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**Variational Techniques for Problems
in Materials Sciences**

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Variational Techniques for Problems in Materials Sciences

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Abstract

In recent years there has been remarkable progress in the mathematical understanding of variational principles for unstable material phenomena. In this paper some of the techniques developed are outlined.

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1 Introduction

In recent years there has been remarkable progress in the understanding of nonconvex variational problems, motivated in part by ongoing research in the analysis of questions in materials science. In this paper we will indicate briefly some of the techniques developed and we will discuss the results obtained.

Among the many underlying physically relevant frameworks, we are particularly interested in the applications of nonconvex variational principles to the study of phase transitions, crystals with defects (such as dislocations),

metastable equilibrium states for crystals, the onset of microstructure and the creation of concentrations, interactions between fracture and damage, image segmentation, and the behavior and defects of liquid crystals (see [3, 7, 10, 14, 15, 26, 39, 40, 41, 42, 58, 59]). Also, amidst the very many powerful mathematical tools recently developed, here we will concentrate on the study of lower semicontinuity and relaxation techniques (see [11, 12, 25, 29, 33, 55, 56, 57]). Part of the analysis will be undertaken in situations where the growth and co-civeness properties of the energy density are not compatible with the use of standard relaxation techniques (see Section 2; see also [1, 2, 27, 28, 30, 34, 43, 45, 48, 49, 50, 52, 53, 54]). As it turns out, this is the natural setting for the study of some coherent phase transformations and cavitation phenomena. Also, in order to accommodate the study of incoherent phase transformations, image processing, etc, we consider functional spaces that allow for discontinuities of the admissible fields (see [4, 5, 6, 13, 17, 18, 19, 20, 22, 23, 24, 46]).

The study of phase transformations for nonlinear elastic materials suggests the extension of the analysis to energy functionals involving higher order derivatives, interfacial energies, and discontinuous (Carathéodory) energy densities. The role played by interfacial energy terms in stabilizing the oscillations developed, as well as the dynamical creation of microstructure, concentrations, their evolution and interaction, is very complex and very little is known on this direction. Surface energy terms may be produced naturally by the bulk energy (see [6, 46]), while in some other models interfacial energies are present in the model from the start (see [3, 13, 39]). In the latter case, one may ask what will be the interaction, if any, between the surface energy provided in the model and the one which underlies the bulk energy term (see Section 3).

The framework of generalized measure-valued solutions may turn out to give some insight into these questions. Thermochemical equilibria for coherent two-phase alloys have been analyzed using Young measures (see [42]), and the underlying framework requires a good understanding of constrained variational problems. This remains a virtually unexplored area of research (see [31, 44]).

A common feature of the above described problems is the study of equilibria through a minimization problem of the type

$$\underset{u \in \mathcal{A}}{\text{minimize}} I(u),$$

where

$$I(u) := \int_{\Omega} f(x, u, \nabla u) dx + \dots, \quad (1.1)$$

and where the unspecified terms in the energy $I(\cdot)$ account for body forces, lower order terms, interfacial energies, surface energies, etc. Here $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, it represents the reference configuration of a certain material body, and $u : \Omega \rightarrow \mathbb{R}^d$ stands for the deformation, mass density, etc.

The main question of the calculus of variations focus on the search of necessary and sufficient conditions satisfied by f which will guarantee (weak) lower

semicontinuity of the energy $I(\cdot)$ in some functional space. Usually, due to coercivity and growth conditions verified by f , this space turns out to be a Sobolev space $W^{1,p}$. It is well known (see [2, 11, 55, 57]) that if

$$0 \leq f(x, u, \xi) \leq C(1 + |\xi|^q) \quad (1.2)$$

and if $p \geq q$, then

$$E(u) := \int_{\Omega} f(x, u, \nabla u) dx \quad (1.3)$$

is $W^{1,p}$ -weak lower semicontinuous,

$$u_n \rightharpoonup u \text{ in } W^{1,p} \Rightarrow E(u) \leq \liminf_{n \rightarrow \infty} E(u_n),$$

if and only if $f(x, u, \cdot)$ is *quasiconvex*, i.e.

$$f(x, u, A) \leq \int_{(0,1)^N} f(x, u, A + \nabla \varphi(y)) dy$$

for all matrices $A \in \mathbb{M}^{d \times N}$ and all $\varphi \in W_0^{1,p}((0,1)^N, \mathbb{R}^d)$. Here, and in what follows, $\mathbb{M}^{d \times N}$ denotes the vector space of $d \times N$ real-valued matrices. We recall that in the scalar case ($N = 1$ or $d = 1$) a function is quasiconvex if and only if it is convex. In cases where f is not quasiconvex, and in order to study limiting energies and stress states of the system, we consider the *relaxed energy*

$$\mathcal{F}(u, \Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx : u_n \rightharpoonup u \text{ in } W^{1,p} \right\}.$$

One of the central questions of the relaxation theory is to find the new relaxed bulk energy density \bar{f} , i.e. an integral representation for $\mathcal{F}(u, \cdot)$ of the form

$$\mathcal{F}(u, \Omega) := \int_{\Omega} \bar{f}(x, u, \nabla u) dx.$$

We recall that, under suitable regularity and growth assumptions, $\mathcal{F}(\cdot, \Omega)$ is now a lower semicontinuous functional, and the relaxation theory (see [29]) guarantees that

$$\inf_{u \in \mathcal{A}} \int_{\Omega} f(x, u, \nabla u) dx = \min_{u \in \mathcal{A}} \int_{\Omega} \bar{f}(x, u, \nabla u) dx,$$

where $\bar{f}(x, u, \cdot)$ is the *quasiconvexification* $Qf(x, u, \cdot)$ of $f(x, u, \cdot)$, precisely

$$Qf(x, u, A) := \inf \left\{ \int_{(0,1)^N} f(x, u, A + \nabla \varphi(y)) dy : \varphi \in W_0^{1,p}((0,1)^N, \mathbb{R}^d) \right\}.$$

Often, the regularity or the growth conditions required to apply the above standard relaxation techniques fail. In this paper we will overview new mathematical approaches which will allow us to study some of the situations not covered under the usual assumptions. In Section 2 we concentrate on the analysis of the bulk energy term, precisely on the lower semicontinuity properties of (1.3) when the function f satisfies the bound (1.2), and the relaxation is taken among sequences $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$, $u_n \rightarrow u$ in $W^{1,p}$ weak, with $p < q$. On Section 3 we extend the analysis to include the relaxation in BV of the same functionals in the case where coerciveness only guarantees $W^{1,1}$ bounds for minimizing sequences. Section 4 addresses the relaxation in BV of energies of the type (1.1) which involve bulk and interfacial energy contributions, and where the energy densities have linear growth. The analysis of the superlinear growth case is undertaken partially on Section 5.

2 Growth conditions versus Sobolev bounds

In collaboration with Marcellini [45] and with Malý [43], we studied $W^{1,p}$ -weak lower semicontinuity properties of the functional $E(\cdot)$ introduced in (1.3), where f satisfies (1.2) and $p < q$. A prototype example is given by

$$G(u) := \int_{\Omega} (h(\nabla u) + g(\det \nabla u)) \, dx, \quad (2.1)$$

where

$$\frac{1}{C_1} |\xi|^p - C_2 \leq h(\xi) \leq C_1(1 + |\xi|^p), \quad \frac{1}{C_1} |t| - C_2 \leq g(t) \leq C_1(1 + |t|)$$

for some $C_1 > 0, C_2 \geq 0, N - 1 < p < N$ and for all $\xi \in \mathbb{M}^{N \times N}, t \in \mathbb{R}$. Integrands of this type are considered in nonlinear elasticity and the condition $p < q = N$ plays a fundamental role in cavitation analysis (see [11]).

The (weak) lower semicontinuity problem in $W^{1,p}$ for *polyconvex* integrands (i.e., suprema of rank-one affine functions) and for p below the growth exponent q was first considered by Marcellini [55]. In particular, if we restrict to our prototype example (2.1), Marcellini proved the lower semicontinuity for $p > \frac{N^2}{N+1}$. This result was extended to the case $p > N - 1$ by Dacorogna and Marcellini [33]; the borderline case $p = N - 1$ was considered in [52] with a partial success, and it completely established by Acerbi, Dal Maso and Sbordone [1, 34] (see also [28, 48, 49]).

With Marcellini [45] we studied a class of quasiconvex integrands $f = f(\xi)$ satisfying some structure conditions naturally verified by (2.1). We proved lowersemicontinuity of the energy (1.3) for $p > q - 1$. For non quasiconvex integrands, we define the relaxed energy

$$\mathcal{F}^{q,p}(u, \Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} f(\nabla u_n) : u_n \in W^{1,q}(\Omega, \mathbb{R}^d), \right.$$

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^d) \Big\}$$

and we proved that

$$\mathcal{F}^{q,p}(u) \geq \int_{\Omega} QW(\nabla u(x)) dx.$$

Equality holds provided $u \in W^{1,q}(\Omega; \mathbb{R}^d)$. Recently, Malý [54] extended the later result to the borderline case $p = q - 1$, assuming always supplementary structural conditions on f .

In collaboration with Malý [43] we developed a new variational approach which allows us to eliminate the above mentioned additional structural assumptions if the growth condition is (1.2) and $p > q \frac{N-1}{N}$. We defined a new relaxed energy

$$\mathcal{F}_{\text{loc}}^{q,p}(u, \Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) : u_n \in W_{\text{loc}}^{1,q}(\Omega, \mathbb{R}^d), \right. \\ \left. u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^d) \right\}.$$

The behaviors of $\mathcal{F}_{\text{loc}}^{q,p}$ and $\mathcal{F}^{q,p}$ may be rather different, and their values depend, in a strikingly complicated way, on the ranges of p , q , and on the regularity properties of u . Consider the example where $N = d$, $f(x, u, \xi) = f(\xi) = |\det \xi|$. Notice that f is polyconvex and the growth condition

$$0 \leq f(\xi) \leq |\xi|^N$$

is satisfied. It is well known that

$$\mathcal{F}^{q,p}(u, \Omega) \geq \int_{\Omega} |\det \nabla u| dx \quad (2.2)$$

if $p, q \geq N$. Recently, this result has been improved to include the case where $q \geq N$ and $p \geq N - 1$ (see [28]). If $u \in W^{1,N}(\Omega, \mathbb{R}^N)$, then we get equality in (2.2), whereas for $u \notin W^{1,N}(\Omega, \mathbb{R}^N)$ it is difficult to describe $\mathcal{F}^{q,p}(u, \Omega)$. We obtain

$$\mathcal{F}^{q,p}(u, \Omega) = 0 \quad (2.3)$$

if $q < N$ (see [12]) or if $p < N - 1$ (see [43, 52]).

As before, in order to identify $\mathcal{F}^{q,p}(u, \Omega)$ and $\mathcal{F}_{\text{loc}}^{q,p}(u, \Omega)$, in [43] we started by obtaining a lower bound for the relaxed energy,

$$\mathcal{F}^{q,p}(u, \Omega) \geq \int_{\Omega} Qf(x, u, \nabla u) dx, \quad (2.4)$$

which amounts to proving a lower semicontinuity result for quasiconvex integrals, namely

$$\liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx \geq \int_{\Omega} f(x, u, \nabla u) dx \quad (2.5)$$

if $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^d)$, $f(x, u, \cdot)$ is quasiconvex, and $p > q \frac{N-1}{N}$. In view of (2.3), the inequality (2.5) may no longer be valid if $p < q \frac{N-1}{N}$.

Further, we investigated the dependence of $\mathcal{F}^{q,p}(u, U)$ and $\mathcal{F}_{\text{loc}}^{q,p}(u, U)$ on the open subsets $U \subset \Omega$. Assuming that

$$0 \leq f(x, \zeta, \xi) \leq C(1 + |\zeta|^r + |\xi|^q),$$

we proved that if $p > \max \left\{ q \frac{N-1}{N}, \frac{rN^2 - rN}{N^2 + rN - r} \right\}$, and if $\mathcal{F}^{q,p}(u, \Omega) < \infty$, then there exists a finite, nonnegative, Radon measure μ such that

$$\mathcal{F}^{q,p}(u, U) = \mu(U) \quad (2.6)$$

for all open sets $U \subset \Omega$ with $\mu(\partial U) = 0$. In addition,

$$\mathcal{F}_{\text{loc}}^{q,p}(u, U) = \int_U Qf(x, u, \nabla u) dx + \lambda(U)$$

holds for all open sets $U \subset \Omega$, where λ is some finite, nonnegative, Radon measure.

The representation formula (2.6) may fail if $p \leq q \frac{N-1}{N}$, as illustrated by an example of the class (2.1), provided by Acerbi and Dal Maso [1] : if $f(\xi) := |\xi|^{N-1} + |\det \xi|$ and setting $p = N - 1$, then $\mathcal{F}^{q,p}(u, \cdot)$ is not even subadditive. Here $q = d = N$, $r = p = N - 1$, $u(x) = \frac{x}{|x|} \in W^{1,s}(B, \mathbb{R}^N)$ for all $s < N$, in particular for $s = p$. Then

$$\rho \mapsto \mathcal{F}^{q,p}(u, \rho B) - \int_{\rho B} f(\nabla u)$$

is of order ρ at 0, whereas

$$\mathcal{F}^{q,p}(u, B \setminus \rho B) - \int_{B \setminus \rho B} f(\nabla u) = 0.$$

Hence $\mathcal{F}^{q,p}(u, \cdot)$ is not be subadditive.

In this example, the additivity property failed due to the fact that $p \leq q \frac{N-1}{N}$. Now we will see that, in spite of requiring $p > q \frac{N-1}{N}$, the measure representation (2.6) may not hold for open sets U with $\mu(\partial U) > 0$. Let $q = d = N$ and $u(x) := \frac{x}{|x|}$, but now $N > p > N - 1$, and

$$f(\xi) := |\det \xi|.$$

Let $\mu := \mathcal{L}^N(B) \delta_0$ be the $\mathcal{L}^N(B)$ -multiple of the Dirac measure at 0. Then (see [45], Theorem 4.1),

$$\mathcal{F}^{q,p}(u, U) = \mu(U) \quad (2.7)$$

if $\mu(\partial U) = 0$. For $U = \{x \in B : x_1 > 0\}$ we have

$$\mathcal{F}^{q,p}(u, U) = \mu(U) < \mu(\bar{U}), \quad (2.8)$$

and if $U := B \setminus \{0\}$ then

$$\mathcal{F}^{q,p}(u, U) = \mu(B) = \mu(\bar{U}) > \mu(U),$$

because each $v \in W^{1,q}(U)$ is also in $W^{1,q}(B)$ (the point 0 is a removable singularity). Clearly, $\mathcal{F}^{q,p}(u, \cdot)$ cannot be a measure since in this case, and by (2.7), it would have to be the measure μ , contradicting (2.8).

Moreover, we showed that (2.6) holds for all open sets $U \subset \Omega$ if and only if it exists a Radon measure ν such that

$$\mathcal{F}^{q,p}(u, U) \leq \nu(U)$$

for all open subset $U \subset \Omega$.

Under standard continuity hypotheses on f with respect to x and ζ (see [13]), the lower semicontinuity result (2.4) implies the estimate

$$\mu_a \geq Qf(x, u, \nabla u) \mathcal{L}^N$$

for the absolutely continuous part μ_a of μ , where \mathcal{L}^N is the N -dimensional Lebesgue measure. Actually, in all known examples the equality

$$\mu_a = Qf(x, u, \nabla u) \mathcal{L}^N$$

holds.

The main novelty of the paper [43] lies on the construction of a linear operator Tu from $W^{1,p}$ into $W^{1,q}$ which conserves boundary values and improves integrability of u and ∇u . Namely, the $W^{1,q}$ -norm of Tu is estimated in terms of a special maximal function if $p > q \frac{N-1}{N}$. We use this extension operator to "connect" two functions across a thin transition layer and to estimate the increase of the energy. We remark that the standard way to perform this connection, by means of convex combinations using cut-off functions, would not achieve a comparable result, namely an arbitrarily small increase of the energy on an arbitrarily thin transition layer, since the admissible sequences may not remain bounded in $W^{1,q}(\Omega, \mathbb{R}^d)$.

3 Interactions between bulk and interfacial energies : bulk generated surface energy

Several problems in Mathematical Physics may be modelled by functionals of the form (1.1), where the underlying function spaces should allow discontinuous vector-valued functions u . In particular, relaxation techniques for these energy

functionals have been used in the study of variational models in fracture mechanics, where u is the displacement field and the discontinuity set $S(u)$ represents the crack site (see [26, 39]); in the theory of computer vision and image segmentation, where the function u represents the so-called image reconstruction and $S(u)$ is the edge contour (see [3, 4, 5, 7, 36, 58, 59]). As a common feature to these problems, we seek to minimize (1.1) on the class BV of functions of bounded variation, or on some suitable subclass of it.

We recall that the distributional derivative of a function u in $BV(\Omega; \mathbb{R}^d)$ is a finite Radon measure on Ω , and it may be decomposed as

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu H^{N-1} \llcorner S(u) + C(u),$$

where ∇u is a $L^1(\Omega; \mathbb{M}^{d \times N})$ function (the density of the absolutely continuous part of the measure Du), $S(u)$, the jump set of u , is an $N - 1$ rectifiable hypersurface with normal vector ν , u^+ and u^- , the traces of u on each side of $S(u)$, are such that, for H^{N-1} -a.e. x_0 in $S(u)$,

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^N} \int_{\{y \in B(x_0, \epsilon) : (y - x_0) \cdot \nu_{x_0} > (<) 0\}} |u(y) - u^\pm(x_0)|^{N/(N-1)} dy = 0, \quad (3.1)$$

and $C(u)$, the Cantor part of the measure Du , satisfies

$$H^{N-1}(B) < +\infty \Rightarrow |C(u)|(B) = 0$$

for any Borel subset B of Ω (see e.g. [37, 38, 60]).

In collaboration with Müller [46] (see also [6]), we studied the lower semi-continuity and relaxation properties of bulk energies, i.e. when (1.1) reduces to (1.3), and in the case where the density $f(x, u, \cdot)$ has linear growth. Clearly, the natural space for performing the relaxation is BV , rather than $W^{1,1}$. This study was undertaken in the scalar case by Dal Maso [32].

Let $u \in BV(\Omega, \mathbb{R}^d)$, and let $\mathcal{F}(u, \cdot)$ be the relaxation in BV of the functional (1.3), i.e.

$$\mathcal{F}(u, U) := \inf_{\{u_n\}} \left\{ \liminf_{n \rightarrow \infty} \int_U f(x, u_n, \nabla u_n) dx : u_n \rightharpoonup u \text{ in } L^1(\Omega, \mathbb{R}^d) \right\},$$

where U is an open subset of Ω . Under suitable continuity hypotheses on f , and assuming that $c g(x, u)(1 + \|\xi\|) \leq f(x, u, \xi) \leq C g(x, u)(1 + \|\xi\|)$ for some nonnegative function g , we obtained the following representation

$$\begin{aligned} \mathcal{F}(u) &= \int_{\Omega} Qf(x, u(x), \nabla u(x)) dx \\ &+ \int_{S(u)} K_f(x, u^-(x), u^+(x), \nu(x)) dH^{N-1}(x) \\ &+ \int_{\Omega} (Qf)^\infty(x, u(x), dC(u)). \end{aligned}$$

The surface energy density $K_f : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \rightarrow [0, +\infty)$ is given by (see also [21, 24])

$$K_f(x, a, b, \nu) = \inf \left\{ \int_{Q_\nu} f^\infty(x, v(y), \nabla v(y)) dy : v \in \mathcal{A}(a, b, \nu) \right\} \quad (3.2)$$

where Q_ν is a unit cube centered at the origin with two of its faces normal to ν , and

$$\mathcal{A}(a, b, \nu) := \left\{ v \in W^{1,1}(Q_\nu; \mathbb{R}^d) : \begin{array}{l} v(y) = a \text{ if } y \in \partial Q_\nu, y \cdot \nu \leq 0, \\ v(y) = b \text{ if } y \in \partial Q_\nu, y \cdot \nu \geq 0 \end{array} \right\}.$$

If f does not depend explicitly on u , then it turns out that

$$K_f(x, a, b, \nu) = (Qf)^\infty(x, (b-a) \otimes \nu)$$

(see [6] and [46], Remark 2.17), where $(Qf)^\infty$ is the *recession function* of f , namely

$$(Qf)^\infty(x, A) := \limsup_{t \rightarrow \infty} \frac{QF(x, tA)}{t}.$$

The method introduced to obtain this integral representation, the *blow-up method*, may be summarized as follows. Suppose that we want to represent $\mathcal{F}(u, \Omega)$ as

$$\begin{aligned} \mathcal{F}(u, \Omega) &= \int_{\Omega} \bar{f}(x, u, \nabla u), dx \\ &+ \int_{S(u)} K_f(x, u^-(x), u^+(x), \nu(x)) dH^{N-1}(x) \\ &+ \int_{\Omega} G(x, u(x), dC(u)). \end{aligned} \quad (3.3)$$

In order to identify the energy densities \bar{f}, K_f, G , we claim that it suffices to characterize $\mathcal{F}(v, Q)$ when Q is a unit cube and v is obtained as the blow-up around a point x_0 of the function u . Precisely, let $\{u_n\}$ be a minimizing sequence for $\mathcal{F}(u, \Omega)$, i.e. $u_n \rightarrow u$ in L^1 and

$$\mathcal{F}(u, \Omega) = \lim_{n \rightarrow \infty} \int_{\Omega} f(x, u_n, \nabla u_n) dx,$$

and define the sequence $\{\mu_n\}$ of Radon measures by

$$\mu_n := f(x, u_n, \nabla u_n) \mathcal{L}^N.$$

Assuming that $\mathcal{F}(u, \Omega) < +\infty$, it follows that $\sup_n |\mu_n|(\Omega) < +\infty$ and so there exists a subsequence (still denoted μ_n) and a finite Radon measure μ such that $\mu_n \xrightarrow{*} \mu$ in the sense of measures, i.e. for every $\psi \in C_0(\Omega)$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \psi(x) f(x, u_n(x), \nabla u_n(x)) dx = \int_{\Omega} \psi(x) d\mu(x).$$

By the Radon-Nikodym Theorem, we may decompose μ as the sum of four mutually singular nonnegative measures

$$\mu = \mu_a \mathcal{L}^N + \mu_J H^{N-1} \lfloor S(u) + \mu_c |C(u)| + \mu_s.$$

The equality (3.3) is achieved provided one can prove that

$$\begin{aligned} \mu_a(x_0) &= \bar{f}(x_0, u(x_0), \nabla u(x_0)) \text{ for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega, \\ \mu_J(x_0) &= K_f(x_0, u^-(x_0), u^+(x_0), \nu(x_0)) \text{ for } H_{N-1} \text{ a.e. } x_0 \in \Omega \cap S(u), \\ \mu_c(x_0) &= G\left(x_0, u(x_0), \frac{dC(u)}{d|C(u)|}(x_0)\right) \text{ for } |C(u)| \text{ a.e. } x_0 \in \Omega, \\ \mu_s &= 0. \end{aligned}$$

To show that

$$\mu_a(x_0) = \bar{f}(x_0, u(x_0), \nabla u(x_0)) \text{ for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega,$$

we select a point $x_0 \in \Omega$ such that the following hold:

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{B(x_0, \epsilon)} |u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)| dx = 0, \quad (3.4)$$

$$\mu_a(x_0) = \lim_{\epsilon \rightarrow 0^+} \frac{\mu(Q(x_0, \epsilon))}{\mathcal{L}^N(Q(x_0, \epsilon))} \text{ exists and is finite,} \quad (3.5)$$

where $Q(x_0, \epsilon) := x_0 + \epsilon Q$, $Q = (0, 1)^N$. Denote by D the (at most countable) set of all $\epsilon > 0$ such that the boundary of $x_0 + \epsilon Q$ is not μ -negligible. For every $\epsilon \notin D$, we have

$$\begin{aligned} \mu_a(x_0) &= \lim_{\epsilon \rightarrow 0^+, \epsilon \notin D} \frac{\mu(Q(x_0, \epsilon))}{\epsilon^N} \\ &= \lim_{\epsilon} \frac{1}{\epsilon^N} \lim_{n \rightarrow +\infty} \int_{x_0 + \epsilon Q} f(x, u_n(x), \nabla u_n(x)) dx \\ &= \lim_{\epsilon} \lim_{n \rightarrow +\infty} \int_Q f(x_0 + \epsilon y, u_n(x_0 + \epsilon y), \nabla u_n(x_0 + \epsilon y)) dy \\ &= \lim_{\epsilon} \lim_{n \rightarrow +\infty} \int_Q f(x_0 + \epsilon y, u_{n, \epsilon}(y), \nabla u_{n, \epsilon}(y)) dy, \end{aligned}$$

where

$$u_{n, \epsilon}(y) := \frac{u_n(x_0 + \epsilon y) - u(x_0)}{\epsilon}.$$

Clearly, by (3.4)

$$\lim_{\epsilon \rightarrow 0^+} \lim_{n \rightarrow +\infty} \|u_{n, \epsilon} - u_0\|_{L^1(Q)} = 0$$

with $u_0(y) := \nabla u(x_0)y$, and so, due to the continuity of f , and after extracting a diagonal subsequence, we conclude that

$$\mu_a(x_0) = \lim_{k \rightarrow \infty} \int_Q f(x_0, u(x_0), \nabla v_k(y)) dy$$

where v_k is a sequence bounded in $W^{1,1}(Q, \mathbb{R}^d)$ such that $v_k \rightarrow u_0$ in L^1 . The advantage of this method is that we reduced the relaxation problem for an arbitrary BV function to the relaxation problem when the target is a homogeneous function. In the same way, when searching for a characterization of the interfacial energy density K_f , the blow-up method, together with (3.1), allows us to focus on the case where the macroscopic limit is the piecewise constant function

$$u_1(y) := \begin{cases} u^+(x_0) & \text{if } y \cdot \nu(x_0) > 0 \\ u^-(x_0) & \text{if } y \cdot \nu(x_0) < 0. \end{cases}$$

The density G on the Cantor part is obtained following a similar argument.

4 Interactions between bulk and interfacial energies : the linear growth case

The next step was taken in collaboration with Barroso, Bouchitté and Buttazzo [13], where we studied the relaxation of (1.1) in the case where there is an interfacial contribution from the start. Precisely, we studied the relaxation $\mathcal{F}(u, \Omega)$ with respect to the L^1 convergence of the functional defined in $SBV(\Omega; \mathbb{R}^d)$ by

$$u \mapsto I(u) := \int_{\Omega} f(x, \nabla u(x)) dx + \int_{S(u)} \varphi(x, [u](x), \nu(x)) dH^{N-1}(x) \quad (4.1)$$

where we assume that $f(x, \cdot)$ is quasiconvex and has linear growth, and that $\varphi(x, \cdot, \nu)$ grows also linearly,

$$c\|A\| \leq f(x, A) \leq C(1 + \|A\|), \quad c_1|\xi| \leq \varphi(x, \xi, \nu) \leq C_1|\xi|. \quad (4.2)$$

The space $SBV(\Omega; \mathbb{R}^d)$ of *special functions of bounded variation* was firstly introduced in [35]; a function $u \in BV(\Omega; \mathbb{R}^d)$ is said to be of special bounded variation if $C(u) = 0$, i.e. the distributional derivative of u can be written as $Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu H^{N-1} \llcorner S(u)$. Under some technical continuity conditions and using the blow-up method described in Section 3, we obtained the integral representation

$$\begin{aligned} \mathcal{F}(u, \Omega) &= \int_{\Omega} g(x, \nabla u(x)) dx \\ &+ \int_{S(u)} h(x, [u](x), \nu(x)) dH^{N-1}(x) \\ &+ \int_{\Omega} g^{\infty}(x, dC(u)) \end{aligned}$$

in $BV(\Omega; \mathbb{R}^d)$, where g is the quasiconvexification of the *inf-convolution* of f and φ_0 , $g = Q(f \nabla \varphi_0)$, the inf-convolution is defined by

$$f \nabla \varphi_0(x, A) := \inf \left\{ f(x, A - a \otimes b) + \varphi_0(x, a, b) : a \in \mathbb{R}^d, b \in S^{N-1} \right\},$$

and h is given by

$$h(x_0, \eta, \nu) := \inf \left\{ \int_{Q_\nu} f^\infty(x_0, \nabla u(x)) dx + \int_{S(u) \cap Q_\nu} \varphi(x_0, [u](x), \nu(x)) dH^{N-1}(x) : u \in \mathcal{A}(\eta, \nu) \right\}.$$

Here Q_ν is any unit cube centered at the origin and with two of its faces normal to ν ,

$$\mathcal{A}(\eta, \nu) := \left\{ v \in SBV_{loc}(S_\nu; \mathbb{R}^d) : v(y) = 0 \text{ if } y \cdot \nu = -\frac{1}{2}, \right. \\ \left. v(y) = \eta \text{ if } y \cdot \nu = \frac{1}{2}, \right. \\ \left. v \text{ is 1-periodic in the directions of } \nu_1, \dots, \nu_{N-1} \right\},$$

$\{\nu_1, \dots, \nu_{N-1}, \nu\}$ forms an orthonormal basis of \mathbb{R}^N , and S_ν stands for the strip $S_\nu := \{y \in \mathbb{R}^N : |y \cdot \nu| < \frac{1}{2}\}$. In the above, as usual f^∞ (resp. g^∞) denotes the recession function of f (resp. g), and φ_0 is the positively homogeneous of degree one function defined by

$$\varphi_0(x, \eta, \nu) := \lim_{t \rightarrow 0^+} \frac{\varphi(x, t\eta, \nu)}{t}.$$

Bouchitté, Braides and Buttazzo [16] studied the scalar version of this problem, and they extended the results to the case of linear growth in f or φ , although they needed to make an isotropy assumption. Recently, Braides and Coscia [22] obtained an integral representation of the relaxation with respect to the L^1 topology of the functional

$$u \mapsto \int_{\Omega} f(\nabla u(x)) dx + \int_{S(u)} \varphi([u](x) \otimes \nu(x)) dH_{N-1}(x),$$

under the assumption that φ is positively homogeneous of degree one. Very mild restrictions are placed on f .

5 Interactions between bulk and interfacial energies : the superlinear growth case

The analysis in [13] was carried out under the linear growth assumptions (4.2). These prove to be too exclusive in some settings. As an example, in problems

in image segmentation and interaction between fracture and damage, it is commonly assumed that the bulk energy density $f(x, u, \cdot)$ grows superlinearly.

In the study of variational models in fracture mechanics, u is the displacement field and $S(u)$ represents the fracture region (see [26, 39]). In collaboration with Francfort [39] we studied variationally equilibria for materials that experienced brutal partial damage. To this end, we used the model introduced in [47], where it is assumed that the material is only allowed to drastically drop from its healthy state to its damaged state, the latter retaining some positive definite stiffness. Further, the material may undergo fracture, in which case material discontinuities at a macroscopic level will develop. The quasistatic evolution of both damage and fracture is governed by a yield criterion, commonly accepted in fracture mechanics. After the work of Griffith (see [51], Chapter 4), the criterion compares the decrease in potential energy due to either damage (in which case it is a local decrease) or fracture (in which case it is a global decrease) to the resulting increment of energy dissipated through either process. Both processes are further assumed to be irreversible. In other words, self-healing is absent from both the damaged part of the material and the cracks through that material.

The adopted model results in a time indexed sequence of partial minimization problems. At each time step the potential energy to be minimized is of the form

$$E(u, \Omega) := \int_{\Omega} W(\nabla u) dx + \lambda H^{N-1}(S(u)) - \int_{\Omega} f \cdot u dx,$$

where $W(\xi)$ is the "elastic" energy, $\lambda > 0$ is a dissipation rate and f represents the body loadings. The density W is non quasiconvex, and it satisfies

$$\alpha |\xi|^p \leq W(\xi) \leq \beta(\delta + |\xi|^p), \quad \xi \in \mathbb{M}^{d \times N},$$

$$|W(\xi) - W(\eta)| \leq \gamma(1 + |\eta|^{p-1} + |\xi|^{p-1})|\xi - \eta|, \quad \xi, \eta \in \mathbb{M}^{d \times N},$$

where $\alpha, \beta, \gamma > 0$, $\delta \geq 0$ and $1 < p < +\infty$. If $u \in SBV(\Omega, \mathbb{R}^d)$, and setting

$$\mathcal{F}(u, U) := \inf_{\{u_n\}} \left\{ \liminf_{n \rightarrow \infty} E(u_n, U) : u_n \rightarrow u \text{ in } L^1(\Omega, \mathbb{R}^d) \right\},$$

where U is an open subset of Ω , then we showed in [39] that

$$\mathcal{F}(u, U) := \int_U QW(\nabla u) dx + H^{N-1}(S(u) \cap U). \quad (5.1)$$

The analysis relies heavily on the blow-up method (see Section 3; see also [46]) and on Ambrosio's lower semi-continuity result in $SBV(\Omega; \mathbb{R}^d)$ for quasiconvex Carathéodory integrands with superlinear growth (see [3]). In [39] we used (5.1) to investigate the quasistatic evolution of damage and fracture at discretized times.

An important question that remains to be answered regards the regularity of the interface $S(u)$ and of u , whenever u minimizes the relaxed energy, and under suitable boundary conditions or constraints. With the exception of the case where $QW(\xi) = |\xi|^p$ (see [8, 9, 36]), this question remained virtually untouched until very recently. Ongoing work on this direction is being carried out in collaboration with Francfort.

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