^3 \times \mathbf{R}$. Also, by using an estimate of Solonnikov [S01], they proved the existence of the "suitable weak solution" in their paper.

The main technical devices in [CKN] are an induction argument and a decay estimate. The induction argument is a localized version of Scheffer's argument [S4], which gives the $\mathbf{P}^{\frac{5}{3}}$ estimate on the singular sets. However, due to one missing estimate for the pressure (see the proposed conjecture in [CKN] page 780), the argument in [CKN] becomes very difficult. However they still managed to get the conclusion $\mathbf{P}^1(\mathbf{S}) = 0$.

Motivated by the results of [CKN] and the relation between the system (1.1)—(1.6) and the Navier-Stokes equation (1.7)—(1.9) which was established in [LL], we will show the following partial regularity results for the system (1.1)—(1.6).

Main Theorem. *If the domain and the initial-boundary conditions in problem (1.1)—(1.6) are smooth enough, then there exists a suitable weak solution such that the singular set of this solution has one-dimension Hausdorff measure zero in space-time.*

Like what Scheffer [S1] and Caffarelli,Kohn Nirenberg [CKN] did for the Navier-Stokes equation, the "suitable weak solution" here will have the following "generalized energy inequality":

$$\begin{aligned}
&2\int_0^T\int_\Omega(|\nabla \mathbf{v}|^2+|\nabla^2\mathbf{d}|^2)\phi\,dxdt \\
&\leq \int_0^T\int_\Omega(|\mathbf{v}|^2+|\nabla\mathbf{d}|^2)(\phi_t+\Delta\phi)dxdt \\
&+\int_0^T\int_\Omega(|\mathbf{v}|^2+|\nabla\mathbf{d}|^2+2P)\mathbf{v}\cdot\nabla\phi\,dxdt \\
&+\int_0^T\int_\Omega((\mathbf{v}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\phi\,dxdt+\mathbf{R}(f,\phi)
\end{aligned} \tag{1.11}$$

for any ϕ which is a smooth function and has compact support in $\Omega \times (0, T)$.

The second to the last term in (1.11) represents the following:

$$\int_0^T\int_\Omega((\mathbf{v}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\phi\,dxdt = \int_0^T\int_\Omega \mathbf{v}^l\nabla_l\mathbf{d}^k\nabla_i\mathbf{d}^k\nabla_i\phi\,dxdt$$

The term $\mathbf{R}(f,\phi)$ is given by $\mathbf{R}(f,\phi) = \int_0^T\int_\Omega \nabla f(\mathbf{d})\nabla\mathbf{d}\phi\,dxdt$. Under very mild assumption on $f(\mathbf{d})$, one can easily show that $\mathbf{d}$ is bounded. Hence the term $\mathbf{R}(f,\phi)$ can be bounded by other terms in the inequality.

The generalized energy inequality can be formally obtained as follows. Multipling (1.1) by $\mathbf{v}\phi$, integrating by part, we have

$$\int_0^T \int_\Omega (-\frac{1}{2}|\mathbf{v}|^2\phi_t - \frac{1}{2}|\mathbf{v}|^2\mathbf{v}\nabla\phi - P\mathbf{v}\cdot\nabla\mathbf{v})dxdt$$
$$= \int_0^T \int_\Omega (-|\nabla\mathbf{v}|^2\phi + \frac{1}{2}|\mathbf{v}|^2\Delta\phi + \frac{1}{2}|\nabla\mathbf{d}|^2\mathbf{v}\cdot\nabla\phi - \Delta\mathbf{d}\nabla\mathbf{d}\mathbf{v}\phi)dxdt$$

Then take derivative of (1.3) with respect to the spatial variables,

$$\nabla\mathbf{d}_t + \nabla\mathbf{v}\cdot\nabla\mathbf{d} + \mathbf{v}\cdot\nabla\nabla\mathbf{d} = \Delta\nabla\mathbf{d} + \nabla f(\mathbf{d})$$

and multipling it by $\nabla\mathbf{d}\phi$, integrating by part, one obtains

$$\int_0^T \int_\Omega (-\frac{1}{2}|\nabla\mathbf{d}|^2\phi_t + \nabla\mathbf{v}\cdot\nabla\mathbf{d}\nabla\mathbf{d}\phi + \mathbf{v}\cdot\nabla\nabla\mathbf{d}\nabla\mathbf{d}\phi)dxdt$$
$$= \int_0^T \int_\Omega (-|\nabla^2\mathbf{d}|^2\phi + \frac{1}{2}|\nabla\mathbf{d}|^2\Delta\phi)dxdt + R(f,\mathbf{d})$$

Since

$$\int_0^T \int_\Omega (\nabla\mathbf{v}\cdot\nabla\mathbf{d}\nabla\mathbf{d}\phi + \mathbf{v}\cdot\nabla\nabla\mathbf{d}\nabla\mathbf{d}\phi)dxdt = \int_0^T \int_\Omega (-\Delta\mathbf{d}\nabla\mathbf{d}\mathbf{v}\phi - \mathbf{v}\cdot\nabla\mathbf{d}\nabla\mathbf{d}\nabla\phi)dxdt$$

We finally arrive at (1.11) by adding these two result.

This paper follows very closely to [CKN]. In Section 2, we will prove several key estimates. These estimates form a frame of the induction argument. After taking out the terms involving $\mathbf{d}$, these estimates are just those for the Navier-Stokes equation. After using a stronger estimate for the pressure due to Von Wahl, our arguments are some what simpler than that of [CKN].

In Section 3, we will first prove the key decay estimate. Combining this decay estimate and the results from Section 2, we will be able to obtain the main theorem.

Viscousity constants ν, λ, γ play no role in the results of this paper. For this reason we shall simply assume them to be all 1.

§2. Notations And Basic Estimates.

Suppose Ω is an open, smooth domain in $\mathbf{R}^3$, we define a cylinder with the top center point (x,t) to be

$$\mathbf{Q}_r(x,t) = \{(y,\tau)||y-x| < r, t-r^2 < \tau < t\} \tag{2.1}$$

For any $X \subset \mathbf{R}^3 \times \mathbf{R}, k \leq 0$, we define

$$\mathbf{P}^k(X) = \lim_{\delta\to 0^+} \mathbf{P}^k_\delta(X) \tag{2.2}$$

where $\mathbf{P}^k_\delta(X) = \inf\{\sum_{i=1}^\infty r^k | X \subset \bigcup_{i=1}^\infty \mathbf{Q}_{r_i}, r_i < \delta\}$ here $\mathbf{Q}_{r_i}$ represents a parabolic cylinder. We know that $\mathbf{P}^k$ is an outer measure. All Borel sets are $\mathbf{P}^k$ measurable, and $\mathbf{P}^k$ is Borel regular (cf. [Fed]).

Also, we have

$$\mathbf{H}^k \leq c(k)\mathbf{P}^k \tag{2.3}$$

where $\mathbf{H}^k$ is the Hausdorff measure with respect to the parabolic matrix $d((x,t),(y,\tau)) = [|x-y|^2 + |t-\tau|^2]^{\frac{1}{2}}$.

Next, we define another kind of cylinder

$$\mathbf{Q}^*_r(x,t) = \{(y,\tau) | |y-x| < r, t - \frac{7}{8}r^2 < \tau < t + \frac{1}{8}r^2\} \tag{2.4}$$

As we have seen in [LL], for the system (1.1)—(1.6), $\mathbf{v}$ has scaling dimension -1, $\mathbf{P}$ has dimension -2, $\mathbf{d}$ has dimension 0, while x has dimension 1 and time t has dimension 2.

We define $\mathbf{L}^p(0,T;\mathbf{L}^q(\Omega))$ to be the closure of C^∞ functions under the following norm

$$||\mathbf{v}||_{\mathbf{L}^p(0,T;\mathbf{L}^q(\Omega))} = (\int_0^T (\int_\Omega |\mathbf{v}|^q dx)^{\frac{p}{q}} dt)^{\frac{1}{p}} \tag{2.5}$$

In the case of $T = \infty$, we simply write it as $\mathbf{L}^p(\mathbf{L}^q(\Omega))$. If $p = q$, we will write it as $\mathbf{L}^p(\Omega)$.

The following lemmas will be very important in this and the next section.

Lemma 2.1 (Poincare Inequality). *If Ω is a bounded smooth domain, then*

$$||\phi||_{\mathbf{H}^{m,p}(\Omega)} \leq c(\Omega, m, p)||\nabla^m \phi||_{\mathbf{L}^p(\Omega)} \tag{2.6}$$

for all $\phi \in \mathbf{H}^{m,p}_0(\Omega)$.

Lemma 2.2. *For a bounded, smooth domain Ω, for any function $\mathbf{v} \in \mathbf{L}^\infty(0,T;\mathbf{L}^m(\Omega)) \cap \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$, and $\mathbf{v}$ vanishes at the boundary, then there exists a constant c depending only on n, the space dimension, m and p, such that*

$$\begin{aligned} &\int_0^T \int_\Omega |\mathbf{v}(x,t)|^q dx dt \\ &\leq c^q (\int_0^T \int_\Omega |D\mathbf{v}(x,t)|^p dx dt)(\sup_{0<t<T} \int_\Omega |\mathbf{v}(x,t)|^m dx)^{\frac{p}{n}} \end{aligned} \tag{2.7}$$

where $q = \frac{p(m+n)}{n}$.

Note that in the case $n = 3, m = 2, p = 2$, the lemma gives the bound of the $\mathbf{L}^{\frac{10}{3}}$ norm of the function. Since the weak solutions obtained in [LL] satisfy:

$$\mathbf{v} \in \mathbf{L}^2(\mathbf{H}^1) \cap \mathbf{L}^\infty(\mathbf{L}^2), \mathbf{d} \in \mathbf{L}^2(\mathbf{H}^2) \cap \mathbf{L}^\infty(\mathbf{H}^1)$$

we have, by Lemma 2.2, that

$$\mathbf{v} \in \mathbf{L}^{\frac{10}{3}}, \nabla \mathbf{d} \in \mathbf{L}^{\frac{10}{3}}.$$

Lemma 2.3. *For* $\mathbf{v} \in \mathbf{H}^1(\mathbf{R}^3)$,

$$\int_{B^r} |\mathbf{v}|^q \le C(\int_{B^r} |\nabla \mathbf{v}|^2)^a)(\int_{B^r} |\mathbf{v}|^2)^{\frac{q}{2}-a} + \frac{C}{r^{2a}}(\int_{B^r} |\mathbf{v}|^2)^{\frac{q}{2}} \tag{2.8}$$

where C *is independent of* r,

$$2 \le q \le 6, a = \frac{3}{4}(q-2).$$

The proof of this lemma can be found in [CKN]. One interesting case is when $q = \frac{10}{3}, a = 1$, we recover the $\mathbf{L}^{\frac{10}{3}}$ norm stated above.

Let us consider the global weak solution of (1.1)—(1.6) so that,

$$\begin{aligned} &\mathbf{v} \in \mathbf{L}^\infty(0,T;\mathbf{L}^2) \cap \mathbf{L}^2(0,T;\mathbf{H}^1) \\ &\mathbf{d} \in \mathbf{L}^\infty(0,T;\mathbf{H}^1) \cap \mathbf{L}^2(0,T;\mathbf{H}^2) \end{aligned} \tag{2.9}$$

and satisfies (1.1)—(1.6) in the weak sense.

the next Theorem will give an estimate of the pressure $\mathbf{P}$. It follows from that of Sohr and von Wahl [SW] for the Navier-Stokes equations.

Theorem 2.5. *Let* $\Omega \subset \mathbf{R}^3$ *be a smooth, bounded domain.* $\mathbf{v}_0, \mathbf{d}_0$ *are smooth enough.* $s, p \in (1, \infty)$ *with* $n \le \frac{2}{s} + \frac{n}{p}, \frac{1}{p} + \frac{1}{n} < 1, \frac{1}{q} = \frac{1}{p} + \frac{1}{n}$, *such that*

$$\begin{aligned} &\mathbf{v} \in \mathbf{L}^\infty(0,T;\mathbf{L}^2) \cap \mathbf{L}^2(0,T;\mathbf{H}^1) \\ &\mathbf{d} \in \mathbf{L}^\infty(0,T;\mathbf{H}^1) \cap \mathbf{L}^2(0,T;\mathbf{H}^2) \end{aligned}$$

Then there exists a $\mathbf{P} \in \mathbf{L}^s(0,T;\mathbf{L}^p(\Omega))$ *with* $\nabla \mathbf{P} \in \mathbf{L}^s(0,T;\mathbf{L}^q(\Omega))$ *which satisfies (1.1)—(1.6) together with* $\mathbf{v}, \mathbf{d}$.

Remark. In Theorem 2.5, if we take $s = p = \frac{5}{3}, n = 3, q = \frac{15}{14}$, we get that $\mathbf{P} \in \mathbf{L}^{\frac{5}{3}}(0,T;\mathbf{L}^{\frac{5}{3}})$. And this proves the conjecture of [CKN] page 780.

There are several ways to prove the theorem. Besides the method used by Sohr and von Wahl [SW], one can also use the estimate by Solonnikov [Sol] which was actually used by Caffarelli, Kohn, Nirenberg in [CKN] to prove the existence of the suitable weak solution of Navier-Stokes equation. We will just sketch the idea of the proof here.

Since we have that

$$\mathbf{v}\cdot\nabla\mathbf{v}\in\mathbf{L}^{\infty}([0,T],\mathbf{L}^2(\Omega))\cap\mathbf{L}^2([0,T],\mathbf{H}^1(\Omega))$$

which implies that

$$\mathbf{v}\in\mathbf{L}^{\frac{10}{3}}([0,T],\mathbf{L}^{\frac{10}{3}}(\Omega)),\nabla\mathbf{v}\in\mathbf{L}^2([0,T],\mathbf{L}^2(\Omega)) \tag{2.10}$$

By Holder's inequality, we have that

$$\mathbf{v}\cdot\nabla\mathbf{v}\in\mathbf{L}^{\frac{5}{4}}([0,T],\mathbf{L}^{\frac{5}{4}}(\Omega)) \tag{2.11}$$

The same argument works for the term $\nabla(\nabla\mathbf{d}\odot\nabla\mathbf{d})$.

We rewrite the equation (1.1) in the form:

$$\mathbf{v}_t-\nu\Delta\mathbf{v}+\nabla\mathbf{P}=-(\mathbf{v}\cdot\nabla)\mathbf{v}-\lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) \tag{1.1$'$}$$

The left hand side is the Stokes operator, while the right hand side is in $\mathbf{L}^{\frac{5}{4}}$ space-time. The Solonnikov [Sol] gives

$$\nabla\mathbf{P}\in\mathbf{L}^{\frac{5}{4}}([0,T],\mathbf{L}^{\frac{5}{4}}(\Omega)) \tag{2.12}$$

which in turn gives:

$$\mathbf{P}\in\mathbf{L}^{\frac{5}{4}}([0,T],\mathbf{L}^{\frac{15}{7}}(\Omega)) \tag{2.13}$$

see [CKN].

If we use Lemma 2.3 with

$$q=\frac{10}{3},a=1$$

we obtain:

$$\int|\mathbf{v}|^{\frac{10}{3}}\le C(\int|\nabla\mathbf{v}|^2)(\int|\mathbf{v}|^2)^{\frac{2}{3}}+\frac{C}{r^2}(\int|\mathbf{v}|^2)^{\frac{5}{3}} \tag{2.14}$$

But if we choose

$$q=\frac{30}{13},a=\frac{3}{4}(\frac{30}{13}-2)=\frac{3}{13}$$

we have

$$\int|\mathbf{v}|^{\frac{30}{13}}\le C(\int|\nabla\mathbf{v}|^2)^{\frac{3}{13}}(\int|\mathbf{v}|^2)^{\frac{12}{13}}+\frac{c}{r^{\frac{6}{13}}}(\int|\mathbf{v}|^2)^{\frac{15}{13}} \tag{2.15}$$

Applying the Holder's inequality again:

$$\begin{aligned}||\nabla\mathbf{v}\cdot\mathbf{v}||_{\frac{15}{14}}^{\frac{5}{3}}&\le C||\mathbf{v}||_{\frac{30}{13}}^{\frac{5}{3}}||\nabla\mathbf{v}||_2^{\frac{5}{3}}\\&\le C_1||\nabla\mathbf{v}||_2^2+c_2||\mathbf{v}||_{\frac{30}{13}}^{10}\end{aligned} \tag{2.16}$$

To estimate the the second term, we note that:

$$(\int |\mathbf{v}|^{\frac{30}{13}})^{\frac{13}{30}} \leq C(\int |\nabla \mathbf{v}|^2)^{\frac{1}{10}}(\int |\mathbf{v}|^2)^{\frac{2}{5}} + \frac{C}{r^{\frac{1}{5}}}(\int |\mathbf{v}|^2)^{\frac{1}{2}} \tag{2.17}$$

thus,

$$\|\mathbf{v}\|^{10}_{\frac{30}{13}} \leq (\int |\nabla \mathbf{v}|^2)(\int |\mathbf{v}|^2)^4 + \frac{C}{r^2}(\int |\mathbf{v}|^2)^5 \tag{2.18}$$

All the above integrals are taken with respect to the space. And we see that

$$\mathbf{v} \cdot \nabla \mathbf{v} \in \mathbf{L}^{\frac{5}{3}}([0,T], \mathbf{L}^{\frac{15}{14}}(\Omega)) \tag{2.19}$$

After we do the same estimates fot $\nabla(\nabla \mathbf{d} \bigodot \nabla \mathbf{d})$, we get,

$$\nabla \mathbf{P} \in \mathbf{L}^{\frac{5}{3}}([0,T], \mathbf{L}^{\frac{15}{14}}(\Omega)) \tag{2.20}$$

which, by the Sobolev's embedding theorem, implies,

$$\mathbf{P} \in \mathbf{L}^{\frac{5}{3}}([0,T], \mathbf{L}^{\frac{5}{3}}(\Omega)) \tag{2.21}$$

Now we can give the definition of the "suitable weak solutions" of the system (1.1)—(1.6). Since the estimate of the pressure in Theorem 2.5, we don't need the unnatural restraint $\mathbf{P} \in \mathbf{L}^{\frac{5}{4}}(\Omega \times (0,T))$ as in the definition of [CKN].

Definition. $(\mathbf{v}, \mathbf{d})$ *is called a suitable weak solution of the system (1.1)—(1.6) on an open set* $\mathbf{D} \subset \mathbf{R}^3 \times \mathbf{R}$ *if the following conditions are true:*

1. $(\mathbf{v}, \mathbf{d})$ *satisfies the system (1.1)—(1.6) in the weak sense, i.e., it is a weak solution.*

2. There exist constants E_1, E_2 *such that,*

$$\begin{aligned} &\int_{\mathbf{D}_t} [|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2 + F(\mathbf{d})] dx < E_1 \\ &\int \int_{\mathbf{D}} [|\nabla \mathbf{v}|^2 + |\Delta \mathbf{d} - f(\mathbf{d})|^2] dx dt < E_2 \end{aligned} \tag{2.22}$$

where $\mathbf{D}_t = \mathbf{D} \cap (\mathbf{R}^3 \times t)$.

3. For any $\phi \in \mathbf{C}^\infty(\mathbf{D}), \phi > 0$, *the generalized energy inequality (1.11) holds.*

Remark. The existence of the suitable weak solution can be shown by the exactly the same method as that of [CKN] and [LL1], together with Theorem 2.5 and we will omit the detail here.

The key theorem of this section is the following,

Theorem 2.6. *There exist constants* $\epsilon, C_1 > 0$, *such that, if* $(\mathbf{v}, \mathbf{d}, \mathbf{P})$ *is a suitable weak solution of (1.1)—(1.6) on* $\mathbf{Q}_1$ *with the following property,*

$$\int\int_{\mathbf{Q}_1} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3 + |\mathbf{P}|^{\frac{3}{2}})dxdt < \epsilon \tag{2.23}$$

Then

$$|\mathbf{v}(x,t)| + |\nabla \mathbf{d}(x,t)| \leq C$$

for Lebesgue almost every point $(x,t) \in \mathbf{Q}_{\frac{1}{2}}$.

Definition. A point x, t is called a regular point of the solution if $|\mathbf{v}(x,t)| + |\nabla \mathbf{d}(x,t)| \leq C$for Lebesgue almost every point $(x,t) \in \mathbf{Q}_{\frac{1}{2}}$. The complement of the set of all the regular points will be called the singular set.

Remark. Suppose that Theorem 2.6 is true. Let V be a neighbourhood of S, which is the singular set of a solution in $D = \Omega \times [0,T]$. For each point $(x,t) \in S$, we choose $\mathbf{Q}^*_r(x,t) \subset V$ such that, for any δ, we can find $r < \delta$, and

$$r^{-\frac{5}{3}} \int\int_{\mathbf{Q}^*_r(x,t)} (|\mathbf{v}|^{\frac{10}{3}} + |\nabla \mathbf{d}|^{\frac{10}{3}} + |\mathbf{P}|^{\frac{5}{3}}) > \epsilon$$

This is because otherwise, by using Holder's inequality, we see (2.23) will be true and (x,t) will be a regular point.

Applying a Vitali-type covering lemma, we obtain a disjoint subfamily $\mathbf{Q}^*_{r_j}(x,t)$, such that

$$S \subset \cup_i \mathbf{Q}^*_{5r_j}(x,t), r_i < \delta$$

and we see that

$$\begin{aligned}\sum r_i^{\frac{5}{3}} &\leq \epsilon^{-1} \sum_i \int\int_{\mathbf{Q}^*_{r_j}(x,t)} (|\mathbf{v}|^{\frac{10}{3}} + |\nabla \mathbf{d}|^{\frac{10}{3}} + |\mathbf{P}|^{\frac{5}{3}})dxdt \\ &\leq 5\epsilon^{-1} \int\int_V (|\mathbf{v}|^{\frac{10}{3}} + |\nabla \mathbf{d}|^{\frac{10}{3}} + |\mathbf{P}|^{\frac{5}{3}})dxdt\end{aligned}$$

Since δ is arbitrary, we get that S has Lebesgue measure zero. and also

$$\mathbf{P}^{\frac{5}{3}}(S) \leq \frac{5}{\epsilon} \int\int_V (|\mathbf{v}|^{\frac{10}{3}} + |\nabla \mathbf{d}|^{\frac{10}{3}} + |\mathbf{P}|^{\frac{5}{3}})dxdt$$

for every neighbourhood V of S.

Since

$$\int\int (|\mathbf{v}|^{\frac{10}{3}} + |\nabla \mathbf{d}|^{\frac{10}{3}} + |\mathbf{P}|^{\frac{5}{3}})dxdt < \infty$$

and since V is arbitrary neighbourhood of S, We have

$$\mathbf{P}^{\frac{5}{3}}(S) = 0$$

The proof of the Theorem 2.6 is built upon an induction argument which was used in [S1] and [CKN].

We pick a point $(a, s) \in \mathbf{Q}_{\frac{1}{2}}(0,0)$, such that,

$$\int\int_{\mathbf{Q}_{\frac{1}{2}}(a,s)} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3 + |\mathbf{P}|^{\frac{3}{2}}) dxdt < \epsilon \tag{2.24}$$

Here we notice the fact that $\mathbf{Q}_{\frac{1}{2}}(a, s) \subset \mathbf{Q}_1(0,0)$.

Let $\mathbf{Q}^n = \mathbf{Q}_{r_n}(a, s)$, where $r_n = 2^{-n}$.

The induction argument follows from the following Lemmas.

Let $(\mathbf{v}, \mathbf{d}, \mathbf{P})$ be a suitable solution of (1.1)—(1.6) satisfying (2.23), (2.24).

Lemma 2.7. *If* $2 \le k \le n$,

$$\begin{aligned}&\sup_{s-r_k{}^2 \le t \le s} \fint_{|x-a| \le r_k} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2) dx \\ &+ r_k^{-3} \int\int_{\mathbf{Q}^k} (|\nabla \mathbf{v}|^2 + |\nabla^2 \mathbf{d}|^2) dxdt \\ &\le C\epsilon^{\frac{2}{3}}\end{aligned} \tag{2.25$_k$}$$

then

$$\fint\fint_{\mathbf{Q}^{(n+1)}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3) + r_n^{\frac{1}{3}} \fint\fint_{\mathbf{Q}^{(n+1)}} |\mathbf{v}||\mathbf{P} - \bar{\mathbf{P}}_{n+1}| dxdt \le \epsilon^{\frac{2}{3}} \tag{2.26$_{n+1}$}$$

Lemma 2.8. *if* $3 \le k \le n$,

$$\fint\fint_{\mathbf{Q}^k} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3) + r_n^{\frac{1}{3}} \fint\fint_{\mathbf{Q}^k} |\mathbf{v}||\mathbf{P} - \bar{\mathbf{P}}_k| dxdt \le \epsilon^{\frac{2}{3}} \tag{2.26$_k$}$$

then

$$\begin{aligned}&\sup_{s-r_n < t < s} \fint_{|x-a| < r_{n+1}} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2) dx \\ &+ r_{n+1}^{-3} \int\int_{\mathbf{Q}^n} (|\nabla \mathbf{v}|^2 + |\nabla^2 \mathbf{d}|^2) dxdt \\ &\le C\epsilon^{\frac{2}{3}}\end{aligned} \tag{2.25$_n$}$$

Here $\fint$ and $\fint\fint$ denote the averages. Also,

$$\bar{\mathbf{P}}_n = \bar{\mathbf{P}}_n(t) = \fint_{|x-a|<r_n} \mathbf{P} dx$$

Proof of Theorem 2.6. Using the generalized energy inequality, where we choose the test function ϕ to be smooth, positive with value between 0 and 1, and $\phi = 1$ on $\mathbf{Q}_2$ and $\phi = 0$ our of $\mathbf{Q}_1$, and the Holder's inequality, we have that $(2.25)_2$ is true.

Then by applying Lemma 2.7 and Lemma 2.8, we get

$$\fint_{|x-a|<r_n} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2) dx \le C\epsilon_1^{\frac{2}{3}}$$

for any $(a,s) \in \mathbf{Q}_{\frac{1}{2}}(0,0)$, and $r_n = 2^{-n}$.

Then we have $|\mathbf{v}|^2(a,s) + |\nabla \mathbf{d}|^2(a,s) \le C$, for any Lebesgue point (a,s). In particular, $\mathbf{v}$ is essentially bounded in $\mathbf{Q}_{\frac{1}{2}}(0,0)$.

(Q.E.D.)

Proof of Lemma 2.7. From (2.7)and $(2.25)_k$, together with Lemma 2.3, one has,

$$\int_{\mathbf{B}_r} |\mathbf{v}|^3 dx \le (\int_{\mathbf{B}_r} |\nabla \mathbf{v}|^2 dx)^{\frac{3}{4}} (\int_{\mathbf{B}_r} |\mathbf{v}|^2 dx)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} (\int_{\mathbf{B}_r} |\mathbf{v}|^2 dx)^{\frac{3}{2}} \tag{2.27}$$

$$\begin{aligned} \int\int_{\mathbf{Q}_r} |\mathbf{v}|^3 dx dt \le & (\int\int_{\mathbf{Q}_r} |\nabla \mathbf{v}|^2 dx dt)^{\frac{3}{4}} (\int_{-r^2}^{0} (\int_{\mathbf{B}_r} |\mathbf{v}|^2 dx)^3 dt)^{\frac{1}{4}} \\ & + \frac{C}{r^{\frac{3}{2}}} \int_{-r^2}^{0} (\int_{\mathbf{B}_r} |\mathbf{v}|^2 dx)^{\frac{3}{2}} dt \end{aligned} \tag{2.28}$$

Thus we get,

$$\begin{aligned} r_n^5 \fint\fint_{\mathbf{Q}_n} |\mathbf{v}|^3 dx dt \le & C(r_n^3 \epsilon^{\frac{2}{3}})^{\frac{3}{4}} (r_n^2 (r_n^3 C \epsilon^{\frac{2}{3}})^3)^{\frac{1}{4}} + C r_n^{\frac{-3}{2}} r_n^2 (r_n^3 C \epsilon^{\frac{2}{3}})^{\frac{3}{2}} \\ \le & C r_n^5 \epsilon + C r_n^5 \epsilon \end{aligned} \tag{2.29}$$

by using (2.25_k), that is,

$$\fint\fint_{\mathbf{Q}_n} |\mathbf{v}|^3 dx dt \le C\epsilon \le \epsilon^{\frac{2}{3}} \tag{2.30}$$

The same estimates work for the term $\fint\fint_{\mathbf{Q}^n} |\nabla \mathbf{d}|^3$.

For the last term of the left hand side of (2.12_{n+1}), we can use the Green function representation of P

$$\begin{aligned} \phi P &= -\frac{3}{4\pi} \int_{\mathbf{R}^3} \frac{1}{|x-y|} \Delta_y (\phi P) dy \\ &= -\frac{3}{4\pi} \int_{\mathbf{R}^3} \frac{1}{|x-y|} (P\Delta\phi + 2(\nabla\phi, \nabla P) + \phi \Delta P) dy \end{aligned}$$

Note, by equation (1.1), that

$$\Delta P = -\nabla_i \mathbf{v}^j \nabla_j \mathbf{v}^i - \nabla_{ij}(\nabla_i \mathbf{d} \nabla_j \mathbf{d}) \tag{2.31}$$

After integration by part, we see,

$$P = P_3 + P_4 + P_5 \tag{2.32}$$

where

$$P_3 = \frac{3}{4\pi}\int \frac{1}{|x-y|} P(y)\Delta\phi(y)dy + \frac{3}{2\pi}\int \frac{x_i - y_i}{|x-y|^3}\nabla_i \phi P(y)dy, \tag{2.33}$$

$$\begin{aligned} P_4 = & \frac{3}{2\pi}\int \frac{x_i - y_i}{|x-y|^3}\nabla_j \phi(\mathbf{v}^i\mathbf{v}^j + \nabla_i \mathbf{d}\nabla_j \mathbf{d})dy \\ & + \frac{3}{4\pi}\int \frac{1}{|x-y|}\nabla_{ij}\phi(\mathbf{v}^i\mathbf{v}^j + \nabla_i \mathbf{d}\nabla_j \mathbf{d})dy \end{aligned} \tag{2.34}$$

$$P_5 = \frac{3}{4\pi}\int \nabla_i \nabla_j \frac{1}{|x-y|}\phi(y)(\mathbf{v}^i\mathbf{v}^j + \nabla_i \mathbf{d}\nabla_j \mathbf{d})dy \tag{2.35}$$

where ϕ is the cut-off function equal to 1 in $\{|y| \leq \frac{1}{4}\}$ and equal to 0 outside $\{|y| \geq \frac{1}{2}\}$, and

$$|\nabla_i \phi| \leq C, \quad |\nabla_{ij}\phi| \leq C. \tag{2.36}$$

We decompose $P_5 = P_8 + P_9$, where

$$P_8 = \frac{3}{4\pi}\int_{|y|<2r_{n+1}} \nabla_i \nabla_j \frac{1}{|x-y|}\phi(y)(\mathbf{v}^i\mathbf{v}^j + \nabla_i \mathbf{d}\nabla_j \mathbf{d})dy \tag{2.37}$$

$$P_9 = \frac{3}{4\pi}\int_{2r_{n+1}<|y|<\frac{1}{4}} \nabla_i \nabla_j \frac{1}{|x-y|}\phi(y)(\mathbf{v}^i\mathbf{v}^j + \nabla_i \mathbf{d}\nabla_j \mathbf{d})dy \tag{2.38}$$

We note that

$$|P - \bar{P}_r| \leq \sum_{i=3,4,8,9} |P_i - \bar{P}_i|,$$

here

$$\bar{P}_i = \fint_{\mathbf{B}_n} P_i$$

For P_8, we can use the Calderon-Zygmond's inequality to obtain, we get:

$$\int_{\mathbf{B}_{n+1}} |P_8|^{\frac{3}{2}} \leq C\int_{\mathbf{B}_n} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3) \tag{2.39}$$

Therefore

$$\begin{aligned}\int_{\mathbf{B}_{n+1}} |\mathbf{v}||P_8 - \bar{P}_8| &\le C(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3)^{\frac{1}{3}}(\int_{\mathbf{B}_n} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{2}{3}} \\ &\quad + C(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3)^{\frac{1}{3}}(\fint_{\mathbf{B}_n} P_8)r_n^2 \\ &\le C(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3)^{\frac{1}{3}}(\int_{\mathbf{B}_n} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{2}{3}} \\ &\quad + C(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3)^{\frac{1}{3}}(\int_{\mathbf{B}_n} (P_8)^{\frac{3}{2}})^{\frac{2}{3}} r_n^2 \frac{1}{r_n^2} \\ &\le C(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3)^{\frac{1}{3}}(\int_{\mathbf{B}_n} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{2}{3}}\end{aligned}$$

For P_3, P_4, P_9, we bound $|P_i - \bar{P}_i|$ uniformaly in $\mathbf{B}_{n+1}$. Indeed, for $|x| < r_{n+1}$, we have

$$|\nabla P_3| \le C \int_{\mathbf{B}_{\frac{1}{4}}} |P| \tag{2.40}$$

$$|\nabla P_4| \le C \int_{\mathbf{B}_{\frac{1}{4}}} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2) \tag{2.41}$$

$$|\nabla P_9| \le C \int_{r_n < |y| < \frac{1}{4}} (\frac{|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2}{|y|^4})dy \tag{2.42}$$

Thus for $i = 3, 4, 9$, one has,

$$\begin{aligned}\int_{\mathbf{B}_{n+1}} |\mathbf{v}||P_i - \bar{P}_i| &\le Cr_n^2(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}} \sup_{x\in \mathbf{B}_{n+1}} |P_i(x) - \bar{P}_i| \\ &\le Cr_n^3(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}} \sup_{x\in \mathbf{B}_n} |\nabla P_i|\end{aligned}$$

by the mean value theorem.

Remark. We can also use the Poincare Inequality in the last estimateabove. Indeed,

$$\begin{aligned}\int_{\mathbf{B}_{n+1}} |\mathbf{v}||P_i - \bar{P}_i| &\le C(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}}(\int_{\mathbf{B}_{n+1}} |P_i(x) - \bar{P}_i|^{\frac{3}{2}})^{\frac{2}{3}} \\ &\le Cr_{n+1}(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}}(\int_{\mathbf{B}_{n+1}} |\nabla P_i|^{\frac{3}{2}})^{\frac{2}{3}} \\ &\le Cr_n^3(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}} \sup_{x\in \mathbf{B}_n} |\nabla P_i|\end{aligned}$$

and hence we have

$$\begin{aligned}\int_{\mathbf{B}_{n+1}} |\mathbf{v}||P_3 - \bar{P}_3| &\le Cr_n^3(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}}(\int_{\mathbf{B}_2} |P|) \\ &\le Cr_n^3(\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}}(\int_{\mathbf{B}_2} |P|^{\frac{3}{2}})^{\frac{2}{3}}\end{aligned},$$

$$\int_{\mathbf{B}_{n+1}} |\mathbf{v}||P_4 - \bar{P}_4| \le Cr_n^3 (\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}} (\int_{\mathbf{B}_2} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2$$

$$\le Cr_n^3 (\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}} (\int_{\mathbf{B}_2} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{2}{3}}$$

and

$$\int_{\mathbf{B}_{n+1}} |\mathbf{v}||P_9 - \bar{P}_9| \le Cr_n^3 (\int_{\mathbf{B}_{n+1}} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{1}{3}} \int_{r^n < |y| < \frac{1}{4}} (\frac{|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2}{|y|^4}) dy$$

To summerize, we have obtained

$$\int\int_{\mathbf{Q}_{n+1}} |\mathbf{v}||P_8 - \bar{P}_8| dxdt \le C (\int\int_{\mathbf{Q}_{n+1}} |\mathbf{v}|^3)^{\frac{1}{3}} (\int\int_{\mathbf{Q}_n} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{2}{3}}$$

$$\le Cr_{n+1}^5 \epsilon$$

$$\int\int_{\mathbf{Q}_{n+1}} |\mathbf{v}||P_9 - \bar{P}_9| dxdt \le Cr_{n+1}^3 r_{n+1}^{\frac{4}{3}} (\int\int_{\mathbf{Q}_{n+1}} |\mathbf{v}|^3)^{\frac{1}{3}}$$

$$\sup_{s - r_{n+1}^2 < t < s} \int_{r_n < |y| < \frac{1}{4}} (\frac{|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2}{|y|^4}) dy$$

$$\le Cr_{n+1}^{\frac{13}{3}} r_{n+1}^{\frac{5}{3}} \epsilon^{\frac{1}{3}} \sum_{k=2}^{n} r_k^{-4} r_k^3 \epsilon^{\frac{2}{3}} = Cr_{n+1}^6 \epsilon \sum_{k=2}^{n} r_k^{-1} = Cr_{n+1}^5 \epsilon$$

$$\int\int_{\mathbf{Q}_{n+1}} |\mathbf{v}||P_4 - \bar{P}_4| dxdt \le Cr_{n+1}^3 (\int\int_{\mathbf{Q}_{n+1}} |\mathbf{v}|^3)^{\frac{1}{3}} (\int\int_{\mathbf{Q}_2} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{2}{3}}$$

$$\le Cr_{n+1}^3 r_{n+1}^{\frac{5}{3}} \epsilon \le Cr_{n+1}^5 r_{n+1}^{\frac{-1}{3}} \epsilon$$

and

$$\int\int_{\mathbf{Q}_{n+1}} |\mathbf{v}||P_3 - \bar{P}_3| dxdt \le Cr_{n+1}^3 (\int\int_{\mathbf{Q}_{n+1}} |\mathbf{v}|^3)^{\frac{1}{3}} (\int\int_{\mathbf{Q}_2} |P|^{\frac{3}{2}})^{\frac{2}{3}}$$

$$\le Cr_{n+1}^3 r_{n+1}^{\frac{5}{3}} \epsilon \le Cr_{n+1}^5 r_{n+1}^{\frac{-1}{3}} \epsilon$$

In the other words, we have

$$r_{n+1}^{\frac{1}{3}} \fint\fint_{\mathbf{Q}_{n+1}} |\mathbf{v}||P - \bar{P}| \le C\epsilon \le \epsilon^{\frac{2}{3}}$$

(Q.E.D.)

Proof of Lemma 2.8. Without lose of generality, we will assume (a, s) to be just $(0, 0)$.

First, we have to obtain a modified version of the generalized energy inequality (1.11). Take the test function of the form $\phi(x,t)\eta(\frac{t-s}{\epsilon})$, where $\eta = 1$ when $s \ge 1$ and $\eta = 0$ for

$s \leq 0$ and it is a smooth function between 0 and 1. After putting this in (1.11) and taking the limit of ϵ goes to zero, we get the following:

$$
\begin{aligned}
\int_{\Omega\times\{t\}} (|\mathbf{v}|^2 + |\nabla\mathbf{d}|^2)\phi dx + 2\int_0^t \int_\Omega (|\nabla\mathbf{v}|^2 + |\nabla^2\mathbf{d}|^2)\phi dxdt \\
\leq \int_0^t \int_\Omega (|\mathbf{v}|^2 + |\nabla\mathbf{d}|^2)(\phi_t + \Delta\phi) dxdt \\
+ \int_0^t \int_\Omega (|\mathbf{v}|^2 + |\nabla\mathbf{d}|^2 + 2P)\mathbf{v}\cdot\nabla\phi dxdt \\
+ \int_0^t \int_\Omega ((\mathbf{v}\cdot\nabla)\mathbf{d} \odot \nabla\mathbf{d})\cdot\nabla\phi dxdt + \mathbf{R}(f,\phi)
\end{aligned}
\tag{2.43}
$$

Next we choose the test function to be of form $\phi_n = \chi\psi_n$, such that

$$
\psi_n = \frac{1}{(r_n^2 - t)^{\frac{3}{2}}} e^{\frac{-|x|^2}{4(r_n^2 - t)}} \tag{2.44}
$$

and χ is the cut-off function with is smooth, between 0 and 1, and equal to 1 in $\mathbf{Q}_{\frac{1}{4}}(0,0)$ and equal to 0 outside $\mathbf{Q}_{\frac{1}{3}}(0,0)$.

We note that ψ_n is the fundemental solution of the backward heat equation

$$
u_t + \Delta u = 0
$$

with singularity at $(0, r_n^2)$.

After putting these test functions in (2.43), we obtain that:

$$
\begin{aligned}
&\sup_{s-{r_n}^2 \leq t \leq s} \fint_{|x-a|\leq r_n} (|\mathbf{v}|^2 + |\nabla\mathbf{d}|^2) dx + r_n^{-3} \int\int_{\mathbf{Q}^k} (|\nabla\mathbf{v}|^2 + |\nabla^2\mathbf{d}|^2) dxdt \\
&\leq C(\int_0^t \int_{\mathbf{B}_n} (|\mathbf{v}|^2 + |\nabla\mathbf{d}|^2)(\phi_{nt} + \Delta\phi_n) dxdt \\
&+ \int_0^t \int_{\mathbf{B}_n} (|\mathbf{v}|^3 + |\nabla\mathbf{d}|^3)\nabla\phi_n dxdt + \int_0^t \int_{\mathbf{B}_n} (\mathbf{v}P\nabla\phi_n dxdt) \\
&\leq I + II + III
\end{aligned}
\tag{2.45}
$$

Direct calculation shows that:

$$
|\phi_{nt} + \Delta\phi_n| \leq C \tag{2.46}
$$

$$
|\phi_n| \leq C r_n^{-3}, |\nabla\phi_n| \leq C r_n^{-4}, \quad \text{on } \mathbf{Q}^n, \tag{2.47}
$$

$$
|\phi_n| \leq C r_k^{-3}, |\nabla\phi_n| \leq C r_k^{-4}, \quad \text{on } \mathbf{Q}^{k-1} \setminus \mathbf{Q}^k, \tag{2.48}
$$

Therefore, by the Holder inequality, one has the following.

$$I \leq C\int_0^t \int_{\mathbf{B}_n} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2) \leq C(\int\int_{\mathbf{Q}_1} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)^{\frac{2}{3}} \leq c\epsilon^{\frac{2}{3}} \tag{2.49}$$

by (2.24), and

$$II \leq C\sum_{k=1}^{n} r_k^{-4} \int\int_{\mathbf{Q}_k} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3) dxdt \leq C\sum_{k=1}^{n} r_k \epsilon^{\frac{2}{3}} \leq C\epsilon^{\frac{2}{3}} \tag{2.50}$$

For eash $k \geq 1$, take η_k to be a smooth cut off function with value 1 at $\mathbf{Q}_{\frac{7}{8}r^k}$ and 0 out of $\mathbf{Q}_{r_k}$ with the property:

$$|\nabla \eta_k| \leq Cr_k^{-1}$$

$$III \leq C\sum_{k=1}^{n} \int\int_{\mathbf{Q}_1} P\mathbf{v} \cdot \nabla((\eta_k - \eta_{k+1})\phi_n) + \int\int_{\mathbf{Q}_1} P\mathbf{v} \cdot \nabla(\eta_n \phi_n) \tag{2.51}$$

Use the divergence-free property of $\mathbf{v}$,

$$\sum_{k\geq 3}^{n} \int\int_{\mathbf{Q}_1} P\mathbf{v} \cdot \nabla((\eta_k - \eta_{k+1})\phi_n) \leq \sum_{k\geq 3}^{n} \int\int_{\mathbf{Q}_k} (P - \bar{P}_k)\mathbf{v} \cdot \nabla((\eta_k - \eta_{k+1})\phi_n)$$

where $\bar{P}_k = \bar{P}_k(t) = \fint_{bfB_k} P$.

Similarly, one has

$$\int\int_{\mathbf{Q}_1} P\mathbf{v} \cdot \nabla(\eta_n \phi_n) \leq \int\int_{\mathbf{Q}_n} (P - \bar{P}_k)\mathbf{v} \cdot \nabla(\eta_n \phi_n)$$

When $k = 1, 2$, one simply has

$$\int\int_{\mathbf{Q}_k} (P - \bar{P}_k)\mathbf{v} \cdot \nabla((\eta_k - \eta_{k+1})\phi_n) \leq C\int\int_{\mathbf{Q}_1} |P||\mathbf{v}|$$

Thus

$$\begin{aligned} III &\leq C\sum_{k=3}^{n} r_k^{-4} \int\int_{\mathbf{Q}_k} |\mathbf{v}||P - \bar{P}_k| + C\int\int_{\mathbf{Q}_1} |\mathbf{v}||P| \\ &\leq C\sum_{k=3}^{n} r_k^{\frac{2}{3}} \epsilon^{\frac{2}{3}} \leq C\epsilon^{\frac{2}{3}} \end{aligned}$$

and this yields the result. (Q.E.D.)

Remark. From the proof of the Lemmas, we see that, by using the new estimate for the pressure, the proof is somewhat simplier than that in [CKN]. We also see that it is the generalized energy law which relates the quantities of the different scaling dimensions.

§3. Proof Of The Main Theorem.

We will proof the following key decay estimate. This decay estimate and Theorem 2.6 will imply the main theorem.

Lemma 3.1. *If* $(\mathbf{v}, \mathbf{d}, \mathbf{P})$ *is a suitable solution of the system (1.6)—(1.7). Then there exists a constant* ϵ*, such that, if*

$$\limsup_{r\to 0} r^{-1} \int\int_{\mathbf{Q}_r^*(0,0)} (|\nabla \mathbf{v}|^2 + |\nabla^2 \mathbf{d}|^2) dxdt < \epsilon \tag{3.1}$$

Then there exist $\gamma < 1$ *such that if* $r < 1$*, then*

$$\begin{aligned}
&\frac{1}{(\gamma r)^2} \int\int_{\mathbf{Q}_{\gamma r}(0,0)} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3) dxdt + (\frac{1}{(\gamma r)^2} \int\int_{\mathbf{Q}_{\gamma r}(0,0)} |\mathbf{P}|^{\frac{3}{2}} dxdt)^2 \\
&+ (\frac{1}{(\gamma r)^2} \int\int_{\mathbf{Q}_{\gamma r}(0,0)} |\mathbf{v}||\mathbf{v}|^2 - \overline{|\mathbf{v}|^2}| dxdt)^{\frac{3}{2}} \\
&+ (\frac{1}{(\gamma r)^2} \int\int_{\mathbf{Q}_{\gamma r}(0,0)} |\mathbf{v}||\nabla \mathbf{d}|^2 - \overline{|\nabla \mathbf{d}|^2}| dxdt)^{\frac{3}{2}} \\
&+ (\frac{1}{(\gamma r)^2} \int\int_{\mathbf{Q}_{\gamma r}(0,0)} |P||\mathbf{v}| dxdt)^{\frac{3}{2}} \\
&\le \frac{1}{2}[\frac{1}{r^2} \int\int_{\mathbf{Q}_r(0,0)} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3) dxdt + (\frac{1}{(r)^2} \int\int_{\mathbf{Q}_r(0,0)} |\mathbf{P}|^{\frac{3}{2}} dxdt)^2 \\
&+ (\frac{1}{(r)^2} \int\int_{\mathbf{Q}_r(0,0)} |\mathbf{v}||\mathbf{v}|^2 - \overline{|\mathbf{v}|^2}| dxdt)^{\frac{3}{2}} \\
&+ (\frac{1}{(r)^2} \int\int_{\mathbf{Q}_r(0,0)} |\mathbf{v}||\nabla \mathbf{d}|^2 - \overline{|\nabla \mathbf{d}|^2}| dxdt)^{\frac{3}{2}} \\
&+ (\frac{1}{(r)^2} \int\int_{\mathbf{Q}_r(0,0)} |P||\mathbf{v}| dxdt)^{\frac{3}{2}}]
\end{aligned} \tag{3.2}$$

Proof of Lemma 3.1. First, for the convenience of the latter arguments, we define several dimensionless quantities:

$$
\begin{aligned}
A(r) &= \sup_{-\frac{7}{8}r^2<t<\frac{1}{8}r^2} r^{-1} \int_{\mathbf{B}_r\times\{t\}} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2)dx \\
B(r) &= r^{-1} \int\int_{\mathbf{Q}_r^*} (|\nabla \mathbf{v}|^2 + |\nabla^2 \mathbf{d}|^2)dxdt \\
C(r) &= r^{-2} \int\int_{\mathbf{Q}_r^*} (|\mathbf{v}|^3 + |\nabla \mathbf{d}|^3)dxdt \\
D(r) &= r^{-2} \int\int_{\mathbf{Q}_r^*} |\mathbf{P}|^{\frac{3}{2}} dxdt \\
E(r) &= r^{-2} \int\int_{\mathbf{Q}_r^*} |\mathbf{v}||\,|\mathbf{v}|^2 - \overline{|\mathbf{v}|^2}\,|dxdt + r^{-2} \int\int_{\mathbf{Q}_r^*} |\mathbf{v}||\,|\nabla \mathbf{d}|^2 - \overline{|\nabla \mathbf{d}|^2}\,|dxdt \\
F(r) &= r^{-2} \int\int_{\mathbf{Q}_r^*} |P||\mathbf{v}|dxdt
\end{aligned}
\tag{3.3}
$$

We will assume that $r \leq \rho$,

What we want to prove in the Lemma 3.1 is a decay estimate of the quantity $C(r) + D(r)^2 + E(r)^{\frac{3}{2}} + F(r)^{\frac{3}{2}}$.

Like in the proof of the Lemma 5.2 of [CKN], we can get the following estimate by using the interpolating estimate in Lemma 2.3:

$$
C(r) \leq \{(\frac{r}{\rho})^3 A(\rho)^{\frac{3}{2}} + (\frac{\rho}{r})^3 A(\rho)^{\frac{3}{4}} B(\rho)^{\frac{3}{4}}\} \tag{3.4}
$$

Next we make use of the generalized energy inequality again. This time we take the test function ϕ as follows:

ϕ is smooth and compactly supported in $\mathbf{Q}_\rho^*$. It is between 0 and 1. It is equal to 1 in $\mathbf{Q}_r^*$ and 0 out side of $\mathbf{Q}_\rho^*$. It has the properties:

$$
|\nabla \phi| \leq \frac{C}{\rho}, \quad |\phi_t| + |\Delta \phi| \leq \frac{C}{\rho^2}. \tag{3.5}
$$

By using this test function, we immediately get, for $-\frac{7}{8}r^2 < t < \frac{1}{8}r^2$, that

$$
\begin{aligned}
\int_{\mathbf{B}_r\times\{t\}} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2)dx &\leq \int\int_{\mathbf{Q}_\rho^*} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2)(\phi_t + \Delta\phi)dxdt \\
&+ \int\int_{\mathbf{Q}_\rho^*} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2)\mathbf{v}\cdot\nabla\phi dxdt + \int\int_{\mathbf{Q}_\rho^*} (2P)\mathbf{v}\cdot\nabla\phi dxdt \\
&+ \int\int_{\mathbf{Q}_\rho^*} ((\mathbf{v}\cdot\nabla)\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\phi dxdt + \mathbf{R}(f,\phi) \\
&\leq IV + V + VI + VII + VIII
\end{aligned}
\tag{3.6}
$$

We now estimate the right hand side of (3.6) in the following way. By using the Holder inequality,

$$IV \leq \frac{C}{\rho^2}\int\int_{\mathbf{Q}_\rho^*}(|\mathbf{v}|^2+|\nabla\mathbf{d}|^2) \leq \frac{C}{\rho^2}(\int\int_{\mathbf{Q}_\rho^*}(|\mathbf{v}|^3+|\nabla\mathbf{d}|^3)^{\frac{2}{3}}\rho^{\frac{5}{3}} \leq C\rho^1 C^{\frac{2}{3}}(\rho) \tag{3.7}$$

While by the divergence free property of $\mathbf{v}$, Poincare inequality and Holder inequality, we have :

$$V = \int\int_{\mathbf{Q}_\rho^*}|\mathbf{v}||\mathbf{|v|}^2+|\nabla\mathbf{d}|^2-\overline{|\mathbf{v}|^2+|\nabla\mathbf{d}|^2}|dxdt \leq C\rho^1 E(\rho)$$

where $\overline{g}$ above means the average of g in $\mathbf{Q}_\rho^*$. The reason of introducing it is to lower the power of $\nabla|\mathbf{v}|^2$ as we will see later.

The term VI which has the pressure P is nothing else but $F(\rho)$.

We now deal with term VII

$$\begin{aligned} VII \leq & C\int\int_{\mathbf{Q}_\rho^*}\mathbf{d}(\mathbf{v}\cdot\nabla)\nabla\mathbf{d}\nabla\phi dxdt + C\int\int_{\mathbf{Q}_\rho^*}\mathbf{d}\nabla\mathbf{d}(\mathbf{v}\cdot\nabla)\nabla\phi dxdt \\ \leq & C\frac{1}{\rho}\int(\int_{\mathbf{B}_\rho^*}|\mathbf{v}|^3dx)^{\frac{1}{3}}(\int_{\mathbf{B}_\rho^*}|\nabla\nabla\mathbf{d}|^{\frac{3}{2}}dx)^{\frac{2}{3}}dt \\ & + C\frac{1}{\rho^2}\int(\int_{\mathbf{B}_\rho^*}|\mathbf{v}|^2dx)^{\frac{1}{2}}(\int_{\mathbf{B}_\rho^*}|\nabla\mathbf{d}|^2dx)^{\frac{1}{2}}dt \\ \leq & C\frac{1}{\rho}(\int\int_{\mathbf{Q}_\rho^*}|\mathbf{v}|^3dx)^{\frac{1}{3}}(\int\int_{\mathbf{Q}_\rho^*}|\nabla\nabla\mathbf{d}|^2dxdt)^{\frac{1}{2}}\rho^{\frac{5}{4}} \\ & + C\frac{1}{\rho^2}(\int\int_{\mathbf{Q}_\rho^*}|\mathbf{v}|^2+|\nabla\mathbf{d}|^2dx) \\ \leq & C\rho^{\frac{5}{3}}C(\rho)^{\frac{1}{3}}B(\rho)^{\frac{1}{2}} + C\rho^1 C(\rho)^{\frac{3}{2}} \end{aligned} \tag{3.8}$$

and

$$VIII \leq C\rho^2\int\int_{\mathbf{Q}_\rho^*}|\mathbf{v}|^2+|\nabla\mathbf{d}|^2dxdt \leq C\rho C(\rho)^{\frac{2}{3}}$$

From above we see that:

$$A(r) \leq C[(\frac{\rho}{r})C^{\frac{2}{3}}(\rho)+(\frac{\rho}{r})E(\rho)+(\frac{\rho}{r})F(\rho)+(\frac{\rho}{r})C(\rho)^{\frac{1}{3}}B(\rho)^{\frac{1}{2}}] \tag{3.9}$$

Here we can estimate the term $E(r)$ by using the methods as we mentioned earlier:

$$
\begin{aligned}
r^2E(r) &\leq \int (\int_{\mathbf{B}_r^*} |\mathbf{v}|^3 dx)^{\frac{1}{3}} [(\int_{\mathbf{B}_r^*} ||\mathbf{v}|^2 - \overline{|\mathbf{v}|^2}|^{\frac{3}{2}} dx)^{\frac{2}{3}} \\
&+ (\int_{\mathbf{B}_r^*} ||\nabla \mathbf{d}|^2 - \overline{|\nabla \mathbf{d}|^2}|^{\frac{3}{2}} dx)^{\frac{2}{3}}] dt \\
&\leq \int (\int_{\mathbf{B}_r^*} |\mathbf{v}|^3 dx)^{\frac{1}{3}} (\int_{\mathbf{B}_r^*} (\nabla |\mathbf{v}|^2 + \nabla |\nabla \mathbf{d}|^2) dx) dt \\
&\leq \int [(\int_{\mathbf{B}_r^*} |\mathbf{v}|^3 dx)^{\frac{1}{3}} (\int_{\mathbf{B}_r^*} (|\mathbf{v}|^2 + |\nabla \mathbf{d}|^2) dx)^{\frac{1}{2}} \\
&(\int_{\mathbf{B}_r^*} (|\nabla \mathbf{v}|^2 + |\nabla \nabla \mathbf{d}|^2) dx)^{\frac{1}{2}}] dt \\
&\leq (rA(r))^{\frac{1}{2}} (\int \int_{\mathbf{Q}_r^*} |\mathbf{v}|^3 dx dt)^{\frac{1}{3}} (\int \int_{\mathbf{Q}_r^*} (|\nabla \mathbf{v}|^2 + |\nabla \nabla \mathbf{d}|^2) dx dt)^{\frac{1}{2}} r^{\frac{1}{3}} \\
&\leq r^2 (C(r)^{\frac{2}{3}} + A(r)B(r))
\end{aligned}
\tag{3.10}
$$

What we need next is the estimate of the pressure P term, which is probabily the most difficult part, we decompose it into

$$
P = P_1 + P_2 \tag{3.11}
$$

where

$$
P_1 = \frac{3}{4\pi} \int \frac{1}{|x-y|} P(y) \Delta \phi(y) dy + \frac{3}{2\pi} \int \frac{x_i - y_i}{|x-y|^3} \nabla_i \phi P(y) dy, \tag{2.12}
$$

$$
P_2 = \frac{3}{4\pi} \int \frac{1}{|x-y|} \phi(y) \Delta P dy. \tag{3.13}
$$

This can be obtained from the Green's representation of P, and integrate by part. The test function ϕ here is equal to 1 in $\{|y| \leq \frac{3}{4}\rho\}$ and equal to 0 out of $\{|y| \geq \rho\}$. with the property:

$$
|\nabla_i \phi| \leq C\rho^{-1}, |\nabla_{ij} \phi| \leq C\rho^{-2}. \tag{3.14}
$$

Realizing that these are of convolution form, one uses Young's inequality and Holder's inequality,

$$
\int_{\mathbf{B}_r} |P_2|^{\frac{3}{2}} \leq \rho^{\frac{3}{2}} (\int_{\mathbf{B}_\rho} (|\nabla \mathbf{v}|^2 + |\nabla \nabla (\nabla \mathbf{d} \nabla \mathbf{d})|^2) dx)^{\frac{3}{2}} \tag{3.15}
$$

Also, by using the properties of the test function ϕ and the fact that ϕ is constant near x,

$$
|P_1| \leq C \frac{1}{\rho^3} \int_{\mathbf{B}_\rho} |P| \tag{3.16}
$$

$$\int_{\mathbf{B}_r} |P_1|^{\frac{3}{2}} \le r^3 \frac{1}{\rho^{\frac{9}{2}}} (\int_{\mathbf{B}_\rho} |P|)^{\frac{3}{2}} \le r^3 \frac{1}{\rho^{\frac{9}{2}}} \rho^{\frac{3}{2}} \int_{\mathbf{B}_\rho} |P|^{\frac{3}{2}}$$
$$\le \frac{r^3}{\rho^3} \int_{\mathbf{B}_\rho} |P|^{\frac{3}{2}} \tag{3.17}$$

Hence,

$$\frac{1}{r^2} \int\int_{\mathbf{Q}_r^*} |P_1|^{\frac{3}{2}} \le \frac{r}{\rho^3} \int\int_{\mathbf{Q}_\rho^*} |P|^{\frac{3}{2}} \tag{3.18}$$

Realizing that (3.15) have too much derivatives on $\mathbf{d}$ in the right hand side, we have to decomposit P_2 further by again use

$$\Delta P = -\nabla_i \mathbf{v} \nabla_j \mathbf{v} + \nabla_i \nabla_j (\nabla \mathbf{d} \nabla \mathbf{d}) \tag{3.19}$$

Put (3.19) into (3.13) and integrating by parts once more, we have

$$P_2 = P_6 + P_7 \tag{3.20}$$

where

$$P_6 = -\frac{3}{4\pi} \int \frac{x_i - y_i}{|x-y|^3} \phi(\mathbf{v}\nabla\mathbf{v} + \nabla\mathbf{d}\nabla\nabla\mathbf{d} + \nabla\mathbf{d}\Delta\mathbf{d}) dy \tag{3.21}$$

$$P_7 = -\frac{3}{4\pi} \int \frac{1}{|x-y|} \nabla\phi(\mathbf{v}\nabla\mathbf{v} + \nabla\mathbf{d}\nabla\nabla\mathbf{d} + \nabla\mathbf{d}\Delta\mathbf{d}) dy \tag{3.22}$$

It follows that

$$|P_7| \le \frac{C}{\rho^2} \int_{\mathbf{B}_\rho} (|\mathbf{v}||\nabla\mathbf{v}| + |\nabla\mathbf{d}||\nabla\nabla\mathbf{d}|) \tag{3.23}$$

which in turns implies,

$$|P_7|^{\frac{3}{2}} \le \frac{C}{\rho^3} (\int_{\mathbf{B}_\rho} (|\mathbf{v}|^2 + |\nabla\mathbf{d}|^2))^{\frac{3}{4}} (\int_{\mathbf{B}_\rho} (|\nabla\mathbf{v}|^2 + |\nabla\nabla\mathbf{d}|^2))^{\frac{3}{4}}$$

and

$$\int_{\mathbf{B}_r} |P_7|^{\frac{3}{2}} \le C\frac{r^3}{\rho^3} (\int_{\mathbf{B}_\rho} (|\mathbf{v}|^2 + |\nabla\mathbf{d}|^2))^{\frac{3}{4}} (\int_{\mathbf{B}_\rho} (|\nabla\mathbf{v}|^2 + |\nabla\nabla\mathbf{d}|^2))^{\frac{3}{4}}$$

Finally we do integration with respect to time to obtain,

$$\int\int_{\mathbf{Q}_r^*} |P_7|^{\frac{3}{2}} \le C\frac{r^3}{\rho^3} \rho^{\frac{3}{4}} A(\rho)^{\frac{3}{4}} \int (\int_{\mathbf{B}_\rho} (|\nabla\mathbf{v}|^2 + |\nabla\nabla\mathbf{d}|^2))^{\frac{3}{4}}$$
$$\le C\frac{r^3}{\rho^3} \rho^{\frac{3}{4}} A(\rho)^{\frac{3}{4}} (\int\int_{\mathbf{Q}_\rho^*} (|\nabla\mathbf{v}|^2 + |\nabla\nabla\mathbf{d}|^2))^{\frac{3}{4}} \rho^{\frac{1}{2}} \tag{3.24}$$
$$\le C\frac{r^3}{\rho^3} \rho^2 A(\rho)^{\frac{3}{4}} B(\rho)^{\frac{3}{4}}$$

For the P_6 term, since we do not have the gradient on ϕ, we simply use Young's inequality,

$$(\int_{\mathbf{B}_r} |P_6|^{\frac{3}{2}} dx)^{\frac{2}{3}} \leq C(\int_{\mathbf{B}_\rho} (\frac{1}{r^2})^{\frac{4}{3}})^{\frac{3}{4}} (\int_{\mathbf{B}_\rho} |\mathbf{v}||\nabla \mathbf{v}|)^{\frac{12}{11}})^{\frac{11}{12}}$$
$$\leq C\rho^{\frac{1}{4}} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^{\frac{1}{2}} |\mathbf{v}|^{\frac{1}{2}} |\nabla \mathbf{v}|)^{\frac{12}{11}})^{\frac{11}{12}}$$
$$\leq C\rho^{\frac{1}{4}} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^2)^{\frac{1}{4}} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^3)^{\frac{1}{6}} (\int_{\mathbf{B}_\rho} |\nabla \mathbf{v}|^2)^{\frac{1}{2}}$$

Thus

$$\int_{\mathbf{B}_r} |P_6|^{\frac{3}{2}} dx \leq C\rho^{\frac{3}{8}} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^2)^{\frac{3}{8}} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^3)^{\frac{1}{4}} (\int_{\mathbf{B}_\rho} |\nabla \mathbf{v}|^2)^{\frac{3}{4}}$$

and

$$\int\int_{\mathbf{Q}_r^*} |P_6|^{\frac{3}{2}} dxdt \leq C\rho^{\frac{3}{2}} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^2)^{\frac{3}{8}} (\int\int_{\mathbf{Q}_\rho^*} |\mathbf{v}|^3)^{\frac{1}{4}} (\int\int_{\mathbf{Q}_\rho^*} |\nabla \mathbf{v}|^2)^{\frac{3}{4}}$$
$$\leq Cr^4 (\int\int_{\mathbf{Q}_\rho^*} |\mathbf{v}|^3)^{\frac{1}{2}} + r^{-4} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^2)^{\frac{3}{4}} (\int\int_{\mathbf{Q}_\rho^*} |\nabla \mathbf{v}|^2)^{\frac{3}{2}} \tag{3.25}$$

We therefore obtain

$$D(r)^2 \leq C[(\frac{r}{\rho})^2 D(\rho)^2 + (\frac{r}{\rho})^2 A(\rho)^{\frac{3}{4}} B(\rho)^{\frac{3}{2}}$$
$$+ (\frac{r}{\rho})^2 C(\rho) + (\frac{r}{\rho})^{-6} A(\rho)^{\frac{3}{2}} B(\rho)^3] \tag{3.26}$$

Finally, we estimate the term $F(r)$.

By the property of P_1 (3.16), one concludes,

$$\int_{\mathbf{B}_r} |P_1||\mathbf{v}|dx \leq \int_{\mathbf{B}_r} \mathbf{v}dx \frac{1}{\rho^3} \int_{\mathbf{B}_\rho} Pdx$$
$$\leq \frac{1}{\rho^3} \int_{\mathbf{B}_r} |\mathbf{v}|^{\frac{1}{2}} |\mathbf{v}|^{\frac{1}{2}} dx \int_{\mathbf{B}_\rho} Pdx$$
$$\leq \frac{1}{\rho^3} r^{\frac{7}{4}} (\int_{\mathbf{B}_r} |\mathbf{v}|^2)^{\frac{1}{4}} (\int_{\mathbf{B}_r} |\mathbf{v}|^3)^{\frac{1}{6}} (\int_{\mathbf{B}_\rho} |P|^{\frac{3}{2}})^{\frac{2}{3}} \rho$$

thus

$$\int\int_{\mathbf{Q}_r^*} |P_1||\mathbf{v}|dxdt \leq \frac{1}{\rho^2} r^{\frac{7}{4}} (\int_{\mathbf{B}_r} |\mathbf{v}|^2)^{\frac{1}{4}} (\int\int_{\mathbf{Q}_r^*} |\mathbf{v}|^3)^{\frac{1}{6}} (\int\int_{\mathbf{Q}_\rho^*} |P|^{\frac{3}{2}})^{\frac{2}{3}} r^{\frac{1}{3}}$$
$$\leq \frac{1}{\rho^1} (r^{\frac{29}{12}}) [(\rho^{-1} \int_{\mathbf{B}_\rho} |\mathbf{v}|^2) + (r^{-2} \int\int_{\mathbf{Q}_r^*} |\mathbf{v}|^3)^{\frac{2}{3}} \tag{3.27a}$$
$$+ (\rho^{-2} \int\int_{\mathbf{Q}_\rho^*} |P|^{\frac{3}{2}})^{\frac{4}{3}}]$$

On the other hand, one has

$$
\begin{aligned}
\int_{\mathbf{B}_r} |P_2||\mathbf{v}|dx &\le \int_{\mathbf{B}_r} |P_6+P_7||\mathbf{v}|dx \\
&\le \frac{C}{\rho^2}\int_{\mathbf{B}_r} |\mathbf{v}|dx \int_{\mathbf{B}_\rho} (|\mathbf{v}||\nabla \mathbf{v}| + |\nabla \mathbf{d}||\nabla\nabla \mathbf{d}|) \\
&+ (\int_{\mathbf{B}_r} |\mathbf{v}|^3)^{\frac{1}{3}} (\int_{\mathbf{B}_\rho} |P_6|^{\frac{3}{2}})^{\frac{2}{3}} \\
&\le \frac{C}{\rho^2}(\int_{\mathbf{B}_r} |\mathbf{v}|^2)^{\frac{1}{2}} r^{\frac{3}{2}} (\int_{\mathbf{B}_r} |\mathbf{v}|^2 + |\nabla \mathbf{d}|^2)^{\frac{1}{2}} (\int_{\mathbf{B}_\rho} |\nabla \mathbf{v}|^2 + |\nabla\nabla \mathbf{d}|^2)^{\frac{1}{2}} \\
&+ \rho^{\frac{3}{4}} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^3)^{\frac{1}{2}} (\int_{\mathbf{B}_\rho} |\mathbf{v}|^2))^{\frac{1}{4}} (\int_{\mathbf{B}_\rho} |\nabla \mathbf{v}|^2))^{\frac{1}{2}} \\
&\le \frac{r^{\frac{3}{2}}}{\rho^{\frac{3}{2}}}(\int_{\mathbf{B}_\rho} |\mathbf{v}|^2 + |\nabla \mathbf{d}|^2)(\int_{\mathbf{B}_\rho} |\nabla \mathbf{v}|^2 + |\nabla\nabla \mathbf{d}|^2)^{\frac{1}{2}} \\
&+ r^4(\int_{\mathbf{B}_\rho} |\mathbf{v}|^3)^{\frac{2}{3}} + r^{-12}(\int_{\mathbf{B}_\rho} |\mathbf{v}|^2))(\int_{\mathbf{B}_\rho} |\nabla \mathbf{v}|^2))^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
\int\int_{\mathbf{Q}_r^*} |P_2||\mathbf{v}|dxdt \le \frac{r^{\frac{3}{2}}}{\rho^{\frac{3}{2}}}(\int_{\mathbf{B}_\rho} |\mathbf{v}|^2 + |\nabla \mathbf{d}|^2)(\int\int_{\mathbf{Q}_\rho^*} |\nabla \mathbf{v}|^2 + |\nabla\nabla \mathbf{d}|^2)^{\frac{1}{2}} \\
+ r^4(\int\int_{\mathbf{Q}_\rho^*} |\mathbf{v}|^3)^{\frac{2}{3}} + r^{-12}(\int_{\mathbf{B}_\rho} |\mathbf{v}|^2))(\int\int_{\mathbf{Q}_\rho^*} |\nabla \mathbf{v}|^2))^2
\end{aligned}
\tag{3.27b}
$$

Therefore,

$$
\begin{aligned}
F(r) \le (\frac{r}{\rho})^{\frac{5}{12}}(A(\rho) + C(r)^{\frac{2}{3}} + D(\rho)^{\frac{4}{3}}) \\
+ (\frac{\rho}{r})^{-2}A(\rho)B(\rho) + (\frac{r}{\rho})^2 C(\rho)^{\frac{2}{3}} + (\frac{\rho}{r})^{-14}A(\rho)B(\rho)^2
\end{aligned}
\tag{3.28}
$$

Now we can combine all the above estimates to get the followings.

$$
C(\gamma\rho) \le \gamma^3 A(\rho)^{\frac{3}{2}} + \gamma^{-3}A(\rho)^{\frac{3}{2}}B(\rho)^{\frac{3}{2}}
$$

$$
D(\gamma\rho)^2 \le \gamma^2 D(\rho)^2 + \gamma^2 A(\rho)^{\frac{3}{2}} B(\rho)^{\frac{3}{2}} + \gamma^2 C(\rho) + \gamma^{-6}A(\rho)^{\frac{3}{2}} B(\frac{1}{2}\rho)^3
$$

Finally we have

$$
\begin{aligned}
& C(\gamma\rho) + D(\gamma\rho)^2 + E(\gamma\rho)^{\frac{3}{2}} + F(\gamma\rho)^{\frac{3}{2}} \\
& \le \gamma^3 A(\frac{1}{2}\rho)^{\frac{3}{2}} + \gamma^{-3}A(\frac{1}{2}\rho)^{\frac{3}{4}}B(\frac{1}{2}\rho)^{\frac{3}{4}} + \gamma^2 D(\frac{1}{2}\rho)^2 + \gamma^2 A(\frac{1}{2}\rho)^{\frac{3}{2}}B(\frac{1}{2}\rho)^{\frac{3}{2}} \\
& + \gamma^4 C(\frac{1}{2}\rho) + \gamma^{-6}A(\frac{1}{2}\rho)^{\frac{3}{2}}B(\frac{1}{2}\rho)^3 \\
& + C(\gamma\rho) + A(\gamma\rho)^{\frac{3}{2}}B(\gamma\rho)^{\frac{3}{2}} + \gamma^{\frac{5}{8}}A(\frac{1}{2}\rho)^{\frac{3}{2}} + (\gamma^{\frac{5}{8}})C(\gamma\rho) + \gamma^3 C(\frac{1}{2}\rho) + \gamma^{\frac{5}{8}}D(\frac{1}{2}\rho)^2 \\
& + \gamma^{-3}A(\frac{1}{2}\rho)^{\frac{3}{2}}B(\frac{1}{2}\rho)^{\frac{3}{2}} + \gamma^{-21}A(\frac{1}{2}\rho)^{\frac{3}{2}}B(\frac{1}{2}\rho)^3
\end{aligned}
$$

and using the estimates of $A(r)$, one has concluded that,

$$C(\gamma\rho)+D(\gamma\rho)^2+E(\gamma\rho)^{\frac{3}{2}}+F(\gamma\rho)^{\frac{3}{2}}$$
$$\le G(\gamma)[C(\rho)+E(\rho)^{\frac{3}{2}}+F(\rho)^{\frac{3}{2}}+D(\rho)^2+C(\rho)^{\frac{1}{2}}B(\frac{1}{2}\rho)^{\frac{3}{4}}]+C(\rho)^{\frac{1}{2}}B(\rho)^{\frac{1}{2}}$$
$$+H(\gamma)A(\frac{1}{2}\rho)^{\frac{3}{2}}B(\frac{1}{2}\rho)^{\frac{3}{4}}$$

here $H(\gamma)$ is a smooth function for γ not equal to zero, and $G(\gamma)$ is a function for γ which is some linear combination of the positive power of γ. We will put (3.9) again in the last term of the right hand side.

Now we see, if we take γ small enough to make $G(\gamma)$ small, then take $B(\rho)$ which is the ϵ small enough to make $H(\gamma)B(\frac{1}{2}\rho)^{\frac{3}{4}}$ small, we will get (3.1).

Remark. In the above argument, we see some special features of system (1.1)—(1.6). For the Navier-Stokes equation, ΔP is equal to the quadratic form of $\nabla \mathbf{v}$, but for the system (1.1)—(1.6), it als depends on the multiplication of $\nabla \mathbf{d}$ and $\nabla^3\mathbf{d}$! (see (3.15)) We overcome this difficulty by integrating by part (see (3.20)).

The immediate result consequence of Lemma 3.1 and Theorem 2.6 is the following

Theorem 3.1. *There exist a absolute constant $\epsilon > 0$, such that, if $(\mathbf{v}, \mathbf{d}, P)$ is a suitable weak solution of (1.1)—(1.6), and*

$$\limsup_{r\to 0} r^{-1}\int\int_{\mathbf{Q}_r^*(x,t)}(|\nabla \mathbf{v}|^2+|\nabla^2\mathbf{d}|^2)dxdt<\epsilon \tag{3.29}$$

then (x,t) is a regular point.

The following somewhat standard covering argument can be used to prove the Main Theorem.

Let V be a neighbourhood of S, which is the singular set of a solution in $D=\Omega\times[0,T]$. For each point $(x,t)\in S$, we choose $\mathbf{Q}_r^*(x,t)\subset V$ such that, for any δ, we can find $r<\delta$, and

$$r^{-1}\int\int_{\mathbf{Q}_r^*(x,t)}(|\nabla \mathbf{v}|^2+|\nabla^2\mathbf{d}|^2)dxdt>\epsilon \tag{3.30}$$

Applying a Vitali-type covering lemma, we obtain a disjoint subfamily $\mathbf{Q}_{r_j}^*(x,t)$, such that

$$S\subset \cup_i \mathbf{Q}_{5r_j}^*(x,t) \tag{3.31}$$

and we got that

$$\sum r_i \le \epsilon^{-1}\sum_i\int\int_{\mathbf{Q}_{r_j}^*(x,t)}(|\nabla \mathbf{v}|^2+|\nabla^2\mathbf{d}|^2)dxdt\le\epsilon^{-1}\int\int_V(|\nabla \mathbf{v}|^2+|\nabla^2\mathbf{d}|^2)dxdt \tag{3.32}$$

Since δ is arbitrary, we get that S has Lebesgue measure zero. Also we know that

$$\mathbf{P}^1(S) \leq \frac{5}{\epsilon} \int \int_V (|\nabla \mathbf{v}|^2 + |\nabla^2 \mathbf{d}|^2) dx dt \tag{3.33}$$

for every neighbourhood V of S. and

$$\int \int (|\nabla \mathbf{v}|^2 + |\nabla^2 \mathbf{d}|^2) dx dt < \infty \tag{3.34}$$

Since V is arbitrary, we have

$$\mathbf{P}^1(S) = 0 \tag{3.35}$$

which completes the proof of the Main Theorem!

(Q.E.D.)

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