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Maximum Entropy States and Coherent Structures in Two-Dimensional Microtearing Turbulence

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(I). Introduction

One of the most striking features of many turbulent fluid and plasma systems is the emergence and persistence of large-scale organized states, or coherent structures, amidst the small-scale turbulent fluctuations. In large Reynolds number two-dimensional hydrodynamics, a coherent state typically appears in the form of a large-scale mean vortical flow, while the vorticity field itself develops fine-scale spatial oscillations [1,2,3]. In slightly dissipative two-dimensional magnetofluids, the magnetic field and velocity field fluctuate wildly on small scales, while coherent structures emerge in the form of macroscopic magnetic and kinetic islands [4,5,6]. In this paper, we will discuss another turbulent system, the two-dimensional microtearing system of plasma physics, which has the tendency to develop persistent organized macroscopic states in the midst of microscopic disorder. This behavior of the microtearing system has been demonstrated by the numerical simulations of Craddock *et al.* [7].

In this note, we propose a model that predicts the properties of coherent structures in the two-dimensional microtearing system. This feature of the turbulence is naturally modeled as a statistical equilibrium phenomenon. Our approach is information-theoretic in spirit [8,9], and is very much in accord with the Jaynesian theory of statistical mechanics and statistical inference [8]. In characterizing the relaxation of the system into a coherent state, we appeal to the general principle that entropy is maximized subject to constraints imposed by the underlying dynamics. These constraints are dictated by the global conserved quantities associated with the ideal (i.e., nondissipative) microtearing dynamics. By solving the constrained maximum entropy problem, we obtain a most probable macrostate, which quantifies both the large-scale mean field and mean electron density, and the small-scale fluctuations present in the turbulent relaxed state.

The essence of our approach is to introduce a macroscopic description of the turbulent state of the microtearing system. A macrostate is taken to be a local joint probability distribution, or Young measure [10,11], p_x on the values of the magnetic field and the electron density at each point x in the spatial domain D. In other words, at a given point x in D, we no longer have well determined values of the magnetic field and the electron density, but only a probability distribution on the possible values. Because of the extremely intricate small-scale behavior of the system, this macroscopic description is intuitively appealing. Further justification for the introduction of this macroscopic description will be given below.

Our model for coherent structures in the microtearing system is largely motivated by the recent statistical theories of Robert *et al.* [12,13] and Miller *et al* [14,15] of coherent structures in ideal two-dimensional hydrodynamics. In these theories, a macrostates is taken to be a local probability distribution, or Young measure, on the values of the fluctuating scalar vorticity. In the Robert theory, the most probable macrostate is determined as a maximizer of an appropriate entropy functional subject to constraints imposed by the conservation of energy, and the infinite family of generalized enstrophy integrals under the Euler dynamics. The model of Miller *et al.*, which is based on a canonical Gibbs ensemble theory and also incorporates the complete family of integrals conserved by the two-dimensional Euler dynamics, is conceptually different from the Robert approach, but yields identical predictions.

We have applied similar techniques to model coherent structures in two-dimensional magnetohydrodynamic turbulence [16,17,18]. Other recently proposed models of coherent structures in magnetohydrodynamics have been developed by Gruzinov and Isichenko [19,20] and Kinney *et al.* [21].

(II). The two-dimensional microtearing system

The model equations for ideal two-dimensional microtearing relaxation can be expressed neatly in appropriately normalized variables as

$$\frac{\partial A}{\partial t}=\partial(N,A), \tag{1}$$

$$\frac{\partial N}{\partial t}=\partial(J,A). \tag{2}$$

Here, A is the magnetic flux function, or vector potential, N is the electron density, and J is the current density defined by $J=-\Delta A$. The Poisson bracket $\partial(\cdot,\cdot)$ is given by

$$\partial(f,g)=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \tag{3}$$

These equations are assumed to hold in a simply connected bounded spatial domain $D\subset\mathbb{R}^2$ with smooth boundary ∂D. For simplicity, we assume that the normal component of the magnetic field $B=(A_y,-A_x)$ vanishes on ∂D; that is

$$B.n|_{\partial D}=0, \qquad (4)$$

where n is the outwardly directed normal to the boundary. We also impose the boundary conditions

$$A=0,\ N=0 \ on\ \partial D. \qquad (5)$$

The system (1)-(2) describes relaxation in a finite pressure, nonelectrostatic isothermal two-dimensional plasma [7,22,23]. Equation (1) is simply an expression of Ohm's law, and Equation (2) corresponds to electron continuity. For a derivation of these equations, and discussion of their physical relevance, the reader is referred to [7,23]. It is of interest to note that the microtearing system is also a special case of electron magnetohydrodynamics, which is a three-dimensional evolution equation describing the dynamics of the large-scale magnetic field of a fast moving plasma [22,24], viz,

$$\frac{\partial B}{\partial t}=\nabla\times((\nabla\times B)\times B). \qquad (6)$$

Indeed, choosing $B=(A_y,-A_x,N)$ in Equation (6) formally leads to the Equations (1)-(2).

A classical solution of the ideal microtearing system conserves flux, cross-correlation, and energy. These quantities are given by, respectively,

$$F_i=\int_D f_i(A)\,dx, \tag{7}$$

$$H_i=\int_D Nf_i(A)\,dx, \tag{8}$$

$$E=\frac{1}{2}\int_D(|\nabla A|^2+N^2)\,dx=\frac{1}{2}\int_D(|B|^2+N^2)\,dx. \tag{9}$$

The functions f_i in (7) and (8) must satisfy certain regularity conditions, but are otherwise arbitrary. Thus, there is an infinite family of conserved flux integrals, and an infinite family of conserved cross-correlation integrals. That these functionals are invariant under the dynamics can be verified directly by differentiating with respect to time, and using the boundary conditions (4)-(5), the equations of motion (1)-(2), and properties of the Poisson bracket (3). To the best of our knowledge, these are the only invariants of the motion.

As these conserved quantities play a central role in the statistical model developed below, we now briefly comment on their physical significance. The meaning of the energy functional is obvious. The flux and cross-correlation integrals are most readily interpreted by choosing $f_i(s)=1$ for $s>\sigma_i$, $f_i(s)=0$ otherwise, for an indexed family of constants σ_i. In the topologically trivial case in which the flux tubes form a regular nested family, F_i and H_i, are equal to the total area and total electron density within flux tube i. Thus, under the evolution (1)-(2), each flux tube distorts in a highly convoluted and intricate manner, while preserving its area and electron density.

Despite the constraints imposed by the conservation of energy, flux and cross-correlation, the state variable $U(x,t)=(B(x,t),N(x,t))$ may evolve in a very complex and irregular fashion, with the magnetic field and electron density characterized by intermittency and intricate small-scale fluctuations, much like the behavior of the magnetic field and the velocity field in two-dimensional magnetohydrodynamics [4,5,6], or the vorticity field in the two-dimensional Euler system [1,2,3]. The numerical simulations of Craddock *et al.* [7], which address a dissipative version of equations (1)-(2) in which the collisional resistivity is negligible compared with the cross-field particle diffusivity, clearly demonstrate the complicated behavior of the magnetic field. They find that starting from random initial conditions, the field evolves to a state consisting of long-lived isolated current filaments and small-scale fluctuations. The evolution is further characterized by a cascade of magnetic flux to large scales and a cascade of magnetic energy to small scales. Under the ideal dynamics (or in the situation in which the collisional resistivity and the cross-field diffusivity are of the same order of magnitude, but extremely small), we expect that the electron density would also exhibit intricate small-scale fluctuations together with persistent large-scale organized states. The model developed below attempts to capture, at least partially, both the intricate small-scale oscillations and the macroscopic coherent structures that are present in this turbulent relaxed state.

III. Microscopic and macroscopic descriptions

The key idea behind the statistical model is to introduce a macroscopic description of the microtearing system. The state variable $U=(B,N)$, which evolves according the equations (1)-(2), is viewed as a microscopic description of the system. Due to its tendency to display complicated behavior on increasingly small scales as time evolves, the microstate U does not furnish a useful description of the long-time behavior of the system. It is natural, therefore, to introduce a macroscopic, or coarse grained, description that represents the state of the system in a more suitable manner. Such a description is furnished by a local probability distribution, $(p_x)_{x\in D}$, on the values of fluctuating microstate $U(x)$ at each $x\epsilon D$. Thus, for each $x\epsilon D$, p_x is a probability measure on $\mathbb{R}^3$, the range of the values of $U(x)$. Intuitively, for any (Borel) set $A\subset\mathbb{R}^3$,

$$p_x(A) = \lim_{\ell\to 0}\frac{|x'\in N_\ell(x) : U(x')\in A|}{|N_\ell(x)|},$$

where $N_l(x)$ is a neighborhood of x in D, with diam $N_l(x)\leq l$, and $|S|$ denotes the volume (or Lebesque measure) of the set S. Thus, $p_x(A)$ represents the probability that (or frequency with which) a macrostate U takes values in the set A when sampled in an infinitesimal neighborhood of x. The macrostate p varies slowly with x, while U varies rapidly with x, and for any infinitesimal cell dx over which p is effectively constant, U behaves like a random variable with distribution p. The macrostate p has the advantage that it encodes only partially the infinitesimal scale fluctuations of the microstate, as it ignores the extremely complex local arrangements realized by these fluctuations. The family of local probability distribution $(p_x)_{x\in D}$ is referred to in the context of nonlinear analysis as a Young measure [10]. The Young measure arises naturally in the study of fine structures in crystalline solids and other materials with order [25,26]. As mentioned in the introduction, Robert *et al.* [12,13] and Miller *et al.* [14,15] utilized the Young measure to characterize coherent vortex structures in two-dimensional Euler turbulence.

There is a natural description of the system by Young measures, at least for finite time. Indeed, the trivial macrostates

$$p_X^t(\cdot)=\delta_{U(x,t)}(\cdot), \qquad (10)$$

provide such a description. However, the macroscopic description is really intended to capture the long-time behavior of the system. Thus, a macrostate $p=\{p_x\}_{x\in D}$ can be conceptualized as a possible limit as $t\to\infty$ of (perhaps a subsequence of) the Dirac masses p^t. It is not difficult to show that, because of the conservation of energy, the family of measures $\{p^t\}_{t\in(0,\infty)}$ is tight (or weakly relatively compact) [27], so that any sequence of the $\{p^t\}$ has a weakly convergent subsequence. That is, for any bounded continuous function $g(x,u)$, and for any sequence of the p^t on $D\times\mathbb{R}^3$, we have (at least for a subsequence)

$$\lim_{t\to\infty}\int_D g(x,U(x,t))\,dx=\lim_{t\to\infty}\int_{D\times\mathbb{R}^3} g(x,u)\,dp_x^t(u)\,dx$$
$$=\int_D\int_{\mathbb{R}^3} g(x,u)\,dp_x(u)\,dx \quad , \qquad (11)$$

Note that the Young measures are closed in the space of bounded Radon measures on $D\times\mathbb{R}^3$ with respect to weak convergence [11,12]. Thus, any weak limit must be a Young measure. These theoretical considerations justify, at least partially, the introduction of the parameterized measure $(p_x)_{x\in D}$ as a description of the long-time behavior of the microtearing system.

IV. Admissible macrostates and the maximum entropy principle

The conservation of energy, flux, and cross-correlation impose important constraints on the state variable U, as was discussed in Section II. Indeed, as the behavior of the system becomes increasingly complicated with time, the invariance of the functionals (7)-(9) furnishes perhaps the only tangible information about the system after some period of time. The invariance of these functionals translates under the weak convergence into corresponding constraints on the possible long-time macroscopic states p, as we shall now demonstrate.

For a parameterized measure $p=\{p_x\}_{x\in D}$, we define the functionals

$$E(p)=\frac{1}{2}\int_D\int_{\mathbb{R}^3}(b^2+n^2)\,dp_x(u)\,dx=\frac{1}{2}\int_{D\times\mathbb{R}^3}(b^2+n^2)\,dp, \qquad (12)$$

$$H_i(p)=\int_D \overline{N}(x)\,f_i(\overline{A}(x))\,dx, \qquad (13)$$

$$F_i(p)=\int_D f_i(\overline{A}(x))\,dx, \qquad (14)$$

where $u=(b,n)$ with $b\epsilon\mathbb{R}^2, n\epsilon\mathbb{R}$ running over the ranges of the magnetic field and electron density, respectively, and

$$\overline{N}(x)=\int_{\mathbb{R}^3} n\,dp_x(u), \qquad (15)$$

is the local mean electron density. We have also defined the local mean magnetic field:

$$\overline{B}(x)=\int_{\mathbb{R}^3} b\,dp_x(u). \qquad (16)$$

Since, for an arbitrary parameterized measure p_x, the corresponding $\overline{B}(x)$ need not be divergence free or satisfy the boundary condition $\overline{B}\cdot n|_{\partial D}=0$, some care must be taken in defining the local mean flux function $\overline{A}(x)$. We note that if the "energy" $E(p)$ of the measure p is finite, then $\overline{B}(x)\epsilon L^2(D:\mathbb{R}^2)$. $\overline{A}(x)$ is then defined by the relationship

$$\overline{A}(x)=curl^{-1}(Proj_H\overline{B}(x)), \qquad (17)$$

where $Proj_H$ is the projection onto the closed subspace H of $L^2(D:\mathbb{R}^2)$, consisting of divergence free vector fields whose normal components vanish on ∂D [28], i.e.,

$$H=\{B\in L^2(D:\mathbb{R}^2):\nabla\cdot B=0,\,B\cdot n|_{\partial D}=0\}. \qquad (18)$$

The operator $curl^{-1}$ is a compact operator from H to the Sobolev space $H^1_0(D)$ [28,29]. Thus, $\bar{A}$ is the unique (up to a set of measure zero) flux function in $H^1_0(D)$ corresponding to the projection of $\bar{B}$ on H.

Now if a sequence (or subsequence) of the trivial Young measure $p^t_x = \delta_{U(x,t)}$ converges weakly in the sense of Equation (11) to a limit macrostate $\{p_x\}_{x \in D}$, then the limit necessarily satisfies

$$E(p) \le E^0, \tag{19}$$

$$F_i(p) = F_i^0, \tag{20}$$

$$H_i(p) = H_i^0, \tag{21}$$

where E°, F_i°, and H_i° are the constant values of the functionals (7)-(9) fixed by the initial data. Thus, we take as the admissible class W of macrostates all Young measure p on $D \times \mathbf{R}^3$ that satisfy constraints (19)-(21).

The forms of the constraints (19)-(21) on the macrostates are simple consequences of the definition of weak convergence given by (11), and the compactness of the operator $curl^{-1}$. The analysis given in [17] can easily be adapted to rigorously establish these constraints. The methods of [17] can also be modified to show that the constraint set

$$W = \{p \in M : E(p) \le E^0, F_i(p) = F_i^0, H_i(p) = H_i^0\},$$

is closed in the space of bounded Radon measures on $D \times \mathbf{R}^3$ with respect to the weak convergence defined by (11). Here, M is our notation for the space of Young measures on $D \times \mathbf{R}^3$. For more details on properties of M, the reader may consult [11], and the references therein.

The inequality in (19) may be a bit disturbing at first sight, as it represents a potential loss of information in shifting to the macroscopic description of the long-time behavior of the system. Indeed, for any finite time, we have

$$E^0=\frac{1}{2}\int_D(|B^2(x,t)|+N^2(x,t))\,dx=E(p^t)\,.$$

But under the weak convergence of p^t to p, we could very well have $E(p)<E^0$ (the difficulty is that the function $g(b,n)=(1/2)(|b|^2+n^2)$ is not bounded, so that the functional $E(p)$ defined by (12) is not continuous under weak convergence). In Section VI, we will show, however, that the "most probable" macrostate p actually satisfies equality in (19), and hence it contains all of the information afforded by the conservation of energy, flux, and cross-correlation under the ideal dynamics.

The constraints (20)-(21), which can also be expressed as

$$\int_D f_i(\overline{A}(x))\,dx=\int_D f_i(A^0(x))\,dx=F_i^0,$$
$$\int_D \overline{N}f_i(\overline{A}(x))\,dx=\int_D N_0 f_i(A^0(x))\,dx=H_i^0,$$

where A^0 and N_0 are the initial flux and electron density, have the important interpretation that the flux integrals and cross-correlation integrals are determined entirely by the mean field in the long-time limit.

V. The maximum entropy principle

Now that we have determined the class W of admissible macrostates, we seek to determine the most probable of these macrostates. To this end, we introduce the Kullback entropy of a macrostate p [11,12,13,30]:

$$K(p:\pi) = -\int_D \int_{\mathbb{R}^3} \log \frac{dp_x}{d\pi^0} dp_x(u)\, dx. \tag{22}$$

Here, π° is a (x-independent) probability measure on $\mathbb{R}^3$, $\pi = dx \otimes \pi^\circ$ is a spatially homogenous Young measure on $D \times \mathbb{R}^3$, and $dp_x/d\pi^\circ$ is the density (or Radon-Nikodym derivative) of p_x with respect to π°. If p_x is not absolutely continuous with respect to π° for almost every x, then $K(p:\pi)$ is set equal to $-\infty$. As an integral over $\mathbb{R}^3$, $K(p:\pi)$ has either of the standard interpretations as the (logarithm) of the number of microscopic realizations of the macrostate p or the uncertainty in the macrostate p_x [31]. The functional $-K(p:\pi)$ is a measure of the "statistical distance" from the macrostate p to the spatially homogeneous measure π [30]. The form of K as an integral over D implies that the fluctuation at two distinct parts in D are treated as independent. This implicit assumption is a hypothesis of the model, reflecting the supposed ergodicity of the local mixing of the turbulent microtearing system. The most probable macrostate is then found by maximizing $K(p:\pi)$ over the constraint set W, once an appropriate reference measure π° has been chosen.

The assumption of ergodicity in the dynamic evolution of the state variable $U(x,t)$ implies that, in the absence of information that constrains spatial variations of the macrostates, the most probable macrostate should be spatially homogeneous. It is evident that the constraints (20)-(21) on flux and cross-correlation do impose spatial structure on the admissible macrostates, whereas the energy constraint (19) does not. For these reasons we choose the reference measure π to be the macrostate whose local distribution is for all $x \in D$,

$$d\pi^0(u) = \frac{1}{(2\pi)^{3/2}\sigma^3}\exp\{-\frac{1}{2\sigma^2}(b^2+n^2)\}du,$$

with

$$\sigma = \left(\frac{2}{3}\frac{E^0}{|D|}\right)^{\frac{1}{2}}.$$

π is the most probable spatially homogeneous macrostate in the sense that it maximizes the Boltzmann-Gibbs-Shannon entropy functional [31]

$$K(p) = -\int_D\int_{\mathbb{R}^3}\log\frac{dp_x}{du}dp_x(u)\,dx,$$

subject to the constraint $E(p) \le E^\circ$. In fact, π actually satisfies $E(\pi)=E^\circ$. Thus, in the absence of constraints on cross-correlation and flux the most probable macrostate is a spatially homogeneous Gaussian measure with mean 0 and with energy equal to the initial value of energy. To determine the most probable macrostate that satisfies all of the constraints (19)-(21), we solve the constrained maximum entropy problem:

$$K(p:\pi) \to max \quad subject\ to\ p \in W. \qquad (MEP)$$

The maximum entropy principle (MEP) selects those admissible macrostates that minimize the statistical distance to the homogeneous measure π. Unlike π, a measure which solves (MEP) exhibits spatial variations characteristic of a coherent structure.

While it may be conventional wisdom that maximizers of entropy are in some sense most probable, the sense in which solutions to (MEP) are the most probable elements in W can be made more precise by appealing to Robert's Concentration Theorem [11,12,13], which is a convenient restatement of results from the theory of large deviations [30]. This program has been carried out by the author for the problem of characterizing coherent structures in two-dimensional magnetohydrodynamic turbulence [17], and the arguments given there can be easily modified to the present situation.

VI. Solutions of (MEP)

The functional $K(p:\pi)$ is upper semicontinuous with respect to the topology of weak convergence and has compact level sets [30]. Therefore, it achieves its maximum over the closed, nonempty set W. We calculate the maximizer(s) by applying the Kuhn-Tucker theory of constrained optimization with equality and inequality constraints [32]. To overcome the analytical difficulties associated with the infinite families of conserved flux and cross-correlation integrals, we choose an appropriate finite collection of basis functions $f_1, f_2, \ldots, f_m$ in the constraints (20)-(21). Technically, the f_i should be chosen so that the f_i' are linearly independent in order to apply the Kuhn-Tucker theory [32]. We also assume that $|f_i(s)| \le C|s|^q$, for some $q<\infty$. We could choose, for example, the basis functions used in [32]. We further assume that the initial energy E° is larger than E_{min}, where E_{min} is the minimum value of the energy functional

$$E(B,N) = \frac{1}{2}\int_D (|B^2| + N^2)\, dx, \tag{23}$$

consistent with the constraints

$$F_i = \int_D f_i(A)\, dx = F_i^0,\ i=1,\dots,m, \tag{24}$$

$$H_i = \int_D N f_i(A)\, dx = H_i^0,\ i=1,\dots,m. \tag{25}$$

That the variational problem $E(B,N)\to min$, subject to the constraints (24) and (25) has a solution (in $L^2(D:\mathbb{R}^2)\times L^2(D))$ can be established by the direct methods of the calculus of variations [33,34]. An efficient numerical algorithm to compute solutions to variational problems of this type, together with an associated Lagrange multiplier rule, is presented in [33].

Under the above assumption, there are admissible macrostates in W that have finite entropy (as will be demonstrated below), so that the entropy maximizer(s) must also have finite entropy (because $-\infty\le K(p:\pi)\le 0$). In this case, any maximizer $(p^*_\cdot)_{x\in D}$ has density $\rho^*(x,u)$ with respect to π°. For macrostates $(p_x)_{x\in D}$ that have densities $\rho(x,u)$ with respect to π°, the constraint functionals (19)-(21) and the entropy functional (22) can be expressed in terms of ρ as follows:

$$K(\rho)=-\int_D\int_{\mathbb{R}^3}\rho(x,u)\log\rho(x,u)\,d\pi^0(u)\,dx, \tag{26}$$

$$E(\rho)=\frac{1}{2}\int_D\int_{\mathbb{R}^3}(|b|^2+n^2)\,\rho(x,u)\,d\pi^0(u)\,dx\leq E^0, \tag{27}$$

$$F_i(\rho)=\int_D f_i(\overline{A})\,dx=F_i^0,\ i=1,\dots,m, \tag{28}$$

$$H_i(\rho)=\int_D \overline{N}f_i(\overline{A})\,dx=H_i^0,\ i=1,\dots,m, \tag{29}$$

$$\textit{where}\quad \overline{n}(x)=\int_{\mathbb{R}^3}n\rho(x,u)\,d\pi^0(u),$$

and

$$\overline{B}(x)=\int_{\mathbb{R}^3}b\rho(x,u)\,d\pi^0(u),$$

The most probable macrostate $p_x^*=\rho^*(x,u)d\pi^0(u)$ maximizes $K(\rho)$ over $L^1(\pi)$ subject to the constraints (27)-(29) and

$$\int_{\mathbb{R}^3}\rho^*(x,u)\,d\pi^0(u)=1,\ \ \rho^*(x,u)\geq 0\quad a.e.\ x\in D. \tag{30}$$

According to the Kuhn-Tucker rule [32], there exists Lagrange multipliers $\beta,\alpha_i,i=1,\dots,m,$ and $\zeta_i,i=1,\dots,m$ such that

$$K'(\rho^*)=\beta E'(\rho^*)+\sum\alpha_i F_i'(\rho^*)+\sum\zeta_i H_i'(\rho^*). \tag{31}$$

The multiplier β satisfies the side conditions

$$\beta\geq 0,\ \ \beta(E(p^*)-E^0)=0. \tag{32}$$

In particular, if $\beta>0$, then $E(\rho^*)=E^0$.

The functional derivatives appearing in (30) can be readily calculated, and the resulting equilibrium equation is

$$\rho^*(x,u)=Z^{-1}(x)\exp\{\frac{-\beta}{2}(b^2+n^2)-b\cdot\Sigma\alpha_i\, curl\, Gf_i'(\bar{A})-n\Sigma\zeta_i f_i(\bar{A})-b\cdot\Sigma\zeta i\, curl\, G(\bar{N}i_i'(\bar{A}))\},$$

where $Z(x)$ is the partition function which enforces the normalization constraint $(30)_1$, and G is the inverse of the operator $-\Delta$ on D corresponding to homogeneous boundary conditions. After a straightforward but tedious calculation, we arrive at the following expressions for the maximum entropy macrostate $p_x^*(du)=\rho^*(x,u)\pi^\circ(du)$:

$$p_x^*(du)=\frac{\exp\{-\frac{1}{2}\bar{\beta}(b-\bar{B}(x)^2\}}{2\pi/\bar{\beta}}\;\frac{\exp\{-\frac{1}{2}\bar{\beta}(n-\bar{N}(x))^2\}}{(2\pi/\bar{\beta})^{1/2}}du, \tag{33}$$

where

$$\bar{\beta}=\beta+\frac{3}{2}|D|(E^0)^{-1}, \tag{34}$$

$$\bar{B}(x)=-\frac{1}{\bar{\beta}}\Sigma\, curl\,(G(\alpha_i f_i'(\bar{A})+\zeta_i\bar{N}f_i'(\bar{A})), \tag{35}$$

$$\bar{N}(x)=-\frac{1}{\bar{\beta}}\Sigma\zeta_i f_i(\bar{A}). \tag{36}$$

By taking the curl of equation (35), we obtain the following expression for the mean flux function $\bar{A}$

$$-\Delta\bar{A}=-\frac{1}{\bar{\beta}}\Sigma(\alpha_i f_i'(\bar{A})+\zeta_i\bar{N}f_i'(\bar{A})). \tag{37}$$

Notice that the mean magnetic field $\bar{B}$ is divergence free, and that $(\bar{A},\bar{N})$ is a stationary solution of the ideal microtearing equations (1)-(2). In fact, $(\bar{B},\bar{N})$ is a critical point of the energy functional $E(B,N)$ given by equation (23) subject to the constraints (24)-(25) on cross-correlation and flux, as can be readily verified. We now prove the following important results.

(i). Any maximum entropy macrostate p^* satisfies $E(p^*)=E^\circ$. Consequently, p^* contains all of the information furnished by the conservation of energy, flux, and cross-correlation under the ideal microtearing dynamics.

(ii). The mean field and the mean electron density $(\overline{B}(x),\overline{N}(x))$ corresponding to a solution p^* of (MEP) is an absolute minimizer of the energy functional (23) subject to the constraints (24),(25). That is, $E(\overline{B},\overline{N})=E_{min}$.

To establish (i), we calculate the energy $E(p^*)$ of p^*. A direct substitution of (33) into (27) yields

$$E(p^*)=\frac{3}{2}|D|\beta^{-1}+\overline{E}\leq E^0, \tag{37}$$

where $\overline{E}\equiv E(\overline{B},\overline{N})$. As $0<\beta<\infty$ (by equation (34)), it follows from (37) that $E^\circ>\overline{E}$. Substituting (34) into (37) gives

$$\beta\geq\frac{3|D|}{2}\,\frac{\overline{E}}{E^0(E^0-\overline{E})},$$

Therefore, $\beta>0$ because $\overline{E}\geq E_{min}>0$. From (32), it follows that $E(p^*)=E^\circ$, as claimed.

As for assertion (ii) let us note that any macrostate of the form

$$p_x(du) = \frac{\exp\{-\frac{r}{2}[(b-B(x))^2+(n-N(x))^2]\}}{(2\pi/r)^{3/2}}du, \qquad (38)$$

with $B(x)$ and $N(x)$ satisfying the constraints (24)-(25), and satisfying $E(B,N)+(3/2)r^{-1}|D|=E^\circ$, $r>0$, is an element of the admissible class W. The entropy of such a macrostate can be calculated directly from either equation (26) or equation (22). The result is

$$K(p:\pi) = \frac{3}{2}|D|\log\left(\frac{E^0-E(B,N)}{E^0}\right). \qquad (39)$$

The macrostate p^* is of the from (38) (with $r=\bar{\beta}$, $(B,N)=(\bar{B},\bar{N})$, and $E(\bar{B},\bar{N})=\bar{E}$), and since p^* maximizes $K(p:\pi)$ over all p in W, it follows from (39) that $\bar{E}=E_{min}$, the minimum possible value of the energy (23) consistent with the constraints on flux (24) and cross-correlation (25).

We remark that, in general, the solution to (MEP) is not unique. In fact, the analysis of this section has shown that there are as many solutions to (MEP) as there are absolute minimizers of the energy functional (23) subject to the constraints (24)-(25) on flux and cross-correlation.

VII. Predictions of the model

The analysis of the previous section shows that the model predicts that, under the ideal dynamics, the microtearing system will evolve to a turbulent relaxed state consisting of a coherent macroscopic mean magnetic field and mean electron density coupled with turbulent local Gaussian fluctuations. The mean field-density

$(\bar{B}(x),\bar{N}(x))$ is a stationary solution of the evolution equations (1)-(2) of the microtearing system, and, in fact, $(\bar{B},\bar{N})$ minimizes the energy (23) subject to the constraints (24)-(25) on flux and cross-correlation. A closer inspection of the expression (33) for the maximum entropy macrostate p^*, reveals that the variance of each of the components $B_1(x), B_2(x), N(x)$ is $\bar{\beta}^{-1}$, which has been shown to be equal to $2(E^\circ - E_{min})/3|D|$ for each x in D. Thus the variance of each component is independent of x. Furthermore, the components $B_1(x), B_2(x)$ and $N(x)$ are statistically independent for each x in D.

While the flux and cross-correlation integrals are determined entirely by the mean field-density, the energy is divided into mean and fluctuating components. Indeed, this is demonstrated by equation (37). The contribution of the fluctuations to the energy is

$$\frac{3}{2}|D|\bar{\beta}^{-1} = E_0 - E_{min},$$

and the contribution of the mean field-density is, of course, $\bar{E}=E_{min}$. We might say that flux and cross-correlation are cascaded to large-scales, while energy is cascaded to small (infinitesimal) scales. In particular, these predictions are in agreement with the numerical simulations of Craddock *et al.* [7], which were discussed in Section II of this paper.

Two regimes are of particular interest. In the high energy regime $(E^\circ >> E_{min})$, the variance $2(E^\circ - E_{min})/3|D|$ of the distributions p_x^* is very large, so we expect that the mean field-density will be obscured by large fluctuations. The system will be highly turbulent. On the other hand, when $E^{\circ} \approx E_{min}$, the variance is close to zero, so that the fluctuations about the coherent mean field-density are very small. In this low-energy regime, therefore, we expect the system to relax to a quasistationary state consisting of clearly discernible large-scale organized structure, which minimizes the energy (23) subject to constraints on flux and cross-correlation (24)-(25).

While there is qualitative agreement of the predictions of the model with the numerical simulations of Craddock *et al.*, [7], it must be emphasized that those simulations addressed a particular dissipative version of the microtearing equations in which collisional resistivity is negligible compared with the cross-field particle diffusivity. In other words, Craddock *et al.* considered dynamics of the form

$$\frac{\partial A}{\partial t} = \partial(N,A) + \eta \Delta A, \quad \frac{\partial \eta}{\partial t} = \partial(J,A) + \nu \Delta n,$$

with $\eta << \nu$. Our model is intended to apply to the case when $\eta \approx \nu$ are both extremely small. Numerical simulations of the dynamics in this dissipative regime will be needed to fully test the predictions of the model.

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