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# RELAXATION OF MULTIPLE INTEGRALS IN SOBOLEV SPACES BELOW THE GROWTH EXPONENT FOR THE ENERGY DENSITY

IRENE FONSECA[1] AND JAN MALÝ[2]

ABSTRACT

The integral representation of the relaxed energies

$$\mathcal{F}^{q,p}(u,\Omega) := \inf_{\{u_n\}}\left\{\liminf_{n\to\infty}\int_\Omega F(x,u_n,\nabla u_n)\,dx : u_n \in W^{1,q}(\Omega,\mathbb{R}^d),\right.$$
$$\left. u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega,\mathbb{R}^d)\right\},$$

$$\mathcal{F}^{q,p}_{\mathrm{loc}}(u,\Omega) := \inf_{\{u_n\}}\left\{\liminf_{n\to\infty}\int_\Omega F(x,u_n,\nabla u_n)\,dx : u_n \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^d),\right.$$
$$\left. u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega,\mathbb{R}^d)\right\}$$

of a functional

$$E : u \mapsto \int_\Omega F(x,u,\nabla u)\,dx, \qquad u \in W^{1,q}(\Omega,\mathbb{R}^d),$$

where $0 \le F(x,\zeta,\xi) \le C(1+|\zeta|^r+|\xi|^q)$ and $\max\left\{1, r\frac{N-1}{N+r}, q\frac{N-1}{N}\right\} < p \le q$, is studied. In particular, $W^{1,p}$-sequential weak lower semicontinuity of $E(\cdot)$ is obtained in the case where $F = F(\xi)$ is a quasiconvex function and $p > q(N-1)/N$.

**Keywords**: lower semicontinuity of multiple integrals, quasiconvexity, relaxation, trace-preserving operators

## 1. INTRODUCTION

We study lower semicontinuity properties of functionals

$$(1.1) \qquad u \mapsto \int_\Omega F(x,u,\nabla u)\,dx, \qquad u \in W^{1,p}(\Omega,\mathbb{R}^d),$$

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bounded above by $F$. As it turns out, $PF \leq QF$ and we say that $F$ is *polyconvex* when $PF = F$.

In this paper we will treat the case where $q$ is the growth exponent of $F$ and $p < q$. As a first step towards obtaining an integral representation for $\mathcal{F}^{q,p}(u, \Omega)$, we aim at identifying a lower bound for the relaxed energy, precisely

$$\mathcal{F}^{q,p}(u, \Omega) \geq \int_\Omega QF(\nabla u)\, dx, \tag{1.4}$$

assuming that the growth assumption

$$0 \leq F(\xi) \leq C(1 + |\xi|^q) \tag{1.5}$$

is verified. In view of (1.4), we need to establish a lower semicontinuity result for quasiconvex integrals (see Theorem 4.1), namely

$$\liminf_{n \to \infty} \int_\Omega F(\nabla u_n)\, dx \geq \int_\Omega F(\nabla u)\, dx \tag{1.6}$$

if $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^d)$ and $F$ is quasiconvex. It is well known that this inequality holds true when $p \geq q$ (see [2, 4, 20, 21]). As indicated by (1.3), we remark that the inequality (1.6) may no longer be valid if $p < q$.

The study of lower semicontinuity properties for (1.1) when $p < q$ finds its motivation on questions in nonlinear elasticity. As an example, in the case where $F$ is the polyconvex function $F(\xi) := |\det \xi|$, the condition $p < N$ plays a fundamental role in the study of cavitation, as it allows deformations to be discontinuous (see [3]). It can be shown that, within the class of polyconvex energy densities, and under suitable structure conditions, if $u_n \in W^{1,N}(\Omega, \mathbb{R}^N)$ converges to $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ weakly in $W^{1,p}$, then

$$\liminf_{n \to \infty} \int_\Omega F(\nabla u_n)\, dx \geq \int_\Omega F(\nabla u)\, dx$$

provided $p \geq N - 1$. This result was first found by Marcellini [19] for $p > \frac{N^2}{N+1}$, then extended by Dacorogna and Marcellini [8] for $p > N - 1$. The borderline case $p = N - 1$ was considered in [15] with a partial success, and completely established by Acerbi, Dal Maso and Sbordone [1], [9]. Improvements are due to Gangbo [13] and Celada and Dal Maso [6]. An elementary approach has been found by Fusco and Hutchinson [12].

The quasiconvex case is more general. Under the growth condition (1.5), and some additional structure conditions, the lower semicontinuity property was proved by Marcellini [19] for $p > q\,\frac{N}{N+1}$, by Carbone and De Arcangelis [5] in some further special cases, by Fonseca and Marcellini [11] for $p > q - 1$. Recently, Malý [17] extended the later result to the borderline case $p = q - 1$. Notice that all the above mentioned results need some additional assumptions. Our approach allows to eliminate additional assumptions if the growth is (1.5) and $p > q\frac{N-1}{N}$, and it is where $\Omega \subset \mathbb{R}^N$ is a bounded, open set and $F : \Omega \times \mathbb{R}^d \times \mathbb{M}^{d\times N} \to \mathbb{R}$ is a nonnegative Carathéodory function. Here, and in what follows, $\mathbb{M}^{d\times N}$ denotes the set of real-valued $d \times N$ matrices.

We are interested in problems where there is lack of convexity, which leads us to considering various types of relaxed energies. Let $p, q \in [1, \infty]$ and let $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. We introduce the functionals

$$\mathcal{F}^{q,p}(u,\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \int_\Omega F(x, u_n, \nabla u_n)\, dx : u_n \in W^{1,q}(\Omega, \mathbb{R}^d), \right.$$
$$\left. u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^d) \right\},$$
$$\mathcal{F}^{q,p}_{\rm loc}(u,\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \int_\Omega F(x, u_n, \nabla u_n)\, dx : u_n \in W^{1,q}_{\rm loc}(\Omega, \mathbb{R}^d), \right.$$
$$\left. u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^d) \right\}.$$

The value of the functional $\mathcal{F}^{q,p}$ may depend, in a rather complicated way, on the values of $p$, $q$, and on the regularity properties of $u$. Consider the example where $N = d$, $F(x, \zeta, \xi) = F(\xi) = |\det \xi|$. Notice that $F$ is polyconvex, hence is quasiconvex (see the definitions below), and the growth condition

$$0 \le F(\xi) \le |\xi|^N$$

is satisfied. It is well known that

$$\mathcal{F}^{q,p}(u,\Omega) \ge \int_\Omega |\det \nabla u|\, dx \tag{1.2}$$

if $p, q \ge N$ ([2,3,7,21]). Recently, (1.2) was shown to hold also for $q \ge N$ and $p \ge N - 1$ (see Celada and Dal Maso [6]; for related work, we refer to [1, 8, 9, 12, 15, 18]). If $u \in W^{1,N}(\Omega, \mathbb{R}^d)$, then we get equality in (1.2), whereas for $u \notin W^{1,N}(\Omega, \mathbb{R}^d)$ it is difficult to describe $\mathcal{F}^{q,p}(u,\Omega)$ (for partial results on this direction, see Remark 3.3 and [1, 11]). We obtain

$$\mathcal{F}^{q,p}(u,\Omega) = 0 \tag{1.3}$$

if $q < N$ (see [4, 14]) or if $p < N - 1$ (see [15] and [10]).

As it is usual, the relaxed energy is related to the *quasiconvexification* of $F$. We recall that, when $F(x, \eta, \xi) = F(\xi)$, the *quasiconvex envelope* of $F$ is defined by (see [7, 22])

$$QF(\xi) := \inf \left\{ \int_{(0,1)^N} F(\xi + \nabla\varphi(x))\, dx \colon \varphi \in C_c^\infty((0,1)^N, \mathbb{R}^d) \right\}.$$

It is clear that $QF \le F$, and $F$ is said to be *quasiconvex* if $QF = F$. Also, the *polyconvex envelope*, $PF$, of $F$ is the supremum of all rank-one affine functions based on a method presented by Malý in [16], where lower semicontinuity is shown to hold in the context of $W^{1,p}$-weak convergence of $C^1$-functions.

Further, we investigate the dependence of $\mathcal{F}^{q,p}(u,U)$ and $\mathcal{F}^{q,p}_{\text{loc}}(u,U)$ on the open subsets $U \subset \Omega$. We assume that

$$0 \leq F(x,\zeta,\xi) \leq C(1+|\zeta|^r+|\xi|^q). \tag{1.7}$$

We prove that if $p > \max\left\{q\frac{N-1}{N}, r\frac{N-1}{N+r}\right\}$, and if $\mathcal{F}^{q,p}(u,\Omega) < \infty$, then there exists a finite, nonnegative, Radon measure $\mu$ such that

$$\mathcal{F}^{q,p}(u,U) = \mu(U) \tag{1.8}$$

at least for open sets $U \subset \Omega$ with $\mu(\partial U) = 0$. In addition, we can show that

$$\mathcal{F}^{q,p}_{\text{loc}}(u,U) = \lambda(U) \tag{1.9}$$

holds for all open sets $U \subset \Omega$, where $\lambda$ is some finite, nonnegative, Radon measure. The representation formula (1.8) may fail if $p \leq q\frac{N-1}{N}$, as illustrated by an example provided by Celada and Dal Maso [6]: if $F(\xi) := |\xi|^{N-1} + |\det \xi|$ and if $p = N-1$, then $\mathcal{F}^{q,p}(u,\cdot)$ is not even subadditive (see Remark 3.3 (i)).

If $F$ is independent of $x$ and $\zeta$, then the lower semicontinuity result (1.6) implies the estimate $\mu_a \geq QF(\nabla u)\mathcal{L}^N$ for the absolutely continuous part, $\mu_a$, of $\mu$. Actually, in all known examples the equality $\mu_a = QF(\nabla u)\mathcal{L}^N$ holds.

This paper is organized as follows:

In Section 2 we construct a linear operator $Tu$ from $W^{1,p}$ into $W^{1,p}$ which conserves boundary values and improves integrability of $u$ and $\nabla u$. Namely, the $W^{1,q}$-norm of $Tu$ is estimated in terms of a special maximal function if $p > q\frac{N-1}{N}$. We use this trace-preserving operator to "connect" two functions across a thin transition layer and to estimate the increase of the energy. We remark that the standard way to perform this connection, by means of convex combinations using cut-off functions, would not achieve a comparable result, namely an arbitrarily small increase of the energy on an arbitrarily thin transition layer, since the admissible sequences may not remain bounded in $W^{1,q}(\Omega,\mathbb{R}^d)$.

In Section 3 we prove that $\mathcal{F}^{q,p}_{\text{loc}}(u,\cdot)$ is a Radon measure and we obtain a representation of $\mathcal{F}^{q,p}(u,\cdot)$ by means of a Radon measure $\mu$, in the sense described above (see (1.8), (1.9)). Moreover, we show that (1.8) holds for all open sets $U \subset \Omega$ provided it exists a Radon measure $\nu$ such that

$$\mathcal{F}^{q,p}(u,U) \leq \nu(U) \tag{1.10}$$

for all open subset $U \subset \Omega$. In Remark 3.3 we provide a couple of examples to illustrate the sharpness of these results.

In Section 4 we establish that (1.1) is lower semicontinuous in $W^{1,p}$-weak if $F$ is quasiconvex (see Theorem 4.1). This enables us to obtain a lower bound for $\mathcal{F}^{q,p}(u,U)$,

$$\mathcal{F}^{q,p}(u,U) \geq \int_U QF(\nabla u(x))\,dx.$$

In particular, when (1.10) holds then the absolutely continuous part of $\mu$ with respect to $\mathcal{L}^N$, $\mu_a$, satisfies

$$\mu_a(x) \geq QF(\nabla u(x))$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$. Here, and in what follows, $\mathcal{L}^N$ denotes the $N$-dimensional Lebesgue measure. If, in addition, $u \in W^{1,q}(\Omega, \mathbb{R}^d)$, then we obtain the usual relaxation result,

$$\mathcal{F}^{q,p}(u,U) = \int_U QF(\nabla u(x))\,dx.$$

The extension of this lower semicontinuity result to more general energy density functions $F = F(x, \zeta, \xi)$ is addressed in Remark 4.3 and Example 4.4.

## 2. Trace-preserving operators

Throughout this section we consider fixed exponents $r, q \geq 1$, and

$$p > \max\left\{1,\ r\frac{N-1}{N+r},\ q\frac{N-1}{N}\right\}.$$

Further, let $\eta \in C_c^\infty(\Omega)$ be a nonnegative function, and $[t_1, t_2] \subset (0, ||\eta||_\infty)$. Suppose that $0 < |\nabla \eta| \leq A$ on $\{t_1 \leq \eta \leq t_2\}$. Given a subinterval $(a, b) \subset (t_1, t_2)$, we write $Z_a^b := \{a < \eta < b\}$.

In the sequel we will need an operator on $W^{1,p}(\Omega)$ which improves the integrability properties of a function and its gradient in $Z_a^b$, while conserving the function values elsewhere.

Fix $t_0 \in (t_1, t_2)$ and consider the level set $\Gamma_{t_0} := \{\eta = t_0\}$. There exists a diffeomorphism $\Phi_{t_0}$ of $\Gamma_{t_0} \times [t_1, t_2]$ onto $\overline{Z}_{t_1}^{t_2}$ such that

$$\begin{cases} \Phi_{t_0}(z, t_0) = z \\ \eta(\Phi_{t_0}(z,t)) = t \end{cases} \tag{2.1}$$

for all $z \in \Gamma_{t_0}, t \in [t_1, t_2]$. Precisely, given $z \in \Gamma_{t_0}$ it suffices to consider the flow $h_z$ verifying

$$\begin{cases} \frac{dh_z}{dt} = \frac{\nabla\eta(h(t))}{|\nabla\eta(h(t))|^2} \\ \\ h_z(t_0) = z \end{cases}$$

and set $\Phi_{t_0}(z,t) := h_z(t)$. The mapping $\Phi_{t_0}$ satisfies the bi-Lipschitz condition and the jacobians of $\Phi_{t_0}$ and $\Phi_{t_0}^{-1}$ are bounded. Also, using $\Phi_{t_0}$ and by virtue of the Sobolev imbedding theorem on smooth $N-1$-dimensional manifolds, one can show that if $v$ is a smooth function then

$$\left(\int_{\{\eta=t_0\}} |v|^r \, dH^{N-1}\right)^{1/r} \leq C\left(\int_{\{\eta=t_0\}} (|v|^\beta + |\nabla v|^\beta)\, dH^{N-1}\right)^{1/\beta}, \tag{2.2}$$

where $1 \leq \beta$, and either $\beta \geq N-1$ or $r \leq \frac{\beta(N-1)}{N-1-\beta}$, and $C = C(N, \beta, r, \eta, t_1, t_2)$.

**2.1. Lemma.** *Consider $s \in (t_1, t_2)$ and $\rho > 0$ such that $[s-\rho, s+\rho] \subset (t_1, t_2)$. Let $f$ be a nonnegative measurable function on $\Omega$. Then*

$$\int_{\{\eta=s\}} \Big( \int_{B(z,\frac{\rho}{A})} f(y)\,dy \Big)\, dH^{N-1}(z) \le C\rho^{N-1} \int_{Z_{s-\rho}^{s+\rho}} f(y)\,dy$$

*where $C = C(N, \eta, t_1, t_2)$.*

*Proof.* It can be seen easily that if $z \in \Gamma_s$ then $B(z, \frac{\rho}{A}) \subset Z_{s-\rho}^{s+\rho}$. Hence, using the change of variables $y = \Phi_s(z,t)$ and (2.1), we obtain

$$\begin{aligned}
&\int_{\{\eta=s\}} \Big( \int_{B(z,\frac{\rho}{A})} f(y)\,dy \Big)\, dH^{N-1}(z) \\
&\le C \int_{\Gamma_s} \Big( \int_{s-\rho}^{s+\rho} \Big( \int_{\{\sigma\, \in\Gamma_s : |\Phi_s(\sigma,t)-\Phi_s(z,s)|<\frac{\rho}{A}\}} f\circ\Phi_s(\sigma,t)\, dH^{N-1}(\sigma) \Big)\, dt \Big)\, dH^{N-1}(z) \\
&= C \int_{\Gamma_s} \Big( \int_{s-\rho}^{s+\rho} \Big( \int_{\{z\in\Gamma_s :\ |\Phi_s(\sigma,t)-\Phi_s(z,s)|<\frac{\rho}{A}\}} f\circ\Phi_s(\sigma,t)\, dH^{N-1}(z) \Big)\, dt \Big)\, dH^{N-1}(\sigma) \\
&\le C \int_{\Gamma_s\times(s-\rho,s+\rho)} H^{N-1}(\{z\in\Gamma_s : |\Phi_s(\sigma,t)-\Phi_s(z,s)| < \frac{\rho}{A}\})\ f\circ\Phi_s(\sigma,t)\, dH^N(\sigma,t) \\
&\le C\rho^{N-1} \int_{Z_{s-\rho}^{s+\rho}} f(y)\,dy,
\end{aligned}$$

since, due to the Lipschitz continuity of the mapping $\Phi_s^{-1}$,

$$H^{N-1}(\{z \in \Gamma_s : |\Phi_s(\sigma,t) - \Phi_s(z,s)| < \frac{\rho}{A}\}) \le C\rho^{N-1}.$$

**2.2. Lemma.** *Let $t_1 < a < b < t_2$. There exists a linear operator $T: W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ such that $Tu = u$ on $\Omega \setminus Z_a^b$, and*

$$\begin{aligned}
\|Tu\|_{W^{1,q}(Z_a^b)} + \|Tu\|_{L^r(Z_a^b)} \le C(b-a)^\tau \Big( &\sup_{t\in(a,b)} (t-a)^{-1/p}\, \|u\|_{W^{1,p}(Z_a^t)} \\
&+ \sup_{t\in(a,b)} (b-t)^{-1/p}\, \|u\|_{W^{1,p}(Z_t^b)} \Big),
\end{aligned} \tag{2.3}$$

*where $C = C(N, p, q, r, \eta, t_1, t_2)$ and $\tau = \tau(N, p, q, r) > 0$.*

*Proof.* Set

$$Tu(x) := \fint_{B(0,1)} u(x + \theta(x)y)\,dy,$$

where

$$\theta(x) := \frac{1}{2A} \max\{0, \min\{\eta(x) - a, b - \eta(x)\}\} = \begin{cases} 0 & \text{if } \eta(x) \ge b \\ \frac{b-\eta(x)}{2A} & \text{if } \frac{a+b}{2} < \eta(x) < b \\ \frac{\eta(x)-a}{2A} & \text{if } a < \eta(x) \le \frac{a+b}{2} \\ 0 & \text{if } \eta(x) \le a. \end{cases}$$

It is clear that $Tu(x) = x$ if $x \notin Z_a^b$, and

$$Tu = \fint_{B(x,\theta(x))} u(z)\, dz$$

for $x \in Z_a^b$. Let $c := \frac{a+b}{2}$ and denote

$$M_0 := \sup_{t\in(a,b)} (t-a)^{-1} \int_{Z_a^t} |u|^p\, dy,$$
$$M_1 := \sup_{t\in(a,b)} (t-a)^{-1} \int_{Z_a^t} (|u|^p + |\nabla u|^p)\, dy.$$

Assume, first, that $u$ is smooth and fix $\alpha \geq p$. If $\rho \in (0, \frac{1}{4}(b-a))$ and if $z \in \{\eta = a + 2\rho\}$, then $\theta(z) = \frac{\rho}{A}$ and $B(z, \theta(z)) \subset Z_{a+\rho}^{a+3\rho}$. Thus,

$$\begin{aligned}|Tu(z)|^\alpha &\leq C\rho^{-N\alpha}\Big(\int_{B(z,\frac{\rho}{A})} |u(y)|\, dy\Big)^\alpha \\ &\leq C\rho^{-N\alpha}\rho^{N\alpha(1-\frac{1}{p})}\Big(\int_{B(z,\frac{\rho}{A})} |u(y)|^p\, dy\Big)^{\frac{\alpha}{p}} \\ &\leq C\rho^{-\frac{N\alpha}{p}}\Big(\int_{Z_{a+\rho}^{a+3\rho}} |u(y)|^p\, dy\Big)^{\frac{\alpha}{p}-1}\Big(\int_{B(z,\frac{\rho}{A})} |u(y)|^p\, dy\Big).\end{aligned}$$

Using Lemma 2.1, we obtain

$$\begin{aligned}&\int_{\{\eta=a+2\rho\}} |Tu|^\alpha(z)\, dH^{N-1}(z) \\ &\leq C\rho^{-\frac{N\alpha}{p}}\Big(\int_{Z_{a+\rho}^{a+3\rho}} |u(y)|^p\, dy\Big)^{\frac{\alpha}{p}-1}\int_{\{\eta=a+2\rho\}}\Big(\int_{B(z,\frac{\rho}{A})} |u(y)|^p\, dy\Big)\, dH^{N-1}(z) \\ &\leq C\rho^{-\frac{N\alpha}{p}}\Big(\int_{Z_{a+\rho}^{a+3\rho}} |u(y)|^p\, dy\Big)^{\frac{\alpha}{p}-1}\rho^{N-1}\Big(\int_{Z_{a+\rho}^{a+3\rho}} |u(y)|^p\, dy\Big) \\ &= C\rho^{-\frac{N\alpha}{p}+N-1}\Big(\int_{Z_{a+\rho}^{a+3\rho}} |u(y)|^p\, dy\Big)^{\frac{\alpha}{p}}.\end{aligned} \tag{2.4}$$

By virtue of the co-area formula and (2.4) for $\alpha = q$, and since $|\nabla\eta|$ is bounded away from zero, we obtain

$$\begin{aligned}\int_{Z_a^c} |Tu|^q(x)\, dx &\leq C\int_0^{\frac{1}{4}(b-a)}\Big(\int_{\{\eta=a+2\rho\}} |Tu|^q(z)\, dH^{N-1}(z)\Big)\, d\rho \\ &\leq C\int_0^{\frac{1}{4}(b-a)} \rho^{-\frac{Nq}{p}+N-1}\Big(\int_{Z_{a+\rho}^{a+4\rho}} |u(y)|^p\, dy\Big)^{\frac{q}{p}}\, d\rho.\end{aligned}$$

The latter inequality has been proven for smooth functions $u$. Using a standard approximation argument, together with Fatou's Lemma, it can be seen easily that it is still valid for any $u \in L^p(\Omega)$. In addition, and since

$$\int_{Z_{a+\rho}^{a+3\rho}} |u(x)|^p\, dx \leq CM_0\rho,$$

we have

$$\int_{Z_a^c} |Tu|^q(x)\,dx \le CM_0^{\frac{q}{p}} \int_0^{\frac14(b-a)} \rho^{-\frac{Nq}{p}+N-1+\frac{q}{p}}\,d\rho \le CM_0^{\frac{q}{p}}(b-a)^{q\tau_1}, \tag{2.5}$$

where

$$\tau_1 := -\frac{N-1}{p} + \frac{N}{q}.$$

By means of an entirely similar argument we conclude that

$$\int_{Z_c^b} |Tu|^q(x)\,dx \le CM_0^{\frac{q}{p}}(b-a)^{q\tau_1}.$$

Now we obtain estimates on the gradient of $Tu$. We have

$$\frac{\partial Tu}{\partial x_i}(x) = \fint_{B(0,1)} \Big(\frac{\partial u}{\partial x_i}(x+\theta(x)y) + \sum_{j=1}^N \frac{\partial u}{\partial x_j}(x+\theta(x)y)y_j\frac{\partial\theta}{\partial x_i}(x)\Big)\,dy$$

and thus

$$|\nabla Tu| \le CT|\nabla u|. \tag{2.6}$$

It follows that the $L^q$ estimate (2.5) holds also for derivatives, so that

$$\|Tu\|_{W^{1,q}(Z_a^b)} \le C(b-a)^{\tau_1}\Big(\sup_{t\in(a,b)}(t-a)^{-1/p}\|u\|_{W^{1,p}(Z_a^t)} + \sup_{t\in(a,b)}(b-t)^{-1/p}\|u\|_{W^{1,p}(Z_t^b)}\Big),$$

Note, however, that the right hand side of the above inequality may not be finite. Next, using the co-area formula, (2.4) with $\alpha = p$, and (2.6), we obtain, for smooth functions $u$,

$$\begin{aligned}
&\int_{Z_a^c} (|Tu|^p + |\nabla Tu|^p)\,dy \\
&\le C\int_0^{\frac14(b-a)} \Big(\int_{\{\eta=a+2\rho\}} |Tu|^p(z) + |\nabla Tu|^p(z)\,dH^{N-1}(z)\Big)d\rho \\
&\le C\int_0^{\frac14(b-a)} \Big(\int_{Z_{a+\rho}^{a+3\rho}} \rho^{-1}(|u(y)|^p + |\nabla u(y)|^p)\,dy\Big)d\rho \\
&\le C\int_0^{\frac14(b-a)} \Big(\int_{a+\rho}^{a+3\rho}\Big(\int_{\{\eta=t\}} \rho^{-1}(|u(z)|^p + |\nabla u(z)|^p)\,dH^{N-1}(z)\Big)\,dt\Big)\,d\rho \\
&= C\int_a^b \Big(\int_{\{\eta=t\}}\Big(\int_{\frac{t-a}{3}}^{\min\{t-a,\frac{b-a}{4}\}} \rho^{-1}(|u(z)|^p + |\nabla u(z)|^p)\,d\rho\Big)\,dH^{N-1}(z)\Big)\,dt \\
&\le C\int_{Z_a^b} (|u(y)|^p + |\nabla u(y)|^p)\,dy.
\end{aligned} \tag{2.7}$$

A similar bound holds for

$$\int_{Z_c^b} (|Tv|^p + |\nabla Tv|^p)\, dy.$$

It is easy to see that $Tu$ is weakly differentiable on $\Omega$ (and thus, by the above estimates, $Tu \in W^{1,p}(\Omega)$ and (2.7) holds with $Z_a^c$ and $Z_a^b$ replaced by $\Omega$) if $u$ is smooth enough. If $v \in W^{1,p}(\Omega)$ and if $\{u_n\}$ is a sequence of smooth functions converging to $v$ in $W^{1,p}(\Omega)$, then clearly $Tu_n(x) \underset{n\to\infty}{\rightarrow} Tv(x)$ for all $x \in \Omega$, while, by (2.7), $\{Tu_n\}$ is bounded in $W^{1,p}(\Omega)$. Thus, a subsequence of $\{Tu_n\}$ converges weakly in $W^{1,p}(\Omega)$ to $v$ and by (2.7) we have

$$\int_\Omega (|Tv|^p + |\nabla Tv|^p)\, dy \le C \int_\Omega (|v(y)|^p + |\nabla v(y)|^p)\, dy,$$

and we conclude that $T$ is a linear continuous map from $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)$. It remains to prove the $L^r$-estimate. Fix $\beta \ge 1$ such that

$$\text{(2.8)} \qquad \frac{1}{p} - \frac{1}{r(N-1)} < \frac{1}{\beta} \le \min\left\{\frac{1}{p}, \frac{1}{r} + \frac{1}{N-1}\right\}.$$

Given a smooth function $u$, by (2.2), by (2.4) with $\alpha = \beta$, and by (2.6), we have

$$\begin{aligned}\Big(\int_{\{\eta=a+2\rho\}} |Tu|^r(z)\, dH^{N-1}(z)\Big)^{\beta/r} &\le C \int_{\{\eta=a+2\rho\}} (|Tu|^\beta(z) + |\nabla Tu|^\beta(z))\, dH^{N-1}(z) \\ &\le C\rho^{-\frac{N\beta}{p}+N-1}\Big(\int_{Z_{a+\rho}^{a+3\rho}} (|u(y)|^p + |\nabla u(y)|^p)\, dy\Big)^{\frac{\beta}{p}};\end{aligned}$$

hence, just as in the proof of (2.5), we obtain

$$\int_{Z_a^c} |Tu|^r \le C \int_0^{\frac{b-a}{4}} \rho^{-\frac{Nr}{p}+\frac{r(N-1)}{\beta}} \Big(\int_{Z_{a+\rho}^{a+3\rho}} (|u(y)|^p + |\nabla u(y)|^p\, dy\Big)^{\frac{r}{p}}\, d\rho.$$

Using a density argument we conclude that this inequality is still valid for $u \in W^{1,p}(\Omega)$, from which we obtain

$$\begin{aligned}\int_{Z_a^c} |Tu|^r(x) dx &\le CM_1^{\frac{r}{p}} \int_0^{\frac{1}{4}(b-a)} \rho^{-\frac{Nr}{p}+\frac{r(N-1)}{\beta}+\frac{r}{p}}\, d\rho \\ &\le CM_1^{\frac{r}{p}}(b-a)^{r\tau_2},\end{aligned}$$

where $\tau_2 := \frac{1}{r} - (N-1)(\frac{1}{p} - \frac{1}{\beta}) > 0$. This concludes the proof.

**2.3. Elementary lemma.** *Let $\psi$ be a continuous nondecreasing function on an interval $[a,b]$, $a<b$. There exist $a'\in[a,a+\frac{1}{3}(b-a)]$, $b'\in[b-\frac{1}{3}(b-a),b]$, such that $a\le a'<b'\le b$, and*

$$\begin{aligned}\frac{\psi(t)-\psi(a')}{t-a'}&\le 3\frac{\psi(b)-\psi(a)}{b-a},\\ \frac{\psi(b')-\psi(t)}{b'-t}&\le 3\frac{\psi(b)-\psi(a)}{b-a}\end{aligned}\tag{2.9}$$

*for all $t\in(a',b')$.*

*Proof.* Without loss of generality, we may assume that $a=0$, $\psi(a)=0$. Let $a'$ be a point of $[0,b]$ where

$$\varphi(t):=\psi(t)-3t\frac{\psi(b)}{b}$$

attains its maximum and let $b'$ be a point of $[0,b]$ where $\varphi$ attains its minimum. It is clear that formulas (2.9) hold. To show that $a'\le\frac{b}{3}$, it suffices to remark that $\varphi(0)=0$, while $\varphi(t)<0$ whenever $t>\frac{b}{3}$. Indeed, as $\psi$ is nondecreasing, $3t\frac{\psi(b)}{b}>\psi(b)\ge\psi(t)$. In a similar way, one can show that $b'\ge b-\frac{1}{3}b$.

**2.4. Lemma.** *Let $V\subset\subset\Omega$ and $W\subset\Omega$ be open sets, $\Omega=V\cup W$, $v\in W^{1,q}(V)$ and $w\in W^{1,q}(W)$. Let $m\in\mathbb{N}$. There exist a function $z\in W^{1,q}_{\mathrm{loc}}(\Omega)$ and open sets $V'\subset V$ and $W'\subset W$, such that $V'\cup W'=\Omega$, $z=v$ in $\Omega\setminus W'$, $z=w$ on $\Omega\setminus V'$,*

$$\mathcal{L}^N(V'\cap W')\le Cm^{-1}\tag{2.10}$$

*and*

$$\begin{aligned}&\|z\|_{L^r(V'\cap W')}+\|z\|_{W^{1,q}(V'\cap W')}\\ &\le Cm^{-\tau}\Big(\|v\|_{W^{1,p}(V\cap W)}+\|w\|_{W^{1,p}(V\cap W)}+m\,\|w-v\|_{L^p(V\cap W)}\Big),\end{aligned}\tag{2.11}$$

*where $C=C(p,q,r,V,W)$ and $\tau=\tau(N,p,q,r)>0$.*

*Proof.* Let $\eta\in C_c^\infty(\Omega)$ be such that

$$\eta=0\quad\text{on }\Omega\setminus V\quad\text{and}\quad\eta=1\quad\text{on }\Omega\setminus W.\tag{2.12}$$

By Sard's Lemma, the image of the set of all critical points of $\eta$ is a closed set of measure zero; hence, there is an nondegenerate interval $[a,b]\subset(0,1)\setminus\eta(\{\nabla\eta=0\})$. Choose $m\in\mathbb{N}$ and define

$$f:=1+|\nabla v|^p+|\nabla w|^p+|v|^p+|w|^p+m^p|w-v|^p.$$

Since $\{a<\eta<b\}\subset V\cap W$, we may find $k\in\{1,\dots,m\}$ such that

$$\int_{\{a_k<\eta<b_k\}}f\,dx\le\frac{1}{m}\int_{V\cap W}f\,dx,\tag{2.13}$$

where $a_k := a + \frac{(k-1)(b-a)}{m}$, $b_k := a + \frac{k(b-a)}{m}$. Using Lemma 2.3, with

$$\psi(t) := \int_{\eta<t} f\,dx,$$

we find $[a', b'] \subset [a_k, b_k]$ such that $b' - a' \geq \frac{1}{3}(b_k - a_k)$, and

$$\begin{aligned} &\int_{\{a'<\eta<t\}} f\,dx \leq 3\,\frac{t-a'}{b'-a'} \int_{\{a'<\eta<b'\}} f\,dx, \\ &\int_{\{t<\eta<b'\}} f\,dx \leq 3\,\frac{b'-t}{b'-a'} \int_{\{a'<\eta<b'\}} f\,dx \end{aligned} \tag{2.14}$$

for all $t \in (a', b')$. Set

$$V' := \Omega \cap \{\eta > a'\}, \quad W' := \Omega \cap \{\eta < b'\},$$

$$u := \begin{cases} v, & \text{on } \{\eta \geq b'\}, \\ \frac{(\eta-a')v+(b'-\eta)w}{b'-a'} & \text{in } \{a' < \eta < b'\}, \\ w, & \text{on } \{\eta \leq a'\}. \end{cases}$$

By (2.12), it is clear that $V' \subset V$, $W' \subset W$, and $V' \cup W' = \Omega$. Also, (2.10) holds because $|\nabla\eta|$ is bounded away from zero on $\{a < \eta < b\}$ and $b' - a' \leq \frac{b-a}{m}$. A direct computation shows that

$$|u|^p + |\nabla u|^p \leq Cf$$

on $\{a' < \eta < b'\}$. Using (2.13), (2.14) and Lemma 2.2, we find a function $z \in W^{1,p}(\Omega)$ such that $z = u = v$ on $\{\eta \geq b'\} = \Omega \backslash W'$, $z = u = w$ on $\{\eta \leq a'\} = \Omega \backslash V'$ and (2.11) is satisfied.

## 3. The relaxed energy: dependence on the domain

Let $\mu$ be a Radon measure on $\overline{\Omega}$. We say that $\mu$ (strongly) *represents* $\mathcal{F}^{q,p}(u,\cdot)$ if

$$\mu(U) = \mathcal{F}^{q,p}(u,U)$$

for all open sets $U \subset \Omega$. We say that $\mu$ *weakly represents* $\mathcal{F}^{q,p}(u,\cdot)$ if

$$\mu(U) \leq \mathcal{F}^{q,p}(u,U) \leq \mu(\overline{U})$$

for all open sets $U \subset \Omega$. Strong and weak measure representations for $\mathcal{F}^{q,p}_{\text{loc}}(u,\cdot)$ are defined in an similar way. In this section we will study measure representation properties of the relaxed functional $\mathcal{F}^{q,p}_{\text{loc}}(u,\cdot)$ for a functional (1.1) satisfying (1.7). We show that if

$$\begin{aligned} &r, q \geq p, \\ &p > \max\left\{1, r\frac{N-1}{N+r}, q\frac{N-1}{N}\right\}, \end{aligned} \tag{3.1}$$

then $\mathcal{F}^{q,p}_{\text{loc}}(u,\cdot)$ can be represented by a Radon measure and $\mathcal{F}^{q,p}(u,\cdot)$ is weakly represented (see Theorems 3.1, 3.2). We characterize the case where strong measure representation for $\mathcal{F}^{q,p}(u,\cdot)$ occurs. We include an example of weak measure representation which is not strong and an example which illustrates that measure representation properties may fail altogether if the condition (3.1) is violated.

First we state the main results which will be proved later in this section.

**3.1. Theorem.** *Let $F$ be a Carathéodory function satisfying (1.7) and let $p, q, r$ verify (3.1), $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. If $\mathcal{F}^{q,p}_{\text{loc}}(u, \Omega) < \infty$, then there exists a nonnegative, finite Radon measure $\lambda$ on $\Omega$ which represents $\mathcal{F}^{q,p}_{\text{loc}}(u, \Omega)$.*

**3.2. Theorem.** *Let $F$ be a Carathéodory function satisfying (1.7) and let $p, q, r$ verify (3.1), $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. If $\mathcal{F}^{q,p}(u, \Omega) < \infty$ then there exists a nonnegative, Radon measure $\mu$ on $\overline{\Omega}$ which weakly represents $\mathcal{F}^{q,p}(u, \Omega)$.*

**3.3. Remark.**
(i) The latter result is sharp, in that we may find $p = q\frac{N-1}{N}$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ such that $\mathcal{F}^{q,p}(u, \cdot)$ cannot be weakly represented by a measure. Indeed, let $B$ stand for the unit ball in $\mathbb{R}^N$, let $q = d = N$, $r = p = N - 1$, $u(x) := \frac{x}{|x|}$ and

$$F(\xi) := |\xi|^{N-1} + |\det \xi|.$$

Then $u \in W^{1,s}(B, \mathbb{R}^N)$ for all $s < N$, in particular for $s = p$,

$$\rho \mapsto \mathcal{F}^{q,p}(u, \rho B) - \int_{\rho B} F(\nabla u) dx$$

is of order $\rho$ at 0, whereas

$$\mathcal{F}^{q,p}(u, B \setminus \rho B) - \int_{B \setminus \rho B} F(\nabla u) dx = 0.$$

Hence, $\mathcal{F}^{q,p}(u, \cdot)$ cannot be additive. The same argument works here also for $\mathcal{F}_{\text{loc}}(u, \cdot)$. This example is essentially due to Acerbi and Dal Maso [1].
(ii) In (i) the additivity property failed due to the fact that $p \le q\frac{N-1}{N}$. Now we will see that, in spite of requiring $p > q\frac{N-1}{N}$, the measure representation may not be strong. Let $q = d = N$ and $u(x) := \frac{x}{|x|}$, but now $p > N - 1$ (which is the case in which Theorems 3.1 and 3.2 are valid), and

$$F(\xi) := |\det \xi|.$$

Let $\mu := \mathcal{L}^N(B)\, \delta_0$ be the $\mathcal{L}^N(B)$-multiple of the Dirac measure at 0. Then (see [11], Theorem 4.1),

$$\mathcal{F}^{q,p}(u, U) = \mu(U) \tag{3.2}$$

if $\mu(\partial U) = 0$. If $U = \{x \in B : x_1 > 0\}$, then we have

$$\mathcal{F}^{q,p}(u, U) = \mu(U) < \mu(\overline{U})$$

(this can be seen using the approximation $u_n(x) = u\left(x + \frac{1}{n} e_1\right)$). In the case where $U := B \setminus \{0\}$, we have

$$\mathcal{F}^{q,p}(u, U) = \mu(B) = \mu(\overline{U}) > \mu(U), \tag{3.3}$$

as each $v \in W^{1,q}(U,\mathbb{R}^d)$ is also in $W^{1,q}(B,\mathbb{R}^d)$ (the point 0 is a removable singularity). Clearly, $\mathcal{F}^{q,p}(u,\cdot)$ cannot be a measure since in this case, and by (3.2), it would have to be the measure $\mu$, contradicting (3.3).

Theorem 3.1 ensures that a similar example does not exist for the relaxed energy $\mathcal{F}^{q,p}_{\rm loc}(u,\cdot)$-situation; notice that it may happen that $v \in W^{1,q}_{\rm loc}(U,\mathbb{R}^d)$ and $v \notin W^{1,q}_{\rm loc}(B,\mathbb{R}^d)$.

(iii) If $U \subset\subset V \subset \Omega$, then, obviously,

$$\mathcal{F}^{p,q}_{\rm loc}(u,U) \le \mathcal{F}^{p,q}(u,U) \le \mathcal{F}^{p,q}_{\rm loc}(u,V).$$

Hence, if the measures $\mu$ and $\lambda$ from Theorems 3.1 and 3.2 exist, then $\lambda = \mu\lfloor\Omega$.

**3.4. Lemma.** *Let $F$ be a Carathéodory function satisfying (1.7), and let $p,q,r$ verify (3.1). Let $V, W \subset \Omega$ be open sets, $V \subset\subset \Omega$ and $\Omega = V \cup W$, and let $u \in W^{1,p}(\Omega,\mathbb{R}^d)$. Then*

$$\mathcal{F}^{q,p}(u,\Omega) \le \mathcal{F}^{q,p}(u,V) + \mathcal{F}^{q,p}(u,W).$$

*Proof.* Choose $\varepsilon > 0$. We find open sets $V' \subset V$ and $W' \subset W$ such that $\Omega = V' \cup W'$ and $\overline{V' \cap W'} \subset V \cap W$. Using the definition of relaxation and Rellich's compact imbedding theorem, we find $v_n \in W^{1,q}(V,\mathbb{R}^d)$ and $w_n \in W^{1,q}(W,\mathbb{R}^d)$ such that

$$\begin{aligned}
&v_n \rightharpoonup u \text{ weakly in } W^{1,p}(V,\mathbb{R}^d),\\
&\|v_n - u\|_{L^p(V'\cap W')} \le \frac{1}{n},\\
&\int_V F(x,v_n,\nabla v_n)\,dx \le \mathcal{F}^{q,p}(u,V) + \varepsilon,\\
&w_n \rightharpoonup u \text{ weakly in } W^{1,p}(W,\mathbb{R}^d),\\
&\|w_n - u\|_{L^p(V'\cap W')} \le \frac{1}{n},\\
&\int_W F(x,w_n,\nabla w_n)\,dx \le \mathcal{F}^{q,p}(u,W) + \varepsilon.
\end{aligned}$$

By virtue of Lemma 2.4, we may find open sets $V_n \subset V'$, $W_n \subset W'$, and functions $z_n \in W^{1,q}(\Omega,\mathbb{R}^d)$, such that $V_n \cup W_n = \Omega$, $z_n = v_n$ on $\Omega \setminus W_n$, $z_n = w_n$ on $\Omega \setminus V_n$, and, by (1.7),

$$\begin{aligned}
\int_{V_n\cap W_n} F(x,z_n,\nabla z_n)\,dx &\le C\int_{V_n\cap W_n} (1 + |z_n|^r + |\nabla z_n|^q)\,dx\\
&\le Cn^{-p\tau},
\end{aligned}$$

$$\mathcal{L}^N(V_n \cap W_n) \le \frac{C}{n},$$

where $\tau$ is as in Lemma 2.4. It follows that

$$\int_\Omega F(x,z_n,\nabla z_n)\,dx \le \int_V F(x,v_n,\nabla v_n)\,dx + \int_W F(x,w_n,\nabla w_n)\,dx + Cn^{-p\tau};$$

hence

$$\liminf_{n\to\infty} \int_\Omega F(x, z_n, \nabla z_n)\, dx \le \mathcal{F}^{q,p}(u, V) + \mathcal{F}^{q,p}(u, W) + 2\varepsilon.$$

It remains to prove that $z_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. It is easy to check that the sequence is bounded in $W^{1,p}(\Omega, \mathbb{R}^d)$. Furthermore, taking into account that $\mathcal{L}^N(V_n \cap W_n) \to 0$ and Rellich's compact imbedding theorem, we see that each subsequence of $z_n$ contains a sub-subsequence converging to $u$ a.e. It follows that $z_n \rightharpoonup u$ which concludes the proof.

**3.5. Remark.** A similar assertions holds for $\mathcal{F}^{q,p}_{\mathrm{loc}}(u, \cdot)$, with essentially the same proof.

*Proof of Theorem 3.2.* We write

$$\mathcal{F}(U) := \mathcal{F}^{q,p}(u, U).$$

First we assume that the coercivity assumption

$$F(x, \zeta, \xi) \ge c(|\zeta|^p + |\xi|^p) \tag{3.4}$$

is satisfied. Let $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$ be a minimizing sequence such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^d)$, and

$$\lim_{n\to\infty} \int_\Omega F(x, u_n, \nabla u_n)\, dx = \mathcal{F}(\Omega).$$

Passing to a subsequence, if necesssary, there exists a nonnegative Radon measure $\mu$ on $\overline{\Omega}$ such that

$$w^*\text{-} \lim_{n\to\infty} F(\nabla u_n)\mathcal{L}^N \lfloor \Omega = \mu$$

(weak* convergence in measures on $\overline{\Omega}$). In particular, we have

$$\mu(\overline{\Omega}) = \mathcal{F}(\Omega) \tag{3.5}$$

and for every open set $V \subset \Omega$

$$\mathcal{F}(V) \le \liminf_{n\to\infty} \int_V F(x, u_n, \nabla u_n)\, dx \le \mu(\overline{V}). \tag{3.6}$$

Conversely, let $V \subset \Omega$ be an open set and fix $\varepsilon > 0$. We find an open set $Z \subset\subset V$ such that

$$\mu(V) - \mu(Z) < \varepsilon.$$

Then, using Lemma 3.4, (3.5), (3.6), we have

$$\mu(V) \le \mu(Z) + \varepsilon = \mu(\overline{\Omega}) - \mu(\overline{\Omega} \setminus Z) + \varepsilon \le \mathcal{F}(\Omega) - \mathcal{F}(\Omega \setminus \overline{Z}) + \varepsilon \le \mathcal{F}(V) + \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain

$$\mu(V) \le \mathcal{F}(V).$$

Now, we remove the assumption (3.4). By the above part of the proof, and for every $\varepsilon > 0$, we obtain a measure $\mu_\varepsilon$ representing the relaxation $\mathcal{F}_\varepsilon$ of the functional

$$\int_\Omega (F(x, u, \nabla u) + \varepsilon|u|^p + \varepsilon|\nabla u|^p)\,dx.$$

Since

$$\mu_\varepsilon(\overline{\Omega}) = \mathcal{F}_\varepsilon(\Omega) \le \mathcal{F}(\Omega) + \varepsilon \sup_n ||u_n||_{W^{1,p}} \le C,$$

we may select $\varepsilon_k \to 0$ such that the subsequence $\mu_{\varepsilon_k}$ converges weak* in the sense of measures to a finite, nonnegative, Radon measure $\mu$. Let $U \subset \Omega$ be open. Then, obviously,

$$\mathcal{F}(U) \le \mu_\varepsilon(\overline{U}),$$

and passing to the weak* limit,

$$\mathcal{F}(U) \le \mu(\overline{U}).$$

Conversely, given $\varepsilon' > 0$, there exists a sequence $v_n$ such that $v_n \rightharpoonup u$ weakly in $W^{1,p}(U)$ and

$$\int_U F(x, v_n, \nabla v_n)\,dx \le \mathcal{F}(U) + \varepsilon'.$$

Then, for $k$ large enough, we have

$$\int_U (F(x, v_n, \nabla v_n) + \varepsilon_k|v_n|^p + \varepsilon_k|\nabla v_n|^p)\,dx \le \mathcal{F}(U) + 2\varepsilon',$$

thus

$$\mu_{\varepsilon_k}(U) \le \mathcal{F}(U) + 2\varepsilon'.$$

Passing to the weak* limit and letting $\varepsilon' \to 0$ we conclude the proof.

We show that (1.10) is a necessary and sufficient condition for strong representation. This will be a consequence of the following lemma.

**3.6. Lemma.** *Let $F$ be a Carathéodory function satisfying (1.7) and let $p, q, r$ verify (3.1), $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. Let $U$ be an open subset of $\Omega$. If $\mu$ is a Radon measure on $\overline{\Omega}$ weakly representing $\mathcal{F}^{q,p}(u, \cdot)$ then*

$$\mu(U) = \mathcal{F}^{q,p}(u, U),$$

*provided*

$$\inf_K \{\mathcal{F}^{q,p}(u, U \setminus K) \colon K \subset U \text{ is compact}\} = 0. \tag{3.7}$$

*Proof.* We need to establish the inequality $\mathcal{F}^{q,p}(u, U) \le \mu(U)$. Fix $\varepsilon > 0$ and, by virtue of (3.7), let $K \subset U$ be a compact set such that

$$\mathcal{F}^{q,p}(u, U \setminus K) < \varepsilon.$$

Choosing an open set $W$ such that $K \subset W \subset\subset U$, by Lemma 3.4 we have

$$\begin{aligned}\mathcal{F}^{q,p}(u, U) &\le \mathcal{F}^{q,p}(u, W) + \mathcal{F}^{q,p}(u, U \setminus K)\\ &\le \mathcal{F}^{q,p}(u, W) + \varepsilon\\ &\le \mu(\overline{W}) + \varepsilon\\ &\le \mu(U) + \varepsilon,\end{aligned}$$

and this concludes the proof.

**3.7. Corollary.** *Let $F$ be a Carathéodory function satisfying (1.7) and let $p, q, r$ verify (3.1), $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. If $\mu$ is a finite Radon measure on $\overline{\Omega}$ weakly representing $\mathcal{F}^{q,p}(u, \cdot)$, then $\mu$ represents $\mathcal{F}^{q,p}(u, \cdot)$ if and only if (1.10) is satisfied.*

*Proof.* If (1.10) is satisfied, then clearly (3.7) holds for any open set $U \subset \Omega$. Thus, by Lemma 3.6, $\mu$ represents $\mathcal{F}^{q,p}(u, \cdot)$. The converse implication is trivial.

**3.8. Remark.** If $u \in W^{1,q}(\Omega, \mathbb{R}^d)$, then the hypotheses of Corollary 3.7 are fulfilled by setting

$$\nu(U) := \int_U F(x, u, \nabla u)\, dx.$$

As we will see in Corollary 4.5, in this case, and if $F$ does not depend on $x$ and $\zeta$, we have $\mu = QF(\nabla u)\mathcal{L}^N$, and, in particular,

$$\mathcal{F}^{q,p}(u, \Omega) = \int_\Omega QF(\nabla u)\, dx.$$

We conclude this section proving that $\mathcal{F}^{q,p}_{\mathrm{loc}}(u, \Omega)$ admits always a measure representation.

*Proof of Theorem 3.1.* Assume, in addition, that the coercivity condition (3.4) is satisfied. As in the proof of Theorem 3.2, we find a Radon measure $\lambda$ on $\overline{\Omega}$ such that

$$\lambda(U) \le \mathcal{F}^{q,p}_{\mathrm{loc}}(u, U) \le \lambda(\overline{U})$$

for every open set $U \subset \Omega$. Given an open set $U \subset \Omega$, we are going to show that

$$\lambda(U) \ge \mathcal{F}^{q,p}_{\mathrm{loc}}(u, U).$$

Consider an increasing sequence of open, bounded, smooth sets $U_h \subset\subset U$, $h \in \mathbb{N}$, such that $\overline{U}_h \subset U_{h+1}$ for all $h$ and $U = \bigcup_{h=1}^{\infty} U_h$. By the definition of relaxed energy, for $h \ge 3$ there exists a sequence $u_{h,n} \in W^{1,q}_{\mathrm{loc}}(U_h \setminus \overline{U}_{h-2}, \mathbb{R}^d)$ such that

$$u_{h,n} \underset{n\to\infty}{\rightharpoonup} u \text{ weakly in } W^{1,p}(U_h \setminus \overline{U}_{h-2}, \mathbb{R}^d),$$

and

$$\int_{U_h \setminus \overline{U}_{h-2}} F(x, u_{h,n}, \nabla u_{h,n}) \le \mathcal{F}^{q,p}_{\mathrm{loc}}(u, U_h \setminus \overline{U}_{h-2}) + 2^{-h}. \tag{3.8}$$

Fix positive integers $\alpha_h$, to be determined later in the proof, and after extracting a subsequence from $u_{h,n}$ (still denoted by $u_{h,n}$), we may assume that $u_{h,n} \underset{n\to\infty}{\to} u$ a.e. in $U_h \setminus \overline{U}_{h-2}$ and

$$\|u_{h,n} - u\|_{L^p(U_h \setminus \overline{U}_{h-2})} \le 2^{-h-n}\alpha_h^{-1}.$$

We make use of Lemma 2.4 to connect $u_{h,n}$ to $u_{h+1,n}$ across $U_h \setminus \overline{U}_{h-1}$. There exist open sets $V^+_{h,n}, V^-_{h+1,n}$ such that $V^+_{h,n} \subset U_h \setminus \overline{U}_{h-2}$, $V^-_{h+1,n} \subset U_{h+1} \setminus \overline{U}_{h-1}$, $U_{h+1} \setminus \overline{U}_{h-2} = V^+_{h,n} \cup V^-_{h+1,n}$,

$$\mathcal{L}^N(V^+_{h,n} \cap V^-_{h+1,n}) \leq C_h 2^{-h-n}\alpha_h^{-1},$$

and there exist functions $z_{h,n} \in W^{1,q}(U_{h+1,n} \setminus \overline{U}_{h-1}, \mathbb{R}^d)$ such that $z_{h,n} = u_{h,n}$ in $(U_h \setminus \overline{U}_{h-1}) \setminus V^-_{h+1,n}$, $z_{h,n} = u_{h+1,n}$ in $(U_{h+1} \setminus \overline{U}_{h-1}) \setminus V^+_{h,n}$, and

$$\begin{aligned}\int_{V^+_{h,n}\cap V^-_{h+1,n}} F(x, z_{h,n}, \nabla z_{h,n})\,dx &\leq C\int_{V^+_{h,n}\cap V^-_{h+1,n}} (1 + |z_{h,n}|^r + |\nabla z_{h,n}|^q)\,dx \\ &\leq C_h\alpha_h^{-\tau p}2^{-p\tau\,(n+h)},\end{aligned}$$

where $\tau$ is as in Lemma 2.4 and $C_h$ depends on $h$. Now we specify the choice of $\alpha_h$ so that $\alpha_h^{-\tau q}C_h \leq 1$. Let $z_n \in W^{1,q}_{\mathrm{loc}}(\Omega \setminus U_1, \mathbb{R}^d)$ be given by $z_n = z_{h,n}$ on $V^+_{h,n} \cap V^-_{h+1,n}$, $z_n = u_{h+1,n}$ on $(U_{h+1} \setminus U_{h-1}) \setminus (V^+_{h,n} \cup V^-_{h+2,n})$. Fix $k \in \mathbb{N}, k \geq 2$. We have

$$\begin{aligned}\int_{\Omega\setminus\overline{U}_k} F(x, z_n, \nabla z_n)\,dx &\leq \sum_{h=k+1}^{\infty}\int_{U_h\setminus\overline{U}_{h-1}} F(x, z_n, \nabla z_n)dx \\ &\leq \sum_{h=k+1}^{\infty}\left\{\int_{U_{h+1}\setminus\overline{U}_{h-1}} F(x, u_{h+1,n}, \nabla u_{h+1,n})\,dx\right. \\ &\quad\left.+\int_{U_h\setminus\overline{U}_{h-1}} F(x, u_{h,n}, \nabla u_{h,n})\,dx + \int_{V^+_{h,n}\cap V^-_{h+1,n}} F(x, z_{h,n}, \nabla z_{h,n})\,dx\right\} \\ &\leq \sum_{h=k+1}^{\infty}(2\mathcal{F}^{q,p}_{\mathrm{loc}}(u, U_{h+1} \setminus \overline{U}_{h-1}) + 2^{-h+1}) + \sum_{h=k+1}^{\infty} 2^{-q\tau\,(n+h)} \\ &\leq \sum_{h=k+1}^{\infty} 2\lambda(U_{h+2} \setminus U_{h-1}) + 2^{-k-1} + C2^{-q\tau\,(n+k)} \\ &\leq 6\lambda(U \setminus U_{k-1}) + 2^{-k+1} + C2^{-q\tau\,(n+k)}.\end{aligned}$$

Due to (3.4), (3.8), and since $\mathcal{F}^{q,p}_{\mathrm{loc}}(u, \Omega) < \infty$, the sequence $z_n$ is bounded in $W^{1,p}(U \setminus \overline{U}_k)$, and, as in Lemma 3.4, we show that $z_n \to u$ weakly in $W^{1,p}(U \setminus \overline{U}_k)$. We infer that

$$\mathcal{F}^{q,p}_{\mathrm{loc}}(u, U \setminus \overline{U}_k) \leq 6\lambda(U \setminus U_{k-1}) + C2^{-k+1}.$$

Hence, (3.7) is verified and, by virtue of Lemma 3.6, we conclude that

$$\lambda(U) = \mathcal{F}^{q,p}_{\mathrm{loc}}(u, U).$$

Now, using the same argument as in the proof of Theorem 3.1, we remove the additional assumption (3.4).

**3.9. Remark.** The growth condition (1.7) can be further weakened. The constant 1 may be replaced by an integrable function. If $p > N - 1$, then, in view of the Sobolev imbedding theorem, the function $z$ in Lemma 2.4 is bounded. In this case, it is enough to assume, instead of (1.7), that

$$0 \leq F(x, \zeta, \xi) \leq c(|\zeta|)(1 + |\xi|^q)$$

for some increasing function $c$.

## 4. THE RELAXED ENERGY: A LOWER BOUND

**4.1. Theorem.** *Suppose that $q \geq 1$ and $p > q(N-1)/N$. Let $F$ be a quasiconvex function on $\mathbb{M}^{d\times N}$ satisfying (1.5). Let $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^d)$. Then*

$$\liminf_{n\to\infty} \int_\Omega F(\nabla u_n)\, dx \geq \int_\Omega F(\nabla u)\, dx.$$

*Proof.* The proof will be carried out in two steps.
*Step 1.* Suppose that $\Omega = B = B(0,1)$ and $u$ is linear, $u(x) = \xi_0 x$ for $\xi_0 \in \mathbb{M}^{d\times N}$. In view of Rellich's compact imbedding theorem, passing to a subsequence we may assume that

$$\|u_n - u\|_p \leq n^{-1}.$$

Let $R < 1$, and set $\rho := \frac{1+R}{2}$. We apply Lemma 2.4 to $v := u_n$, $w := u$, $V = \rho B$ and $W = B \setminus RB$ to obtain functions $z_n \in W^{1,q}(B, \mathbb{R}^d)$ and open sets $V_n \subset\subset V$ and $W_n \subset W$ such that $V_n \cup W_n = B$, $z_n = u_n$ on $B \setminus W_n$, $z_n = u$ on $B \setminus V_n$ and

$$\mathcal{L}^N(V_n \cap W_n) \leq \frac{C(R)}{n}$$
$$\int_{V_n\cap W_n} |\nabla z_n|^q\, dx \leq C(R)\, n^{-q\tau},$$

$\tau > 0$. Since $z_n - u \in W_0^{1,q}(B, \mathbb{R}^d)$, due to the growth condition (1.5) it is legitimate to test the quasiconvexity of $F$ with $z_n$ and we obtain

$$\int_B F(\nabla u)\, dx \leq \int_B F(\nabla z_n)\, dx.$$

It follows that

$$\begin{aligned}
&\int_B F(\nabla u)\, dx - \int_B F(\nabla u_n)\, dx \leq \int_B F(\nabla z_n) - F(\nabla u_n)\, dx \\
&= \int_{B\setminus V_n} (F(\nabla u) - F(\nabla u_n))\, dx + \int_{V_n\cap W_n} (F(\nabla z_n) - F(\nabla u_n))\, dx \\
&\leq \int_{B\setminus V_n} F(\nabla u)\, dx + \int_{V_n\cap W_n} F(\nabla z_n)\, dx \\
&\leq C\left(\mathcal{L}^N(B\setminus V_n) + \int_{V_n\cap W_n} (1 + |\nabla z_n|^q)\, dx\right) \\
&\leq C\left(\mathcal{L}^N(B\setminus RB) + C(R)\, n^{-q\tau}\right) \\
&\leq C\,(1-R) + C(R)\, n^{-q\tau}.
\end{aligned}$$

To conclude, it suffices to let first $n \to \infty$ and then $R \to 1$.
*Step 2.* Let $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, $u_n \in W^{1,q}(\Omega, \mathbb{R}^N)$, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^N)$. Without loss of generality, we may assume that

$$\sup_n \int_\Omega F(\nabla u_n)\, dx < \infty.$$

Passing, if necessary, to a subsequence, we obtain the existence of finite, Radon, nonnegative measures $\mu$ and $\nu$ such that

$$\liminf_{n\to\infty} \int_\Omega F(\nabla u_n)\,dx = \lim_{n\to\infty} \int_\Omega F(\nabla u_n)\,dx$$

and

$$\mu = w^*\text{-}\lim_{n\to\infty} F(\nabla u_n)\,\mathcal{L}^N\,,$$
$$\nu = w^*\text{-}\lim_{n\to\infty} |\nabla u_n|^p\,\mathcal{L}^N\,.$$

We are going to show that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0+} \frac{\mu(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon)} \geq F(\nabla u(x_0)) \tag{4.1}$$

holds true for almost every $x_0 \in \Omega$. Assuming that (4.1) is verified, for any $\varphi \in \mathcal{C}_c(\Omega)$, $0 \leq \varphi \leq 1$, we have

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega F(\nabla u_n)\,dx &\geq \lim_{n\to\infty} \int_\Omega \varphi F(\nabla u_n)\,dx \\ &= \int_\Omega \varphi\,d\mu \\ &\geq \int_\Omega \varphi \frac{d\mu}{d\mathcal{L}^N}\,dx \\ &\geq \int_\Omega \varphi\,F(\nabla u)\,dx.\end{aligned}$$

It suffices to let $\varphi$ to converge increasingly to 1 and to apply Lebesgue's monotone convergence theorem, to conclude that

$$\lim_{n\to\infty} \int_\Omega F(\nabla u_n)\,dx \geq \int_\Omega F(\nabla u)\,dx.$$

It remains to prove (4.1). To this end, we consider $x_0 \in \Omega$ such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0+} \frac{\mu(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))} \text{ exists and is finite,} \tag{4.2}$$

$$\frac{d\nu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0+} \frac{\nu(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))} \text{ exists and is finite} \tag{4.3}$$

and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \fint_{B(x,\varepsilon)} \big|u(y) - u(x_0) - \nabla u(x_0)(y - x_0)\big|\,dy = 0. \tag{4.4}$$

We select $\varepsilon_k \to 0^+$ such that $\mu(\partial B(x_0, \varepsilon_k)) = 0, \nu(\partial B(x_0, \varepsilon_k)) = 0$. It is well known that conditions (4.2), (4.3) and (4.4) are satisfied by all points $x_0 \in \Omega$, except maybe on a set of $\mathcal{L}^N$ measure zero. Then

$$\lim_{k\to\infty} \frac{\mu(B(x_0, \varepsilon_k))}{\mathcal{L}^N(B(x_0, \varepsilon_k))} = \lim_{k\to\infty} \lim_{n\to\infty} \fint_{B(x_0,\varepsilon_k)} F(\nabla u_n(x))\, dx$$

$$= \lim_{k\to\infty} \lim_{n\to\infty} \fint_{B(0,1)} F(\nabla u_{n,k}(y))\, dy,$$

where

$$u_{n,k}(y) := \frac{u_n(x_0 + \varepsilon_k y) - u(x_0)}{\varepsilon_k}.$$

Then $u_{n,k} \in W^{1,q}(B(0,1), \mathbb{R}^d)$,

$$\lim_{k\to\infty} \lim_{n\to\infty} \|u_{n,k} - u_0\|_{L^1(B(0,1))} = 0$$

and

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|\nabla u_{n,k}\|_{L^p(B)} \le \frac{d\nu}{d\mathcal{L}^N}(x_0) < \infty,$$

where $u_0(x) := \nabla u(x_0)x$. Hence, we may extract a subsequence $v_k = u_{n_k,k}$ such that (passing, if necessary, to a subsequence) $v_k \rightharpoonup u_0$ weakly in $W^{1,p}(B(0,1), \mathbb{R}^d)$, and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty} \fint_{B(0,1)} F(\nabla v_k(y))\, dy.$$

From Step 1, we deduce that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty} \fint_{B(0,1)} F(\nabla v_k(y))\, dy \ge F(\nabla u(x_0)).$$

This shows (4.1), and thus it concludes the proof.

**4.2. Corollary.** *Suppose that $q \ge 1$ and $p > q(N-1)/N$. Let $F$ be a function on $\mathbb{M}^{d\times N}$ satisfying (1.5), and let $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. Then*

$$\mathcal{F}^{q,p}(u, \Omega) \ge \int_\Omega QF(\nabla u)\, dx.$$

*Proof.* Since $0 \le QF(\xi) \le F(\xi)$, it follows that $QF$ satisfies the growth condition (1.5), i.e.

$$0 \le QF(\xi) \le C(1 + |\xi|^q).$$

Hence, if $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$ and if $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^d)$, then by Theorem 4.1

$$\liminf_{n\to\infty} \int_\Omega F(\nabla u_n)\, dx \ge \liminf_{n\to\infty} \int_\Omega QF(\nabla u_n)\, dx$$

$$\ge \int_\Omega QF(\nabla u)\, dx.$$

Taking the infimum over all such sequences we obtain

$$\mathcal{F}^{q,p}(u) \ge \int_\Omega QF(\nabla u)\, dx$$

as required.

**4.3. Remark.** It is easy to verify that the blow-up argument of Theorem 4.1, Step 2, can be used to prove that

$$\mathcal{F}^{q,p}(u,\Omega) \geq \int_\Omega QF(x,u,\nabla u)\,dx, \tag{4.5}$$

in the case where $F(x,\zeta,\xi) = a(x)g(\xi)$, $a$ is nonnegative, continuous, and $g$ satisfies (1.5). The generalization of (4.5) to more general energy density functions $F = F(x,\zeta,\xi)$ can be obtained under some smallness assumptions on

$$|F(x,\zeta,\xi) - F(x',\zeta',\xi)|.$$

However, these conditions are far from being 'natural'. By analogy with the case where $p \geq q$, we consider to be 'natural' those conditions of the form

$$|F(x,\zeta,\xi) - F(x',\zeta',\xi)| \leq \omega(|x-x'| + |\zeta-\zeta'|)(1+|\xi|^q),$$

where $\omega$ is a bounded modulus of continuity. The latter ensures (4.5) if $p \geq q$.

We recall that Gangbo [13] proved that lower semicontinuity holds when $d = N = q$, $p > N-1$, $F(x,\zeta,\xi) = a(x,\zeta)g(x,\xi)$, $a$ is continuous, nonnegative and bounded away from zero, $g$ is continuous, and $g(x,\cdot)$ is a polyconvex function for all $x \in \Omega$. In that same paper, Gangbo used heavily the fact that $d = N$, without which lower semicontinuity may fail (see Example 4.4). In addition, he showed that the continuity of the integrand function is an important feature. Indeed, he exhibited an example where $F(x,\xi) = \chi_K(x)\det\xi$, $\chi_K$ is the characteristic function of a compact set $K$, and where, given $N-1 < p < N$,

$$u \mapsto \int_K |\det \nabla u|\,dx$$

is lower semicontinuous in $W^{1,p}(\Omega,\mathbb{R}^N)$ if and only if $\mathcal{L}^N(\partial K) = 0$.

**4.4. Example.** This example is similar to examples by Ball and Murat [4] and Malý [15].

Here $Q$ denotes the cube $(-1,1)^N$ in $\mathbb{R}^N$, and $N-1 < p < N$. Let $u_n : Q \to \mathbb{R}^{N+1}$ be a $2/n$-periodic function, given by $u_n(x) := (\varphi_n(|x|)x, \psi_n(|x|))$ if $x \in [-1/n,1/n]^N$, where

$$\varphi_n(r) := \begin{cases} 1, & r \geq n^{-1}, \\ 1/(nr), & n^{-k} < r < n^{-1}, \\ n^{k-1}, & r \geq n^{-k}, \end{cases}$$

and

$$\psi_n(r) := \begin{cases} 1, & r \geq n^{-k+1} \\ k + \frac{\ln r}{\ln n}, & n^{-k} < r < n^{-k+1}, \\ 0, & r \leq n^{-k}. \end{cases}$$

The integer $k$ is fixed so that $\{u_n\}$ remains bounded in $W^{1,p}(\Omega,\mathbb{R}^{N+1})$, precisely $(k-1)(N-p) > N$. Then $u_n$ are Lipschitz-continuous, and $u_n \rightharpoonup u := (x,1)$ in $W^{1,p}(\Omega,\mathbb{R}^{N+1})$.

Now, define $F : \mathbb{R}^{N+1} \times \mathbb{M}^{(N+1)\times N} \to [0, +\infty)$ by

$$F(\zeta, \xi) := b(\zeta_{N+1})\, |\det(\xi_{i,j})_{i,j=1,\dots,N}|\,,$$

with

$$b(t) := \frac{1+3t^2}{1+t^2}.$$

As it turns out,

$$\begin{aligned}\int_\Omega F(u, \nabla u)\, dx &= 2^N\, b(1)\\ &> 2^N\, b(1) + \alpha_N\, (b(0) - b(1)) = \liminf_{n\to+\infty} \int_\Omega F(u_n, \nabla u_n)\, dx,\end{aligned}$$

where $\alpha_N$ denotes the volume of the unit ball in $\mathbb{R}^N$. Note that $a(\zeta) := b(\zeta_{N+1})$ is continuous, bounded away from zero, and

$$|F(\zeta, \xi) - F(\zeta', \xi)| \le \omega(|\zeta - \zeta'|)(1 + |\xi|^N),$$

where $\omega$ is a bounded modulus of continuity. It is well know that the latter condition provides $W^{1,N}(\Omega, \mathbb{R}^{N+1})$ weak lower semicontinuity.

**4.5. Corollary.** *Suppose that $q \ge 1$ and $p > q(N-1)/N$. Let $F$ be a function on $\mathbb{M}^{d\times N}$ satisfying (1.5), and let $u \in W^{1,q}(\Omega, \mathbb{R}^d)$. Then*

$$\mathcal{F}^{q,p}(u, \Omega) = \int_\Omega QF(\nabla u)\, dx.$$

*Proof.* Since $u \in W^{1,q}(\Omega, \mathbb{R}^d)$ and (1.5) holds, the standard relaxation results apply (see [2, 7]). Thus, we may find a sequence of functions $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \int_\Omega F(\nabla u_n)\, dx \to \int_\Omega QF(\nabla u)\, dx,$$

and so

$$\mathcal{F}^{q,p}(u, \Omega) \le \int_\Omega QF(\nabla u)\, dx,$$

which, together with Corollary 4.2, yields the desired representation.

**4.6. Remark.** Notice that if $QF$ is convex, i.e. $QF = F^{**}$ where $F^{**}$ denotes the lower convex envelope of $F$, then

$$\mathcal{F}^{q,p}(u, U) = \int_U QF(\nabla u)\, dx,$$

for every $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, for every $p \ge 1$, and every open set $U \subset\subset \Omega$.

The result is trivial in the case where $p \ge q$, since the standard relaxation theorems can be applied (see [7]).

Suppose now that $p < q$. If the sequence $\{u_n\} \subset W^{1,q}(\Omega, \mathbb{R}^d)$ converges to $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ in $W^{1,1}$-weak, since $F^{**}$ is convex we have

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega F(\nabla u_n)\, dx &\geq \lim_{n\to\infty} \int_\Omega F^{**}(\nabla u_n) \\ &\geq \int_\Omega F^{**}(\nabla u)\, dx.\end{aligned}$$

Conversely, consider a smooth kernel $\omega \geq 0$ in $\mathbb{R}^N$ with support on $\overline{B(0,1)}$, $\int_{\mathbb{R}^N} \omega(x) dx = 1$, and given $k \in \mathbb{N}$ we set $\omega_k(x) := k^N \omega(kx)$. For each $k \in \mathbb{N}$ select a sequence $v_{k,n} \in W^{1,q}(U, \mathbb{R}^d)$ such that

$$v_{k,n} \underset{n\to\infty}{\rightharpoonup} \omega_k * u \text{ in } W^{1,q}(U, \mathbb{R}^d) \text{ weak}$$

and

$$\lim_{n\to\infty} \int_U F(\nabla v_{k,n}) dx = \int_U QF(\nabla(\omega_k * u)) dx.$$

As $p < q$ we may extract a diagonal subsequence $u_k := v_{k,n(k)}$ such that

$$\begin{aligned}&\|u_k - \omega_k * u\|_{W^{1,p}} \leq \frac{1}{k}, \\ &\left| \int_U F(\nabla u_k) dx - \int_U F(\nabla(\omega_k * u)) dx \right| \leq \frac{1}{k}.\end{aligned}$$

Therefore $u_k \to u$ in $W^{1,p}(U, \mathbb{R}^d)$, and

$$\begin{aligned}\mathcal{F}^{q,p}(u, U) &\leq \liminf_{k\to\infty} \int_U F(\nabla u_k) dx \\ &= \liminf_{k\to\infty} \int_U QF(\nabla(\omega_k * u)) dx.\end{aligned}$$

However, since $QF$ is convex and as the measure $\mu_x^k$, given by

$$< \mu_x^k, \varphi >:= \int_U \omega_k(x - y) \varphi(y)\, dy,$$

is a probability measure, using Jensen's inequality we have

$$\begin{aligned}\liminf_{k\to\infty} \int_U QF(\nabla(\omega_k * u)) dx &= \liminf_{k\to\infty} \int_U QF\left(< \mu_x^k, \nabla u >\right) dx \\ &\leq \liminf_{k\to\infty} \int_U < \mu_x^k, QF(\nabla u) > dx \\ &= \int_U QF(\nabla u(x)) dx.\end{aligned}$$

We conclude that

$$\mathcal{F}^{q,p}(u, U) = \int_U QF(\nabla u) dx.$$

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