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# A New Variational Method for Crystals with Defects

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## 1 Introduction

In this paper we follow a model of Del Piero and Owen [23] providing a continuum mechanics basis for the application of techniques in the calculus of variations to non-classical deformations of continua. These *structured* deformations are suitable for describing deformations of materials whose kinematics warrants analysis at both the macroscopic and microscopic levels.

The motivation for this work lies on the study of equilibrium configurations of crystals with defects. In a defective crystal, the macroscopic deformation together with the referential (Bravais) lattice configuration do not suffice to describe fully the configuration of a deformed body; phenomena such as slips, vacancies, and dislocations may be present in the deformed (Bravais) lattice basis, thus preventing the use of the Cauchy-Born hypothesis, as described below.

In a perfect crystal, it is postulated that the crystal lattice consists of identical atoms located at all positions vectors

$$x = m_1a_1 + m_2a_2 + m_3a_3,$$

where $a_i \in \mathbb{R}^3$ and $m_i \in \mathbb{Z}$. The $a_i$ are called *lattice vectors* and the matrix $L$, whose columns consist of the $a_i$, is referred to as the *lattice matrix* or *lattice basis*. At each point $x$ in the crystal there exists a tensor $L(x)$ representing average values over microscopic regions of lattice vectors which define locally the position of the atoms. The Cauchy-Born hypothesis (see Ericksen [24]) establishes the behavior of the lattice basis field under an elastic deformation, and it asserts that an orientation preserving map $u : \Omega \to \Omega^*$ leads to a new lattice basis $L^*$ given by

$$L^*(u(x)) = \nabla u(x)L(x) \qquad x \in \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ represents the referential position of the crystal, and $\Omega^* = u(\Omega)$ is the deformed configuration.

Suppose that we start with a perfect, cubic crystal whose lattice basis field is identically the identity matrix $\mathbb{I}$. It often happens that, after undergoing some deformation, a new lattice basis is observed which does not coincide with $\nabla u$ (see for example Hill [35]). This discrepancy is viewed as the creation of defects. In [17], [18], and [19], Davini and Parry proposed a continuum model for defectiveness and introduced the notion of defect preserving configurations. They studied pairs

$$(u(\Omega), \bar{L}(x)),$$

where $\bar{L}(x)$ stands for the matrix of lattice vectors at $u(x)$. A complete list of measures of defectiveness, including a generalization of the classical *Burger's vectors*, were given in [19]. These measures consist of line, surface, and bulk integrals of certain functionals depending on $\bar{L}(x)$ and on its spatial derivatives; as it turns out, these functionals agree on configurations which are elastically related, in the sense of (1.1). These measures of defectiveness partition the set of configurations, or equivalently, deformations, into equivalence classes, and the equivalence class containing the perfect cubic crystal $(\Omega, \mathbb{I})$ is called the class of *neutral deformations*. This class was found to be strictly larger than the class of elastic deformations $\{(u(\Omega), \nabla u(x))\}$ of a perfect cubic crystal. Indeed, the lattice basis field of a neutral deformation may include a "plastic" part, accounting for the discrepancy between $\bar{L}$ and $\nabla u$, in spite of the fact that the deformation is defect preserving. Fonseca and Parry [34] pursued this idea and found that neutral deformations may be represented as

$$(u(\Omega), \nabla u \{\nabla v\}^{-1}), \tag{1.2}$$

where $u$ and $v$ belong to some appropriate Sobolev spaces, and $\det \nabla v = 1$ a.e. The function $u$ is interpreted as the *macroscopic deformation* and $v$ as the *plastic* part of the deformation, or, simply, the *slip*. Within this framework, the use of variational principles on neutral deformations was undertaken in [34], under the assumption that, among all neutral deformations of a perfect cubic crystal $(\Omega, \mathbb{I})$, equilibria correspond to minima of some appropriate energy. The energy functional studied in [34] was given by

$$E(u,v) := \int_\Omega W\left(\nabla u(x)\{\nabla v(x)\}^{-1}\right) dx, \tag{1.3}$$

where the bulk energy density $W$ was the Helmholz free energy satisfying appropriate symmetry properties, as considered by Chipot and Kinderlehrer [15] in the study of nondefective elastic crystals. In [16], variational problems consisting of minimizing $E(u,v)$ over some appropriate subclass of neutral deformations were referred to as *variation of the domain* since, formally, if $v$ is invertible then (1.3) can be written as

$$\int_{v(\Omega)} W(\nabla w(y))\, dy,$$

where $w := u \circ v^{-1}$. Several mathematical and physical difficulties were encountered within this model. The fact that Sobolev functions with nonzero Jacobians are not necessarily locally invertible prevented the use of the direct method of the calculus of variations; in addition, bounds on $L = \nabla u\{\nabla v\}^{-1}$ in no way imply bounds on $\nabla u$ and $\nabla v$. Moreover, lower semicontinuity with respect to an appropriate notion of weak convergence was established only under certain restrictive growth conditions on $W$ (see [30]). Using different analytical methods, Dacorogna and Fonseca [16] addressed the case where $W = |\cdot|^r$. Existence of minima was obtained for $r \geq N$, and for $r < N = 2$ it was shown that

$$\inf\left\{E(u,v) : u \in W^{1,\infty}, v \in W^{1,\infty}, u(x)|_{\partial\Omega} = x, \det \nabla v = 1 \text{ a.e. }\right\} = 0;$$

hence, the infimum is not attained in spite of the convexity of $W$. Note also that this model associates zero energy to a *rearrangement* of a natural state of the crystal, which is a particular type of neutral deformation where $u$ is invertible, $v = u$, thus $\bar{L} = \mathbb{I}$, and the lattice vectors retain their orientation. We take these results as an indication that the energy defined in (1.3) is "too low", in that it neglects terms which may account for the presence of microscopic slips.

In physical terms, Fonseca and Parry [34] studied stress in equilibrium configurations in the case where neutral deformations were admissible. Via the theory of Young measures, it was shown that certain symmetry properties of $W$ imply that the average stress associated with an infimizing sequence is a hydrostatic pressure; hence, the crystal is weak at equilibrium, since it cannot sustain non zero averaged shear stresses. This result had been previously obtained by Chipot and Kinderlehrer [15] in the case where only elastic deformations were allowed to compete (see, also, a similar result of Ericksen [25] for elastic crystals). When defective configurations are admissible, the latter result is regarded as an indication that frictional effects due to slips should be represented in the energy functional to be minimized (see Parry [37]). The question now is: how should we introduce an energy penalization due to slips, or to more general defects? Intuitively, we expect that the total energy should include a measure of the discrepancy

$$\int_\Omega |\bar{L}(x) - \nabla u(x)|\, dx,$$

or, more generally, some functional of $(\bar{L}(x) - \nabla u(x))$, depending on the interaction between $W$ and some surface energy associated with slips. Our goal in this paper is to obtain specific information on the total effective energy which incorporates bulk and surface terms accounting for slips. We provide a description of the energy functional that should be minimized, in the hope that this information will help determining the (meta)stable states, or (local) minimizers (see Corollary 5.4, Proposition 5.6, and the subsequent discussion after it for some partial results in this direction).

In recent years much attention has been given to variational methods addressing discontinuous classes of functions with energies which include both bulk and interfacial terms. Consider the functional

$$E(u) := \int_\Omega W(\nabla u)\, dx + \int_{\text{crack site}} \psi([u], \nu)\, dH^{N-1}, \tag{1.4}$$

where $H^{N-1}$ denotes the $N-1$ dimensional Hausdorff measure, $\nu$ stands for the unit normal to the crack (jump discontinuity) site, and $[u]$ is the size, or amplitude, of the jump discontinuity. Here there is a direct penalization of jump discontinuities in $u$, and a to macroscopic slip is assigned a precise energy via the density $\psi$. Functionals having this form have been studied in relation to problems in fracture mechanics, phase transitions, image segmentation and pattern recognition (see for example, [21],[29]). In this paper, we discuss a mechanism for taking into account microscopic defects via limits of configurations with (small) interfaces which diffuse in the limit, disappear at the macroscopic level, and contribute in some way to the effective "bulk" energy. This approach lies on a model proposed by Del Piero and Owen [23], which we now outline.

The theory of Del Piero and Owen deals with three types of deformations. For simplicity, we take the reference configuration $\Omega$ to be a bounded, open subset of $\mathbb{R}^N$ and rephrase slightly the definitions in [23].

- *Simple deformations* are pairs $(K, g)$ where $K \subset \Omega$ consists of a finite union of Lipschitz sets of Hausdorff dimension $N-1$, and $g|_{\Omega\setminus K}$ is a one to one differentiable function. We set $\nabla g := (\nabla g|_{\Omega\setminus K}) \cdot \chi_{\Omega\setminus K}$.
- A triple $(K, g, G)$ is a *limit of simple deformations* if $K \subset \Omega$, $g \in L^\infty(\Omega, \mathbb{R}^N)$, $G \in L^\infty(\Omega, \mathbb{M}^{N\times N})$, and there exists a sequence of simple deformations $(K_n, f_n)$ such that

$$K := \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} K_n, \quad \lim_{n\to\infty} |g - f_n|_{L^\infty(\Omega,\mathbb{R}^N)} = 0, \quad \lim_{n\to\infty} |G - \nabla f_n|_{L^\infty(\Omega,\mathbb{M}^{N\times N})} = 0. \tag{1.5}$$

- A triple $(K, g, G)$ is a *structured deformation* if $(K, g)$ is a simple deformation, $G : \Omega\setminus K \to \mathbb{M}^{N\times N}$ is continuous and there exists $m > 0$ such that for all $x \in \Omega\setminus K$, $m < \det G(x) \le \det \nabla g(x)$.

Here, and in what follows, $\mathbb{M}^{d\times N}$ stands for the vector space of $d \times N$ matrices. One of the central results of the theory of Del Piero and Owen is that every structured deformation is a limit of simple deformations (see [23], Theorem 5.8). We give two simple examples from [23] illustrating the convergence in (1.5). First, we consider the so called *broken ramp* sequence. Let $N = 1$, $\Omega = (0,1)$, $K = \emptyset$, $g(x) = 2x$, and $G(x) = 1$. This structured deformation can be approximated by

$$f_n(x) := x + \frac{k}{n} \qquad \text{for } \frac{k}{n} \leq x < \frac{k+1}{n}, \qquad k = 0, \ldots\ldots, n-1,$$

because $f_n(x) \to 2x$ and $\nabla f_n(x) \to 1$ in $L^\infty(0,1)$; hence, $(0, 2, \mathbb{I})$ is a limit of simple deformations. In terms of the total distributional derivative, we have $Df_n \to Dg$ in the sense of distributions, and

$$Df_n = 1 + \sum_{k=1}^{n-1} \frac{1}{n} \delta_{\frac{k}{n}},$$

where $\delta_a$ is the Dirac mass at $x = a$. Thus, the part of $Df_n$ corresponding to jumps will converge, in the sense of distributions, to the difference between $G$ and $\nabla g$ which, in this case, is the constant function 1. Note the relation between $\sum_{k=1}^{n-1} \frac{1}{n} \delta_{\frac{k}{n}}$ and the Riemann sum for $f(x) = 1$.

The second example is particularly illuminating in the context of the microscopic slip mentioned in (1.2), and is referred to as the *deck of cards*. Let $N = 3$, $\Omega = (0,1)^3$, $K = \emptyset$, $g(x)$ is the simple shear $g(x) = g(x_1, x_2, x_3) = (x_1 + x_3, x_2, x_3)$, and $G(x) = \mathbb{I}$. An approximating sequence is given by

$$f_n(x) := \left(x_1 + \frac{k}{n}, x_2, x_3\right) \quad \text{for } \frac{k}{n} \leq x_3 < \frac{k+1}{n}, \quad k = 0, \ldots, n-1.$$

Within the framework adopted in [17], [18], [19], and [34], $(g, \mathbb{I})$ represents a particular type of *rearrangement* of the crystal, namely a *slip*, and it is a neutral deformation in the sense of (1.2), with $u = g$ and $v = g$. The notion of microscopic slip has the interpretation of a limit of decreasing displacements along glide planes which are diffusing throughout the body.

As the last example suggests, one may consider $g$ as the macroscopic deformation of a defective crystal with cubic symmetry, $K$ as the macroscopic crack site, and $G(x)$ as the referential description of the averaged lattice basis field in the deformed configuration. The constructions of Del Piero and Owen support the interpretation of $G\mathbf{e}_i$ ($\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ denotes the standard orthonormal basis in $\mathbb{R}^N$) as being a limit of averages of discrete lattice bases. To see this, approach a purely microscopic structured deformation $(\emptyset, id, G)$ ($id$ stands for the identity deformation) by simple deformations $(K_n, f_n)$ such that $\{f_n\}$ are piecewise affine, and so $\nabla f_n\, \mathbf{e}_i$ is interpreted as a set of discrete lattice bases for all atomic sites in the deformed state determined by $(K_n, f_n)$. Then, for every $x \in \Omega$,

$$\lim_{n\to\infty} \frac{\int_{B(x,(n+1)^{-1})} \nabla f_n(y)\, \mathbf{e}_i\, dy}{\mathcal{L}^N\left(B\left(x,(n+1)^{-1}\right)\right)} = G(x)\, \mathbf{e}_i, \tag{1.6}$$

where $B(x, a)$ denotes the ball with centre $x$ and radius $a$, and $\mathcal{L}^N$ is the $N$-dimensional Lebesgue measure. See Section 7c of [23] for details. In the phenomenalogical theories of plasticity (see [35],Si), $G$ corresponds to the elastic component $F^e$ of the total deformation gradient $\nabla g$, i.e., $G$ represents the deformation of the lattice basis. The well-known elastic-plastic decomposition takes the form

$$F = \nabla g = G(G^{-1}\nabla g) = F^e F^p,$$

where $F^p$ is the plastic component of the macroscopic gradient $F$.

In this paper we will consider a framework for structured deformations which will encompass the use of modern techniques in nonlinear analysis and the calculus of variations. In particular, the principal fields will be allowed to oscillate, which is in contrast with the notions of convergence considered in (1.5). We will work in the *space of functions of special bounded variation, SBV*, introduced by De Giorgi and Ambrosio in [20], and consisting of integrable functions $u$ whose distributional derivatives are Radon measures $\mu = \mu_a + \mu_s$, where $\mu_a$ is absolutely continuous with respect to $\mathcal{L}^N$ and $\mu_s$ is absolutely continuous with respect to N-1 dimensional Hausdorff measure $H^{N-1}$ restricted to the set where the function $u$ experiences jump discontinuities. We denote $\mu_a$ by $\nabla u\, \mathcal{L}^N$, $\nabla u$ being the Radon-Nikodym derivative of $Du$ with respect to $\mathcal{L}^N$. A *structured deformation* will be represented by a pair $(g, G)$, where the macroscopic deformation $g$ is in $SBV(\Omega, \mathbb{R}^d)$ and $G$ is an integrable tensor field in $\Omega$. A theorem of Alberti [1] allows us to recover the approximation theorem of Del Piero and Owen (Theorem 5.8 of [23]). That is (see Theorem 2.12), given any structured deformation $(g, G)$ there exist deformations $u_n$ in $SBV(\Omega, \mathbb{R}^d)$ such that

$$u_n \to g \text{ in } L^1 \qquad \text{and} \qquad \nabla u_n \overset{*}{\rightharpoonup} G \quad \text{in } \mathcal{M}(\Omega), \tag{1.7}$$

where $\mathcal{M}(\Omega)$ denotes the space of Radon measures on $\Omega$. Given the lack of information on the convergence of the jump set of $u_n$, this is a weaker statement than the theorem of Del Piero and Owen (see (1.5) and Theorem 5.8 in [23]). Assume, for simplicity, that $g \in W^{1,1}$, i.e, there are no macroscopic cracks. $Du_n$ consists of a part absolutely continuous with respect to Lebesgue measure, $\nabla u_n \mathcal{L}^N$, and a singular part, $J(u_n)$, which is supported on the jump set of $u_n$, denoted by $S(u_n)$. From (1.7) we have that $Du_n \to Dg$ in the sense of distributions; hence, $J(u_n) \to \nabla g - G$ in the sense of distributions, and so the difference between macro and microscopic bulk is achieved by a limit of singular measures. However, under certain additional conditions, a compactness theorem of Ambrosio [2] for $SBV$ guarantees that $\nabla g = G\, a.e.$ in $\Omega$, unless $H^{N-1}(S(u_n)) \to \infty$, i.e., there is a diffusion of cracks whose amplitude is tending to zero (see Remark 2.13 for details, also see Theorem 5.10 of [23]). This fact prevents $\psi$ from being bounded away from zero, if we are to consider $\nabla g \neq G$ on a set of positive measure, together with the convergence (1.7).

Given $u \in SBV$, we associate an energy functional of the form $E(u)$ introduced in (1.4). We define the energy of a structured deformation $(g, G)$ as the most economical way to build up the deformation using the approximations in $SBV$, i.e.,

$$I(g, G) := \inf_{\{u_n\}} \left\{ \liminf_{n \to \infty} E(u_n) : u_n \to (g, G) \text{ in the sense of (1.7)} \right\}. \tag{1.8}$$

Clearly, this class of admissible sequences includes the limits of simple deformations in the sense of (1.5) provided $g$ and $G$ are sufficiently smooth (see [23], Theorem 5.8).

The energy (1.8) is in relaxed form due to its own definition, and the first question we ask concerns the description of the resulting interaction between the initial interfacial and bulk densities, $\psi$ and $W$, appearing in $E$. For example, if the macroscopic deformation $g$ is smooth and $\nabla g \neq G$, as mentioned above this discrepancy is realized by the diffusion of jumps in the approximating sequences. Thus, the resulting energy should involve a new bulk density depending on $\nabla g$ and $G$, via some combination of the initial densities $W$ and $\psi$. Characterizing this new function amounts to finding an integral representation for $I$. Integral representations for similar relaxed energy functionals have been the focus of extensive research in the calculus of variations over the past decade, for example see [5], [7], [9], [12], [14], [32], [33]. In these cases, relaxation of $E$ is taken with respect to the $L^1$ ($BV$ weak) topology, whereas in our present situation we relax with respect to a more restrictive topology where gradients are constrained.

In the context of defective crystals, we interpret (1.7) and (1.8) as a means to realize the deformed crystal by piecing together elastic crystals at a finer and finer scale; that is, the creation of the non-trivial microstructure is achieved naturally by rearrangements within the crystal at a very fine scale. We expect that there will be an interfacial energy associated with this process, in addition to the bulk (Helmholz free) energy, and we prescribe that the overall energy of the deformation should be lowest among all such possible rearrangements which give rise to the same macroscopic and microscopic configuration. In this paper, we characterize this total energy. In doing so, we are not taking the particular view that the integral in (1.4) which contains $\psi$ corresponds to energy which is dissipated during the structured deformation, nor are we ruling out such an interpretation. The functional (1.8) is the energy associated with deforming the crystal, and it may be that energy corresponding to small interfaces is stored in the deformed configuration. For now, we leave open these possibilities.

It is well known that the bulk energy $W$ (the Helmholtz free energy) associated with a crystal may have potential wells (at matrices where $W$ vanishes) centered at matrices of a material symmetry (point) group (see [15], [24], [28]). Thus, it is desirable not to impose a coercivity condition on $W$ but only a growth condition, $0 \le W(A) \le C(1+|A|^p)$ for some $p \ge 1$, constant $C$, and for all $A \in \mathbb{M}^{d\times N}$. For $p > 1$, we require admissible sequences to satisfy $\nabla u_n \rightharpoonup G$ in $L^p$. This, of course, follows from (1.7) if $W$ is $p$-coercive, i.e., if there exists a constant $c$ such that $c|A|^p \le W(A)$ for all $A \in \mathbb{M}^{d\times N}$, and if $\lim_n E(u_n) < \infty$. On the other hand, if $p = 1$, and even under the coercivity hypothesis, the sequence $\{\nabla u_n\}$ may develop concentrations. To accomodate this fact, we will assume that $\nabla u_n \overset{*}{\rightharpoonup} m$, where $m \in \mathcal{M}(\Omega)$ and $\frac{dm}{d\mathcal{L}^N} = G$. Based on the above considerations, for $W : \mathbb{M}^{d\times N} \to [0,\infty)$ and $\psi : \mathbb{R}^d \times S^{N-1} \to [0,\infty)$ continuous functions, where $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, we consider the following energies:

$$I_0(g,G,\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \left[ \int_\Omega W(\nabla u_n)\,dx + \int_{S(u_n)\otimes} \psi([u_n],\nu_{u_n})\,dH^{N-1} \right] : u_n \in SBV(\Omega,\mathbb{R}^d), \right.$$
$$\left. u_n \to g \text{ in } L^1(\Omega,\mathbb{R}^d),\ \nabla u_n \overset{*}{\rightharpoonup} m,\ m \in \mathcal{M}(\Omega) \text{ and } \frac{dm}{d\mathcal{L}^N} = G \right\},$$

and for $p \ge 1$,

$$I_p(g,G,\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \left[ \int_\Omega W(\nabla u_n)\,dx + \int_{S(u_n)\cap\Omega} \psi([u_n],\nu_{u_n})\,dH^{N-1} \right] : u_n \in SBV(\Omega,\mathbb{R}^d), \right.$$
$$\left. u_n \to g \text{ in } L^1(\Omega,\mathbb{R}^d),\ \nabla u_n \overset{*}{\rightharpoonup} G,\ \sup_n |\nabla u_n|_{L^p(\Omega,\mathbb{M}^{d\times N})} < \infty \right\}.$$

The main results of this paper are the following (see Theorems 2.16 and 2.17). Assuming that the initial bulk density is Lipschitz, with $p = 1$, if $\psi$ has linear growth, is subadditive, and is homogeneous of degree 1, then the following integral representations for $I_1$ and $I_0$ hold:

$$I_1(g,G,\Omega) = \int_\Omega H_1(\nabla g(x), G(x))\,dx + \int_{S(g)\cap\Omega} h_1([g],\nu_g)\,dH^{N-1},$$

and

$$I_0(g,G,\Omega) = \int_\Omega H_1(\nabla g(x), G(x))\,dx + \mu_s(\Omega), \tag{1.9}$$

for some Radon measure $\mu_s$ absolutely continuous with respect to $\mathcal{H}^{N-1}\lfloor S(g)$. The new bulk and crack density are defined below. If $p > 1$, then, under some additional hypotheses, the following representation for $I_p$ holds:

$$I_p(g,G,\Omega) = \int_\Omega H_p(\nabla g(x), G(x))\,dx + \int_{S(g)} h([g])\,dH^{N-1}, \tag{1.10}$$

where, for $A, B \in \mathbb{M}^{d\times N}$, $\lambda \in \mathbb{R}^d$, $\nu \in S^{N-1}$,

$$H_p(A,B) := \inf_u \left\{ \int_Q W(\nabla u)\,dx + \int_{S(u)\cap Q} \psi([u],\nu)\,dH^{N-1} : u \in SBV(Q,\mathbb{R}^d), u|_{\partial Q} = Ax, \right.$$
$$\left. |\nabla u| \in L^p(Q),\ \int_Q \nabla u\,dx = B \right\},$$

$$h_1(\lambda,\nu) := \inf_u \left\{ \int_{Q_\nu} W^\infty(\nabla u)dx + \int_{S(u)\cap Q} \psi([u],\nu_u)dH^{N-1} : u \in SBV(Q_\nu,\mathbb{R}^d), u|_{\partial Q_\nu} = u_{\lambda,\nu}, \int_{Q_\nu} \nabla u dx = 0 \right\},$$

and

$$h(\lambda) := \inf_u \left\{ \int_{S(u)\cap Q} \psi([u],\nu_u)dH^{N-1} : u \in SBV(Q,\mathbb{R}^D), u|_{\partial Q} = u_{\lambda,e_N}, \nabla u(x) = 0\,\mathcal{L}^N a.e. \right\}.$$

Here $\nu$ denotes the normal to the jump set $S(u)$, $e_N$ is the standard basis vector $(0,\ldots 0,1) \in \mathbb{R}^N$, $Q$ denotes the open unit cube $(-\frac{1}{2},\frac{1}{2})^N$, and $Q_\nu$, $u_{\lambda,\nu}$ are defined in (2.1) and (2.2). The *recession function* of $W$ (see

(2.11)), $W^\infty$, captures the linear behavior of $W$ at infinity. The new bulk density $H_p$ is, essentially, the same for all $p \geq 1$, and it exhibits the interaction between the initial bulk density $W$ and the initial interfacial density $\psi(\lambda, \nu)$. This is hardly surprising in view of the fact that at points away from the macro-fractures $S(g)$, the jumps in the $u_n$ are diffusing as their amplitudes tend to zero (see Remark 2.13). If admissible sequences are taken so that $\{|\nabla u_n|\}$ is bounded in $L^p$, $p > 1$, then the new crack (interfacial) density $h$ is independent of $W$. Loosely speaking, in these cases it is cheaper to approximate jumps with jumps rather than with sharp gradients. If $p = 1$ and if we only require $L^1$ bounds on $\{|\nabla u_n|\}$, then there is a contribution of $W$, via $W^\infty$, in the new crack density $h_1$.

Just as it was important not to assume coercivity on $W$, coercivity and homogeneity of $\psi$ may rule out certain important physical settings. If we include the extra condition on admissible sequences that they must remain bounded in the $BV$ norm, then we do not have to assume coercivity, while, if $p > 1$, we may also relax the homogeneity assumption. In this case, in the new bulk $H$ the density $\psi$ is replaced by $\psi_0$, where

$$\psi_0(\lambda, \nu) := \limsup_{t \to 0^+} \frac{\psi(t\lambda, \nu)}{t}.$$

It is the linear behavior in fixed directions at (amplitude equal to) zero of the initial interfacial energy density $\psi$ which contributes to the new bulk density.

As it turns out, using our results we may recover some of the recently obtained integral representations for relaxed energies (in the $L^1$ topology) of functionals consisting of bulk and interfacial terms. In particular, by taking the infimum over all $G \in L^1(\Omega, \mathbb{M}^{d\times N})$ on both sides of (1.9) and (1.10), we obtain some of the representations of [7] and [12]. Also, in the context of crystalline solids, for a given macroscopic deformation $g \in W^{1,1}(\Omega, \mathbb{R}^d)$ the energy associated with the optimal microstructure is given by the relaxation of $E(g)$ in the $L^1$ (BV weak) topology. Moreover, if we assume coercivity on $W$, the direct methods of the calculus of variations can be implemented to show that the infimum over all microstructures is achieved (for details, see Section 5).

Lastly, we remark that in the $SBV$ setting, we have the following analogue of (1.6). If $\nabla u_n \rightharpoonup G$ in $L^p$, then there exists a sequence $m(n)$ such that for a.e. $x$

$$\lim_{n\to\infty} \frac{\int_{B(x,n^{-1})} \nabla u_{m(n)}(y)\, \mathbf{e}_i\, dy}{\mathcal{L}^N\left(B\left(x, n^{-1}\right)\right)} = G(x)\, \mathbf{e}_i,$$

and so, as before, we interpret the lattice basis $G(x)$ as a limit of averages of lattice bases resulting from elastic deformations.

This paper is organized as follows: in Section 2 we briefly review properties of functions of bounded variation, we introduce the notion of structured deformations (see Definition 2.11), we state the main representation theorems, Theorems 2.16 and 2.17, and we prove that $I_p(g, G, \cdot)$ is a finite Radon measure (see Proposition 2.22). Section 3 is dedicated to the characterization of the effective bulk energy $H_p$, i.e., the Radon-Nikodym derivative of $I_p(g, G, \cdot)$ with respect to $\mathcal{L}^N$, while in Section 4 we identify the $N-1$ dimensional part of that measure, precisely, we obtain a characterization of the new surface energy density, $h_p$. Finally, in Section 5 we study some properties of $H_p$ and $h_p$, and we relate our relaxation result to others previously obtained (see Corollary 5.4 and Proposition 5.6).

## 2 The Spaces $BV$, $SBV$, and $SD$. Statement of the Main Results

Let $N$ and $d$ denote positive integers. Let $\Omega$ be an open, bounded subset of $\mathbb{R}^N$, $\bar{\Omega}$ its closure, and let $Q$ denote the open unit cube $(-\frac{1}{2}, \frac{1}{2})^N$ and $Q(a, r)$ the open cube centered at $a$ with side length $r$, i.e., $Q(a, r) = a + r\,Q$. We identify the space of $d \times N$ matrices, $\mathbb{M}^{d\times N}$, with $\mathbb{R}^{dN}$, $|x|$ denotes the standard Euclidean norm of $x$, and $|f|_{L^p}$ is the $L^p$ norm of a function $f$. For integrable functions $u_n, u : \Omega \to \mathbb{R}^d$, $u_n \overset{*}{\rightharpoonup} u$ stands for weak star convergence in the sense of measures, i.e, for any $\phi \in C_0(\Omega)$

$$\int_\Omega \phi(x) u_n(x)\, dx \to \int_\Omega \phi(x) u(x)\, dx.$$

Let $\mathcal{M}(\Omega)$ stand for the space of Radon measures on $\Omega$. We allow for the fact that $\mu \in \mathcal{M}(\Omega)$ maybe matrix valued, and denote by $||\mu||$ its total variation measure. Throughout this paper, $C\,(C')$ is a generic constant which may vary from line to line. Let $\nu \in S^{N-1}$, and let $Q_\nu$ be an open unit cube centered at the origin with two of its faces normal to $\nu$, i.e.,

$$Q_\nu := \left\{x \in \mathbb{R}^N : |x \cdot \nu_i| < \frac{1}{2}, |x \cdot \nu| < \frac{1}{2}, i = 1, \ldots, N-1\right\} \tag{2.1}$$

for some orthonormal basis of $\mathbb{R}^N$, $\{\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu\}$. We write $Q_\nu(a, r) := a + rQ_\nu$, $a \in \mathbb{R}^N, r > 0$. Given $\lambda \in \mathbb{R}^d$, let $u_{\lambda,\nu}$ be the $\mathbb{R}^d$-valued function defined in $Q_\nu$ by

$$u_{\lambda,\nu}(x) := \begin{cases} 0 & \text{if} \quad -\frac{1}{2} \leq x \cdot \nu < 0 \\ \lambda & \text{if} \quad 0 \leq x \cdot \nu < \frac{1}{2}. \end{cases} \tag{2.2}$$

We state some basic definitions and properties of functions of bounded variation, $BV$, and of functions of special bounded variation, $SBV$, which will be needed in the sequel. For more details, see Ambrosio [2], Evans and Gariepy [26], Federer [27].

**Definition 2.1** *A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of* bounded variation, $u \in BV(\Omega; \mathbb{R}^d)$, *if for all $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, N\}$, there exists a finite Radon measure $\mu_{ij}$ such that*

$$\int_\Omega u_i(x) \frac{\partial \psi}{\partial x_j}(x)\, dx = -\int_\Omega \psi(x)\, d\mu_{ij}$$

*for every $\psi \in C_0^1(\Omega)$. The distributional derivative $Du$ is the matrix-valued measure with components $\mu_{ij}$. We denote by $||Du||$ the total variation of the gradient measure, i.e., $||Du||(\Omega) := \sum_{i=1}^d ||Du_i||(\Omega)$ where*

$$||Du_i||(\Omega) := \sup_\psi \left\{ \int_\Omega u_i \operatorname{div} \psi\, dx \, : \, \psi \in C_0^1(\Omega, \mathbb{R}^N), |\psi|_\infty \leq 1 \right\}.$$

The space $BV$ is a Banach space equipped with the norm

$$|u|_{BV(\Omega,\mathbb{R}^d)} := |u|_{L^1(\Omega,\mathbb{R}^d)} + ||Du||(\Omega),$$

and it is well known that $C_0^\infty(\Omega; \mathbb{R}^d)$ is dense in $BV$ in the following sense.

**Proposition 2.2** *Let $u \in BV(\Omega)$. There exist $u_n \in C_0^\infty(\Omega)$ such that*

$$\lim_{n\to\infty} \int_\Omega |u_n - u|\, dx = 0 \;\; \textit{and} \;\; \lim_{n\to\infty} ||Du_n||(\Omega) = ||Du||(\Omega).$$

**Definition 2.3** *A set $A \subset \Omega$ is said to be of* finite perimeter *in $\Omega$ if $\chi_A \in BV(\Omega)$, where $\chi_A$ denotes the characteristic function of $A$. The* perimeter *of $A$ in $\Omega$ is defined by*

$$\mathrm{Per}_\Omega(A) := ||D\chi_A||(\Omega) = \sup\left\{ \int_A \operatorname{div} \psi(x)\, dx : \psi \in C_0^1(\Omega; \mathbb{R}^N),\ |\psi|_\infty \leq 1 \right\}.$$

Given $u \in BV(\Omega;\mathbb{R}^d)$, the *approximate upper* and *lower limit* of each component $u_i$, $i \in \{1,\dots,d\}$, are given by

$$u_i^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\mathcal{L}^N\left(\{u_i > t\} \cap Q(x,\varepsilon)\right) = 0\right\}$$

and

$$u_i^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\mathcal{L}^N\left(\{u_i < t\} \cap Q(x,\varepsilon)\right) = 0\right\}.$$

The set

$$S(u) := \bigcup_{i=1}^{d} \{x \in \Omega : u_i^-(x) < u_i^+(x)\}$$

is called the *singular set*, or *jump set* of $u$, and the value $\tilde{u}(x) := \frac{u^+(x)+u^-(x)}{2}$ is defined for every $x \in \Omega$. It is well known that $S(u)$ is $N-1$ rectifiable, i.e.,

$$S(u) = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $H^{N-1}(E) = 0$ and $K_n$ is a compact subset of a $C^1$ hypersurface for each $n$. If $u \in BV(\Omega;\mathbb{R}^D)$, we write

$$Du = \nabla u\, \mathcal{L}^N + D_s u,$$

where $\nabla u$ is the Radon-Nikodym derivative of $Du$ with respect to $\mathcal{L}^N$, and $D_s u$ and $\mathcal{L}^N$ are mutually singular.

**Theorem 2.4** *If $u \in BV(\Omega;\mathbb{R}^d)$ then*

*i) for $\mathcal{L}^N$ a.e. $x \in \Omega$*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N+1}} \left\{\int_{Q(x,\varepsilon)} |u(y) - u(x) - \nabla u(x)\cdot(y-x)|^{\frac{N}{N-1}} dy\right\}^{\frac{N-1}{N}} = 0;$$

*ii) for $H^{N-1}$ a.e. $x \in S(u)$, there exists a unit vector $\nu(x) \in S^{N-1}$, normal to $S(u)$ at $x$, and there exist vectors $u_-(x), u_+(x) \in \mathbb{R}^d$, such that*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_{\nu(x)}(x,\varepsilon) : (y-x)\cdot\nu(x) > 0\}} |u(y) - u_+(x)|^{\frac{N}{N-1}} dy = 0,$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_{\nu(x)}(x,\varepsilon) : (y-x)\cdot\nu(x) < 0\}} |u(y) - u_-(x)|^{\frac{N}{N-1}} dy = 0;$$

*iii) for $H^{N-1}$ a.e. $x_0 \in \Omega \setminus S(u)$*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} |u(x) - \tilde{u}(x_0)| dx = 0.$$

We remark that, in general, $(u_i)^\pm \neq (u_\pm)_i$. In the following, we shall denote by $[u](x)$ the jump of $u$ at $x$, defined by

$$[u](x) := u_+(x) - u_-(x).$$

If $u \in BV(\Omega;\mathbb{R}^d)$, then the measure $Du$ may be represented as

$$Du = \nabla u\, \mathcal{L}^N + (u_+ - u_-) \otimes \nu\, H^{N-1} \lfloor S(u) + C(u), \tag{2.3}$$

where $\nabla u$ is the density of the absolutely continuous part of $Du$ with respect to $\mathcal{L}^N$, and $C(u)$ is the so-called *Cantor part.* The three measures in (2.3) are mutually singular: if $H^{N-1}(B) < +\infty$ then $||C(u)||(B) = 0$, and there exists a Borel set E such that $\mathcal{L}^N(E) = 0$ and $||C(u)||(X) = ||C(u)||(X \cap E)$ for all Borel sets $X \subset \Omega$.

The following subspace of $BV$ was introduced by De Giorgi and Ambrosio in [20].

**Definition 2.5** *A function $u \in BV(\Omega, \mathbb{R}^d)$ is said to be of* special bounded variation *if $C(u) = 0$. We write $u \in SBV(\Omega; \mathbb{R}^d)$.*

Next we state a generalization of the Besicovitch Differentiation Theorem, due to Ambrosio and Dal Maso ([5], Proposition 2.2).

**Theorem 2.6** *If $\lambda$ and $\mu$ are Radon measures in $\Omega$, $\mu \geq 0$, then there exists a Borel set $E \subset \Omega$ such that $\mu(E) = 0$ and for every $x \in \operatorname{supp}\mu \setminus E$*

$$\frac{d\lambda}{d\mu}(x) := \lim_{\varepsilon \to 0^+} \frac{\lambda(x + \varepsilon C)}{\mu(x + \varepsilon C)}$$

*exists and is finite whenever $C$ is a bounded, convex, open set containing the origin.*

The following $SBV$ compactness theorem of Ambrosio (see [2]) will impose restrictions on the growth conditions of bulk and interfacial energies that we will consider in the sequel (see Remark 2.13).

**Theorem 2.7** *Let $\Phi : [0, \infty) \to \mathbb{R}$ and $\Theta : (0, \infty] \to \mathbb{R}$ be convex and concave respectively, nondecreasing, and satisfying*

$$\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty, \qquad \lim_{t \to 0^+} \frac{\Theta(t)}{t} = \infty.$$

*Let $\{u_n\}$ be a sequence of functions in $SBV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$ such that $\sup_n |u_n|_\infty < \infty$ and*

$$\sup_n \left\{ \int_\Omega \Phi(|\nabla u_n|)\, dx + \int_{S(u_n)} \Theta(|[u_n]|)\, dH^{N-1} \right\} < \infty.$$

*Then there exists a subsequence $\{u_{n_i}\}$ and a function $u \in SBV(\Omega, \mathbb{R}^d)$ such that*

$$u_{n_i} \to u \text{ strongly in } L^1 \quad \text{and} \quad \nabla u_{n_i} \rightharpoonup \nabla u \text{ weakly in } L^1.$$

The theorem below was obtained by Alberti [1].

**Theorem 2.8** *Let $f \in L^1(\Omega, \mathbb{R}^{d \times N})$. There exists $u \in SBV(\Omega, \mathbb{R}^d)$ and a Borel function $g : \Omega \to \mathbb{R}^{d \times N}$ such that*

$$Du = f \cdot \mathcal{L}^N + g \cdot H^{N-1} \lfloor S(u), \quad \text{and} \int_{S(u) \cap \Omega} |g|\, dH^{N-1} \leq C|f|_{L^1(\Omega, \mathbb{R}^{d \times N})}, \tag{2.4}$$

*$C$ depends only on $N$.*

The next lemma is a simple corollary of the co-area formula (see Evans and Gariepy [26]), and a similar result may be found in [12], Proposition 3.1, Step 1.

**Lemma 2.9** *Let $u \in BV(\Omega, \mathbb{R}^d)$. There exist $u_n$ piecewise constant, $u_n \in SBV$, such that $u_n \to u$ in $L^1(\Omega, \mathbb{R}^d)$ and*

$$\begin{aligned} ||Du||(\Omega) &= \lim_{n\to\infty} ||Du_n||(\Omega) \\ &= \lim_{n\to\infty} \int_{S(u_n)} |[u_n](x)|\, dH^{N-1}(x). \end{aligned}$$

**Proof.** By Proposition 2.2 we may assume, without loss of generality, that $u$ is a $C_0^\infty$ scalar-valued function. Further, suppose that $u$ is nonnegative; the general case follows by considering the positive and negative parts of $u$. Let $E_t := \{x \in \Omega : u(x) > t\}$ and define

$$u_n(x) := \sum_{i=0}^{\infty} \sum_{j=1}^{n-1} \frac{1}{n} \chi_{E_{\left(i+\frac{j}{n}\right)}}(x).$$

The above sum is finite, and it is easy to check that $u_n \to u$ in $L^1(\Omega)$. Also,

$$||Du_n||(\Omega) \leq \sum_{i=0}^{\infty} \sum_{j=1}^{n-1} \frac{1}{n} ||D\chi_{E_{\left(i+\frac{j}{n}\right)}}||(\Omega),$$

where the right hand side of the above formula is simply a Riemann sum for $\int_0^\infty ||D\chi_{E_t}||(\Omega)dt$, which, by the co-area formula, equals $||Du||(\Omega)$. Thus, by the lower semicontinuity of the total variation and the fact that $u \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} ||Du||(\Omega) &\leq \liminf_{n\to\infty} ||Du_n||(\Omega) \\ &\leq ||Du||(\Omega). \end{aligned}$$

□

**Lemma 2.10** *Let $u \in BV(Q, \mathbb{R}^d)$ satisfy $u|_{\partial Q} = u_0$ for some $u_0 \in C(\overline{Q}, \mathbb{R}^d)$. Then, for every $\varepsilon > 0$ there exists $0 < r_\varepsilon < 1$ such that $r_\varepsilon \to 1^-$ as $\varepsilon \to 0$, and*

$$\int_{\partial Q(0,r_\varepsilon)} |u(x) - u_0(x)|\, dH^{N-1}(x) < \varepsilon.$$

**Proof.** Without loss of generality, assume $d = 1$. Let $\operatorname{tr} u$ denote the trace operator. If $u \in BV(Q)$ then we have (see Ziemer [39] Theorem 5.10.7)

$$\begin{aligned} \int_{\partial Q} |\operatorname{tr} u|\, dH^{N-1} &\leq C|u|_{BV(Q)} \\ &= C\left[||Du||(Q) + \int_Q |u|\, dx\right] \end{aligned} \tag{2.5}$$

for some constant $C$. Fix $\varepsilon > 0$. Since $||Du||$ is a Radon measure, we may choose $\delta$ such that $C > \delta > 0$ and

$$C||Du||\left(Q \backslash Q(0, 1 - 2\delta)\right) < \frac{\varepsilon}{16}. \tag{2.6}$$

Let $\varphi_\delta \in C^\infty(\bar{Q})$ be such that $0 \leq \varphi_\delta \leq 1$, $\varphi_\delta(x) = 0$ if $x \in Q(0, 1-\delta)$, $\varphi_\delta(x) = 1$ if $x \in \partial Q$, and $|\nabla\varphi_\delta|_{L^\infty} = O\left(\delta^{-1}\right)$. Given $\lambda \in (0,1)$ define $u_\lambda(x) := u(\lambda x)$ for $x \in Q$. Clearly, for a.e. $x \in Q$, $u(\lambda x) \to u(x)$

as $\lambda \to 1^-$. This, combined with the fact that $|u_\lambda|_{BV}$ is uniformly bounded, implies that $u_\lambda \to u$ in $L^1$. Now, choose $\lambda = \lambda(\delta,\varepsilon) \in (0,1)$ such that

$$\lambda > \max\left\{\frac{1-2\delta}{1-\delta}, 2^{\frac{1}{N}}\right\}, \int_{\partial Q} |u_0(\lambda x) - u_0(x)|\, dH^{N-1} < \frac{\varepsilon}{4}, \frac{C}{\delta}\int_Q |u(\lambda x) - u(x)|\, dx < \frac{\varepsilon}{4}. \tag{2.7}$$

We have

$$\begin{aligned}\int_{\partial Q(0,\lambda)} |\mathrm{tr}\, u(x) - u_0(x)|\, dH^{N-1} &= \lambda^{N-1}\int_{\partial Q} |\mathrm{tr}\, u(\lambda x) - u_0(\lambda x)|\, dH^{N-1}\\ &\leq \lambda^{N-1}\int_{\partial Q} |\mathrm{tr}\,(\varphi_\delta(u(\lambda x) - u_0(x)))\,|\, dH^{N-1}\\ &\quad +\lambda^{N-1}\int_{\partial Q} |u_0(\lambda x) - u_0(x)|\, dH^{N-1}.\end{aligned}$$

By (2.5) and $(2.7)_2$ we have

$$\begin{aligned}\int_{\partial Q(0,\lambda)} |\mathrm{tr}\, u(x) - u_0(x)|\, dH^{N-1} &\leq C\left\{||D(\varphi_\delta(u_\lambda - u))||(Q) + \int_Q |u(\lambda x) - u(x)|\, dx\right\} + \frac{\varepsilon}{4}\\ &\leq C\Big\{||Du_\lambda||\,(Q\backslash Q(0,1-\delta)) + ||Du||\,(Q\backslash Q(0,1-\delta)) +\\ &\quad +\frac{1}{\delta}\int_{\{x:1-\delta<|x|<1\}} |u(\lambda x) - u(x)|\, dx\Big\} + \frac{\varepsilon}{2},\end{aligned} \tag{2.8}$$

and by $(2.7)_1$

$$\begin{aligned}||Du_\lambda||\,(Q\backslash Q(0,1-\delta)) &= \lambda^{-N}||Du||\,(Q(0,\lambda)\backslash Q(0,\lambda(1-\delta)))\\ &\leq 2||Du||\,(Q\backslash Q(0,1-2\delta))\,.\end{aligned}$$

Hence (2.6), $(2.7)_3$ and (2.8) yield

$$\int_{\partial Q(0,\lambda)} |u(x) - u_0(x)|\, dH^{N-1} < \varepsilon.$$

□

Now we introduce the space of structured deformations within the $SBV$ framework.

**Definition 2.11** *The space of structured deformations, $SD(\Omega)$, consists of pairs $(g,G)$ where*

$$g \in SBV(\Omega;\mathbb{R}^d) \quad \text{and} \quad G \in L^1(\Omega;\mathbb{M}^{d\times N}).$$

We use the result of Alberti (Theorem 2.8) to recover the approximation theorem of Del Piero and Owen (Theorem 5.8 of [23] ).

**Theorem 2.12** *Let $(g,G) \in SD(\Omega)$. Then there exist $u_n \in SBV(\Omega,\mathbb{R}^d)$ such that*

$$u_n \to g \text{ in } L^1(\Omega,\mathbb{R}^d) \quad \text{and} \quad \nabla u_n \overset{*}{\rightharpoonup} G \text{ in } \mathcal{M}(\Omega). \tag{2.9}$$

**Proof.** We construct a sequence $\{u_n\}$ such that for every $n$, $\nabla u_n = G\ \mathcal{L}^N$ a.e. By Theorem 2.8, there exists $h \in SBV(\Omega, \mathbb{R}^d)$ such that

$$\nabla h = G \quad \mathcal{L}^N \text{ a.e.}$$

Let $\{\tilde{u}_n\}$ be a piecewise constant, $L^1$ - approximation of $g - h$ on a rectangular grid, so that

$$\tilde{u}_n \to g - h \text{ in } L^1(\Omega, \mathbb{R}^d) \quad \text{and} \quad \nabla \tilde{u}_n = 0, \mathcal{L}^N \text{ a.e.}$$

Set $u_n := \tilde{u}_n + h$. Clearly $u_n \in SBV(\Omega, \mathbb{R}^d)$, $u_n \to g$ in $L^1(\Omega, \mathbb{R}^d)$, and for all $n$, $\nabla u_n = \nabla h = G\ \mathcal{L}^N$ a.e.

□

**Remark 2.13** Note that we must have $D_s u_n \to (\nabla g - G)\mathcal{L}^N + D_s g$ in the sense of distributions and so, if $\nabla g \neq G$ we are forced, regardless of whether or not $g \in W^{1,1}$, to consider in (2.9) functions $u_n \in SBV \backslash W^{1,1}$. Moreover, suppose that $|\nabla u_n|$ are uniformly bounded in $L^p, p > 1$. This is the case when $\{u_n\}$ is an admissible sequence for the energy $I_p$ with $p > 1$ (see (2.10)). Then Theorem 2.7 implies that in any open subset $E$ of $\Omega$ such that $\nabla g(x) \neq G(x)$ for a.e. $x \in E$,

$$H^{N-1}(S(u_n) \cap E) \to \infty \text{ as } n \to \infty.$$

The jump discontinuities of $u_n$ diffuse throughout the part of the body where $\nabla g(x) \neq G(x)$ which, following Del Piero and Owen [23], is called the *micro-fractured zone.* Moreover, Theorem 2.7 and Lemma 2.20 prevent us to consider surface energy densities with sublinear growth in the case where $W$ has superlinear growth. Due to these considerations, in this paper we restrict our attention to interfacial energy densities $\psi$ with linear growth at infinity.

**Definition 2.14** *Let $W : \mathbb{M}^{d \times N} \to [0, \infty)$ and $\psi : \mathbb{R}^d \times S^{N-1} \to [0, \infty)$ be continuous functions. Given $(g, G) \in SD(\Omega)$, we define the following energies:*

$$I_0(g, G, \Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n \to \infty} \left[ \int_\Omega W(\nabla u_n)\, dx + \int_{S(u_n) \cap \Omega} \psi([u_n], \nu_{u_n})\, dH^{N-1} \right] : u_n \in SBV(\Omega, \mathbb{R}^d), \right.$$
$$\left. u_n \to g \text{ in } L^1(\Omega, \mathbb{R}^d),\ \nabla u_n \overset{*}{\rightharpoonup} m,\ m \in \mathcal{M}(\Omega),\ \frac{dm}{d\mathcal{L}^N} = G \right\}.$$

*For $p \geq 1$, set*

$$I_p(g, G, \Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n \to \infty} \left[ \int_\Omega W(\nabla u_n)\, dx + \int_{S(u_n) \cap \Omega} \psi([u_n], \nu_{u_n})\, dH^{N-1} \right] : u_n \in SBV(\Omega, \mathbb{R}^d), \right.$$
$$\left. u_n \to g \text{ in } L^1(\Omega, \mathbb{R}^d),\ \nabla u_n \overset{*}{\rightharpoonup} G,\ \sup_n |\nabla u_n|_{L^p(\Omega, \mathbb{M}^{d \times N})} < \infty \right\}, \tag{2.10}$$

*and if $p > 1$, $g \in SBV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$, we define*

$$I_p^\infty(g, G, \Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n \to \infty} \left[ \int_\Omega W(\nabla u_n)\, dx + \int_{S(u_n) \cap \Omega} \psi([u_n], \nu_{u_n})\, dH^{N-1} \right] : u_n \in SBV(\Omega, \mathbb{R}^d), \right.$$
$$u_n \to g \text{ in } L^1(\Omega, \mathbb{R}^d),\ \sup_n |u_n|_{L^\infty(\Omega, \mathbb{R}^d)} < \infty,\ \nabla u_n \overset{*}{\rightharpoonup} G,$$
$$\left. \sup_n |\nabla u_n|_{L^p(\Omega, \mathbb{M}^{d \times N})} < \infty \right\}.$$

**Remark 2.15** The uniform $L^p$ bounds on admissible sequences $\{\nabla u_n\}$ will allow us to consider bulk densities $W$ which may not be coercive, and for $p > 1$ they are equivalent to requiring $\nabla u_n \rightharpoonup u$ in $L^p$, while, in view of the Principle of Uniform Boundedness, these bounds are redundant in the case $p = 1$. The uniform $L^\infty$ bounds for $\{u_n\}$ are useful for proving that the energy is a Radon measure (see Lemma 2.21 and Proposition 2.22). However, using a truncation argument in Lemma 2.20 for $p > 1$ and with $g \in SBV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$, we have $I_p(g, G, \Omega) = I_p^\infty(g, G, \Omega)$, and so we may work simply with $I_p$. Also note that, by virtue of the particular construction of $\{u_n\}$ in Theorem 2.12, $I_0, I_p$, and $I_p^\infty$ are well defined. Finally, we may avoid a coercivity assumption on $\psi$ (*cf.* ($\mathcal{H}2$) ) by requiring admissible sequences to satisfy $\sup_n ||Du_n||(\Omega) < \infty$ (see Remark 3.3 for details).

Let $p \geq 1$, $W : \mathbb{M}^{d\times N} \to [0, +\infty)$ and $\psi : \mathbb{R}^d \times S^{N-1} \to [0, +\infty)$ be continuous functions satisfying the following hypotheses:

- $(\mathcal{H}1)_p$ there exists a constant $C$ such that
$$|W(A) - W(B)| \leq C|A - B| \left(1 + |A|^{p-1} + |B|^{p-1}\right)$$
for any $A, B \in \mathbb{M}^{d\times N}$;
- ($\mathcal{H}2$) there exist constants $c_1$, $C_1 > 0$, such that for all $(\lambda, \nu) \in \mathbb{R}^d \times S^{N-1}$,
$$c_1|\lambda| \leq \psi(\lambda, \nu) \leq C_1|\lambda|;$$
- ($\mathcal{H}3$) $\psi(\cdot, \nu)$ is a positively homogeneous of degree 1 function;
- ($\mathcal{H}4$) $\psi$ is *subadditive*, i.e., for all $\lambda_1, \lambda_2 \in \mathbb{R}^d$ and $\nu \in S^{N-1}$,
$$\psi(\lambda_1 + \lambda_2, \nu) \leq \psi(\lambda_1, \nu) + \psi(\lambda_2, \nu).$$

We recall that the *recession function of W* is defined by

$$W^\infty(A) := \limsup_{t\to+\infty} \frac{W(tA)}{t}. \tag{2.11}$$

If $p = 1$ then we assume further that

- ($\mathcal{H}5$) there exist constants $c, L > 0$, $0 < m < 1$, such that
$$\left|W^\infty(A) - \frac{W(tA)}{t}\right| \leq c\frac{1}{t^m}$$
for every $A \in \mathbb{M}^{d\times N}$ with $|A| = 1$, and for all $t > 0$ such that $t > L$.

It can be shown that if $W$ is Lipschitz then $W^\infty$ is Lipschitz, and positively homogeneous of degree 1 (see [33]).

We now state two of the main results of this paper.

**Theorem 2.16** *Let $(g, G) \in SD$ and assume that $W$ and $\psi$ satisfy hypotheses $(\mathcal{H}1)_1$, $(\mathcal{H}2)$ - $(\mathcal{H}5)$. Then*

$$I_1(g, G, \Omega) = \int_\Omega H_1(\nabla g(x), G(x))\, dx + \int_{S(g)} h_1([g], \nu_g)\, dH^{N-1},$$

*where, for* $A, B \in \mathbb{M}^{d\times N}$,

$$H_1(A,B) := \inf_u \left\{ \int_Q W(\nabla u)dx + \int_{S(u)\cap Q} \psi([u],\nu)dH^{N-1} : u \in SBV(Q,\mathbb{R}^d), u|_{\partial Q} = Ax, \int_Q \nabla u\, dx = B \right\},$$

*and for* $\lambda \in \mathbb{R}^d$, $\nu \in S^{N-1}$,

$$h_1(\lambda,\nu) := \inf_u \left\{ \int_{Q_\nu} W^\infty(\nabla u)\, dx + \int_{S(u)} \psi([u],\nu_u)\, dH^{N-1} : u \in SBV(Q_\nu,\mathbb{R}^d), \right. \tag{2.12}$$

$$\left. u|_{\partial Q_\nu} = u_{\lambda,\nu}, \int_{Q_\nu} \nabla u\, dx = 0 \right\}. \tag{2.13}$$

*In addition, we have*

$$I_0(g,G,\Omega) = \int_\Omega H_1(\nabla g(x), G(x))\, dx + \mu_s(\Omega) \tag{2.14}$$

*for some Radon measure* $\mu_s$ *absolutely continuous with respect to* $H^{N-1}\lfloor S(g)$.

**Theorem 2.17** *Let* $p > 1$ *and let* $(g,G) \in SD$ *with* $G \in L^p(\Omega,\mathbb{M}^{d\times N})$, *and assume that* $W$ *and* $\psi$ *satisfy hypotheses* $(\mathcal{H}1)_p$, $(\mathcal{H}2)$ - $(\mathcal{H}4)$. *Then*

$$I_p(g,G,\Omega) = \int_\Omega H_p(\nabla g(x), G(x))\, dx + \int_{S(g)} h([g])\, dH^{N-1}, \tag{2.15}$$

*where, for* $A, B \in \mathbb{M}^{d\times N}$,

$$H_p(A,B) := \inf_u \left\{ \int_Q W(\nabla u)\, dx + \int_{S(u)\cap Q} \psi([u],\nu)\, dH^{N-1} : u \in SBV(Q,\mathbb{R}^d), u|_{\partial Q} = Ax, \right.$$

$$\left. |\nabla u| \in L^p(Q), \int_Q \nabla u\, dx = B \right\}, \tag{2.16}$$

*and for* $\lambda \in \mathbb{R}^d$,

$$h(\lambda) := \inf_u \left\{ \int_{S(u)} \psi([u],\nu_u)\, dH^{N-1} : u \in SBV(Q,\mathbb{R}^d),\ u|_{\partial Q} = u_{\lambda,e_N}, \nabla u(x) = 0\ \mathcal{L}^N a.e. \right\}. \tag{2.17}$$

Note that in the definition of $h$, $Q$ may be replaced by any $Q_\nu$, for $\nu \in S^{N-1}$, i.e., for $p > 1$ the new relaxed crack density is *isotropic*. As was mentioned in the introduction, it is possible to relax the assumptions of coercivity and homogeneity on $\psi$, still obtaining the representation of Theorem 2.17. For simplicity, we prove the theorem under the original hypotheses and refer the reader to Remark 3.3 for the appropriate modifications.

We divide the proof of Theorems 2.16 and 2.17 into several parts. First, using properties of $I_p^\infty$ we show that $I_0(g,G,\cdot)$ and $I_p(g,G,\cdot)$ are non negative Radon measures, absolutely continuous with respect to $\mathcal{L}^N + |D_s g|$. Then, using techniques such as the *blow up method* (e.g. , [7], [32], [33]), we proceed to characterizing the densities

$$\frac{dI(g,G,\cdot)}{d\mathcal{L}^N} \quad \text{and} \quad \frac{dI(g,G,\cdot)}{d(|g^+ - g^-| H^{N-1}\lfloor S(g))}.$$

The next lemma provides an upper bound for the energies. Let $I$ denote either $I_p$ ($p \geq 1$) or $I_0$.

**Lemma 2.18** *Let $W : \mathbb{M}^{d\times N} \to [0,+\infty)$ be a continuous function, and let $\psi : \mathbb{R}^d \times S^{N-1} \to [0,+\infty)$ be continuous with $0 \le \psi(\lambda,\nu) \le C|\lambda|$ for some constant independent of $\lambda,\nu$. Then for every $(g,G) \in SD(\Omega)$ and $p > 1$*

$$I(g,G,\Omega) \le C\left\{\int_\Omega W(G)\,dx + |G|_{L^1(\Omega,\mathbb{M}^{d\times N})} + ||Dg||(\Omega)\right\},$$

*where $C$ is a constant independent of $\Omega$.*

**Proof.** By Theorem 2.8 there exists $h \in SBV(\Omega,\mathbb{R}^d)$ such that $\nabla h = G\ \mathcal{L}^N$ a.e. and $||Dh||(\Omega) \le C_1||G||_{L^1}$. By Lemma 2.9 there exist $\{\tilde{u}_n\}$ piecewise constant such that

$$\tilde{u}_n \to g - h \quad \text{and} \quad ||D\tilde{u}_n||(\Omega) \to ||Dg - Dh||(\Omega).$$

Define $u_n := \tilde{u}_n + h$. Clearly $\nabla u_n(x) = G(x)$ for $\mathcal{L}^N$ a.e. $x$ and $u_n \to g$ in $L^1$. Thus

$$\begin{aligned}
I(g,G,\Omega) &\le \liminf_n \left\{\int_\Omega W(\nabla u_n(x))\,dx + \int_{S(u_n)} \psi([u_n](x),\nu_{u_n}(x))dH^{N-1}(x)\right\}\\
&\le C\liminf_n \left\{\int_\Omega W(G)\,dx + ||Du_n||(\Omega)\right\}\\
&\le C\liminf_n \left\{\int_\Omega W(G)\,dx + ||D\tilde{u}_n||(\Omega) + |G|_{L^1}\right\}\\
&= C\left\{\int_\Omega W(G)\,dx + ||Dg - Dh||(\Omega) + |G|_{L^1}\right\}\\
&\le C\left\{\int_\Omega W(G)\,dx + |G|_{L^1} + ||Dg||(\Omega)\right\}.
\end{aligned}$$

□

**Remark 2.19** Lemma 2.18 implies that for all $(g,G) \in SD(\Omega)$, $I_p(g,G,\Omega) < \infty$ (and also $I_0(g,G,\Omega)$ if $p = 1$) provided that $\int_\Omega W(G)\,dx < \infty$.

Before we establish that $I_p(g,G,\cdot)$ and $I_0(g,G,\cdot)$ are traces of Radon measures, we prove that, for $g \in L^\infty(\Omega,\mathbb{R}^d)$, the additional $L^\infty$ bounds on admissible sequences do not increase the energy $I_p$, $p > 1$.

**Lemma 2.20** *Let $p > 1$, $g \in SBV(\Omega,\mathbb{R}^d) \cap L^\infty(\Omega,\mathbb{R}^d)$, and assume that $(\mathcal{H}1)_p$ and $(\mathcal{H}2)$ hold. If $(g,G) \in SD(\Omega)$ then*

$$I_p(g,G,\Omega) = I_p^\infty(g,G,\Omega).$$

**Proof.** Clearly, it suffices to prove that $I_p^\infty(g,G,\Omega) \le I_p(g,G,\Omega)$. We apply a truncation argument in the same spirit as in [7] (see Lemma 3.7) and [29] (see Proposition 2.8). Let $\phi_i \in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$ be such that

$$\phi_i(x) := \begin{cases} x & \text{if } |x| < e^i \\ 0 & \text{if } |x| \ge e^{i+1}, \end{cases}$$

and $|\nabla\phi_i|_{L^\infty} \le 1$. Since $g \in L^\infty(\Omega,\mathbb{R}^d)$ there exists an $i_0$ such that for $i \ge i_0$, $|g|_\infty \le e^i$ and $\phi_i(g) = g\ \mathcal{L}^N$ a.e. Let $i \ge i_0$ and define $w_n^i(x) := \phi_i(u_n(x))$, where $u_n \to g$ in $L^1$, $\nabla u_n \rightharpoonup G$ in $L^p$, and

$$\lim_n \left\{\int_\Omega W(\nabla u_n)\,dx + \int_{S(u_n)} \psi([u_n](x),\nu_{u_n}(x))\,dH^{N-1}\right\} \le I_p(g,G,\Omega) + \varepsilon$$

for fixed $\varepsilon > 0$. Clearly, $|w_n^i|_{L^\infty} \le e^i$, $S(w_n^i) \subset S(u_n)$, and by the chain rule for $C^\infty$ functions composed with $BV$ functions, it follows that $\nabla w_n^i = \nabla\phi_i(u_n)\,\nabla u_n\ \mathcal{L}^N$ a.e. Moreover, we have

$$|w_n^i(x) - g(x)|_{L^1} = |\phi_i(u_n(x)) - \phi_i(g(x))|_{L^1} \le |u_n(x) - g(x)|_{L^1}.$$

Next, we consider the convergence of $\nabla w_n^i$ as $n \to \infty$. Note that $|\nabla w_n^i(x)|_{L^p} \le |\nabla u_n(x)|_{L^p} \le C$, for $C$ independent of $n$. Let $\xi \in C_0(\Omega)$, then

$$\begin{aligned}\int_\Omega \xi(x)\nabla w_n^i(x)\,dx &= \int_{\{x\,:\,|u_n|<e^i\}} \xi(x)\nabla u_n(x)\,dx + \int_{\{x\,:\,e^i<|u_n|<e^{i+1}\}} \xi(x)\nabla\phi_i(u_n)\nabla u_n(x)\,dx \\ &=: \int_\Omega \xi(x)\nabla u_n(x)\,dx + E_n,\end{aligned}$$

where $|E_n| \le 2|\xi|_{L^\infty}\int_{\{x\,:\,|u_n|>e^i\}} |\nabla u_n(x)|\,dx$. Since $|g|_{L^\infty} < e^i$ and $u_n \to g$ in $L^1$, due to the equi-integrability of the $\{\nabla u_n\}$ we have

$$|E_n| \to 0 \quad \text{as} \quad n \to \infty,$$

and we conclude that

$$\int_\Omega \xi(x)\nabla w_n^i(x)\,dx \to \int_\Omega \xi(x)G(x)\,dx \qquad \text{as } n \to \infty,$$

i.e., $\nabla w_n^i \overset{*}{\rightharpoonup} G$ as $n \to \infty$. Lastly, we compare the energies. Using $(\mathcal{H}1)_p$ and $(\mathcal{H}2)$ we have

$$\begin{aligned}\int_\Omega W(\nabla w_n^i)\,dx &+ \int_{S(w_n^i)} \psi([w_n^i](x), \nu_{w_n^i}(x))\,dH^{N-1} \\ = &\int_{\{x\,:|u_n|<e^i\}} W(\nabla u_n(x))\,dx + \int_{\{x\,:e^i<|u_n|<e^{i+1}\}} W(\nabla\phi_i(u_n)\nabla u_n(x))\,dx \\ &+ \int_{\{x\,:|u_n|>e^{i+1}\}} W(0)\,dx + \int_{S(u_n)\cap\{x\,:|u_n|<e^i\}} \psi([u_n](x), \nu_{u_n}(x))\,dH^{N-1} \\ &+ \int_{S(u_n)\cap\{x\,:e^i<|u_n|<e^{i+1}\}} \psi([w_n^i](x), \nu_{w_n^i}(x))\,dH^{N-1} \\ \le &\int_\Omega W(\nabla u_n(x))\,dx + \int_{S(u_n)} \psi([u_n](x), \nu_{u_n}(x))\,dH^{N-1} + \frac{C|u_n|_{L^1}}{e^{i+1}} \\ &+ C\int_{\{x\,:e^i<|u_n|<e^{i+1}\}} (1 + |\nabla u_n|^p)\,dx + C\int_{S(u_n)\cap\{x\,:e^i<|u_n|<e^{i+1}\}} |[u_n]|\,dH^{N-1},\end{aligned}$$

where we have used the fact that $\mathcal{L}^N(\{x : |u_n| > e^{i+1}\}) \le e^{-(i+1)}\,|u_n|_{L^1}$. Next, for $M > i_0$,

$$\begin{aligned}\frac{1}{M - i_0 + 1}\sum_{i=i_0}^{M} &\left\{ \int_\Omega W(\nabla w_n^i)\,dx + \int_{S(w_n^i)} \psi([w_n^i](x), \nu_{w_n^i}(x))\,dH^{N-1} \right\} \\ \le &\int_\Omega W(\nabla u_n(x))\,dx + \int_{S(u_n)} \psi([u_n](x), \nu_{u_n}(x))\,dH^{N-1} \\ &+ \frac{C}{M - i_0 + 1}\left\{ \sum_{i=i_0}^{M} \frac{1}{e^{i+1}} + \int_\Omega (1 + |\nabla u_n|^p)\,dx + \int_{S(u_n)} |[u_n]|\,dH^{N-1} \right\}. \qquad (2.18)\end{aligned}$$

Clearly, the term inside the parentheses in the last line of (2.18) is bounded independent of $n$, and so we may choose $M$ sufficiently large such that the last line in (2.18) is less than $\varepsilon$. Hence, there exists some $i \in \{i_0, \dots, M\}$ such that

$$\begin{aligned}\int_\Omega W(\nabla w_n^i)\,dx &+ \int_{S(w_n^i)} \psi([w_n^i](x), \nu_{w_n^i}(x))\,dH^{N-1} \\ &\le \int_\Omega W(\nabla u_n(x))\,dx + \int_{S(u_n)\cap\Omega} \psi([u_n](x), \nu_{u_n}(x))\,dH^{N-1} + \varepsilon,\end{aligned}$$

and we conclude that

$$I_p^\infty(g,G,\Omega) \le I_p(g,G,\Omega) + 2\varepsilon.$$

The result follows by letting $\varepsilon \to 0^+$. □

Next, we obtain a subadditivity condition for $I_p$ and $I_0$.

**Lemma 2.21** *If $(\mathcal{H}1)_p$ and $(\mathcal{H}2)$ hold, $p > 1$, and $A, B, C$ are open subsets of $\Omega$ such that $A \subset\subset B \subset\subset C$, then*

$$I_p(g,G,C) \le I_p(g,G,B) + I_p(g,G,C\backslash\bar{A}),$$

*for all $(g,G) \in SD(\Omega)$ with $g \in L^\infty(\Omega,\mathbb{R}^d)$. If $p = 1$ then*

$$I_0(g,G,C) \le I_0(g,G,B) + I_0(g,G,C\backslash\bar{A}) \quad \text{and} \quad I_1(g,G,C) \le I_1(g,G,B) + I_1(g,G,C\backslash\bar{A})$$

*for all $(g,G) \in SD(\Omega)$.*

**Proof.** Fix $\varepsilon > 0$ and and let $I$ denote $I_p$ if $p > 1$, and either $I_0$ or $I_1$ if $p = 1$. Let $u_n \in SBV(B,\mathbb{R}^d)$ and $v_n \in SBV(C\backslash\bar{A},\mathbb{R}^d)$ be "almost minimizing" sequences for I, that is,

$$\lim_n \int_B W(\nabla u_n(x))\,dx + \int_{S(u_n)\cap B} \psi([u_n](x),\nu_{u_n}(x))\,dH^{N-1} \le I(g,G,B) + \varepsilon, \tag{2.19}$$

$$\lim_n \int_{C\backslash\bar{A}} W(\nabla v_n(x))\,dx + \int_{S(v_n)\cap C\backslash\bar{A}} \psi([v_n](x),\nu_{v_n}(x))\,dH^{N-1} \le I(g,G,C\backslash\bar{A}) + \varepsilon, \tag{2.20}$$

$u_n \to g$ in $L^1(B,\mathbb{R}^d)$, $v_n \to g$ in $L^1(C\backslash\bar{A},\mathbb{R}^d)$, $\{|\nabla u_n|_{L^p(B)}\}, \{|\nabla v_n|_{L^p(C\backslash\bar{A})}\}$ are bounded, and

$$\nabla u_n \overset{*}{\rightharpoonup} m_1 \text{ in } \mathcal{M}(B) \quad \nabla v_n \overset{*}{\rightharpoonup} m_2 \text{ in } \mathcal{M}(C\backslash\bar{A}) \quad \text{with } \frac{dm_i}{d\mathcal{L}^N} = G, \quad i = 1,2.$$

In the case where $I = I_p$, $p \ge 1$, we have $m_1 = \chi_B G$ and $m_2 = \chi_{C\backslash\bar{A}}G$. Moreover, by Lemma 2.20 if $p > 1$ we may assume that the sequences $\{u_n\},\{v_n\}$ and $\{\nabla u_n\},\{\nabla v_n\}$ are uniformly bounded in $L^\infty$ and $L^p$, respectively.

Consider

$$\tilde{A} := \{x \in B : \text{dist}\,(x,\bar{A}) < \delta\},$$

where, by virtue of the countable additivity property of the Radon measures, $\delta$ is choosen such that $||m_i||(\partial\tilde{A}) = 0$. Define

$$\alpha_n := |u_n - v_n|_{L^1(B\backslash\bar{A},\mathbb{R}^d)}^{-\frac{1}{2p}} \quad \text{and} \quad \kappa_n := \left[\left[|u_n - v_n|_{L^1(B\backslash\bar{A},\mathbb{R}^d)}^{-\frac{1}{4p}}\right]\right],$$

where $[[\cdot]]$ denotes the greatest integer function. For $i = 0,\ldots,\kappa_n - 1$, define

$$V_i^n := \left\{x \in B\backslash\bar{A} : \delta - \frac{1}{\alpha_n} + \frac{i}{\alpha_n\kappa_n} < \text{dist}\,(x,\bar{A}) < \delta - \frac{1}{\alpha_n} + \frac{i+1}{\alpha_n\kappa_n}\right\}.$$

For each $i$ we introduce cut off functions which are either 1 or 0 on the complement of $V_i^n$, that is, we consider $\phi_i^n \in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $|\nabla\phi_i^n|_{L^\infty} \le C(\alpha_n\kappa_n)$ and

$$\phi_i^n(x) := \begin{cases} 1 & \text{if dist}\,(x,\bar{A}) < \delta - \frac{1}{\alpha_n} + \frac{i}{\alpha_n\kappa_n} \\ 0 & \text{if dist}\,(x,\bar{A}) > \delta - \frac{1}{\alpha_n} + \frac{i+1}{\alpha_n\kappa_n}. \end{cases}$$

For each $i = 0,\ldots,\kappa_n - 1$ define

$$z_n^i := \phi_i^n u_n + (1 - \phi_i^n)v_n,$$

where we have extended $u_n$ by 0 on the complement of $B$ and $v_n$ by 0 on the complement of $C\backslash\tilde{A}$. It is clear that for each $i$

$$|z_n^i(x) - g(x)|_{L^1(C,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Using $(\mathcal{H}1)_p$ and $(\mathcal{H}2)$, we have

$$\begin{aligned}
\int_C W(\nabla z_n^i(x))\,dx \; + & \int_{S(z_n^i)\cap C} \psi([z_n^i](x), \nu_{z_n^i}(x))\,dH^{N-1} \\
\leq & \int_B W(\nabla u_n(x))\,dx + \int_{S(u_n)\cap B} \psi([u_n](x), \nu_{u_n}(x))\,dH^{N-1} \\
& + \int_{C\backslash\tilde{A}} W(\nabla v_n(x))\,dx + \int_{S(v_n)\cap C\backslash\tilde{A}} \psi([v_n](x), \nu_{v_n}(x))\,dH^{N-1} \\
& + C\int_{V_i^n} (1 + |\nabla u_n(x)|^p + |\nabla v_n(x)|^p)\,dx + C(\alpha_n\kappa_n)^p \int_{V_i^n} |u_n(x) - v_n(x)|^p\,dx \\
& + C\int_{S(u_n)\cap V_i^n} |[u_n]|\,dH^{N-1} + C\int_{S(v_n)\cap V_i^n} |[v_n]|\,dH^{N-1}.
\end{aligned}$$

Thus, summing over $i$, and using (2.19), (2.20), we obtain

$$\begin{aligned}
\frac{1}{\kappa_n}\sum_{i=0}^{\kappa_n-1} \int_C W(\nabla z_n^i(x))\,dx \; + & \int_{S(z_n^i)\cap C} \psi([z_n^i](x), \nu_{z_n^i}(x))\,dH^{N-1} \leq I(g,G,B) + I(g,G,C\backslash\tilde{A}) + 2\varepsilon \\
+ & \frac{1}{\kappa_n}\left\{\int_B (1 + |\nabla u_n|^p)\,dx + \int_{C\backslash\tilde{A}} (1 + |\nabla v_n|^p)\,dx + C\int_{S(u_n)\cap B} |[u_n]|\,dH^{N-1}\right. \\
+ & \left. C\int_{S(v_n)\cap C\backslash\tilde{A}} |[v_n]|\,dH^{N-1} + C(\alpha_n\kappa_n)^p |u_n - v_n|^p_{L^p(B\backslash\tilde{A},\mathbb{R}^d)}\right\}. \qquad (2.21)
\end{aligned}$$

If $p > 1$, the $L^\infty$ bounds on $\{u_n\}$ and $\{v_n\}$ yield

$$\begin{aligned}
\frac{1}{\kappa_n}(\alpha_n\kappa_n)^p |u_n - v_n|^p_{L^p} & \leq C\alpha_n^p \kappa_n^{p-1} |u_n - v_n|_{L^1(B\backslash\tilde{A},\mathbb{R}^d)} \\
& = C|u_n - v_n|^a_{L^1(B\backslash\tilde{A},\mathbb{R}^d)},
\end{aligned}$$

where $a > 0$ This, combined with the uniform bounds for $\{\nabla u_n\}$, $\{\nabla v_n\}$, $(\mathcal{H}2)$, and (2.19), (2.20), implies that the last two lines of (2.21 ) tend to 0 as $n \to \infty$. Hence, we may choose $i_n \in \{0, \dots, \kappa_n - 1\}$ such that, setting $w_n := z_n^{i_n}$, we have $w_n \to g$ in $L^1(C, \mathbb{R}^d)$ and

$$\begin{aligned}
I(g,G,C) & \leq \limsup_{n\to\infty} \int_C W(\nabla w_n(x))\,dx + \int_{S(w_n)\cap C} \psi([w_n](x), \nu_{w_n}(x))\,dH^{N-1} \\
& \leq I(g,G,B) + I(g,G,C\backslash\tilde{A}) + 2\varepsilon.
\end{aligned}$$

The result now follows by letting $\varepsilon \to 0^+$, as long as we show that $\sup_n |\nabla w_n|_{L^p(C,\mathbb{R}^d)} < \infty$ and that

$$\nabla w_n \overset{*}{\rightharpoonup} \chi_{\tilde{A}} m_1 + \chi_{C\backslash\tilde{A}} m_2 \quad \text{in } \mathcal{M}(C),$$

since, by definition of $\tilde{A}$,

$$\frac{d\left(\chi_{\tilde{A}} m_1 + \chi_{C\backslash\tilde{A}} m_2\right)}{d\mathcal{L}^N} = \chi_{\tilde{A}} \frac{dm_1}{d\mathcal{L}^N} + \chi_{C\backslash\tilde{A}} \frac{dm_2}{d\mathcal{L}^N} = G \quad \mathcal{L}^N \text{ a.e. } x \in C.$$

To this end, we recall that

$$\int_C |\nabla w_n|^p\,dx \le \int_B |\nabla u_n|^p\,dx + \int_{C\setminus\bar{A}} |\nabla v_n|^p\,dx + (\alpha_n\kappa_n)^p \int_{B\setminus\bar{A}} |u_n - v_n|^p\,dx,$$

and so $\sup_n |\nabla w_n|_{L^p(C,\mathbb{R}^d)} < \infty$. As for the convergence, let $\xi \in C_0(C)$ and consider an increasing sequence of open sets $A_m \subset \bar{A}$ such that $\operatorname{dist}(\bar{A}_m, \partial\bar{A}) = m^{-1}$. Let $\theta_m \in C_0(C,[0,1])$ be a sequence of cut-off functions such that $\theta_m(x) = 1$ if $x \in A_m$ and $\theta_m(x) = 0$ if $x \notin A_{m+1}$. Then

$$\begin{aligned}\lim_{n\to\infty}\int_C \xi\nabla w_n\,dx &= \lim_{n\to\infty}\int_C \left(\xi\phi_n^{i_n}\nabla u_n + \xi(1-\phi_n^{i_n})\nabla v_n + \xi(u_n - v_n)\otimes\nabla\phi_n^{i_n}\right)\,dx\\ &= \lim_{m\to\infty}\lim_{n\to\infty}\left\{\int_B \xi\theta_m\nabla u_n dx + \int_B \xi(\phi_n^{i_n} - \theta_m)\nabla u_n dx + \int_{C\setminus\bar{A}} \xi(1-\theta_m)\nabla v_n dx\right.\\ &\qquad \left. + \int_{C\setminus\bar{A}} \xi(\theta_m - \phi_n^{i_n})\nabla v_n\,dx\right\} + \lim_{n\to\infty} E_n,\end{aligned}$$

where

$$\begin{aligned}|E_n| &= \left|\int_{V_i^n} \xi(u_n - v_n)\otimes\nabla\phi_n^{i_n}\,dx\right|\\ &\le |\xi|_{L^\infty}(\alpha_n\kappa_n)|u_n - v_n|_{L^1} \to 0.\end{aligned}$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\int_C \xi\nabla w_n\,dx &= \lim_{m\to\infty}\lim_{n\to\infty}\left\{\int_B \xi\theta_m\nabla u_n\,dx + \int_{C\setminus\bar{A}} \xi(1-\theta_m)\nabla v_n\,dx\right\} + \lim_{m\to\infty}\lim_{n\to\infty} F_{n,m}\\ &= \lim_{m\to\infty}\left\{\int_B \xi\theta_m\,dm_1 + \int_{C\setminus\bar{A}} \xi(1-\theta_m)\,dm_2\right\} + \lim_{m\to\infty}\lim_{n\to\infty} F_{n,m}\\ &= \int_{\bar{A}} \xi\,dm_1 + \int_{C\setminus\bar{A}} \xi\,dm_2 + \lim_{m\to\infty}\lim_{n\to\infty} F_{n,m},\end{aligned}$$

where

$$F_{n,m} := \int_B \xi(\phi_n^{i_n} - \theta_m)\nabla u_n\,dx + \int_{C\setminus\bar{A}} \xi(\theta_m - \phi_n^{i_n})\nabla v_n\,dx.$$

Finally, we note that $\lim_m \lim_n F_{n,m} = 0$. Indeed, recalling the definition of $\phi_n^i$, for each $m$ we may choose $n >> m$ such that $\phi_n^{i_n}(x) = 1$ if $x \in A_m$, and so

$$\begin{aligned}\lim_{m\to\infty}\limsup_{n\to\infty}\int_B |\phi_n^{i_n} - \theta_m||\nabla u_n|\,dx &\le 2\lim_{m\to\infty} ||m_1||(\bar{A}\setminus A_m)\\ &= 0.\end{aligned}$$

A similar argument gives

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{C\setminus\bar{A}} \xi(\theta_m - \phi_n^{i_n})\nabla v_n\,dx = 0.$$

□

Using Lemmas 2.18, 2.20, and 2.21, we show that if $(g,G) \in SD(\Omega)$, $G \in L^p(\Omega, \mathbb{M}^{d\times N})$, $g \in L^\infty(\Omega,\mathbb{R}^d)$ if $p > 1$, then $I_p(g,G,\cdot)$, $p \ge 1$, (also $I_0(g,G,\cdot)$ if $p = 1$) is a Radon measure, and

$$I_p(g,G,\cdot)\ (\text{also } I_0(g,G,\cdot) \text{ if } p = 1) \ll \mathcal{L}^N + ||D_s g||.$$

**Proposition 2.22** *Assume that* $(\mathcal{H}1)_p$ *and* $(\mathcal{H}2)$ *hold and let* $g \in L^\infty(\Omega, \mathbb{R}^d)$ *if* $p > 1$. *Then, for* $p \geq 1$, $I_p(g, G, \cdot)$ *(also* $I_0(g, G, \cdot)$ *if* $p = 1$*) is the trace on* $\{U \subset \Omega : U$ *is open*$\}$ *of a finite Radon measure on* $\mathcal{B}(\Omega)$.

**Proof.** We proceed separately for $I_p$ and $I_0$. First we consider $I_p$, $p \geq 1$, and we use an argument introduced in Fonseca and Maly [31].

*Step 1.* We assume coercivity, i.e., there exists a constant $C > 0$ such that $W(A) \geq C|A|^p$ for all $A \in \mathbb{M}^{d\times N}$. In this case, by means of a diagonalization procedure we can find a minimizing sequence for $I_p(g, G, \Omega)$, that is, there exist $u_n \in SBV(\Omega, \mathbb{R}^d)$ such that $u_n \to g$ in $L^1$, $\sup_n |\nabla u_n|_{L^p} < \infty$, $\nabla u_n \rightharpoonup G$ in $L^p$, and

$$I_p(g, G, \Omega) = \lim_{n\to\infty} \left\{ \int_\Omega W(\nabla u_n(x))\, dx + \int_{S(u_n)\cap\Omega} \psi([u_n](x), \nu_{u_n}(x))\, dH^{N-1} \right\}.$$

After passing, if necessary, to a subsequence, we may find $\mu \in \mathcal{M}(\overline{\Omega})$ such that

$$W(\nabla u_n(x))dx + \psi([u_n](x), \nu_{u_n}(x))\, dH^{N-1}\lfloor S(u_n) \overset{*}{\rightharpoonup} \mu, \text{ in } \mathcal{M}(\overline{\Omega}),$$

and, in particular,

$$\mu(\overline{\Omega}) = I_p(g, G, \Omega). \tag{2.22}$$

Let $V \subset \Omega$ be open. We must show that $\mu(V) = I_p(g, G, V)$. We always have,

$$\begin{aligned} I_p(g, G, V) &\leq \liminf_{n\to\infty} \int_V W(\nabla u_n(x))\, dx + \int_{S(u_n)\cap V} \psi([u_n](x), \nu_{u_n}(x))\, dH^{N-1} \\ &\leq \mu(\overline{V}). \end{aligned} \tag{2.23}$$

Let $\varepsilon > 0$ and take $W \subset\subset V$ such that $\mu(V \backslash W) < \varepsilon$. By Lemma 2.21, (2.22), and (2.23),

$$\begin{aligned} \mu(V) &\leq \mu(W) + \varepsilon \\ &= \mu(\overline{\Omega}) - \mu(\overline{\Omega}\backslash W) + \varepsilon \\ &\leq I_p(g, G, \Omega) - I_p(g, G, \Omega\backslash \overline{W}) + \varepsilon \\ &\leq I_p(g, G, V) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain

$$\mu(V) \leq I_p(g, G, V). \tag{2.24}$$

On the other hand, Lemma 2.18 implies that

$$I_p(g, G, \cdot) \leq C(1 + |G|^p)\, \mathcal{L}^N + ||Dg||.$$

Denote by $\lambda$ the Radon measure on the right hand side. Let $K \subset\subset V$ be a compact set such that $\lambda(V\backslash K) < \varepsilon$, and choose $W$ open such that $K \subset\subset W \subset\subset V$. Using Lemma 2.21 and (2.23), we have

$$\begin{aligned} I_p(g, G, V) &\leq I_p(g, G, W) + I_p(g, G, V\backslash K) \\ &\leq \mu(\overline{W}) + \lambda(V\backslash K) \\ &\leq \mu(V) + \varepsilon, \end{aligned}$$

and, together with (2.24), the result follows by letting $\varepsilon \to 0^+$.

*Step 2.* We remove the coercivity assumption. Considering in Step 1 the bulk density $W^\varepsilon := W(\cdot) + \varepsilon|\cdot|^p$, we obtain measure representations $\mu_\varepsilon$ for $I_p^\varepsilon(g, G, \Omega)$, where $I_p^\varepsilon$ is the energy in which $W$ is replaced by $W^\varepsilon$. Let $\{u_n\}$ be an admissible sequence, i.e., $u_n \to g$ in $L^1$, $\nabla u_n \rightharpoonup G$ in $L^p$. Then, by Step 1,

$$\begin{aligned} &\mu_\varepsilon(\overline{\Omega}) = I_p^\varepsilon(g, G, \Omega) \\ &\leq \liminf_{n\to\infty} \left\{ \int_\Omega W(\nabla u_n(x)) + |\nabla u_n|^p\, dx + \int_{S(u_n)\cap\Omega} \psi([u_n](x), \nu_{u_n}(x))\, dH^{N-1} \right\} \\ &< \infty, \end{aligned}$$

and so, after extraction of a subsequence, there exists $\mu \in \mathcal{M}(\Omega)$ such that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$, and for every open set $V \subset \Omega$,

$$\begin{aligned} I_p(g, G, V) &\leq I_p^\varepsilon(g, G, V) \\ &= \mu_\varepsilon(V) \\ &\leq \mu_\varepsilon(\overline{V}); \end{aligned}$$

hence,

$$I_p(g, G, V) \leq \mu(\overline{V}). \tag{2.25}$$

Conversely, given $\varepsilon > 0$, there exists a sequence $\{v_n\}$ admissible for $I_p$ such that

$$I_p(g, G, V) + \varepsilon \geq \lim_{n \to \infty} \left\{ \int_V W(\nabla v_n(x))\, dx + \int_{S(v_n) \cap V} \psi([v_n](x), \nu_{v_n}(x))\, dH^{N-1} \right\},$$

and so, for $n$ sufficiently large,

$$\begin{aligned} \mu_\varepsilon(V) &\leq \liminf_\varepsilon \left\{ \int_V W^\varepsilon(\nabla v_n(x))\, dx + \int_{S(v_n) \cap V} \psi([v_n](x), \nu_{v_n}(x))\, dH^{N-1} \right\} \\ &\leq I_p(g, G, V) + \varepsilon + \varepsilon \int_V |\nabla u_n|^p\, dx \\ &\leq I_p(g, G, V) + C\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0^+$ we obtain

$$\mu(V) \leq \liminf_{\varepsilon \to 0^+} \mu_\varepsilon(V) \leq I_p(g, G, V).$$

It remains to prove that $\mu(V) \geq I_p(g, G, V)$. This follows by using the upper bound on $I_p$ (Lemma 2.18), (2.25), and proceeding exactly as in the last part of Step 1.

*Step 3.* The method used in Steps 1 and 2 to prove that $I_p$ is a Radon measure may fail for $I_0$, as we are not able to show that $I_0(g, G, \Omega)$ is realized by some admissible sequence $\{u_n\}$. Thus, we use the De Giorgi-Letta criterion (see [22]) to establish that $I_0(g, G, \cdot)$ is a measure. The following four conditions are necessary and sufficient for guaranteeing that $I_0(g, G, \cdot)$ is the trace of a Borel regular measure on the set of open subsets of $\Omega$. Let $B, C$ be open subsets of $\Omega$.

- (a) if $B \subset C$ then $I_0(g, G, B) \leq I_0(g, G, C)$;
- (b) If $B \cap C = \emptyset$ then $I_0(g, G, B \cup C) = I_0(g, G, B) + I_0(g, G, C)$;
- (c) $I_0(g, G, B \cup C) \leq I_0(g, G, B) + I_0(g, G, C)$;
- (d) $I_0(g, G, B) = \sup \{I_0(g, G, C) : C \subset\subset B\}$.

Conditions (a) and (b) hold trivially. Condition (d) follows by using the upper bound measure $\lambda$ for $I_0$ (Lemma 2.18) and the subadditivity, (Lemma 2.21). This brief argument is given in the last part of Step 1. To prove (c), it suffices to follow Proposition 2.10 of [29], noting that we have already established the subadditivity property, Lemma 2.21. □

# 3 The Bulk Density

We recall the definition of the density function $H_p(A, B), p \geq 1$, introduced in (2.16),

$$H_p(A, B) = \inf_u \left\{ \int_Q W(\nabla u)\, dx + \int_{S(u)} \psi([u], \nu_u)\, dH^{N-1} : u \in SBV(Q, \mathbb{R}^d), u|_{\partial Q} = Ax, \right.$$
$$\left. |\nabla u| \in L^p(Q), \int_Q \nabla u\, dx = B \right\},$$

where $A, B \in \mathbb{M}^{d\times N}$. We give the following limit characterization for $H_p$.

**Proposition 3.1** *Let $p \geq 1$ and assume that $(\mathcal{H}1)_p$,$(\mathcal{H}2)$, and $(\mathcal{H}4)$ hold. Then*

$$\begin{aligned} H_p(A,B) &= \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \left[ \int_Q W(\nabla u_n)\,dx + \int_{S(u_n)\cap Q} \psi([u_n],\nu_{u_n})\,dH^{N-1} \right] : u_n \in SBV(Q,\mathbb{R}^d), \right. \\ &\qquad \left. u_n \to Ax \text{ in } L^1,\ \nabla u_n \overset{*}{\rightharpoonup} B,\ \sup_n |\nabla u_n|_{L^p} < \infty \right\} \\ &=: \tilde{H}_p(A,B). \end{aligned}$$

**Proof.** *Step 1.* We prove that $\tilde{H}_p(A,B) \leq H_p(A,B)$.

Fix $u \in SBV(Q,\mathbb{R}^d)$ such that $u|_{\partial Q} = Ax$, $|\nabla u| \in L^p(Q)$, and $\int_Q \nabla u\,dx = B$. We write $u(x) = Ax+\phi(x)$, where $\phi \in SBV(Q,\mathbb{R}^d)$, $\phi|_{\partial Q} = 0$ , and

$$\int_Q \nabla\phi(x)\,dx = B - A.$$

Extend $\phi$ periodically, with period one, to $\mathbb{R}^N$ and define $u_n(x) := Ax + \frac{1}{n}\phi(nx)$. Then

$$u_n(x) \to Ax \text{ in } L^1, \quad \nabla u_n \overset{*}{\rightharpoonup} B, \quad \text{and } \sup_n |\nabla u_n|_{L^p} < \infty.$$

Thus, using $(\mathcal{H}2)$ we obtain

$$\begin{aligned} \tilde{H}_p(A,B) &\leq \liminf_{n\to\infty} \left\{ \int_Q W(\nabla u_n)\,dx + \int_{S(u_n)\cap Q} \psi([u_n](x),\nu_{u_n})\,dH^{N-1}(x) \right\} \\ &= \liminf_{n\to\infty} \left\{ \int_Q W(A+\nabla\phi(nx))\,dx + \int_{\frac{S(\phi)}{n}\cap Q} \psi\left(\frac{1}{n}[\phi](nx),\nu_\phi\right) dH^{N-1}(x) \right\} \\ &= \liminf_{n\to\infty} \left\{ \frac{1}{n^N}\int_{nQ} W(A+\nabla\phi(y))\,dy + \frac{1}{n^N}\int_{S(\phi)\cap nQ} \psi([\phi](y),\nu_\phi)\,dH^{N-1}(y) \right\} \\ &= \int_Q W(A+\nabla\phi(y))\,dy + \int_{S(\phi)\cap Q} \psi([\phi](y),\nu_\phi)\,dH^{N-1}(y). \end{aligned}$$

Taking the infimum over all such $\phi \in SBV$ we obtain $\tilde{H}_p(A,B) \leq H_p(A,B)$.

*Step 2.* We claim that $\tilde{H}_p(A,B) \geq H_p(A,B)$.

Let $\{u_n\}$ be an admissible sequence in $SBV(Q,\mathbb{R}^d)$, i.e., $u_n \to Ax$ in $L^1$, $\nabla u_n \overset{*}{\rightharpoonup} B$, and $\sup_n |\nabla u_n|_{L^p} < \infty$. Let $Q_k$ be the cube $(-\frac{1}{2}+\frac{1}{k},\frac{1}{2}-\frac{1}{k})^N$. Using the argument given in Lemma 2.21, for each $k$ we can find $Q'_k$ such that $Q_k \subset\subset Q'_k \subset\subset Q$, and $u_n^k$ such that

$$u_n^k(x) = Ax \quad \text{for } x \in \partial Q, \quad u_n^k(x) = u_n(x) \text{ for } x \in Q_k,$$

and for each $k$

$$u_n^k \to Ax, \quad \nabla u_n^k \overset{*}{\rightharpoonup} \chi_{Q'_k}B + \left(\chi_Q - \chi_{Q'_k}\right)A \quad \text{as } n\to\infty, \quad |\nabla u_n^k|_{L^p} \leq C\,(|\nabla u_n|_{L^p} + |A|) + 1.$$

Thus, we may take a diagonal subsequence $v_k := u_{n(k)}^k$ such that $v_k|_{\partial Q} = Ax$ , $v_k \to Ax$ in $L^1$, $\nabla v_k \overset{*}{\rightharpoonup} B$, and

$$\begin{aligned} &\limsup_{k\to\infty} \left\{ \int_Q W(\nabla v_k(x))\,dx + \int_{S(v_k)\cap Q} \psi([v_k],\nu_{v_k})\,dH^{N-1} \right\} \\ &\quad = \liminf_{n\to\infty} \left\{ \int_Q W(\nabla u_n(x))\,dx + \int_{S(u_n)\cap Q} \psi([u_n],\nu_{u_n})\,dH^{N-1} \right\}. \end{aligned} \tag{3.1}$$

Without loss of generality we may assume, upon extracting a subsequence, that $\limsup_k$ in (3.1) is $\lim_k$. Lastly, we modify $v_k$ to accommodate the condition on the average gradient, and we consider two cases.
*Case 1* ($p > 1$): By Lemma 2.10, there exists $r_k \to 1^-$ such that

$$\int_{\partial Q(0,r_k)} |v_k(x) - u_0(x)|\, dH^{N-1}(x) < \frac{1}{k}, \tag{3.2}$$

where $u_0(x) := Ax$. Define

$$w_k(x) := \begin{cases} u_0(x) & \text{if } x \in Q\backslash Q(0,r_k) \\ v_k(x) + C_k x & \text{if } x \in Q(0,r_k), \end{cases}$$

where $C_k$ is chosen such that $\int_Q \nabla w_k(x)\, dx = B$, that is,

$$C_k := \frac{1}{\mathcal{L}^N(Q(0,r_k))}\left[B - \int_{Q(0,r_k)} \nabla v_k\, dx - A\,\mathcal{L}^N(Q\backslash Q(0,r_k))\right].$$

Using the equi-integrability of the sequence $\nabla v_k$ and the fact that $\nabla v_k \overset{*}{\rightharpoonup} B$, we have

$$C_k \to 0 \quad \text{as } k \to \infty. \tag{3.3}$$

Clearly $w_k \to u_0$ in $L^1$, and by $(\mathcal{H}1)_p$, $(\mathcal{H}2)$, we have

$$\begin{aligned} \int_Q W(\nabla w_k)dx \;&+\; \int_{S(w_k)\cap Q} \psi([w_k](x),\nu_{w_k})dH^{N-1}(x) \le \int_Q W(\nabla v_k)dx + C(|A|^p+1)\mathcal{L}^N(Q\backslash Q(0,r_k)) \\ &+\; |C_k|\left\{1 + \int_Q |\nabla v_k|^{p-1}dx + \int_Q |\nabla w_k|^{p-1}dx\right\} + \int_{S(v_k)} \psi([v_k](x),\nu_{v_k})dH^{N-1}(x) \\ &+\; C' \int_{\partial Q(0,r_k)} |\mathrm{tr}\, v_k(x) + C_k x - u_0(x)| dH^{N-1}(x). \end{aligned}$$

Using (3.2), (3.3), and the fact that $\{\nabla v_k\}$ and $\{\nabla w_k\}$ are uniformly bounded in $L^p(Q,\mathbb{R}^D)$, we obtain

$$\begin{aligned} &\lim_{k\to\infty}\left\{\int_Q W(\nabla w_k)\,dx + \int_{S(w_k)} \psi([w_k](x),\nu_{w_k})\,dH^{N-1}(x)\right\} \\ &\quad\le \lim_{k\to\infty}\left\{\int_Q W(\nabla v_k)\,dx + \int_{S(v_k)} \psi([v_k](x),\nu_{v_k})\,dH^{N-1}(x)\right\}. \end{aligned}$$

*Case 2* ($p = 1$): By (3.1), $(\mathcal{H}2)$, using the fact that $\sup_n |\nabla v_k|_{L^1} < \infty$, and after extraction of a subsequence, we may find a Radon measure $\beta$ such that $||Dv_k|| \overset{*}{\rightharpoonup} \beta$. Thus, for all but a countable number of $\varepsilon > 0$,

$$\beta(\partial Q(0,1-\varepsilon)) = 0, \quad \text{and}\, ||Dv_k||(Q(0,1-\varepsilon)) \to \beta(Q(0,1-\varepsilon)) \quad \text{as } k \to \infty. \tag{3.4}$$

Fix such an $\varepsilon$, and define

$$w_{k,\varepsilon}(x) := \begin{cases} u_0(x) & \text{if } x \in Q\backslash Q(0,1-\varepsilon) \\ v_k(x) + C_{k,\varepsilon} x & \text{if } x \in Q(0,1-\varepsilon), \end{cases}$$

where $C_{k,\varepsilon}$ is chosen so that $\int_Q \nabla w_{k,\varepsilon}(x)\, dx = B$, i.e.,

$$C_{k,\varepsilon} := \frac{1}{\mathcal{L}^N(Q(0,1-\varepsilon))}\left[B - \int_{Q(0,1-\varepsilon)} \nabla v_k\, dx - A\,\mathcal{L}^N(Q\backslash Q(0,1-\varepsilon))\right].$$

The weak star convergence (in the sense of measures) of $\nabla v_k$ to $B$ implies that

$$\lim_{\varepsilon\to 0^+}\lim_{k\to\infty}|C_{k,\varepsilon}|=0; \tag{3.5}$$

hence,

$$\lim_{\varepsilon\to 0^+}\lim_{k\to\infty}|w_{k,\varepsilon}-u_0|_{L^1(Q)}=0.$$

Using $(\mathcal{H}1)_1, (\mathcal{H}2)$, we obtain

$$\begin{aligned}
\int_Q W(\nabla w_{k,\varepsilon})dx \;&+\; \int_{S(w_{k,\varepsilon})}\psi([w_{k,\varepsilon}](x),\nu_{w_{k,\varepsilon}(x)})dH^{N-1}(x)\\
&\le \int_Q W(\nabla v_k+C_{k,\varepsilon})dx+\int_{S(v_k)}\psi([v_k](x),\nu_{v_k}(x))dH^{N-1}(x)\\
&\quad +C\,(|A|+1)\mathcal{L}^N(Q\backslash Q(0,1-\varepsilon))+C'\int_{\partial Q(0,1-\varepsilon)}|\mathrm{tr}\, v_k(x)+C_{k,\varepsilon}x-Ax|\;dH^{N-1}\\
&\le \int_Q W(\nabla v_k)\,dx+C|C_{k,\varepsilon}|+\int_{S(v_k)}\psi([v_k](x),\nu_{v_k}(x))\,dH^{N-1}(x)\\
&\quad +C(|A|+1)\,\mathcal{L}^N(Q\backslash Q(0,1-\varepsilon))+C|C_{k,\varepsilon}|H^{N-1}(\partial Q(0,1-\varepsilon))\\
&\quad +C\int_{Q(0,1-\varepsilon)}|\mathrm{tr}\, v_k(x)-Ax|\,dH^{N-1}.
\end{aligned}\tag{3.6}$$

Next, in the spirit of Lemma 2.10, we address the asymptotic behavior of the last term in (3.6). Let $\phi_\delta\in C_0^\infty(Q)$ be a sequence of cut-off functions such that, $0\le\phi_\delta\le 1$, $\phi_\delta=0$ if $x\in Q(0,1-\varepsilon-2\delta)$, $\phi_\delta=1$ if $x\in Q\backslash Q(0,1-\varepsilon-\delta)$, and $|\nabla\phi_\delta|_{L^\infty}=O\left(\frac{1}{\delta}\right)$. By (2.5)

$$\begin{aligned}
\int_{\partial Q(0,1-\varepsilon)}|\mathrm{tr} v_k(x)-Ax|dH^{N-1} &= \int_{\partial Q(0,1-\varepsilon)}|\mathrm{tr}\phi_\delta(x)\,(v_k(x)-Ax)\,|dH^{N-1}\\
&\le \int_{Q(0,1-\varepsilon)}d||D\,(\phi_\delta\cdot(v_k-Ax))\,||+\int_{Q(0,1-\varepsilon)}|\phi_\delta\cdot(v_k-Ax)|dx\\
&\le \int_{Q(0,1-\varepsilon-\delta)\backslash Q(0,1-\varepsilon-2\delta)}d(||Dv_k||+|A|\mathcal{L}^N)+\frac{C}{\delta}\int_Q|v_k-Ax|dx.
\end{aligned}$$

Thus, from (3.4) we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\lim_{k\to\infty}\int_{Q(0,1-\varepsilon)}|\mathrm{tr} v_k(x)-Ax|dH^{N-1} &\le \lim_{\varepsilon\to 0^+}\lim_{\delta\to 0^+}\lim_{k\to\infty}\Big\{||Dv_k||(Q(0,1-\varepsilon-\delta)\backslash Q(0,1-\varepsilon-2\delta))\\
&\quad +|A|\mathcal{L}^N(Q(0,1-\varepsilon-\delta)\backslash Q(0,1-\varepsilon-2\delta))+\frac{C}{\delta}\int_Q|v_k-Ax|dx\Big\}\\
&\le \lim_{\varepsilon\to 0^+}\lim_{\delta\to 0^+}\left\{\beta\left(Q\left(0,1-\varepsilon-\frac{\delta}{2}\right)\backslash Q(0,1-\varepsilon-3\delta)\right)+|A|\mathcal{L}^N(Q(0,1-\varepsilon-\delta)\backslash Q(0,1-\varepsilon-2\delta))\right\}\\
&= \lim_{\varepsilon\to 0^+}\beta(\partial Q(0,1-\varepsilon))=0.
\end{aligned}\tag{3.7}$$

Finally, setting $\varepsilon=1/j$, we take a diagonal sequence of $w_{k,\varepsilon}$, $w_j:=w_{j,k(\frac{1}{j})}$, satisfying

$$w_j(x)|_{\partial Q}=Ax,\quad \int_Q\nabla w_j(x)\,dx=B,$$

and, by (3.5), (3.6), and (3.7),

$$\limsup_{j\to\infty}\left\{\int_Q W(\nabla w_j)\,dx+\int_{S(w_j)}\psi([w_j](x),\nu_{w_j}(x))\,dH^{N-1}(x)\right\}$$
$$\le \liminf_{k\to\infty}\left(\int_Q W(\nabla v_k)\,dx+\int_{S(v_k)}\psi([v_k](x),\nu_{v_k}(x))\,dH^{N-1}(x)\right\}.$$

□

The following characterization of the relaxed bulk density holds for $I_p$, $p\ge 1$, as well as for $I_0$.

**Theorem 3.2** *Let $p\ge 1$ and $W,\psi$ satisfy $(\mathcal{H}1)_p$, $(\mathcal{H}2)-(\mathcal{H}4)$. Then for $(g,G)\in SD(\Omega)$, with $g\in L^\infty$ if $p>1$, we have*

$$\frac{d\,I_p(g,G,\cdot)}{d\,\mathcal{L}^N}(x)=H_p(\nabla g(x),G(x))\quad \mathcal{L}^N \text{ a.e. } x,$$

*where $H_p$ is given by* (2.16). *If $p=1$ then for all $(g,G)\in SD(\Omega)$ we have*

$$\frac{d\,I_0(g,G,\cdot)}{d\,\mathcal{L}^N}(x)=H_1(\nabla g(x),G(x))\quad \mathcal{L}^N \text{ a.e. } x.$$

**Proof.** *Step 1.[Lower Bound]* Let $A\subset\Omega$ be an open set and let $I(g,G,\cdot)$ denote either $I_0(g,G,\cdot)$ or $I_1(g,G,\cdot)$ if $p=1$, and $I(g,G,\cdot)=I_p(g,G,\cdot)$ if $p>1$. We will prove that

$$I(g,G,A)\ge\int_A H_p(\nabla g(x),G(x))\,dx. \tag{3.8}$$

From (3.8) and from Proposition 2.22, it will follow that

$$\frac{d\,I(g,G,\cdot)}{d\,\mathcal{L}^N}(x)\ge H_p(\nabla g(x),G(x))\quad \mathcal{L}^N \text{ a.e. } x.$$

Let $\varepsilon>0$ and let $u_n$ be an admissible sequence for $I$ such that

$$\varepsilon+I(g,G,A)\ge\lim_{n\to\infty}\left\{\int_A W(\nabla u_n(x))\,dx+\int_{S(u_n)\cap A}\psi([u_n](x),\nu_{u_n}(x))\,dH^{N-1}(x)\right\}, \tag{3.9}$$

where $u_n\to g$ in $L^1$, $\sup_n|\nabla u_n|_{L^p}<\infty$, and $\nabla u_n\overset{*}{\rightharpoonup} m$ in $\mathcal{M}(A)$, with $m=G\cdot\mathcal{L}^N+m_s$, $m_s\perp\mathcal{L}^N$. By Theorems 2.4 and 2.6, for $\mathcal{L}^N$ a.e. $x_0\in\Omega$, we have

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N+1}}\int_{Q(x_0,\varepsilon)}|g(x)-g(x_0)-\nabla g(x_0)\cdot(x-x_0)|\,dx=0, \tag{3.10}$$

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}|G(x)-G(x_0)|^p\,dx=0, \tag{3.11}$$

and

$$\lim_{\varepsilon\to 0^+}\frac{d\,m_s}{d\,\mathcal{L}^N}(x_0)=0. \tag{3.12}$$

Choose such a point $x_0$. Upon extraction of a subsequence, which we do not relabel, there exists a non negative Radon measure $\mu$ such that

$$W(\nabla u_n)\,dx+\psi([u_n],\nu_{u_n})\,dH^{N-1}\lfloor S(u_n)\overset{*}{\rightharpoonup}\mu.$$

We claim that proving

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq H_p(\nabla g(x_0), G(x_0)) \tag{3.13}$$

implies the lower bound. Indeed, from (3.9) and (3.13), we have for all $A' \subset\subset A$

$$\begin{aligned}
\varepsilon + I(g, G, A) &\geq \int_{A'} d\mu \\
&\geq \int_{A'} \frac{d\mu}{d\mathcal{L}^N}(x)\, dx \\
&\geq \int_{A'} H_p(\nabla g(x), G(x))\, dx,
\end{aligned}$$

and (3.8) follows by letting $A' \nearrow A$ and $\varepsilon \to 0^+$. It remains to prove (3.13). Using the countable additivity property of $\mu$, choose radii $\varepsilon > 0$, $\varepsilon \to 0^+$, such that $\mu(\partial Q(x_0, \epsilon)) = 0$. By Theorem 2.6 we have

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\left\{\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} W(\nabla u_n)\, dx + \frac{1}{\varepsilon^N}\int_{S(u_n)\cap Q(x_0,\varepsilon)} \psi([u_n](x), \nu_{u_n})\, dH^{N-1}(x)\right\} \\
&= \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\left\{\int_Q W(\nabla u_n(x_0+\varepsilon y))dy + \frac{1}{\varepsilon}\int_{\frac{(S(u_n)-x_0)}{\varepsilon}\cap Q} \psi([u_n](x_0+\varepsilon y), \nu_{u_n})dH^{N-1}(y)\right\}.
\end{aligned}$$

Define

$$u_0(y) := \nabla g(x_0) y \quad \text{and} \quad u_{n,\varepsilon}(y) := \frac{u_n(x_0+\varepsilon y) - g(x_0)}{\varepsilon}.$$

By (3.10) we have

$$\lim_{\varepsilon\to 0^+}\lim_{n\to\infty} |u_{n,\varepsilon}(y) - u_0(y)|_{L^1(Q)} = 0,$$

and, due to the homogeneity of $\psi$,

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\left\{\int_Q W(\nabla u_{n,\varepsilon}(y))\, dy + \int_{S(u_{n,\varepsilon})} \psi([u_{n,\varepsilon}](y), \nu_{u_{n,\varepsilon}}(y))\, dH^{N-1}(y)\right\}. \tag{3.14}$$

*Case 1* : We assume coercivity, i.e., there exists a constant $C$ such that $C|A| \leq W(A)$ for all $A \in \mathbb{M}^{d\times N}$. Then (3.14) implies that

$$\sup_\varepsilon \sup_n |\nabla u_{n,\varepsilon}|_{L^p(Q)} < \infty.$$

Let $\phi \in C_0(Q)$. By (3.11) and (3.12) we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\int_Q [\nabla u_n(x_0+\varepsilon y) - G(x_0)]\,\phi(y)\, dy \\
&= \lim_{\varepsilon\to 0^+}\left\{\int_Q (G(x_0+\varepsilon y) - G(x_0))\,\phi(y)\, dy + \frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} \phi\left(\frac{x-x_0}{\varepsilon}\right) dm_s(x)\right\} \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} \phi\left(\frac{x-x_0}{\varepsilon}\right) d\mu_s(x),
\end{aligned}$$

and

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\left|\int_{Q(x_0,\varepsilon)} \phi\left(\frac{y-x_0}{\varepsilon}\right) dm_s(y)\right| \leq |\phi|_\infty \lim_{\varepsilon\to 0^+}\frac{m_s(Q(x_0,\varepsilon))}{\mathcal{L}^N(Q(x_0,\varepsilon))} = 0.$$

By virtue of the separability property of $C_0(Q)$, we may extract a diagonal subsequence $v_k \in SBV(Q, \mathbb{R}^d)$ such that

$$v_k(y) \to u_0(y) \quad \text{in } L^1(Q, \mathbb{R}^d), \quad \nabla v_k(y) \stackrel{*}{\rightharpoonup} G(x_0), \quad \sup_k |\nabla v_k|_{L^p} < \infty,$$

and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty} \left\{ \int_Q W(\nabla v_k(x))\,dx + \int_{S(v_k)\cap Q} \psi([v_k](x), \nu_{v_k}(x))\,dH^{N-1}(x) \right\}. \tag{3.15}$$

The inequality (3.13) follows from Proposition 3.1 and (3.15).

*Case 2*: Next, we remove the coercivity assumption. To this end, let $W^\varepsilon(\cdot) := W(\cdot) + \varepsilon|\cdot|^p$ and let $\{u_n\}$ be an admissible sequence for $I$ satisfying (3.9). Let $A \subset \Omega$ be open and let $\mu_\varepsilon \in \mathcal{M}(A)$ be such that

$$W^\varepsilon(\nabla u_n)d\mathcal{L}^N + \psi([u_n], \nu_{u_n})\,dH^{N-1}\lfloor S(u_n) \overset{*}{\rightharpoonup} \mu_\varepsilon.$$

By Case 1 we have

$$\frac{d\mu_\varepsilon}{d\mathcal{L}^N}(x_0) \geq H_p^\varepsilon(\nabla g(x_0), G(x_0)) \geq H_p(\nabla g(x_0), G(x_0)),$$

where $H_p^\varepsilon$ is given by (2.16) with $W^\varepsilon$ replacing $W$. This, combined with (3.9) and the uniform $L^p$ bound on $\{u_n\}$, gives for all $A' \subset\subset A$,

$$\begin{aligned}
\varepsilon + I(g, G, A) &\geq \lim_{n\to\infty} \int_A W(\nabla u_n)\,dx + \int_{S(u_n)\cap A} \psi([u_n], \nu_{u_n})\,dH^{N-1} \\
&\geq \lim_{n\to\infty} \int_A W_\varepsilon(\nabla u_n)\,dx + \int_{S(u_n)\cap A} \psi([u_n], \nu_{u_n})\,dH^{N-1} - \varepsilon \int_A |\nabla u_n(x)|^p\,dx \\
&\geq \int_{A'} d\mu_\varepsilon - \varepsilon C \\
&\geq \int_{A'} H_p(\nabla g(x), G(x))\,dx - \varepsilon C.
\end{aligned}$$

Letting $A' \nearrow A$ and then $\varepsilon \to 0^+$, we conclude that

$$I(g, G, A) \geq \int_A H_p(\nabla g(x), G(x))\,dx.$$

*Step 2: [Upper Bound]* Fix $\varepsilon > 0$ and consider an admissible sequence $u_n \in SBV(Q, \mathbb{R}^d)$ for $H_p(A, B)$, i.e., $u_n(x) \to u_0(x) := \nabla g(x_0)x$ in $L^1$, $\nabla u_n \overset{*}{\rightharpoonup} G(x_0)$, $\sup_n |u_n|_{L^p} < \infty$, and

$$\varepsilon + H_p(\nabla g(x_0), G(x_0)) \geq \lim_{n\to\infty} \left\{ \int_Q W(\nabla u_n)\,dx + \int_{S(u_n)\cap Q} \psi([u_n], \nu_{u_n})\,dH^{N-1} \right\}. \tag{3.16}$$

Using the argument given in Proposition 3.1, Step 2, we may assume, without loss of generality, that $u_n|_{\partial Q} = u_0$. Thus, we may write $u_n(x) = u_0(x) + \zeta_n(x)$ where

$$\zeta_n|_{\partial Q} = 0, \quad \zeta_n \to 0 \text{ in } L^1, \quad \sup_n |\nabla\zeta_n|_{L^p} < \infty, \quad \text{and} \quad \nabla\zeta_n \overset{*}{\rightharpoonup} (G(x_0) - \nabla g(x_0)).$$

We extend $\zeta_n$ periodically to all of $\mathbb{R}^N$, with period $Q$. By Theorem 2.8 there exist $h_\varepsilon \in SBV(Q(x_0, \varepsilon), \mathbb{R}^d)$ such that

$$\nabla h_\varepsilon(x) = G(x) - G(x_0) + \nabla g(x_0) - \nabla g(x) \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega,$$

and

$$||Dh_\varepsilon||(Q(x_0, \varepsilon)) \leq C(N) \int_{Q(x_0,\varepsilon)} |G(x) - G(x_0)| + |\nabla g(x_0) - \nabla g(x)|\,dx =: \alpha(\varepsilon),$$

where $x_0$ is chosen such that

$$\frac{1}{\varepsilon^N} \left\{ \int_{Q(x_0,\varepsilon)} |G(x) - G(x_0)|^p + |\nabla g(x) - \nabla g(x_0)|^p\,dx \right\} \to 0 \text{ as } \varepsilon \to 0^+$$

and

$$\frac{d|[g]|H^{N-1}\lfloor S(g)}{d\mathcal{L}^N}(x_0) = 0.$$

Hence,

$$\frac{\alpha(\varepsilon)}{\varepsilon^N} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.17}$$

By Lemma 2.9, there exist $h_{\varepsilon,n}$ piecewise constant such that for each $\varepsilon$,

$$h_{\varepsilon,n} \to -h_\varepsilon \text{ in } L^1(Q(x_0,\varepsilon),\mathbb{R}^d) \quad \text{and} \quad ||Dh_{\varepsilon,n}||(Q(x_0,\varepsilon))||Dh_\varepsilon||(Q(x_0,\varepsilon)) \tag{3.18}$$

as $n \to \infty$. Now define

$$w_{\varepsilon,n}(x) := g(x) + \varepsilon\,\zeta_n\left(\frac{x-x_0}{\varepsilon}\right) + h_\varepsilon + h_{\varepsilon,n}.$$

For each $\varepsilon > 0$, $w_{\varepsilon,n} \underset{n}{\to} g$ in $L^1$, $\sup_n |\nabla w_{\varepsilon,n}|_{L^p(Q(x_0,\varepsilon))} < \infty$, and

$$\nabla w_{\varepsilon,n}(x) = \nabla g(x_0) + \nabla\zeta_n\left(\frac{x-x_0}{\varepsilon}\right) + G(x) - G(x_0) \stackrel{*}{\rightharpoonup} G(x)$$

in $Q(x_0,\varepsilon)$ as $n \to \infty$. Thus $\{w_{\varepsilon,n}\}_n$ is an admissible sequence for $I$ for each $\varepsilon > 0$, and by $(\mathcal{H}4)$ we have

$$\begin{aligned}
\frac{dI(g,G,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to0+}\frac{I(g,G,Q(x_0,\varepsilon))}{\varepsilon^N} \\
&\leq \limsup_{\varepsilon\to0^+}\liminf_{n\to\infty}\left\{\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} W(\nabla w_{\varepsilon,n})\,dx + \frac{1}{\varepsilon^N}\int_{S(w_{\varepsilon,n})\cap Q(x_0,\varepsilon)}\psi([w_{\varepsilon,n}](x),\nu_{w_{\varepsilon,n}})\,dH^{N-1}(x)\right\} \\
&\leq \limsup_{\varepsilon\to0^+}\liminf_{n\to\infty}\left\{\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} W\left(\nabla g(x_0) + \nabla\zeta_n\left(\frac{x-x_0}{\varepsilon}\right) + G(x) - G(x_0)\right)dx\right. \\
&\quad + \frac{1}{\varepsilon^N}\int_{(x_0+\varepsilon S(\zeta_n))\cap Q(x_0,\varepsilon)}\psi\left(\varepsilon[\zeta_n]\left(\frac{x-x_0}{\varepsilon}\right),\nu_{\zeta_n}\right)dH^{N-1} \\
&\quad + \frac{1}{\varepsilon^N}\int_{S(g)\cap Q(x_0,\varepsilon)}\psi([g](x),\nu_g(x))\,dH^{N-1} + \frac{1}{\varepsilon^N}\int_{S(h_\varepsilon)\cap Q(x_0,\varepsilon)}\psi([h_\varepsilon](x),\nu_{h_\varepsilon})\,dH^{N-1} \\
&\quad \left. + \frac{1}{\varepsilon^N}\int_{S(h_{\varepsilon,n})\cap Q(x_0,\varepsilon)}\psi([h_{\varepsilon,n}](x),\nu_{h_{\varepsilon,n}})\,dH^{N-1}\right\} \\
&\leq \limsup_{\varepsilon\to0^+}\liminf_{n\to\infty}\left\{\int_{Q(0,1)} W(\nabla g(x_0) + \nabla\zeta_n(y))\,dy\right. \\
&\quad + \int_{Q(0,1)}\left[W(\nabla g(x_0) + \nabla\zeta_n(y) + G(x_0+\varepsilon y) - G(x_0)) - W(\nabla g(x_0) + \nabla\zeta_n(y))\right]dy \\
&\quad + \frac{1}{\varepsilon^N}\int_{(x_0+\varepsilon S(\zeta_n))\cap Q(x_0,\varepsilon)}\psi\left(\varepsilon[\zeta_n]\left(\frac{x-x_0}{\varepsilon}\right),\nu_{\zeta_n}\right)dH^{N-1} \\
&\quad + \frac{1}{\varepsilon^N}\int_{S(g)\cap Q(x_0,\varepsilon)}\psi([g](x),\nu_g)\,dH^{N-1} \\
&\quad \left. + \frac{1}{\varepsilon^N}\int_{S(h_\varepsilon)\cap Q(x_0,\varepsilon)}\psi([h_\varepsilon](x),\nu_{h_\varepsilon})\,dH^{N-1} + \frac{1}{\varepsilon^N}\int_{S(h_{\varepsilon,n})\cap Q(x_0,\varepsilon)}\psi([h_{\varepsilon,n}](x),\nu_{h_{\varepsilon,n}})\,dH^{N-1}\right\}.
\end{aligned}$$

On the other hand, by $(\mathcal{H}3)$ we have

$$\limsup_{\varepsilon\to0^+}\liminf_{n\to\infty}\frac{1}{\varepsilon^N}\int_{(x_0+\varepsilon S(\zeta_n))\cap Q(x_0,\varepsilon)}\psi\left(\varepsilon[\zeta_n]\left(\frac{x-x_0}{\varepsilon},\nu_{\zeta_n}\right)\right)dH^{N-1}$$

$$\leq \liminf_{n\to\infty} \int_{S(\zeta_n)\cap Q(0,1)} \psi([\zeta_n](y), \nu_{\zeta_n}(y))\, dH^{N-1}(y),$$

and so, by $(\mathcal{H}1)_p$ and (3.18) we conclude that

$$\begin{aligned}\frac{dI(g,G,\cdot)}{d\mathcal{L}^N}(x_0) \quad & \leq \liminf_{n\to\infty} \int_{Q(0,1)} W(\nabla u_n(y))\, dy + \limsup_{\varepsilon\to 0^+} \liminf_{n\to\infty} \int_{Q(0,1)} |G(x_0+\varepsilon y) - G(x_0)| \\ & (1 + |\nabla g(x_0) + \nabla \zeta_n(y) + G(x_0+\varepsilon y) - G(x_0)|^{p-1} + |\nabla g(x_0) + \nabla\zeta_n(y)|^{p-1})\, dy \\ & + \liminf_{n\to\infty} \int_{S(u_n)\cap Q(0,1)} \psi([u_n], \nu_{u_n}(x))\, dH^{N-1} \\ & + \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{S(g)\cap Q(x_0,\varepsilon)} \psi([g](x), \nu_g(x)) dH^{N-1} + C \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} ||Dh_\varepsilon||(Q(x_0,\varepsilon)).\end{aligned}$$

Since $\{\nabla\zeta_n\}$ are uniformly bounded in $L^p$, (3.16) - (3.17) and $(\mathcal{H}2)$ imply that

$$\frac{dI(g,G,\cdot)}{d\mathcal{L}^N}(x_0) \leq \varepsilon + H_p(\nabla g(x_0), G(x_0)) + \limsup_{\varepsilon\to 0^+} \frac{C}{\varepsilon^N} \int_{S(g)\cap Q(x_0,\varepsilon)} |[g](x)|\, dH^{N-1}.$$

The result follows by letting $\varepsilon \to 0^+$. □

**Remark 3.3** If $p > 1$, we may replace hypotheses $(\mathcal{H}2)$ and $(\mathcal{H}3)$ by

- $(\mathcal{H}2)^*$ there exists a constant $c > 0$ such that
$$0 \leq \psi(\lambda,\nu) \leq c|\lambda|$$
for all $(\lambda,\nu) \in \mathbb{R}^d \times S^{N-1}$;
- $(\mathcal{H}3)^*$ there exist constants $C, l, \alpha > 0$ such that
$$\left|\psi_0(\lambda,\nu) - \frac{\psi(t\lambda,\nu)}{t}\right| \leq Ct^\alpha$$
for every $(\lambda,\nu) \in \mathbb{R}^d \times S^{N-1}$ with $|\lambda| = 1$, $0 < t < l$, and where $\psi_0$ is the positively homogeneous of degree 1 function defined as
$$\psi_0(\lambda,\nu) := \limsup_{t\to 0^+} \frac{\psi(t\lambda,\nu)}{t}.$$

We must redefine the energy $I_p$ as follows:

$$\begin{aligned} I_p(g,G,\Omega) := \inf_{\{u_n\}} \Big\{ & \liminf_{n\to\infty} \left[\int_\Omega W(\nabla u_n)\, dx + \int_{S(u_n)} \psi([u_n], \nu_{u_n})\, dH^{N-1}\right] : u_n \in SBV(\Omega, \mathbb{R}^d), \\ & u_n \to g \text{ in } L^1(\Omega,\mathbb{R}^d),\ \nabla u_n \overset{*}{\rightharpoonup} G, \\ & \sup_n \left(|\nabla u_n|_{L^p(\Omega,\mathbb{M}^{d\times N})} + |u_n|_{BV(\Omega,\mathbb{R}^d)}\right) < \infty \Big\}. \end{aligned}$$

The integral representation for $I_p$ provided in Theorem 2.17 holds true, except that the new bulk density (Theorem 3.2) involves $\psi_0$ in place of $\psi$, that is,

$$\begin{aligned} H_p(A,B) := \inf_u \Big\{ & \int_Q W(\nabla u)\, dx + \int_{S(u)} \psi_0([u],\nu)\, dH^{N-1} : u \in SBV(Q,\mathbb{R}^d), u|_{\partial Q} = Ax, \\ & |\nabla u| \in L^p(Q), \int_Q \nabla u\, dx = B \Big\}. \end{aligned}$$

The proof of Theorem 3.2 is carried out with the obvious adaptions. As an example, (3.15) would read

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty}\left\{\int_Q W(\nabla v_k(x))\,dx + \frac{1}{\varepsilon_k}\int_{S(v_k)\cap Q}\psi([\varepsilon_k v_k](x), \nu_{v_k}(x))\,dH^{N-1}(x)\right\},$$

with $v_k \to u_0$ in $L^1$, $\nabla v_k \rightharpoonup G(x_0)$ in $L^p$, and $\sup_k ||Dv_k||(Q) < \infty$. Using the truncation argument introduced in the proof of Lemma 2.20, for all $\delta > 0$ we may find a new sequence $w_k = w_k(\delta)$, with the same convergence properties as $v_k$ and satisfying

$$\sup_k |w_k|_{L^\infty} \le C(\delta), \quad \sup_k \{|\nabla w_k|_{L^p} + ||Dw_k||(Q)\} < \infty,$$

and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \delta + \lim_{k\to\infty}\left\{\int_Q W(\nabla w_k(x))\,dx + \frac{1}{\varepsilon_k}\int_{S(w_k)\cap Q}\psi([\varepsilon_k w_k](x), \nu_{w_k}(x))\,dH^{N-1}(x)\right\}.$$

Since $w_k$ are uniformly bounded in $L^\infty$, and by virtue of $(\mathcal{H}3)^*$, we have

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &\ge \delta + \lim_{k\to\infty}\left\{\int_Q W(\nabla w_k(x))\,dx + \int_{S(w_k)\cap Q}\psi_0([w_k](x), \nu_{w_k}(x))\,dH^{N-1}(x)\right\}\\ &\ge \delta + H_p(\nabla g(x_0), G(x_0)).\end{aligned}$$

It suffices to let $\delta \to 0^+$ to conclude that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge H_p(\nabla g(x_0), G(x_0)).$$

We remark that replacing $(\mathcal{H}2) - (\mathcal{H}3)$ by $(\mathcal{H}2)^* - (\mathcal{H}3)^*$ may accommodate for surface densities $\psi$ which appear naturally in fracture mechanics, for example, functions $\psi(\lambda, \nu)$ which are sublinear in $\lambda$ and approach a constant as $|\lambda| \to \infty$.

# 4 The Crack Density

We will need the following limit characterizations of the functions $h_1$ and $h$.

**Proposition 4.1** *Assuming* $(\mathcal{H}1)_1$, $(\mathcal{H}2)$, $(\mathcal{H}4)$, *and* $(\mathcal{H}5)$, *we have*

$$\begin{aligned}h_1(\lambda,\nu) = \inf_{\{u_n\}}\Bigg\{&\liminf_{n\to\infty}\left[\int_{Q_\nu} W^\infty(\nabla u_n)\,dx + \int_{S(u_n)\cap Q_\nu}\psi([u_n], \nu_{u_n})\,dH^{N-1}\right] : u_n \in SBV(Q_\nu, \mathbb{R}^d),\\ &u_n \to u_{\lambda,\nu} \text{ in } L^1(Q_\nu(0,1), \mathbb{R}^d),\ \nabla u_n \overset{*}{\rightharpoonup} 0\Bigg\}.\end{aligned}$$

**Proof.** The proof of Proposition 4.1 is identical to that of Proposition 3.1. □

**Proposition 4.2** *Let* $p > 1$. *If* $(\mathcal{H}1)_p$, $(\mathcal{H}2)$, $(\mathcal{H}4)$, *and* $(\mathcal{H}5)$ *hold then*

$$\begin{aligned}h(\lambda) &= \inf_{\{u_n\}}\Bigg\{\liminf_{n\to\infty}\int_{S(u_n)\cap Q_\nu}\psi([u_n], \nu_{u_n})\,dH^{N-1} : u_n \in SBV(Q_\nu, \mathbb{R}^d), \nu \in S^{N-1},\\ &\qquad u_n \to u_{\lambda,\nu} \text{ in } L^1(Q_\nu(0,1), \mathbb{R}^d),\ \nabla u_n \to 0 \text{ in } L^p\Bigg\}\\ &=: \tilde{h}(\lambda).\end{aligned}$$

**Proof.** To prove that $\tilde{h} \leq h$, we consider $u = u_{\lambda,e_N} + \phi$ with $\phi|_{\partial Q} = 0, \nabla\phi = 0$ $\mathcal{L}^N$ a.e. Extending $\phi$ periodically to all of $\mathbb{R}^N$ with period $Q$, and setting $u_n := u_{\lambda,e_N} + n^{-1}\phi(nx)$, it is easy to see that $u_n \to u_{\lambda,e_N}$ in $L^1$, $\nabla u_n = 0$ $\mathcal{L}^N$ a.e., and

$$\int_{S(u_n)\cap Q} \psi([u_n], \nu_{u_n})\, dH^{N-1} \to \int_{S(u)\cap Q} \psi([u], \nu_u)\, dH^{N-1} \quad \text{as } n \to \infty.$$

Conversely, let $\nu \in S^{N-1}$ and let $u_n \in SBV(Q_\nu, \mathbb{R}^d)$ be such that $u_n \to u_{\lambda,\nu}$ in $L^1$ and $\nabla u_n \to 0$ in $L^p$ strong. By Theorem 2.8, for each $n$ we choose $f_n \in SBV(Q_\nu(0,1), \mathbb{R}^d)$ such that $\nabla f_n = \nabla u_n$ $\mathcal{L}^N$ a.e. and $||Df_n||(Q_\nu) \leq C|\nabla u_n|_{L^1(Q_\nu}$. By Lemma 2.9, there exist $g_{n,m}$ piecewise constant such that $g_{n,m} \underset{m}{\to} f_n$ and $||Dg_{n,m}||(Q_\nu) \underset{m}{\to} ||Df_n||(Q_\nu)$. Let

$$w_{n,m} := u_n - f_n + g_{n,m}.$$

Clearly, $\nabla w_{n,m} = 0$ $\mathcal{L}^N$ a.e. and $\lim_n \lim_m |w_{n,m} - u_{\lambda,\nu}|_{L^1} = 0$. Moreover, using $(\mathcal{H}2)$ and the fact that

$$||D_s f_n||(Q_\nu) + ||D_s g_{n,m}||(Q_\nu) \leq C \int_{Q_\nu} |\nabla u_n|\, dx \to 0 \quad \text{as } n \to \infty,$$

we have

$$\lim_{n\to\infty} \lim_{m\to\infty} \int_{S(w_{n,m})\cap Q_\nu} \psi([w_{n,m}], \nu_{w_{n,m}})\, dH^{N-1} \leq \lim_{n\to\infty} \int_{S(u_n)\cap Q_\nu} \psi([u_n], \nu_{u_n})\, dH^{N-1}.$$

Hence, we may extract a diagonal sequence in $m, n$, say $v_k$ such that $v_k \to u_{\lambda,\nu}$ in $L^1(Q_\nu, \mathbb{R}^d)$, $\nabla v_k = 0$ a.e., and

$$\lim_{k\to\infty} \int_{S(v_k)\cap Q_\nu} \psi([v_k], \nu_{v_k})\, dH^{N-1} \leq \lim_{n\to\infty} \int_{S(u_n,)\cap Q_\nu} \psi([u_n], \nu_{u_n})\, dH^{N-1}.$$

Next, we amend the sequence $v_k$ to equal $u_{\lambda,\nu}$ on $\partial Q_\nu$. To this end, by Fubini's Theorem there exists $r_k \to 1^-$ such that, upon extracting a subsequence,

$$\int_{\partial Q_\nu(0,1-r_k)} |\text{tr } v_k - u_{\lambda,\nu}|\, dH^{N-1} \underset{k}{\to} 0. \tag{4.1}$$

Define

$$\tilde{v}_k(x) := \begin{cases} v_k & \text{if } x \in Q_\nu(0, 1-r_k) \\ u_{\lambda,\nu} & \text{if } Q_\nu(0,1)\backslash Q_\nu(0, 1-r_k). \end{cases}$$

Clearly $\nabla\tilde{v}_k = 0$ a.e., and by (4.1), $(\mathcal{H}2)$, we have

$$\lim_{k\to\infty} \int_{S(\tilde{v}_k)\cap Q_\nu} \psi([\tilde{v}_k], \nu_{\tilde{v}_k})\, dH^{N-1} \leq \lim_{n\to\infty} \int_{S(u_n)\cap Q_\nu} \psi([u_n], \nu_{u_n})\, dH^{N-1}. \tag{4.2}$$

Let $R$ be a rotation such that $Re_N = \nu$, and set $\theta_k := \tilde{v}_k(Rx)$. It follows that $\{\theta_k\}$ is an admissible sequence for $h$ and

$$\int_{S(\theta_k)\cap Q} \psi([\theta_k], \nu_{\theta_k})\, dH^{N-1} = \int_{S(\tilde{v}_k)\cap Q_\nu} \psi([\tilde{v}_k], \nu_{\tilde{v}_k})\, dH^{N-1},$$

which, together with (4.2), concludes the proof. □

We will also need the following continuity property for $h_1$ and $h$.

**Proposition 4.3** *Let $W, \psi$ satisfy $(\mathcal{H}1)_p - (\mathcal{H}5)$. Then there exists a constant $C$ such that*

$$|h_1(\lambda,\nu) - h_1(\lambda',\nu)| \le C|\lambda - \lambda'| \quad \text{if } p = 1 \quad \text{and} \quad |h(\lambda) - h(\lambda')| \le C|\lambda - \lambda'| \quad \text{if } p > 1.$$

*Also, if $p = 1$, then $h_1$ is upper semicontinuous with respect to $\nu$.*

**Proof.** We start by proving that

$$h_1(\lambda,\nu) \le h_1(\lambda',\nu) + C|\lambda - \lambda'|. \tag{4.3}$$

Fix $\varepsilon > 0$. Using Proposition 4.1, let $\{u_n\}$ be a sequence in $SBV(Q_\nu, \mathbb{R}^d)$ such that $u_n \to u_{\lambda',\nu}$ in $L^1(Q_\nu, \mathbb{R}^d)$, $\nabla u_n \overset{*}{\rightharpoonup} 0$, and

$$\varepsilon + h_1(\lambda',\nu) \ge \lim_{n\to\infty} \left\{ \int_{Q_\nu} W^\infty(\nabla u_n)\, dx + \int_{S(u_n)\cap Q_\nu} \psi([u_n], \nu_{u_n})\, dH^{N-1} \right\}.$$

By Lemma 2.9 we may find a sequence of piecewise functions $\{v_n\}$ such that

$$v_n \to u_{\lambda,\nu} - u_{\lambda',\nu},\ ||Dv_n||(Q_\nu) \to ||D(u_{\lambda,\nu} - u_{\lambda',\nu})||(Q_\nu) = |\lambda - \lambda'|.$$

Then

$$w_n := u_n + v_n \to u_{\lambda,\nu} \quad \text{in } L^1(Q_\nu, \mathbb{R}^d), \quad \nabla w_n \overset{*}{\rightharpoonup} 0,$$

and so, by Proposition 4.1,

$$\begin{aligned} h_1(\lambda,\nu) &\le \liminf_{n\to\infty} \int_{S(w_n)\cap Q_\nu} \psi([w_n], \nu_{w_n})\, dH^{N-1} + \varepsilon \\ &\le \liminf_{n\to\infty} \left\{ \int_{S(u_n)\cap Q_\nu} \psi([u_n], \nu_{u_n})\, dH^{N-1} + \int_{S(v_n)\cap Q_\nu} \psi([v_n], \nu_{v_n})\, dH^{N-1} \right\} + \varepsilon \\ &\le h_1(\lambda',\nu) + \varepsilon + |\lambda - \lambda'|, \end{aligned}$$

where we have used the subadditivity of $\psi$. The inequality converse to (4.3) is proven in the same way. Also, this argument is valid for $h$ as well.

Next, we show that, for fixed $\lambda$,

$$\nu \mapsto h_1(\lambda,\nu) \quad \text{is upper semicontinuous.}$$

We follow the proof of Proposition 3.6, iv, in [7]. By (2.12) we have

$$h_1(\lambda,\nu) = \inf_u \left\{ \int_Q W^\infty(\nabla u\, R^T)\, dx + \int_{S(u)\cap Q} \psi([u], \nu_u)\, dH^{N-1} : R \quad \text{is a rotation}, Re_N = \nu, \right.$$
$$\left. u \in SBV(Q, \mathbb{R}^d),\ u|_{\partial Q} = u_{\lambda, e_N},\ \int_Q \nabla u\, dx = 0 \right\}.$$

Let $\nu_n \to \nu$ and choose a rotation $R$ such that $Re_N = \nu$. Fix $\varepsilon > 0$ and let $u_\varepsilon \in SBV(Q, \mathbb{R}^d)$, $u_\varepsilon = u_{\lambda, e_N}$, $\int_Q \nabla u_\varepsilon\, dx = 0$ and

$$\left| h_1(\lambda,\nu) - \int_Q W^\infty(\nabla u_\varepsilon\, R^T)\, dx + \int_{S(u_\varepsilon)\cap Q} \psi([u_\varepsilon], \nu_{u_\varepsilon})\, dH^{N-1} \right| < \varepsilon.$$

Considering a sequence of rotations $\{R_n\}$ such that $R_n \to R$ and $R_n e_n = \nu_n$, we use the Lipschitz continuity of $W^\infty$ to conclude that

$$\begin{aligned} h(\lambda,\nu) &\le \liminf_{n\to\infty}\left\{\int_Q W^\infty(\nabla u_\varepsilon R_n^T)\,dx + \int_{S(u_\varepsilon)\cap Q}\psi([u_\varepsilon],\nu_{u_\varepsilon})\,dH^{N-1}\right\} \\ &= \int_Q W^\infty(\nabla u_\varepsilon R^T)\,dx + \int_{S(u_\varepsilon)\cap Q}\psi([u_\varepsilon],\nu_{u_\varepsilon})\,dH^{N-1} \\ &\le h(\lambda,\nu)+\varepsilon. \end{aligned}$$

It suffices to let $\varepsilon \to 0^+$. □

**Theorem 4.4** *Let $W,\psi$ satisfy $(\mathcal{H}1)_1$, $(\mathcal{H}2)$, $(\mathcal{H}4)$ and $(\mathcal{H}5)$. Then*

$$\frac{d I_1(g,G,\cdot)}{d(|[g]|\,H^{N-1}\lfloor S(g))}(x) = \frac{1}{|[g](x)|}h_1([g](x),\nu_g(x)) \qquad H^{N-1}\text{ a.e. } x\in S(g), \tag{4.4}$$

*where $h_1$ is given by* (2.12).

**Proof.** *Step 1. [Lower Bound]* Fix $\varepsilon > 0$ and assume that $x_0 \in \Omega\cap S(g)$ satisfies the equalities in Theorem 2.4 ii) with respect to $g$ and, in addition,

$$\lim_{\delta\to 0}\frac{|[g]|H^{N-1}(S(g)\cap Q_{\nu(x_0)})}{\delta^{N-1}} = |[g(x_0)]|, \quad\text{and}\quad \lim_{\delta\to 0^+}\frac{1}{\delta^{N-1}}\int_{Q_\nu(x_0,\delta)}|G(x)|\,dx = 0. \tag{4.5}$$

It is well known that (4.5) can be guaranteed for $H^{N-1}$ a.e. $x_0$ (see Ziemer [Zi] ). Let $A$ be an open subset of $\Omega$ and let $\{u_n\}$ be an admissible sequence for $I_1$ such that

$$\varepsilon + I_1(g,G,A) \ge \lim_n \int_A W(\nabla u_n(x))\,dx + \int_{S(u_n)\cap A}\psi([u_n](x),\nu_{u_n}(x))\,dH^{N-1}(x), \tag{4.6}$$

$u_n \to g$ in $L^1$ and $\nabla u_n \overset{*}{\rightharpoonup} G$. Up to extraction of a subsequence, which we do not relabel, there exists a non negative Radon measure $\mu$ such that

$$W(\nabla u_n)\,\mathcal{L}^N + \psi([u_n],\nu_{u_n})\,dH^{N-1}\lfloor S(u_n) \overset{*}{\rightharpoonup} \mu.$$

By (4.6), the inequality

$$\frac{d\mu}{d(|g^+-g^-|\,H_{N-1}\lfloor S(g))}(x_0) \ge \frac{1}{|[g](x_0)|}h_1([g](x_0),\nu_g(x_0)) \quad \text{for } H^{N-1}\,a.e.\, x_0\in S(g) \tag{4.7}$$

yields the lower bound, after letting $\varepsilon \to 0^+$. Choosing a sequence $\varepsilon \to 0^+$ such that $\mu(\partial B(x_0,\epsilon)) = 0$, we have

$$\begin{aligned} \frac{d\mu}{d(|[g]|\,H^{N-1}\lfloor S(g))}(x_0) &= \frac{1}{|[g(x_0)]|}\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\frac{1}{\varepsilon^{N-1}}\left\{\int_{Q_{\nu(x_0)}(x_0,\varepsilon)} W(\nabla u_n(x))\,dx \right.\\ &\qquad \left. + \int_{S(u_n)\cap Q_{\nu(x_0)}(x_0,\varepsilon)}\psi([u_n](x),\nu_{u_n})\,dH^{N-1}(x)\right\} \\ &= \frac{1}{|[g(x_0)]|}\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\left\{\varepsilon\int_{Q_{\nu(x_0)}} W(\nabla u_n(x_0+\varepsilon y))\,dy \right.\\ &\qquad \left. + \int_{\frac{(S(u_n)-x_0)}{\epsilon}\cap Q_{\nu(x_0)}}\psi([u_n](x_0+\varepsilon y),\nu_n)\,dH^{N-1}(y)\right\}. \end{aligned}$$

Define

$$u_{n,\varepsilon}(y) := u_n(x_0 + \varepsilon y) - g^-(x_0).$$

By Theorem 2.4 ii) we have

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \left| u_{n,\varepsilon}(y) - u_{[g](x_0),\nu(x_0)}(y) \right|_{L^1(Q_{\nu(x_0)})} = 0,$$

and

$$\begin{aligned} \frac{d\mu}{d(|[g]|\, H^{N-1} \lfloor S(g))}(x_0) &= \frac{1}{|[g(x_0)]|} \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \left\{ \varepsilon \int_{Q_{\nu(x_0)}} W\left( \frac{1}{\varepsilon} \nabla u_{n,\varepsilon}(y) \right) dy \right. \\ &\quad \left. + \int_{S(u_{n,\varepsilon}) \cap Q_{\nu(x_0)}} \psi([u_{n,\varepsilon}](y), \nu_{u_{n,\varepsilon}}(y))\, dH^{N-1}(y) \right\}. \end{aligned} \tag{4.8}$$

Now let $\phi \in C_0(Q_{\nu(x_0)}(0,1))$. Using (4.5) we deduce that

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_{Q_{\nu(x_0)}(0,1)} \phi(y) \nabla u_{n,\varepsilon}(y) &= \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{\varepsilon^{N-1}} \int_{Q_{\nu(x_0)}(x_0,\varepsilon)} \phi\left( \frac{x - x_0}{\varepsilon} \right) \nabla u_n(x)\, dx \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{Q_{\nu(x_0)}(x_0,\varepsilon)} \phi\left( \frac{x - x_0}{\varepsilon} \right) G(x)\, dx \\ &= 0. \end{aligned}$$

*Case 1*: Assume that $W$ is coercive, i.e., there exists a constant $C > 0$ such that $C|A| \le W(A)$ for all $A \in \mathbb{M}^{d \times N}$. Using ($\mathcal{H}5$) and the fact that coercivity implies a uniform $L^1$ bound for $\{\nabla u_{n,\varepsilon}\}$, we may follow the arguments given in the proof of Theorem 4.1, Step 3, of [7], to obtain (4.8) with equality replaced by greater than or equal to, and $\varepsilon W\left(\frac{\cdot}{\varepsilon}\right)$ replaced by $W^\infty(\cdot)$. Next, we choose a diagonal sequence in $\varepsilon, n$, and a countable dense collection of functions in $C_0(Q_{\nu(x_0)}(0,1))$ to obtain $v_k$ such that

$$v_k \to u_{[g](x_0),\nu_g(x_0)} \text{ in } L^1, \quad \nabla v_k \overset{*}{\rightharpoonup} 0,$$

and

$$\frac{d\mu}{d\left(|[g]| H_{N-1} \lfloor S(g)\right)}(x_0) \ge \frac{1}{|[g(x_0)]|} \lim_{k \to \infty} \left\{ \int_{Q_{\nu(x_0)}} W^\infty\left(\nabla v_k(x)\right) dx + \int_{S(v_k)} \psi([v_k](y), \nu_{v_k}(y))\, dH^{N-1}(y) \right\}$$

and the result now follows from Proposition 4.1.

*Case 2 :* Proceeding as in Step 1 (Case 2) of Theorem 3.2, we may remove the coercivity assumption. The argument is the same except $\mathcal{L}^N$ is replaced by $|[g]|^{-1} H^{N-1} \lfloor S(g)$.

*Step 2: [Upper Bound]* In view of (4.5), we only need to prove that given $(g, G) \in SD$, for any open $A \subset \Omega$

$$I_1(g, G, A) \le \int_A C(N)(1 + |G(x)|^p)\, dx + \int_{S(g) \cap A} h_1([g](x), \nu(x))\, dH^{N-1}(x). \tag{4.9}$$

Moreover, we claim it suffices to prove (4.9) for $g$ of the form $g = \lambda \chi_E$, where $\chi_E$ is the characteristic function of a set of finite perimeter $E$. This follows from an argument of Ambrosio, Mortola, and Tortorelli given in Proposition 4.8 of [6], and which involves continuity and semi-continuity properties of $h_1$ (see Proposition 4.3).

*Case 1*: Suppose that $E$ is a polygon and $W$ is coercive. We use a Besicovitch covering argument introduced by Braides and Piat [13]. Let $g = \lambda \chi_E$ and $G \in L^1(\Omega, \mathbb{M}^{d \times N})$. Fix $A \subset \Omega$ open, $\delta > 0$, and let $x_0$ be a Lebesgue point for the function $h_1(\lambda, \nu(\cdot))$ with respect to $H^{N-1} \lfloor S(g)$. Then there exists $\varepsilon_{x_0} < \delta$ such that for every $0 < \varepsilon < \varepsilon_{x_0}$,

$$h_1(\lambda, \nu(x_0)) \le \frac{1}{\varepsilon^{N-1}} \int_{S(g) \cap Q_{\nu(x_0)}(x_0,\varepsilon)} h_1(\lambda, \nu(y))\, dH^{N-1}(y) + \delta. \tag{4.10}$$

By the definition of $h_1$ (see(2.12) ), there exists $u_{x_0}$ such that

$$u_{x_0}|_{\partial Q_{\nu(x_0)}(0,1)} = u_{\lambda,\nu(x_0)}, \qquad \int_{Q_{\nu(x_0)}(0,1)} \nabla u_{x_0}\, dx = 0, \tag{4.11}$$

and

$$h_1(\lambda, \nu(x_0)) \geq \int_{Q_{\nu(x_0)}(0,1)} W^\infty(\nabla u_{x_0}(y))\, dy + \int_{S(u_{x_0})\cap Q_{\nu(x_0)}} \psi([u_{x_0}](y))\, dH^{N-1}(y) - \delta. \tag{4.12}$$

Let

$$X := \{x \in A \cap S(g) \,:\, (4.10), (4.12) \text{ hold at } x\}.$$

Note that $H^{N-1}((A\cap S(g))\backslash X) = 0$. Let

$$A' := \bigcup \{Q_{\nu(x)}(x,\varepsilon) \,:\, x \in X, 0 < \varepsilon < \varepsilon_x, Q_{\nu(x)}(x,\varepsilon) \subset A\}.$$

The set of cubes $Q_{\nu(x)}(x,\varepsilon)$ covers $A'$ *finely*, and so by Besicovitch's Covering Theorem there exist $x_i$ and $\varepsilon_i$, $i = 1, \ldots$, such that $A'$ is the disjoint union of $\{Q_{\nu(x_i)}(x_i, \varepsilon_i)\}$. For simplicity of notation, set $u_i := u_{x_i}$, $\nu_i := \nu(x_i)$, $Q_i := Q_{\nu_i}(x_i, \varepsilon_i)$, and $Q_i'$ is the projection of $Q_i$ onto the hyperplane perpendicular to $\nu_i$, passing through $x_i$. Extend by periodicity $u_i(\cdot, y_N)$ to the strip $\left\{x : \left|\left(\frac{x-x_i}{\varepsilon_i}\right)\cdot \nu_i\right| < \frac{1}{2(2k+1)}\right\}$, with period $Q_i'$. Let

$$D_{i,k} := Q_i(x_i,\varepsilon_i) \cap \left\{x : \left|\left(\frac{x-x_i}{\varepsilon_i}\right)\cdot \nu_i\right| < \frac{1}{2(2k+1)}\right\}, \quad \text{and} \quad A_k'' := \bigcup_i D_{i,k}.$$

Note that, due to the polyhedral nature of $E$, for every $k$ we have $S(g)\cap A \subset A_k''$. For $y \in Q_{\nu_i}$, let $y = (y', y_N)$ where $y_N \in R$ is the component of $y$ along $\nu_i$, and define

$$u_{\delta,k}(x) := \begin{cases} \lambda & \text{if } x \notin O'', x \in E \\ u_i\left((2k+1)\frac{x-x_i}{\varepsilon_i}\right) & \text{if } x \in D_{i,k} \\ 0 & \text{if } x \notin O'', x \notin E. \end{cases}$$

We have

$$\begin{aligned} |u_{\delta,k}(x) - g(x)|_{L^1(A)} &= \sum_i \int_{D_{i,k}} |u_{\delta,k}(x) - g(x)|\, dx \\ &\leq |\lambda| \sum_i \varepsilon_i^N + \sum_i \int_{D_{i,k}} |u_{\delta,k}(x)|\, dx, \end{aligned} \tag{4.13}$$

where

$$\sum_i \varepsilon_i^N \leq \delta \sum_i \varepsilon_i^{N-1} \leq \delta H^{N-1}(A\cap S(g)) = O(\delta), \tag{4.14}$$

and

$$\begin{aligned} \sum_i \int_{D_{i,k}} |u_{\delta,k}(x)|\, dx &= \sum_i \varepsilon_i^N \int_{-\frac{1}{2(2k+1)}}^{\frac{1}{2(2k+1)}} \int_{Q_i'(0,1)} |u_i((2k+1)y', (2k+1)y_N)|\, dy' dy_N \\ &= \sum_i \varepsilon_i^N \frac{1}{2k+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Q_i'(0,1)} |u_i((2k+1)y', z)|\, dy' dz, \end{aligned}$$

where $z := (2k+1)y_N$. Since the inner integral tends to

$$\int_{Q_i'(0,1)} |u_i(y', z)|\, dy'$$

as $k \to \infty$, and in view of (4.13) and (4.14), we conclude that

$$\lim_{\delta}\lim_{k} |u_{\delta,k}(x) - g(x)|_{L^1(A)} = 0.$$

Next we show that, for each $\delta > 0$, $\nabla u_{\delta,k} \overset{*}{\rightharpoonup} 0$ as $k \to \infty$. To this end, fix $\delta > 0$ and let $\phi \in C_0(A)$.

$$\int_A \nabla u_{\delta,k}(x)\phi(x)\,dx = \sum_i \int_{D_{i,k}} \frac{2k+1}{\varepsilon_i}\nabla u_i\left((2k+1)\frac{x-x_i}{\varepsilon_i}\right)\cdot\phi(x)\,dx$$

$$= \sum_i {\varepsilon_i}^{N-1} \int_{Q_i(0,1)\cap\left\{y:|y_N|<\frac{1}{2(2k+1)}\right\}} (2k+1)\nabla u_i((2k+1)y)\phi(x_i+\varepsilon_i y)\,dy$$

$$= \sum_i {\varepsilon_i}^{N-1} \int_{-\frac{1}{2(2k+1)}}^{\frac{1}{2(2k+1)}} \int_{Q_i'(0,1)} (2k+1)\nabla u_i((2k+1)y',(2k+1)y_N)\phi(x_i+\varepsilon_i y', x_i+\varepsilon_i y_N)dy'dy_N$$

$$= \sum_i {\varepsilon_i}^{N-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Q_i'(0,1)} \nabla u_i((2k+1)y',z)\phi(x_i+\varepsilon_i y', x_i+\varepsilon_i z(2k+1)^{-1})dy'dz.$$

Let

$$A_i(z) := \int_{Q_i'(0,1)} \nabla u_i(y',z)\,dy'.$$

Due to the periodicity of $\nabla u_i(\cdot,z)$ and the fact that $\phi(x_i-\varepsilon_i y', x_i-\varepsilon_i z(2k+1)^{-1})$ converges uniformly, as $k \to \infty$, to $\phi(x_i-\varepsilon_i y', x_i)$, we have

$$\begin{aligned}\int_A \nabla u_{\delta,k}(x)\phi(x)\,dx \quad &\underset{k\to\infty}{\longrightarrow} \quad \sum_i {\varepsilon_i}^{N-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} A_i(z)\int_{Q_i'(0,1)} \phi(x_i+\varepsilon_i y', x_i)\,dy'dz \\ &= \quad \sum_i {\varepsilon_i}^{N-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} A_i(z)\,dz \int_{Q_i'(0,1)} \phi(x_i+\varepsilon_i y', x_i)\,dy'.\end{aligned}$$

By (4.11), $\int_{-\frac{1}{2}}^{\frac{1}{2}} A_i(z)\,dz = 0$, and so $\nabla u_{\delta,k} \overset{*}{\rightharpoonup} 0$ as $k \to \infty$. Using Theorem 2.8, we may find $h \in SBV(A,\mathbb{R}^D)$ such that

$$\nabla h = G, \quad ||Dh||(A) \le C(N)|G|_{L^1(A,\mathbf{M}^{d\times N})}, \tag{4.15}$$

and, by virtue of Lemma 2.9, we consider $v_k \in SBV(A,\mathbb{R}^D)$ piecewise constant such that $v_k \to h$ in $L^1$ and $||Dv_k||(A) \to ||Dh||(A)$. Set

$$w_{\delta,k}(x) := u_{\delta,k}(x) + h(x) - v_k(x).$$

By the definition of $I_1$, $(\mathcal{H}2)$ and $(\mathcal{H}4)$, we have

$$\begin{aligned} I_1(g,G,A) \;&\le\; \liminf_{\delta\to 0^+}\liminf_{k\to\infty}\left\{\int_A W(\nabla w_{\delta,k}(x))\,dx + \int_{S(w_{\delta,k})\cap A}\psi([w_{\delta,k}(x)])\,dH^{N-1}(x)\right\} \\ &\le\; \liminf_{\delta\to 0^+}\liminf_{k\to\infty}\left\{\int_A W(\nabla u_{\delta,k}(x))\,dx + C\int_A |G|\,dx + \int_{S(u_{\delta,k})\cap A}\psi([u_{\delta,k}(x)])\,dH^{N-1}(x)\right. \\ &\qquad \left. + \int_{S(h)\cap A} C|[h](x)|\,dH^{N-1}(x) + \int_{S(v_k)\cap A} C|[v_k](x)|\,dH^{N-1}(x)\right\} \\ &\le\; C(N)\int_A |G(x)|\,dx + \liminf_{\delta\to 0^+}\liminf_{k\to\infty}\sum_i\left[\int_{D_i} W\left(\frac{(2k+1)}{\varepsilon_i}\nabla u_i\left((2k+1)\left(\frac{x-x_i}{\varepsilon_i}\right)\right)\right)dx\right. \\ &\qquad \left. + \int_{\frac{x_i+\varepsilon_i S(u_i)}{2k+1}\cap D_i} \psi\left([u_i]\left((2k+1)\left(\frac{x-x_i}{\varepsilon_i}\right)\right)\right)dH^{N-1}(x)\right] \end{aligned}$$

$$
\begin{aligned}
&= C(N)\int_A |G(x)|\,dx \\
&\quad + \liminf_{\delta\to 0^+}\liminf_{k\to\infty}\left\{\sum_i \varepsilon_i^N \int_{\frac{-1}{2(2k+1)}}^{\frac{1}{2(2k+1)}} \int_{Q_i'} W\left(\frac{(2k+1)}{\varepsilon_i}\nabla u_i((2k+1)y)\right) dy'dy_N \right.\\
&\quad + \left.\sum_i \varepsilon_i^{N-1}\int_{\frac{S(u_i)}{2k+1}\cap Q_i\cap\left\{y:|y_N|\le\frac{1}{2(2k+1)}\right\}} \psi\left([u_i]\left((2k+1)y\right)\right) dH^{N-1}(y)\right\} \\
&= C(N)\int_A |G(x)|\,dx \\
&\quad + \liminf_{\delta\to 0^+}\liminf_{k\to\infty}\left\{\sum_i \varepsilon_i^{N-1}\frac{\varepsilon_i}{2k+1}\int_{\frac{-1}{2}}^{\frac{1}{2}}\int_{Q_i'} W\left(\frac{(2k+1)}{\varepsilon_i}\nabla u_i((2k+1)y',z)\right) dy'\,dz \right.\\
&\quad + \left.\sum_i \frac{\varepsilon_i^{N-1}}{(2k+1)^{N-1}}\int_{S(u_i)\cap(2k+1)Q_i\cap\left\{y:|y_N|\le\frac{1}{2}\right\}} \psi([u_i](y))dH^{N-1}(y)\right\}. \qquad (4.16)
\end{aligned}
$$

Next, we note that, for each $i$, $\varepsilon_i(\delta)\to 0^+$ as $\delta\to 0^+$, and, due to the coercivity of $W$, $\{(2k+1)\nabla u_i((2k+1)y',z)\}$ are uniformly bounded in $k$; hence, using $(\mathcal{H}5)$ , we may replace in (4.16)

$$\frac{\varepsilon_i}{2k+1}W\left(\frac{2k+1}{\varepsilon_i}\nabla u_i((2k+1)y',t)\right)$$

by

$$\frac{1}{2k+1}W^\infty\left((2k+1)\nabla u_i((2k+1)y',z)\right) = W^\infty\left(\nabla u_i((2k+1)y',z)\right).$$

Using the periodicity of the $u_i$, (4.12), and then (4.10), we obtain

$$
\begin{aligned}
I_1(g,G,A) &\le C(N)\int_A |G(x)|\,dx + \liminf_{\delta\to 0^+}\sum_i\left\{\varepsilon_i^{N-1}\int_{Q_i} W^\infty(\nabla u_i(y))dy + \varepsilon_i^{N-1}\int_{S(u_i)\cap Q_i}\psi([u_i](y))\,dH^{N-1}\right\}\\
&\le C(N)\int_A |G(x)|\,dx + \liminf_{\delta\to 0^+}\sum_i \varepsilon_i^{N-1}\left(h_1([g](x),\nu_i)-\delta\right)\\
&\le C(N)\int_A |G(x)|dx + \liminf_{\delta\to 0^+}\left\{\sum_i\left[\int_{S(g)\cap Q_i(x_i,\varepsilon_i)} h_1([g](x),\nu(x))dH^{N-1}(x)-\delta\right]-\delta\sum_i\varepsilon_i^{N-1}\right\}.
\end{aligned}
$$

By (4.14), $\delta\sum_i \varepsilon_i^{N-1}=O(\delta)$ and thus we conclude that

$$I_1(g,G,O)\le C(N)\int_A |G(x)|\,dx + \int_{S(g)\cap A} h_1([g](x),\nu(x))\,dH^{N-1}(x). \qquad (4.17)$$

*Case 2 :* Let $E$ be an arbitrary set of finite perimeter, and assume that $W$ is coercive. The proof of inequality (4.9) for $g=\lambda\chi_E$ follows from the argument given in the [7] (Step 2d) of the proof of Proposition 5.1), and from the lower semicontinuity of $I(g,G,\Omega)$ for coercive $W$ (see Proposition 5.1). Indeed, consider a sequence of polygons $E_n$ such that $Per_\Omega(E_n)\to Per_\Omega(E)$, $\mathcal{L}^N(E_n\Delta E)\to 0$, and $\chi_{E_n}\to\chi_E$ in $L^1$. In view of the upper semicontinuity of $h_1(\lambda,\cdot)$ (Proposition 4.3), we may apply Proposition 3.6 in [7] to obtain a sequence of continuous functions $h^m:\mathbb{R}^N\to[0,\infty)$ such that

$$h_1(\lambda,y)\le h^m(y)\le C|y|, \quad \text{for every } y\in\mathbb{R}^N$$

and

$$h_1(\lambda,y)=\inf_m h^m(y),$$

where $h_1(\lambda, \cdot)$ has been extended to $\mathbb{R}^N$ as a homogeneous function of degree one. Thus, setting $g_n = \lambda \chi_{E_n}$, using (4.9) and the fact that $\mathcal{L}^N(E_n \Delta E) \to 0$, $\text{Per}(E_n) \to \text{Per}(E)$, we have

$$\begin{aligned} I(g, G, A) &\leq \liminf_{n \to +\infty} I(g_n, G, A \\ &\leq C(N) \int_A |G(x)| \, dx + \lim_n \int_{\partial E_n \cap A} h(\lambda, \nu_n(x)) \, dH^{N-1}(x) \\ &\leq C(N) \int_A |G(x)| \, dx + \lim_n \int_{\partial E_n \cap A} h^m(\nu_n(x)) \, dH^{N-1}(x) \\ &\leq C(N) \int_A |G(x)| \, dx + \int_{\partial E \cap A} h^m(\nu(x)).dH^{N-1}(x) \end{aligned}$$

Letting $m \to +\infty$ and using the Monotone Convergence Theorem, we obtain

$$I(g, G, A) \leq C(N) \int_A |G(x)| \, dx + \int_{S(g) \cap A} h(\lambda, \nu(x)) \, dH^{N-1}(x).$$

*Case 3*: To complete the proof of the upper bound, we remove the coercivity assumption on $W$. Let $W^\varepsilon(\cdot) = W(\cdot) + \varepsilon |\cdot|$. Then, by (4.9) we have

$$\begin{aligned} I_1(g, G, A) &\leq I_1^\varepsilon(g, G, A) \\ &\leq C \int_A 1 + |G(x)| \, dx + \int_A h_1^\varepsilon([g], \nu_g) \, dH^{N-1}, \end{aligned} \tag{4.18}$$

and given $\delta > 0$, by definition of $h_1$ we may find $u \in SBV(Q_\nu(0,1), \mathbb{R}^d)$ such that $u|_{\partial Q_\nu} = u_{\lambda,\nu}$, $\int_{Q_\nu} \nabla u \, dx = 0$, and

$$\delta + h_1(\lambda, \nu) \geq \int_Q W^\infty(\nabla u) \, dx + \int_{S(u)} \psi([u], \nu_u) \, dH^{N-1}.$$

Thus

$$\begin{aligned} h_1^\varepsilon(\lambda, \nu) &\leq \int_Q W^\infty(\nabla u) \, dx + \varepsilon |\nabla u| + \int_{S(u)} \psi([u], \nu_u) \, dH^{N-1} \\ &\leq h_1(\lambda, \nu) + \delta + \varepsilon |\nabla u|_{L^1}, \end{aligned}$$

and we conclude that $\limsup_\varepsilon h_1^\varepsilon(\lambda, \nu) \leq h_1(\lambda, \nu) + \delta$, from which we obtain

$$I_1(g, G, A) \leq C \int_A 1 + |G(x)| \, dx + \int_A h_1([g], \nu_g) \, dH^{N-1}.$$

It suffices to let $\delta \to 0^+$. □

Theorems 3.2 and 4.4 reduce to Theorem 2.16. We now state and prove the counterpart result to Theorem 4.4 for $p > 1$.

**Theorem 4.5** *Let $p > 1$ and $W, \psi$ satisfy $(\mathcal{H}1)_p$, $(\mathcal{H}2)$, and $(\mathcal{H}4)$. If $g \in L^\infty(\Omega, \mathbb{R}^d)$ then*

$$\frac{d I_p(g, G, \cdot)}{d(|g^+ - g^-| H^{N-1} \lfloor S(g))}(x) = \frac{1}{|[g](x)|} h([g](x)),$$

*where $h$ is given by* (2.17).

**Proof.** The proof is very similar to that of Theorem 4.4, except at the following points where the growth of $W$ and the convergence of admissible sequences $\{u_n\}$ become relevant.
*Step 1*: For the lower bound, we apply the same argument to construct $u_{n,\varepsilon}$ and to find a finite, nonnegative Rodon measure $\mu$ such that

$$\frac{d\mu}{d\left(|g^+ - g^-|H^{N-1}\lfloor S(g)\right)}(x_0) = \frac{1}{|[g(x_0)]|}\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\left\{\varepsilon\int_{Q_{\nu(x_0)}} W\left(\frac{1}{\varepsilon}\nabla u_{n,\varepsilon}(y)\right)dy \right.$$
$$\left.\int_{S(u_{n,\varepsilon})\cap Q_{\nu(x_0)}} \psi([u_{n,\varepsilon}](y),\nu_{u_{n,\varepsilon}}(y))\,dH^{N-1}(y)\right\}.$$

Assuming that $W$ is coercive, i.e., there exists a constant $C > 0$ such that $C|A|^p \leq W(A)$ for all $A \in \mathbb{M}^{d\times N}$, and using the fact that the density is finite $H^{N-1}$ a.e. (see Theorem 2.6), we extract a diagonal subsequence, $v_k$, from $u_{n,\varepsilon}(y) := u_n(x_0 + \varepsilon y) - g^-(x_0)$ such that

$$\lim_k |v_k(y) - u_{[g](x_0),\nu(x_0)}(y)|_{L^1(Q_{\nu(x_0)})} = 0, \quad \nabla v_k \to 0 \quad \text{in } L^p(Q_{\nu(x_0)}(0,1))$$

and

$$\frac{d\mu}{d(|g^+ - g^-|H^{N-1}\lfloor S(g))}(x_0) \geq \frac{1}{|[g(x_0)]|}\liminf_{k\to\infty}\int_{S(v_k)}\psi([v_k](y),\nu_{v_k}(y))\,dH^{N-1}(y).$$

The lower bound now follows by Proposition 4.2. Removal of the coercivity assumption can be achieved by means of an argument identical to the one used in Step 1, Case 2, of Theorem 3.2.
*Step 2*. For the upper bound, we proceed with the construction of $u_{\delta,k}(x)$ as in (4.11), (4.12) , noting that, in this case, $\nabla u_{\delta,k}(x) = 0$ a.e. By Theorem 2.8, let $h \in SBV(A,\mathbb{R}^d)$ be such that

$$\nabla h = G, \quad ||Dh||(A) \leq C(N)|G|_{L^1(A,\mathbb{M}^{d\times N})}.$$

By Lemma 2.9 there exist $v_k \in SBV(O,\mathbb{R}^D)$ piecewise constant such that $v_k \to h$ in $L^1$ and $||Dv_k||(A) \to ||Dh||(A)$, and we define

$$w_{\delta,k}(x) := u_{\delta,k}(x) + h(x) - v_k(x).$$

Then

$$I(g,G,A) \leq C\int_A 1 + |G(x)|^p\,dx + \liminf_{\delta\to 0^+}\liminf_{k\to\infty}\int_{S(w_{\delta,k})\cap A}\psi([w_{\delta,k}](x),\nu_{w_{\delta,k}})\,dH^{N-1}(x).$$

The arguments carried out in (4.13) - (4.17), except now involving only the interfacial energy, allow us to conclude that

$$I(g,G,A) \leq C\int_A 1 + |G(x)|^p\,dx + \int_{S(g)\cap A} h([g](x))\,dH^{N-1}(x),$$

for the case when $g = \lambda\chi_E$ and $E$ is a polygon. Since $h$ does not depend on the normal to the jump set, the inequality for $E$ of finite perimeter follows directly by assuming coercivity of $W$, applying Proposition 5.1, and then Lebesgue Dominated Convergence to $\chi_{E_n} \to \chi_E$. To remove the coercivity assumption, we proceed, as in the case where $p = 1$ (Case 3 of Step 2 for Theorem 4.4), to obtain (4.18) with $I_p^\varepsilon$ and $h^\varepsilon$ corresponding to $W^\varepsilon(\cdot) := W(\cdot) + \varepsilon|\cdot|^p$. For $p > 1$, $h^\varepsilon = h$ and the proof is complete. □

From Theorems 3.2 and 4.5, we have Theorem 2.17 for the case when $g \in L^\infty(\Omega,\mathbb{R}^d)$. To complete the proof of Theorem 2.17, we remove this restriction.

**Proof. [Theorem 2.17]** Define

$$J(g,G,\Omega) := \int_\Omega H_p(\nabla g(x),G(x))\,dx + \int_{S(g)} h([g])\,dH^{N-1}.$$

By Theorems 3.2 and 4.5, we know that $I_p(g,G,\Omega) = J(g,G,\Omega)$ if $g \in L^\infty(\Omega,\mathbb{R}^d)$. Let $g \in SBV(\Omega,\mathbb{R}^d)$ be arbitrary.

*Step 1. [Lower bound]* Fix $\delta > 0$ and let $\{u_n\}$ be an admissible sequence such that $u_n \to g$ in $L^1$, $\nabla u_n \rightharpoonup G$ in $L^p$, and

$$\delta + I_p(g,G,\Omega) \geq \lim_{n\to\infty} \left\{ \int_\Omega W(\nabla u)\, dx + \int_{S(u_n)} \psi([u_n],\nu_{u_n})\, dH^{N-1} \right\}.$$

After extracting a subsequence, we may assume that

$$W(\nabla u)\, d\mathcal{L}^N + \psi([u_n],\nu_{u_n})\, dH^{N-1} \overset{*}{\rightharpoonup} \mu, \tag{4.19}$$

where $\mu$ is a finite, Radon measure. The arguments of Theorem 3.2, Step 1, and Theorem 4.5, Step 1, allow us to conclude that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq H_p(\nabla g(x_0), G(x_0)) \quad \mathcal{L}^N \text{ a.e. } x_0 \in \Omega,$$

and

$$\frac{d\mu}{d|[g]|H^{N-1}\lfloor S(g)}(x_0) \geq \frac{1}{|[g](x_0)|} h([g],\nu_g) \quad H^{N-1} \text{ a.e. } x_0 \in S(g). \tag{4.20}$$

Clearly , (4.19) – (4.20) yield

$$\delta + I_p(g,G,\Omega) \geq J(g,G,\Omega).$$

Letting $\delta \to 0^+$, we conclude that

$$I_p(g,G,\Omega) \geq J(g,G,\Omega).$$

*Step 2. [Upper bound]* Conversely, let $n \in \mathbb{N}$ and consider $\phi_n$ as in the proof of Lemma 2.20, i.e., $\phi_n \in C^\infty(\mathbb{R}^d,\mathbb{R}^D)$ such that $|\nabla\phi_n(x)|_{L^\infty} \leq 1$ and

$$\phi_n(x) := \begin{cases} x & \text{if } |x| < e^n \\ 0 & \text{if } |x| \geq e^{n+1}. \end{cases}$$

Since $\phi_n(g) \to g$ in $L^1$, and assuming that $W$ is coercive, Proposition 5.1 implies that

$$\begin{aligned} I_p(g,G,\Omega) &\leq \liminf_{n\to\infty} I_p(\phi_n(g),G,\Omega) \\ &= \liminf_{n\to\infty} J(\phi_n(g),G,\Omega), \end{aligned} \tag{4.21}$$

where we have used the fact that $I_p(g,G,\Omega) = J(g,G,\Omega)$ whenever $g \in L^\infty$. Next, we note that by $(\mathcal{H}2)$, we have for all $\lambda \in \mathbb{R}^d$,

$$h(\lambda) \leq \int_{S(u)} \psi([u_{\lambda,e_N}], e_N)\, dH^{N-1} \leq C|\lambda|,$$

and we claim that there exists a constant $C$ such that for all $A, B \in \mathbb{M}^{d\times N}$, we have

$$H_p(A,B) \leq C(1 + |A| + |B|^p). \tag{4.22}$$

Assuming that (4.22) holds, let

$$\Omega_n := \{x \in \Omega : |g_-(x)| > e^n \text{ or } |g_+(x)| > e^n\} \cap \{x \in \Omega : |g_-(x)| < e^{n+1} \text{ or } |g_+(x)| < e^{n+1}\}.$$

We have

$$J((\phi_n(g),G,\Omega) \leq J(g,G,\Omega) + C\int_{\{x:|g(x)|>e^n\}} (1 + |\nabla(\phi_n(g))| + |G|^p)\, dx + C\int_{\Omega_n} |[g](x)|\, dH^{N-1}(x). \tag{4.23}$$

It can be shown (see (3.19) – (3.22) of [7]) that

$$\sum_{i=n}^{2n} \int_{\Omega_i} |[g](x)|\, dH^{N-1}(x) \le \int_{S(g)\cap\Omega} |[g](x)|\, dH^{N-1}(x),$$

and so there exists $i(n) \in \{n, \ldots, 2n\}$ such that

$$\int_{\Omega_{i(n)}} |[g](x)|\, dH^{N-1}(x) \le \frac{1}{n} \int_{S(g)\cap\Omega} |[g](x)|\, dH^{N-1}(x).$$

Using the fact that the first integrand in (4.23) is bounded independent of $n$, and that $\mathcal{L}^N\{x : |g(x)| > e^n\} \to 0$, we conclude from (4.23) that

$$J(\phi_{i(n)}(g), G, \Omega) \le J(g, G, \Omega) + O\left(\frac{1}{n}\right),$$

which, together with (4.21), yields

$$I_p(g, G, \Omega) \le J(g, G, \Omega).$$

Removal of the coercivity assumption follows the arguments given in Step 2, Case 3, of Theorem 4.4. Here, we apply these arguments to both the densities $H_p$ and $h$.

It remains to prove (4.22). By virtue of Theorem 2.8 and Lemma 2.9, there exist $h \in SBV(Q, \mathbb{R}^d)$ and piecewise constant functions $\tilde{u}_n$ such that $\nabla h = B\ \mathcal{L}^N$ a.e., $||Dh||(Q) \le C|B|$, $\tilde{u}_n \to (Ax - h)$ in $L^1$, and

$$||D\tilde{u}_n||(Q) \longrightarrow |A - B| + ||D_s h||(Q) \le C(|A| + |B|).$$

Let $u_n := \tilde{u}_n + h$. By Proposition 3.1 and $(\mathcal{H}1)_p$, we have

$$\begin{aligned} H_p(A, B) &\le \liminf_{n\to\infty} \int_Q W(\nabla u_n)\, dx + \int_{S(u_n)} \psi([u_n], \nu_{u_n})\, dH^{N-1} \\ &\le W(B) + \lim_{n\to\infty} ||D_s u_n||(Q) \\ &\le C(1 + |A| + |B|^p). \end{aligned}$$

□

# 5 Some Properties of the Energy

In this section we discuss certain properties of the energy $I$. We start with lower semicontinuity with respect to the appropriate topology, and under the assumption that $W$ is coercive.

**Proposition 5.1** *Assume that there exist constants $C, c$, such that $C(|A| - c) \le W(A)$ for all $A \in \mathbb{M}^{d\times N}$. Let $(g_n, G_n), (g, G) \in SD(\Omega)$ with $g_n \to g$ in $L^1(\Omega, \mathbb{R}^d)$, and $G_n \stackrel{*}{\rightharpoonup} G$. Then, for $p \ge 1$,*

$$I_p(g, G, \Omega) \le \liminf_{n\to\infty} I_p(g_n, G_n, \Omega).$$

**Proof.** Without loss of generality, assume that $\liminf_n I_p(g_n, G_n, \Omega) = \lim_n I_p(g_n, G_n, \Omega)$. Due to the coercivity of $W$, we may find a minimizing sequence for $I_p(g_n, G_n, \Omega)$, $u_n^m$, such that

$$I_p(g_n, G_n) = \lim_{m\to\infty} E(u_n^m), \quad u_n^m \underset{m}{\to} g_n \quad \text{in } L^1, \quad \text{and } \nabla u_n^m \underset{m}{\stackrel{*}{\rightharpoonup}} G_n(x)$$

Coercivity of $W$ yields a uniform bound on $\{\nabla u_n^m\}$, and so we may extract a diagonal subsequence in $n, m$, say $v_k := u_{n_k}^{m_k}$, such that $v_k \to g$ in $L^1$, $\nabla v_k \overset{*}{\rightharpoonup} G$, and

$$E(v_k) \leq I_p(g_{n_k}, G_{n_k}, \Omega) + \frac{1}{k}.$$

Thus

$$\begin{aligned} I_p(g, G, \Omega) &\leq \liminf_{k\to\infty} E(v_k) \\ &\leq \liminf_{n\to\infty} I_p(g_n, G_n, \Omega). \end{aligned}$$

□

**Proposition 5.2** *Assume that* $(\mathcal{H}1)_p$, $(\mathcal{H}2)$, *and* $(\mathcal{H}4)$ *hold. Then* $H_p(A,B)$, *defined by* (2.16), *is uniformly continuous in* $A$ *and* $B$.

**Proof.** Let $A_m \to A$. By Lemma 2.9, for each $m$ there exists a sequence of piecewise constant functions $v_n$ defined on $Q$ such that

$$v_n \to (A_m - A)x \quad \text{in } L^1 \quad \text{and} \quad \lim_n ||Dv_n||(Q) = |A_m - A|.$$

Let $\{u_n\}$ be an admissible sequence for the limit description of $H_p(A,B)$ given in Proposition 3.1. Then the sequence $\{u_n + v_n\}$ is admissible for $H_p(A_m, B)$, and using the subadditivity of $\psi$, $(\mathcal{H}4)$, together with the linear growth assumption, we obtain for some constant $C$, independent of $n, m$,

$$E(u_n + v_n) - C||Dv_n||(Q) \leq E(u_n) \leq E(u_n + v_n) + C||Dv_n||(Q).$$

Taking the limit in $n$ and then the infimum over all admissible sequences $u_n$, we obtain

$$H_p(A_m, B) - C|A - A_m| \leq H_p(A,B) \leq H_p(A_m, B) + C|A_m - A|$$

and continuity in $A$ follows by letting $m$ tend to infinity. To prove continuity with respect to $B$, consider $B_m \to B$. By Theorem 2.8 and Lemma 2.9, for each $m$ there exists $h \in SBV(Q, \mathbb{R}^d)$ such that

$$\nabla h = B_m - B, \quad ||Dh|| \leq C|B_m - B|,$$

and there exist piecewise constant functions $v_n$ such that $v_n \to -h$ in $L^1(Q, \mathbb{R}^d)$ and $\lim_n ||Dv_n||(Q) = ||Dh||(Q)$. Let $\{u_n\}$ be an admissible sequence for $H_p(A,B)$. Then the sequence $\{u_n + h + v_n\}$ is admissible for $H_p(A, B_m)$ and, proceeding as before, we obtain

$$H_p(A, B_m) - C|B - B_m| \leq H_p(A,B) \leq H_p(A, B_m) + C|B_m - B|$$

and the result follows. □

In the following proposition we use the notion of *inf-convolution*, precisely, the inf-convolution of $W$ and $\psi$ is given by

$$(W \nabla \psi)(A) := \inf \left\{ W(A - a \otimes b) + \psi_0(x, a, b) : a \in \mathbb{R}^d, b \in S^{N-1} \right\}.$$

Also, given $f : \mathbb{M}^{d\times N} \to \mathbb{R}$, $Qf$ denotes the *quasiconvex envelope* of $f$, that is,

$$Qf(A) := \inf \left\{ \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega f(A + \nabla\phi(x))\, dx : \phi \in W_0^{1,\infty}(\Omega, \mathbb{R}^d) \right\}.$$

**Proposition 5.3** *Let $(g, G) \in SD(\Omega)$ and $p \geq 1$. Assume that there exist constants $C, c$, such that $C(|A|^p - c) \leq W(A)$ for all $A \in \mathbb{M}^{d\times N}$. Then*

$$\inf_{G\in L^p(\Omega,\mathbb{M}^{d\times N})} \int_\Omega H_p(\nabla g(x), G(x))\,dx = \int_\Omega \inf_{B\in\mathbb{M}^{d\times N}} H_p(\nabla g(x), B)\,dx = \int_\Omega Q(W\nabla\psi)(\nabla g(x))\,dx.$$

*In particular,*

$$\inf_{B\in\mathbb{M}^{d\times N}} H_p(A, B) = Q(W\nabla\psi)(A).$$

**Proof.** For fixed $A \in \mathbb{M}^{d\times N}$, and by definition of $H_p$, we have

$$\begin{aligned}\inf_{B\in\mathbb{M}^{d\times N}} H_p(A, B) &= \inf_u \left\{ \int_Q W(\nabla u)\,dx + \int_{S(u)} \psi([u])\,dH^{N-1} : u \in SBV(Q, \mathbb{R}^d), u|_{\partial Q} = Ax \right\} \\ &= Q(W\nabla\psi)(A),\end{aligned}$$

where the last equality was established in [7]. Hence, it suffices to construct for each $\varepsilon > 0$ a function $G \in L^1(\Omega, \mathbb{M}^{d\times N})$ such that

$$H_p(\nabla g(x), G(x)) \leq \inf_B H_p(\nabla g(x), B) + \varepsilon \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega.$$

To this end, let $f_n$ be a sequence of simple functions which converges in $L^1$ to $\nabla g$, satisfies $|f_n(x)| \leq |\nabla g(x)|$, and such that $f_n(x_0) \to \nabla g(x_0)$ for $\mathcal{L}^N$ a.e. $x_0 \in Q$. Assume that $x_0$ is such a point. For every $n$, choose $G^n(x_0) \in \mathbb{M}^{d\times N}$ such that

$$H_p(f_n(x_0), G^n(x_0)) \leq \inf_{B\in\mathbb{M}^{d\times N}} H_p(f_n(x_0), B) + \varepsilon, \tag{5.1}$$

and $G^n(\cdot)$ is a simple function. Define

$$G(x_0) := \limsup_{n\to\infty} G^n(x_0),$$

where, upon extracting a suitable subsequence, the lim sup is taken componentwise. Note that for every $n$, $G^n(\cdot)$ is measurable, and so $G(\cdot)$ is measurable. In order to show that $G(\cdot)$ is integrable, let $u$ be an admissible function for $H_p(f_n(x_0), G^n(x_0))$ such that

$$E(u) \leq H_p(f_n(x_0), G^n(x_0)) + \varepsilon.$$

By $(\mathcal{H}1)_p$ and (5.1) with $B = f_n(x_0)$, we have

$$\begin{aligned}E(u) &\leq H_p(f_n(x_0), G^n(x_0)) + \varepsilon \\ &\leq H_p(f_n(x_0), f_n(x_0)) + 2\varepsilon \\ &\leq C|f_n(x_0)|^p + 2\varepsilon.\end{aligned}$$

Thus, by Jensen's inequality and the coercivity of $W$, we deduce that

$$\begin{aligned}|G^n(x_0)|^p &= \left|\int_Q \nabla u\,dx\right|^p \\ &\leq \int_Q |\nabla u|^p\,dx \\ &\leq CE(u) \\ &\leq C'(|f_n(x_0)|^p + 2\varepsilon) \\ &\leq C'(|\nabla g(x_0)|^p + 2\varepsilon).\end{aligned}$$

Hence, for almost every $x_0$,

$$|G^n(x_0)| \leq C' \left(|\nabla g(x_0)|^p + 2\varepsilon\right)^{\frac{1}{p}},$$

and we conclude that $G \in L^1(Q, \mathbb{M}^{d\times N})$. Finally, by Proposition 5.2 and by virtue of (5.1), for every $B \in \mathbb{M}^{d\times N}$ we have

$$\begin{aligned} H_p(\nabla g(x_0), G(x_0)) &= \lim_{n\to\infty} H_p(f_n(x_0), G^n(x_0)) \\ &\leq \lim_{n\to\infty} H_p(f_n(x_0), B) + \varepsilon \\ &= H_p(\nabla g(x_0), B) + \varepsilon, \end{aligned}$$

and so

$$H_p(\nabla g(x_0), G(x_0)) \leq \inf_B H_p(\nabla g(x_0), B) + \varepsilon.$$

□

As a corollary, we obtain integral representations for the relaxation in the $L^1$ topology of

$$E(g) = \int_\Omega W(\nabla g(x))\, dx + \int_{S(g)} \psi([g](x), \nu_g(x))\, dH^{N-1}(x).$$

Set

$$\mathcal{F}(g) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} E(u_n) \ : u_n \in SBV, u_n \to g \text{ in } L^1(\Omega, \mathbb{R}^d) \right\}.$$

**Corollary 5.4** *Assume that $(\mathcal{H}1)_p$, $(\mathcal{H}2)-(\mathcal{H}4)$ hold, and that there exist constants $C, c$, such that $C(|A|^p - c) \leq W(A)$ for all $A \in \mathbb{M}^{d\times N}$. If $p > 1$, and if $g \in SBV(\Omega, \mathbb{R}^d)$, then*

$$\mathcal{F}(g) = \int_\Omega Q(W\nabla\psi)(\nabla g)\, dx + \int_{S(g)} h([g])\, dH^{N-1},$$

*where $h$ is defined by (2.17). If $p = 1$, and if $g \in W^{1,1}(\Omega, \mathbb{R}^d)$, then*

$$\mathcal{F}(g) = \int_\Omega Q(W\nabla\psi)(\nabla g)\, dx.$$

**Remark 5.5** A representation of $\mathcal{F}(g)$ for $p = 1$ and for all $g \in BV(\Omega, \mathbb{R}^d)$ was obtained directly in [7], precisely

$$\mathcal{F}(g) = \int_\Omega Q(W\nabla\psi)(\nabla g)\, dx + \int_\Omega Q(W\nabla\psi)^\infty (D_s g).$$

**Proof.** Let $p > 1$ and assume that $u_n, g \in SBV(\Omega, \mathbb{R}^d)$, $g \in L^p(\Omega, \mathbb{R}^d)$, $\{\nabla u_n\}$ is uniformly bounded in $L^p$, and $u_n \to g$ in $L^1$. Then, upon extracting a subsequence, there exists $G \in L^p(\Omega, \mathbb{M}^{d\times N})$ such that

$$\nabla u_n \rightharpoonup G \quad \text{in } L^p(\Omega, \mathbb{M}^{d\times N})$$

and so

$$\inf_{G\in L^p} I_p(g, G) \leq \mathcal{F}(g).$$

Hence,

$$\inf_{G\in L^p} I_p(g,G) = \mathcal{F}(g),$$

and the result now follows by virtue of (2.15) and Proposition 5.3.If $p = 1$, and if $\nabla u_n$ uniformly bounded in $L^1$, then, up to a subsequence, there exist $m \in \mathcal{M}(\Omega)$ and $G \in L^1(\Omega, \mathbb{M}^{d\times N})$ such that

$$\nabla u_n \overset{*}{\rightharpoonup} m \quad \text{and} \quad \frac{dm}{d\mathcal{L}^N} = G.$$

Thus

$$\inf_{G\in L^1} I_0(g,G) \le \mathcal{F}(g).$$

Hence

$$\inf_{G\in L^1} I_0(g,G) = \mathcal{F}(g),$$

and the conclusion follows from (2.14), Lemma 2.18, and Proposition 5.3. □

Next, we search for relations between $I(g,\nabla g,\Omega)$ ($I_p$ or $I_0$) and the relaxed energy

$$\int_\Omega QW(\nabla g)\,dx.$$

By Theorem 3.2 and Lemma 2.18, if $g \in W^{1,1}(\Omega,\mathbb{R}^d)$ then $I_0(g,\nabla g) = I_1(g,\nabla g)$. Let $I$ denote $I_p$.

**Proposition 5.6** *i) The function $A \in \mathbb{M}^{d\times N} \mapsto H_p(A,A)$ is quasiconvex and $H_p(A,A) \le QW(A)$. In particular, if $g \in W^{1,1}(\Omega,\mathbb{R}^d)$ then*

$$I(g,\nabla g,\Omega) = \int_\Omega H_p(\nabla g(x),\nabla g(x))\,dx \le \int_\Omega QW(\nabla g(x))\,dx.$$

*ii) Let $g \in W^{1,1}(\Omega,\mathbb{R}^d)$. If $W$ is convex, or if $W$ is quasiconvex, has linear growth (i.e. for some constants $c, C$ $c|A| \le W(A) \le C|A|$), and $\psi(\lambda,\nu) \ge W^\infty(\lambda\otimes\nu)$, then*

$$I(g,\nabla g) = \int_\Omega W(\nabla g(x))\,dx.$$

*iii) $A \in \mathbb{M}^{d\times N}$ is such that $W^{**}(A) < QW(A)$ if and only if there exist a constant $\alpha \in R$ such that*

$$I(g,\nabla g) < \int_\Omega QW(\nabla g(x))\,dx,$$

*where $g(x) = Ax$ and $\psi(\cdot) = \alpha|\cdot|$.*

**Proof.** i) By definition of $H_p$ (see (2.16)) we have

$$H_p(A,A) \le \inf\left\{\int_Q W(\nabla u)\,dx \,:\, u = Ax+\phi,\ \phi\in W_0^{1,\infty}\right\} = QW(A).$$

Therefore, if $g \in W^{1,1}(\Omega,\mathbb{R}^d)$, then by Theorems 2.16 and 2.17 we obtain

$$\begin{aligned} I(g,\nabla g) &= \int_\Omega H_p(\nabla g,\nabla g)\,dx \\ &\le \int_\Omega QW(\nabla g(x))\,dx. \end{aligned}$$

In order to prove that $A \mapsto H_p(A,A)$ is a quasiconvex function, it suffices to apply Theorems 2.16 and 2.17 to $I_p(g,\nabla g)$, to conclude that

$$I_p(g,\nabla g) = \int_\Omega \hat{H}_p(g,\nabla g)\,dx + \int_{S(G)\cap\Omega} \hat{h}([g]\nu_g)\,dH^{N-1},$$

where we have used the lower semicontinuity property of $I_p$ (see Proposition 5.1), and where $\hat{H}_p$ and $\hat{h}_p$ are associated to $H_p$ and to $h$ (or $h_1$ if $p=1$), through the formulas in Theorems 2.16 and 2.17. Thus

$$H_p(A,B) = \hat{H}_p(A,B),$$

and, in particular,

$$\begin{aligned} H_p(A,A) &= \hat{H}_p(A,A) \\ &\leq \inf\left\{\int_Q H_p(\nabla u,\nabla u)\,dx : u \in W^{1,p}(Q,\mathbb{R}^d), u|_{\partial Q} = Ax\right\} \\ &= QH_p(A,A). \end{aligned}$$

ii) For $g \in W^{1,1}$, $I_1(g,G,\Omega) = I_0(g,G,\Omega)$, and hence it suffices to consider $I_p$, $p \geq 1$. Suppose that $W$ is convex, and let $u_n \to g$, $\nabla u_n \overset{*}{\rightharpoonup} \nabla g$. Then, using Jensen's inequality we conclude that

$$\begin{aligned} \liminf_{n\to\infty} E(u_n) &\geq \liminf_{n\to\infty} \int_\Omega W(\nabla u_n)\,dx \\ &\geq \liminf_{n\to\infty} \int_\Omega W(\nabla g)\,dx. \end{aligned}$$

Taking the infimum over all such sequences $\{u_n\}$, we obtain $I_p(g,\nabla g) \geq \int_\Omega W(\nabla g)\,dx$, and the result follows by part i).

Next, assume that $W$ is quasiconvex with linear growth, and that $\psi(\lambda,\nu) \geq W^\infty(\lambda\otimes\nu)$. Take $u_n \to g$, $\nabla u_n \overset{*}{\rightharpoonup} \nabla g$. Then

$$\begin{aligned} \liminf_{n\to\infty} E(u_n) &\geq \liminf_{n\to\infty}\left\{\int_\Omega W(\nabla u_n)\,dx + \int_{S(u_n)} W^\infty([u_n]\otimes\nu_{u_n})\,dH^{N-1}\right\} \\ &=: \liminf_{n\to\infty} \mathcal{G}(u_n). \end{aligned}$$

By a result of Fonseca and Müller (see [33]), $\mathcal{G}(u_n)$ is lower semi-continuous with respect to the $L^1$ topology, and so

$$\begin{aligned} \liminf_{n\to\infty} E(u_n) &\geq \mathcal{G}(g) \\ &= \int_\Omega W(\nabla g)\,dx. \end{aligned}$$

This yields $I(g,\nabla g) \geq \int_\Omega W(\nabla g)\,dx$, and the converse inequality follows from i).

iii) Let $\psi(\cdot) = \alpha|\cdot|$ and suppose that $W^{**}(A) < QW(A)$. Then there exist an $A \in \mathbb{M}^{d\times N}$ and $f \in L^1(Q,\mathbb{M}^{d\times N})$ such that

$$\int_Q f(x)\,dx = 0, \quad \int_Q W(A+f(x))\,dx \leq QW(A) - \varepsilon$$

for some $\varepsilon > 0$. Let $Q_\delta := Q(0,1-\delta)$, where $\delta$ is chosen sufficiently small so that

$$W(A)\,\mathcal{L}^N(Q\backslash Q_\delta) < \frac{\varepsilon}{2}.$$

Set

$$C_\delta := \frac{1}{\mathcal{L}^N(Q_\delta)} \int_{Q\setminus Q_\delta} f(x)\,dx, \quad C_\delta \to 0 \quad \text{as } \delta \to 0^+.$$

By Theorem 2.8, there exists $\phi \in SBV(Q_\delta, \mathbb{R}^d)$ such that $\nabla\phi = f$, $\int_{Q_\delta} \phi = 0$, $||D\phi||(Q_\delta) \le C|f|_{L^1}$, and by (2.5) we have

$$\int_{\partial Q_\delta} |\mathrm{tr}\,\phi|\,dH^{N-1} \le C|f|_{L^1}.$$

Define

$$u(x) := Ax + \begin{cases} 0 & \text{if } x \notin Q_\delta \\ \phi(x) + C_\delta x & \text{if } x \in Q_\delta. \end{cases}$$

Clearly $u|_{\partial Q} = Ax$, and using the fact that $f$ has zero average over $Q$, it follows that $\int_Q \nabla u\,dx = A$. Thus, by definition of $H_p(A,A)$ and by $(\mathcal{H}1)_p$,

$$\begin{aligned} H_p(A,A) &\le \int_Q W(\nabla u)\,dx + \alpha \int_{S(u)} |[u]|\,dH^{N-1} \\ &\le \frac{\varepsilon}{2} + \int_Q W(A + f(x))\,dx + C(C_\delta) + \alpha C'|f|_{L^1} \\ &\le QW(A) - \frac{\varepsilon}{2} + C(C_\delta) + \alpha C'|f|_{L^1}. \end{aligned}$$

Choosing $\alpha$ and $\delta$ sufficiently small so that

$$C(C_\delta) + \alpha C'|f|_{L^1} < \frac{\varepsilon}{2},$$

we obtain

$$H_p(A,A) < QW(A).$$

Conversely, if

$$I_p(g, \nabla g) < \mathcal{L}^N(A) QW(A)$$

then

$$H_p(A,A) < QW(A)$$

and so

$$\begin{aligned} W^{**}(A) &= \inf\left\{ \int_Q W(A+f)\,dx : \int_Q f\,dx = 0 \right\} \\ &\le \inf\left\{ \int_Q W(\nabla u)\,dx + \int_{S(u)\cap Q} \psi([u],\nu)\,dH^{N-1} : u \in SBV(Q,\mathbb{R}^d), \right. \\ &\qquad \left. \nabla u = A + f, \int_Q f\,dx = 0, u|_{\partial Q} = Ax \right\} \\ &= H_p(A,A) \\ &< QW(A). \end{aligned}$$

□

We note that Corollary 5.4 has the interpretation that, for a given macroscopic deformation $g \in W^{1,1}(\Omega, \mathbb{R}^d)$, the energy associated with the optimal microstructure is given by the relaxation of $E(g)$ in the

$L^1$ topology ($BV$ weak). Percisely, by Theorem 2.17, Proposition 5.3, and Corollary 5.4, we have

$$\begin{aligned}\inf_{G\in L^p(\Omega,\mathbb{M}^{p\times N})} I_p(g,G,\Omega) &= \int_\Omega Q(W\nabla\psi)(\nabla g)\,dx + \int_{S(g)\cap\Omega} h([g],\nu_g)\,dH^{N-1} \\ &= \int_\Omega Q(W\nabla\psi)(\nabla g)\,dx.\end{aligned}$$

Moreover, if we assume $W$ to be coercive, the direct method of the calculus of variations can be implemented to show that the infimum over all microstructures is achieved. Indeed, let

$$\inf_{G\in L^p(\Omega,\mathbb{M}^{p\times N})} I_p(g,G,\Omega) = \lim_{n\to\infty} I_p(g,G_n,\Omega).$$

For every $n$ choose $v_n$ such that $|g-v_n|_{L^p(\Omega,\mathbb{R}^d)} < 1/n$, and $I_p(g,G_n,\Omega)\geq E(v_n)-1/n$. Then $\{\nabla v_n\}$ is bounded in $L^p$, and, upon extracting a subsequence, we have $\nabla v_n \rightharpoonup \xi$ in $L^p$, for some $\xi\in L^p(\Omega,\mathbb{M}^{p\times N})$. Finally, by Proposition 4.3 we conclude that

$$\begin{aligned}I_p(g,\xi) &\leq \liminf_{n\to\infty} I_p(v_n,\nabla v_n) \\ &\leq \liminf_{n\to\infty} E(v_n) \\ &= \inf_{G\in L^p(\Omega,\mathbb{M}^{p\times N})} I_p(g,G,\Omega).\end{aligned}$$

There are cases in which

$$I_0(g,G,\Omega)\geq I_0(g,\nabla g,\Omega),$$

for all $G\in L^1(\Omega,\mathbb{M}^{d\times N})$, e.g., if $W$ is quasiconvex with linear growth and $\psi(\lambda,\nu)\geq W^\infty(\lambda\otimes\nu)$. Hence, if variational principles are accepted for this model, we may interprete this result as evidence that for this particular crystal it is energetically more costly to form defects. On the other hand, there are simple examples in which

$$\inf_{G\in L^1} I_0(g,G,\Omega) < I_0(g,\nabla g,\Omega).$$

Consider $W(\cdot)=|\cdot|$ and $\psi(\cdot,\nu)=\alpha|\cdot|$. Using Corollary 5.4, Proposition 5.6 ii), and Theorem 2.14 in [7], we have

$$\begin{aligned}\int_\Omega |\nabla g(x)|\,dx &= I_0(g,\nabla g) \\ &\geq \inf_{G\in L^1} I_0(g,G,\Omega) \\ &= \mathcal{F}(g) \\ &= \int_\Omega \min(\alpha,1)\,|\nabla g(x)|\,dx.\end{aligned}$$

Hence, if $\alpha\geq 1$ the above inequality is in fact an equality, and if $\alpha<1$ then the inequality is strict.

We end with the following conjecture. Fix $p\geq 1$. Then

$$H_p(A,B)=F_1(B)+F_2(A-B),$$

for some functions $F_1$ and $F_2$. Even though we are not able to prove this at the present time, we note that it follows immediatly from its definition that

$$H_p(A,B)\leq H_p(B,B)+\mathcal{H}(A-B),$$

where

$$\mathcal{H}(C) := \inf\left\{\int_{S(u)\cap Q}\psi([u],\nu_u)\,dH^{N-1} : u\in SBV(Q,\mathbb{R}^d), \nabla u(x)=0\,\text{a.e.}, u|_{\partial Q}=Cx\right\}.$$

Proving this conjecture would confirm what was postulated in the introduction. Precisely, the energy functional associated with a structured deformation of a crystal should involve a measure of the discrepancy between the macroscopic and microscopic strains $\nabla g$ , $G$, respectively. Such a result could motivate the use of $H_p$ as the total free (stored) energy in computing stress at equilibrium. Work in this direction has already begun within the classical setting of Del Piero and Owen for structured deformations (see [36]).

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