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REGULARITY RESULTS FOR ANISOTROPIC IMAGE SEGMENTATION MODELS

Irene Fonseca [1] and Nicola Fusco [2]

Abstract - Models involving bulk and interfacial energies have been used to describe phenomena in fracture mechanics, phase transitions, and image segmentation. One of the main mathematical questions involved concerns the regularity of the crack site, or discontinuity set S_u, for local minimizers u of energy functionals of the type

$$\mathcal{G}(v):=\int_\Omega F(\nabla v)\,dx+H^{N-1}(S_v\cap\Omega).$$

The existence of a classical solution in the case where $F(\xi):=|\xi|^p, p>1$, was proven recently by means of compactness results in a somewhat large functional space, followed by a thorough regularity analysis of the jump set S_u of a local minimizer u of $\mathcal{G}$ thus obtained. Here these regularity properties are extended to a class of anisotropic, non homogeneous densities F, with p-growth.

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1. Introduction

Models involving bulk and interfacial energies have been used to describe phenomena in fracture mechanics, phase transitions, and image segmentation (see [BZ], [DGCL], [FF], [MS]). From a simplistic point of view, quasi-static equilibria correspond to minima of an energy functional

$$\mathcal{G}(u) := \int_\Omega F(\nabla u)\,dx + \alpha\int_\Omega |u-g|^q\,dx + \beta H^{N-1}(S_u\cap\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, $g \in L^\infty(\Omega;\mathbb{R}^d)$, $\alpha, \beta > 0$, H^{N-1} stands for the $N-1$-dimensional Hausdorff measure, $u \in BV(\mathbb{R}^N;\mathbb{R}^d)$, S_u is the jump set of u, i.e. the complement of the set of Lebesgue points of u, and the distributional derivative Du is represented by $Du = \nabla u\,\mathcal{L}^N + (u^+ - u^-)\otimes \nu\,H_{N-1}\lfloor S_u + C(u)$, with ν being the normal to S_u.

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Within a fracture mechanics framework, u stands for the deformation, and S_u represents the crack site. Earlier work by Ambrosio and De Giorgi (see [A1], [A2], [DGA]) guarantees the existence of minima, under appropriate boundedness constraints (see [FF]); however, regularity properties of the macroscopic discontinuities, being the next obvious step towards the understanding of the interaction between fracture and damage, cannot be obtained from existing regularity results (see [AFP], [AP], [CL], [DS], [DGCL]), as these apply only to energy densities F of the form $F(\xi) = |\xi|^p$.

In Mumford-Shah model for image segmentation, the energy is a functional of the type (see [BZ], [DMMS], [DGCL], [MS])

$$\mathcal{G}(K,v) := \int_{\Omega\setminus K} F(Dv)\,dx + \alpha \int_{\Omega\setminus K} |v-g|^q\,dx + \beta H^{N-1}(K\cap\Omega),$$

where $q \geq 1$, $\alpha, \beta > 0$, K is a closed set, $v \in W^{1,p}(\Omega\setminus K)$, $g \in L^\infty(\Omega)$, and the main goal is to show the existence of a minimizing pair (K,u) for the functional $\mathcal{G}$. Once again, this can be achieved by means of Ambrosio's existence results (see [A1], [A2]), followed by a regularity analysis of the jump set of the minimizer thus obtained. Here $g(x)$ is a real number representing the "brightness" or "grey level" of the image at a point x (digital image), and K represents the discontinuity set, or "edges" of g.

In this paper we will prove regularity for the jump set S_u of a local minimizer of $\mathcal{G}$, corresponding to a class of anisotropic, non-homogeneous, densities F with p-growth, namely, $H^{N-1}((\overline{S_u}\setminus S_u)\cap\Omega) = 0$, which is a first step towards obtaining $C^{1,\alpha}$ regularity, as it was previously obtained in [AP], [AFP] for scalar-valued functions, and when $F(\xi) = |\xi|^2$ (see also regularity results in [CL] for the vector-valued case, and $F(\xi) = |\xi|^p$). Our proofs are based essentially on the L^∞ gradient estimate obtained in Theorem 2.2 for local minimizers of certain energies corresponding to strictly convex, non-homogeneous, density functions.

2. Preliminary Results

In the sequel Ω denotes a bounded open set of $\mathbb{R}^N$, $B_R(x_0)$ is the ball $\{x \in \mathbb{R}^N : |x-x_0| < R\}$, and if f is an integrable function we define

$$\fint_{B_R(x_0)} f(x)\,dx := \frac{1}{\omega_N R^N}\int_{B_R(x_0)} f(x)\,dx,$$

where ω_N is the Lebesgue measure of the N-dimensional unit ball. We write simply B_R in place of $B_R(x_0)$ when no confusion may arise, Q_1 stands for the unit cube $(0,1)^N$, and we use Einstein's convention for repeated indices. Also, $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbb{R}^N$, and c is a generic constant that may vary from line to line.

Let $F : \mathbb{R}^N \to [0,+\infty)$ be a continuous function, $1 < p < +\infty$, and consider the energy functional

$$\mathcal{F}(v;A) := \int_A F(Dv)\,dx$$

for $v \in W^{1,p}(\Omega)$ and every open set $A \subset \Omega$.

Definition 2.1. *We say that $u \in W^{1,p}(\Omega)$ is a $W^{1,p}$-*local minimizer *of $\mathcal{F}$ if*

$$\mathcal{F}(u; B_R(x_0)) = \min\left\{\mathcal{F}(v; B_R(x_0)) \ : \ v \in u + W_0^{1,p}(B_R(x_0))\right\}$$

for all balls $B_R(x_0) \subset \Omega$.

Now we state the main theorem of this section, which extends regularity results well known in the literature (see [DB], [GM], [M]), but does not seem to have been treated under the general assumptions considered here.

Theorem 2.2. *Let $F : \mathbb{R}^N \to [0, +\infty)$ be a continuous function such that*

$$(i) \qquad (\mu^2 + |z|^2)^{p/2} \leq F(z) \leq L(\mu^2 + |z|^2)^{p/2}$$

for all $z \in \mathbb{R}^N$, where $p > 1$, $0 \leq \mu \leq 1$, and $L > 0$. Suppose, in addition, that F satisfies the following inequality

$$(ii) \qquad \int_{Q_1} F(z + D\varphi)\, dx \geq \int_{Q_1} \left[F(z) + \nu(\mu^2 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\right] dx$$

for every $z \in \mathbb{R}^N$, $\varphi \in C_0^1(Q_1)$, and for some $0 < \nu \leq 1$. If $u \in W^{1,p}(\Omega)$ is a local minimizer of the functional $\mathcal{F}$ then u is locally Lipschitz, and for every $B_R(x_0) \subset \Omega$

$$\sup_{B_{R/2}(x_0)} (\mu^2 + |Du|^2)^{p/2} \leq C \fint_{B_R(x_0)} (\mu^2 + |Du|^2)^{p/2}\, dx$$

where C depends only on N, p, L, ν.

To prove this theorem we give first a precise *sup* estimate for the gradient Du of a local minimizer for $\mathcal{F}$ in the case where F is smooth and satisfies the usual ellipticity assumptions, and then we carry out this estimate to the general case, by means of an approximation argument.

Lemma 2.3. *Let $G : \mathbb{R}^N \to [0, +\infty)$ be a C^2 function such that*

$$(1) \qquad 0 \leq G(z) \leq L(\mu^2 + |z|^2)^{p/2},$$

$$(2) \qquad |D^2 G(z)| \leq \Lambda(\mu^2 + |z|^2)^{\frac{p-2}{2}},$$

$$(3) \qquad D_{ij} G(z) w_i w_j \geq \nu(\mu^2 + |z|^2)^{\frac{p-2}{2}} |w|^2,$$

for every $z, w \in \mathbb{R}^N$, where $L, \Lambda, \mu, \nu > 0$, $p > 1$. If $u \in W^{1,p}(\Omega)$ is a local minimizer of

$$\mathcal{G}(v;A) := \int_A G(Dv)\,dx, \quad A\,open, A \subset \Omega,$$

then there exists a constant $C \equiv C(N,p,L,\nu)$, independent of μ, Λ, such that

$$\sup_{B_{R/2}(x_0)} (\mu^2 + |Du|^2)^{p/2} \le C \fint_{B_R(x_0)} (\mu^2 + |Du|^2)^{p/2}\,dx \tag{2.1}$$

for every $B_R(x_0) \subset \Omega$.

Proof. It follows from standard regularity theory (see [DB], [GM], [M]) that u is a $C^{1,\alpha} \cap W^{2,2}_{loc}$ function, and the estimate (2.1) holds for some constant $C = C(N,p,L,\nu,\mu,\Lambda)$. We claim that C does not depend on μ or Λ.

Replacing $u(x)$ by the function $\tilde{u}(y) := (1/R)u(x_0 + Ry)$, it is clear that $\tilde{u}$ is a local minimizer of $\mathcal{G}$ in $(1/R)(\Omega - x_0)$. Hence, it is not restrictive to suppose that $R = 1$, $x_0 = 0$.

In the Euler equation for $\mathcal{G}$,

$$\int_{B_1} D_i G(Du) D_i\phi\,dx = 0,$$

set $\phi := \eta^2 D_s\psi$, where $s = 1,\dots,N$, $\eta \in C^1_0(B_1), 0 \le \eta \le 1$, and $\psi \in C^2(B_1)$, to obtain

$$\int_{B_1} D_i G(Du) D_s(D_i\psi)\eta^2\,dx = -2\int_{B_1} \eta D_i G(Du) D_s\psi D_i\eta\,dx.$$

Integrating by parts the first integral, we have

$$\begin{aligned}\int_{B_1} & D_{ij}G(Du)D_j(D_s u)(D_i\psi)\eta^2\,dx \\ &= 2\int_{B_1} \eta D_i G(Du) D_s\psi D_i\eta\,dx - 2\int_{B_1} \eta D_i G(Du) D_i\psi D_s\eta\,dx\end{aligned}$$

for all functions $\psi \in W^{1,2}(B_1)$. Note that $\psi := V^\beta D_s u$, where $V(x) := \mu^2 + |Du|^2$, $\beta \ge 0$, is an admissible test function. Therefore, inserting this function in the equation above and noting that (1) and (3) imply that $|DG(z)| \le c(N,p)L(\mu^2 + |z|^2)^{\frac{p-1}{2}}$, we obtain

$$\begin{aligned}\int_{B_1} & D_{ij}G(Du)D_j(D_s u)D_i(D_s u)V^\beta\eta^2\,dx \\ &+ \beta\int_{B_1} D_{ij}G(Du)D_j(D_s u)D_s u D_i(|Du|^2)V^{\beta-1}\eta^2\,dx \\ &\le c(N,p,L)\int_{B_1} V^{\frac{p-1}{2}}\eta|D\eta|\left[V^\beta|D^2u| + \beta V^{\beta-1}|Du|D(|Du|^2)\right]dx.\end{aligned}$$

Summing up this inequality from $s = 1$ to $s = N$ and using (3), we obtain

$$\nu \int_{B_1} V^{\beta+\frac{p-2}{2}} |D^2u|^2\eta^2\,dx + \frac{\nu\beta}{2}\int_{B_1} V^{\beta-1+\frac{p-2}{2}} \left|D(|Du|^2)\right|^2\eta^2\,dx$$
$$\leq c(N,p,L)\int_{B_1} V^{\beta+\frac{p-1}{2}}\eta|D\eta||D^2u|$$
$$+ c(N,p,L)\beta\int_{B_1} V^{\beta-\frac{1}{2}+\frac{p-1}{2}}|D(|Du|^2)|\eta|D\eta|\,dx.$$

Applying Hölder and Young inequalities to the right hand side of the latter formula yields

$$\frac{\nu}{2}\int_{B_1} V^{\beta+\frac{p-2}{2}}|D^2u|^2\eta^2\,dx + \frac{\nu\beta}{4}\int_{B_1} V^{\beta-1+\frac{p-2}{2}}\left|D(|Du|^2)\right|^2\eta^2\,dx$$
$$\leq \frac{c(N,p,L)}{\nu}(1+\beta)\int_{B_1} V^{\beta+\frac{p}{2}}|D\eta|^2\,dx,$$

and, since

$$\int_{B_1} V^{\beta-1+\frac{p-2}{2}}\left|D(|Du|^2)\right|^2\eta^2\,dx \leq c(N)\int_{B_1} V^{\beta+\frac{p-2}{2}}|D^2u|^2\eta^2\,dx,$$

we conclude that

$$\int_{B_1} V^{\beta-1+\frac{p-2}{2}}\left|D(|Du|^2)\right|^2\eta^2\,dx \leq \frac{c(N,p,L)}{\nu^2}\int_{B_1} V^{\beta+\frac{p}{2}}|D\eta|^2\,dx.$$

Setting $\gamma := \frac{p}{4}+\frac{\beta}{2}\geq\frac{p}{4}$, the above inequality becomes

$$\int_{B_1}\left|D(V^\gamma\eta)\right|^2dx \leq c(N,p,L,\nu)\gamma^2\int_{B_1} V^{2\gamma}|D\eta|^2\,dx.$$

Using Poincaré inequality and Sobolev imbedding theorem we deduce that

$$\|V^\gamma\eta\|_{L^{2\chi}(B_1)} \leq c(N,p,L,\nu)\gamma\|V^\gamma D\eta\|_{L^2(B_1)}$$

where $\chi := \frac{N}{N-2}$ if $N\geq 3$, or any number >1 if $N=2$. Now consider the sequence of radii $r_i := \frac{1}{2}+\frac{1}{2^i}$, and for every $i=1,\ldots,$ apply the inequality above to $\gamma=\gamma_i := \frac{p}{4}\chi^{i-1}$ and $\eta\in C_0^1(B_{r_i})$ such that $\eta\equiv 1$ on $B_{r_{i+1}}$, $0\leq\eta\leq 1$, $|D\eta|\leq c\,2^i$. We obtain

$$\|V\|_{L^{2\gamma_{i+1}}(B_{r_{i+1}})} \leq \left(c(N,p,L,\nu)2^i\gamma_i\right)^{\frac{1}{\gamma_i}}\|V\|_{L^{2\gamma_i}(B_{r_i})},$$

for every $i=1,\ldots,$ and iterating the above formula we have

$$\|V\|_{L^{2\gamma_{i+1}}(B_{r_{i+1}})} \leq \prod_{j=1}^{i}\left(c(N,p,L,\nu)2^j\gamma_j\right)^{\frac{1}{\gamma_j}}\|V\|_{L^{p/2}(B_1)},$$

where we used the fact that $2\gamma_1 = \frac{p}{2}$. Letting $i\to+\infty$, and remarking that $\gamma_i\to+\infty$, $B_{\frac{1}{2}}\subset B_{r_i}$ for all i, the result will follow once we show that the sequence $\left\{\prod_{j=1}^{i}\left(c(N,p,L,\nu)2^j\gamma_j\right)^{\frac{1}{\gamma_j}}\right\}_{i=1}^{\infty}$ is bounded. Indeed,

$$\ln\left(\prod_{j=1}^{i}\left(c(N,p,L,\nu)2^j\gamma_j\right)^{\frac{1}{\gamma_j}}\right)=\sum_{j=1}^{i}\frac{4\chi}{p}\frac{1}{\chi^j}\left[\ln\left(\frac{c(N,p,L,\nu)p}{4\chi}\right)+j\ln(2\chi)\right]$$

which is a converging series because $\chi > 1$. $\square$

Next, we present a simple approximation result.

Lemma 2.4. *Let F satisfy the assumptions of Theorem 2.2. There exist a sequence $\{G_h\}$ of $C^2(\mathbb{R}^N)$ functions and a constant $c \equiv c(N,p)$ such that*

$$\text{(1)} \qquad c^{-1}\left(\mu^2+\frac{1}{h^2}+|z|^2\right)^{p/2} \le G_h(z) \le cL\left(\mu^2+\frac{1}{h^2}+|z|^2\right)^{p/2},$$

$$\text{(2)} \qquad |D^2G_h(z)| \le \Lambda_h\left(\mu^2+\frac{1}{h^2}+|z|^2\right)^{\frac{p-2}{2}},$$

$$\text{(3)} \qquad D_{ij}G_h(z)w_iw_j \ge c^{-1}\nu\left(\mu^2+\frac{1}{h^2}+|z|^2\right)^{\frac{p-2}{2}}|w|^2$$

for every $z, w \in \mathbb{R}^N$, and
(4) $G_h \to F$ uniformly on compact sets.

Proof.
Step 1. We show that we may assume, without loss of generality, that F is a C^2 function satisfying (i) and

$$D_{ij}F(z)w_iw_j \ge c^{-1}\nu(\mu^2+|z|^2)^{\frac{p-2}{2}}|w|^2 \tag{2.2}$$

for some μ strictly greater than zero. Let $\rho(z) = \hat{\rho}(|z|)$ be a positive, radially symmetric mollifier, with support equal to $B_1(0)$, $\int_{B_1}\rho(z)\,dz = 1$, $\rho(z) > 0$ if $|z| < 1$, and for every $\varepsilon > 0$ define

$$\begin{aligned} F^\varepsilon(z) &:= (\rho_\varepsilon * F)(z) \\ &= \int_{B_1}\rho(w)F(z+\varepsilon w)\,dw, \end{aligned}$$

where $\rho_\varepsilon(w) := \frac{1}{\varepsilon^N}\hat{\rho}\left(\frac{|w|}{\varepsilon}\right)$. By (ii) F is a convex function, and so F^ε is a C^2 convex function, $F^\varepsilon \to F$ uniformly on compact sets, and we claim that

$$c^{-1}(\mu^2+\varepsilon^2+|z|^2)^{p/2} \le F^\varepsilon(z) \le cL(\mu^2+\varepsilon^2+|z|^2)^{p/2}$$

for some $c > 0$. In fact, using assumption (i) the estimate from above follows immediately, while

$$\begin{aligned}
F^\varepsilon(z) &\geq \int_{B_1} \rho(w)\left(\mu^2+\varepsilon^2|w|^2+|z|^2+2\varepsilon < z, w >\right)^{p/2} dw \\
&\geq \int_{(B_1\setminus B_{1/2})\cap\{<z,w>\geq 0\}} \left(\mu^2+\frac{\varepsilon^2}{4}+|z|^2\right)^{p/2} \rho(w)\,dw \\
&= \frac{1}{2}\left(\mu^2+\frac{\varepsilon^2}{4}+|z|^2\right)^{p/2} \int_{B_1\setminus B_{1/2}} \rho(w)\,dw \\
&\geq c^{-1}\left(\mu^2+\varepsilon^2+|z|^2\right)^{p/2}.
\end{aligned} \tag{2.3}$$

Also, if $z \in \mathbb{R}^N$ and $\varphi \in C_0^1(Q_1)$, using assumption (ii) on F we have

$$\begin{aligned}
&\int_{Q_1} F^\varepsilon(z+D\varphi(x))\,dx \geq F^\varepsilon(z) \\
&\quad + \nu \int_{Q_1} |D\varphi(x)|^2 \left\{\int_{B_1} \rho(w)(\mu^2+|z+\varepsilon w|^2+|D\varphi(x)|^2)^{\frac{p-2}{2}}\,dw\right\} dx \\
&\quad \geq F^\varepsilon(z) + c^{-1}\nu \int_{Q_1} (\mu^2+\varepsilon^2+|z|^2+|D\varphi(x)|^2)^{\frac{p-2}{2}} |D\varphi(x)|^2\,dx,
\end{aligned} \tag{2.4}$$

because

$$\int_{B_1} \rho(w)(\mu^2+|z+\varepsilon w|^2+|D\varphi(x)|^2)^{\frac{p-2}{2}}\,dw \geq c^{-1}(\mu^2+\varepsilon^2+|z|^2+|D\varphi(x)|^2)^{\frac{p-2}{2}}. \tag{2.5}$$

Indeed, if $p \geq 2$, (2.5) follows by virtue of the same argument used to prove (2.3), while, if $1 < p < 2$, then

$$\begin{aligned}
&\int_{B_1} \rho(w)(\mu^2+|z+\varepsilon w|^2+|D\varphi(x)|^2)^{\frac{p-2}{2}}\,dw \\
&\qquad\qquad \geq \int_{B_1} \rho(w)(\mu^2+2|z|^2+2\varepsilon^2|w|^2+|D\varphi(x)|^2)^{\frac{p-2}{2}}\,dw \\
&\qquad\qquad \geq 2^{\frac{p-2}{2}}(\mu^2+\varepsilon^2+|z|^2+|D\varphi(x)|^2)^{\frac{p-2}{2}}.
\end{aligned}$$

It is easy to show that (2.4) implies (2.2), i.e.,

$$D_{ij}F^\varepsilon(z)w_iw_j \geq c^{-1}\nu(\mu^2+\varepsilon^2+|z|^2)^{\frac{p-2}{2}}|w|^2.$$

Step 2. Define

$$F_h(z) := (1-\eta_h(z))F(z) + \eta_h(z)(\mu^2+|z|^2)^{p/2}$$

for $h = 1, \ldots$, where $\eta_h(z) := \eta(\frac{|z|}{h})$, $\eta(t) \in C_0^1(\mathbb{R})$, $\eta(t) = 0$ if $t \leq 1$, $\eta(t) = 1$ if $t \geq 2$. It is clear that F_h satisfies (i). Denoting by $F_h^{**}(z)$ the convex envelope of $F_h(z)$, it follows that

$$F_h^{**}(z) = (\mu^2+|z|^2)^{p/2} \qquad \text{if } |z| \geq 2h. \tag{2.6}$$

We want to show that there exist $M > 2$ and h_0 depending only on N, p, L, such that, for every $h > h_0$,

$$F_h^{**}(z) = F(z) \qquad \text{if } |z| \leq \frac{2h}{M}. \tag{2.7}$$

Notice that, by (i), $F_h(z) \leq F(z)$ for all $z \in \mathbb{R}^N$, and so

$$F_h^{**}(z) \leq F(z).$$

Conversely, it suffices to show that if $|z| \leq \frac{2h}{M}$ and if $w \in \mathbb{R}^N$ then

$$< DF(z), w - z > + F(z) \leq F_h(w).$$

This is always true if $|w| \leq h$, since

$$< DF(z), w - z > + F(z) \leq F(w) = F_h(w),$$

while, if $|w| > h$, and using the fact that convexity and hypothesis (i) imply

$$|DF(z)| \leq C(N,p)L(\mu^2 + |z|^2)^{\frac{p-1}{2}},$$

we have

$$\begin{aligned}
&< DF(z), w - z > + F(z) - F_h(w) \\
&\qquad \leq c(N,p)L(\mu^2 + |z|^2)^{\frac{p-1}{2}}\left[\mu + |w| + |z|\right] - (\mu^2 + |w|^2)^{p/2} \\
&\qquad \leq c(N,p)L\left(\mu^2 + \frac{h^2}{M^2}\right)^{\frac{p-1}{2}}\left[\mu + |w| + \frac{h}{M}\right] - (\mu^2 + |w|^2)^{p/2} \\
&\qquad \leq c(N,p)L\left(\mu^2 + \frac{h^2}{M^2}\right)^{\frac{p-1}{2}}\left[\mu + 2|w|\right] - (\mu^2 + |w|^2)^{p/2} \\
&\qquad \leq c(N,p)L\left(\mu^2 + \frac{h^2}{M^2}\right)^{\frac{p-1}{2}}(\mu^2 + |w|^2)^{1/2} - (\mu^2 + |w|^2)^{p/2} \\
&\qquad < 0
\end{aligned}$$

provided $M \equiv M(N,p,L) > 2$ and $h \geq h_0 \equiv h_0(N,p,L,M)$ are such that

$$c(N,p)\left(\mu^2 + \frac{h^2}{M^2}\right) < \mu^2 + |w|^2$$

for $|w| > h$. Finally, define

$$R_h(z) := \begin{cases} 0 & \text{if } |z| \leq \frac{h}{M} \\ (\mu^2 + |z|^2)^{p/2} - \left(\mu^2 + \frac{h^2}{M^2}\right)^{p/2} & \text{if } |z| \geq \frac{h}{M} \end{cases},$$

$$\tilde{F}_h(z) := F_h^{**}(z) + R_h(z)$$

and

$$G_h(z) := \tilde{F}_h^{1/h}(z) = \int_{B_1} \rho(w)\tilde{F}_h\left(z + \frac{1}{h}w\right) dw = (\rho_h * \tilde{F}_h)(z).$$

Step 3. Now we show that G_h verifies (1), (2), (3) and (4).

By (2.7), $\tilde{F}_h(z) = F(z)$ if $|z| \leq \frac{h}{M}$, and so $G_h \to F$ uniformly on compact sets, proving (4). From (2.6), we have

$$G_h(z) = 2\int_{B_1} \rho(w)\left(\mu^2 + \left|z + \frac{w}{h}\right|^2\right)^{p/2} dw - \left(\mu^2 + \frac{h^2}{M^2}\right)^{p/2}$$

if $|z| \geq 2h + \frac{1}{h}$, and we deduce that

$$|D^2G_h(z)| \leq c(N,p)\int_{B_1} \rho(w)\left(\mu^2 + \left|z + \frac{w}{h}\right|^2\right)^{\frac{p-2}{2}} dw.$$

If $p \geq 2$ then (2) follows immediately from this inequality. If $1 < p < 2$, since $|z| \geq 2h + \frac{1}{h}$ then

$$\begin{aligned}|D^2G_h(z)| &\leq c(N,p)\int_{B_1} \rho(w)\left(\mu^2 + \left(|z| - \frac{1}{h}\right)^2\right)^{\frac{p-2}{2}} dw \\ &\leq c(N,p)\left(\mu^2 + |z|^2 + \frac{1}{h^2}\right)^{\frac{p-2}{2}}.\end{aligned}$$

Since $\tilde{F}_h$ satisfies (i), by (2.3) we have that G_h verifies (1). Finally, by (2.7) and if $|z| < \frac{2h}{M} - \frac{1}{h}$, then

$$G_h(z) = \rho_h * F(z) + \rho_h * R_h(z),$$

and so, since R_h is convex and by (2.2) and (2.5),

$$\begin{aligned}D_{ij}G_h(z)w_iw_j &\geq \rho_h * c^{-1}\nu(\mu^2 + |z|^2)^{\frac{p-2}{2}}|w|^2 \\ &\geq c^{-1}\nu\left(\mu^2 + \frac{1}{h^2} + |z|^2\right)^{\frac{p-2}{2}} |w|^2.\end{aligned}$$

If $|z| > \frac{2h}{M} - \frac{1}{h}$ then, using (2.5),

$$\begin{aligned}D_{ij}G_h(z)w_iw_j &= \left(\rho_h * D_{ij}F_h^{**}\right)(z)w_iw_j \\ &\quad + \left(\rho_h * D_{ij}\left(\mu^2 + |z|^2\right)^{p/2}\right)(z)w_iw_j \\ &\geq \left(\rho_h * D_{ij}\left(\mu^2 + |z|^2\right)^{p/2}\right)(z)w_iw_j \\ &\geq c^{-1}\nu\left(\mu^2 + \frac{1}{h^2} + |z|^2\right)^{\frac{p-2}{2}} |w|^2.\end{aligned}$$

□

In order to prove Theorem 2.2 we need the following convexity property of F.

Proposition 2.5. *If F satisfies*

$$(i') \qquad 0 \le F(z) \le L(\mu^2 + |z|^2)^{p/2}$$

for all $z \in \mathbb{R}^N$, where $p > 1$, $0 \le \mu \le 1$, and $L > 0$, and if

$$(ii) \qquad \int_{Q_1} F(z + D\varphi)\, dx \ge \int_{Q_1} \left[F(z) + \nu\left(\mu^2 + |z|^2 + |D\varphi|^2\right)^{\frac{p-2}{2}} |D\varphi|^2\right] dx$$

for every $z \in \mathbb{R}^N$, $\varphi \in C_0^1(Q_1)$, and for some $0 < \nu \le 1$, then F is convex and

$$F((1-\theta)z_1 + \theta z_2) < (1-\theta)F(z_1) + \theta F(z_2)$$

whenever $\theta \in (0,1)\,, z_1, z_2 \in \mathbb{R}^N, z_1 \ne z_2$.

Proof. Fix $z_1, z_2 \in \mathbb{R}^N$, $\theta \in (0,1)$, with $z_1 - z_2 \ne 0$, and set $z_\theta := (1-\theta)z_1 + \theta z_2, \xi := z_2 - z_1$. Let χ be the characteristic function of the interval $(0,\theta)$, extended periodically to $\mathbb{R}$ with period 1. Then

$$\begin{aligned}(1-\theta)F(z_1) + \theta F(z_2) &= \lim_{n\to\infty} \int_{Q_1} F\left(z_1 + \chi\left(nx.\frac{\xi}{|\xi|}\right)\xi\right) dx \\ &= \lim_{n\to\infty} \int_{Q_1} F(z_\theta + Du_n(x))\, dx,\end{aligned}$$

where

$$u_n(x) := v_n(x) - \int_{Q_1} v_n(y)\, dy, \qquad v_n(x) := \frac{|\xi|}{n} \int_0^{nx.\frac{\xi}{|\xi|}} (\chi(s) - \theta) ds.$$

Since

$$Du_n(x) = \left(\chi\left(nx.\frac{\xi}{|\xi|}\right) - \theta\right)\xi \rightharpoonup 0, \qquad \text{in } L^\infty - w*,$$

we have that $u_n \rightharpoonup 0$ in $W^{1,\infty} - w*$, and using the growth condition (i'), after extracting a subsequence if necessary, we may find cut-off functions $\varphi_n \in C_0^\infty(Q_1; [0,1])$, $\varphi_n(x) = 1$ if $x \in Q_n$, $\mathcal{L}^N(Q_1 \setminus Q_n) \to 0$, such that

$$\int_{Q_1 \setminus Q_n} |D\varphi_n(x)|^p |u_n(x)|^p\, dx \to 0. \tag{2.8}$$

Hence, using (i') we deduce that

$$\lim_{n\to\infty} \int_{Q_1} F(z_\theta + Du_n(x))\, dx \ge \limsup_{n\to\infty} \int_{Q_1} F(z_\theta + D(\varphi_n u_n)(x))\, dx,$$

and by (ii) we conclude that

$$\begin{aligned}&(1-\theta)F(z_1) + \theta F(z_2) \\ &\ge F(z_\theta) + \nu \liminf_{n\to\infty} \int_{Q_1} (\mu^2 + |z_\theta|^2 + |D(\varphi_n u_n)(x)|^2)^{\frac{p-2}{2}} |D(\varphi_n u_n)(x)|^2\, dx \\ &\ge F(z_\theta) + \nu \liminf_{n\to\infty} \int_{Q_n} (\mu^2 + |z_\theta|^2 + |Du_n(x)|^2)^{\frac{p-2}{2}} |Du_n(x)|^2\, dx.\end{aligned}$$

Since, by (2.8), if $q > 1$

$$\begin{aligned}\lim_{n\to\infty}\int_{Q_n}|Du_n(x)|^q\,dx &= \lim_{n\to\infty}\int_{Q_1}|Du_n(x)|^q\,dx\\ &= \lim_{n\to\infty}\int_{Q_1}\left|\chi\left(nx.\frac{\xi}{|\xi|}\right)-\theta\right|^q|z_1-z_2|^q\,dx\\ &= \theta(1-\theta)\left[\theta^{q-1}+(1-\theta)^{q-1}\right]|z_1-z_2|^q,\end{aligned}$$

when $p \geq 2$ we have

$$\begin{aligned}(1-\theta)F(z_1)+\theta F(z_2) \geq F(z_\theta) &+ c(p)\nu\lim_{n\to\infty}\int_{Q_n}(\mu^2+|z_\theta|^2)^{\frac{p-2}{2}}|Du_n|^2\,dx\\ &+ c(p)\nu\lim_{n\to\infty}\int_{Q_n}|Du_n|^p\,dx\\ = F(z_\theta) &+ c(p)\nu(\mu^2+|z_\theta|^2)^{\frac{p-2}{2}}\theta(1-\theta)|z_1-z_2|^2\\ &+ c(p)\nu\theta(1-\theta)\left[\theta^{p-1}+(1-\theta)^{p-1}\right]|z_1-z_2|^p\\ > F(z_\theta)&,\end{aligned}$$

while, if $1 < p < 2$, since $|Du_n| \leq |z_1 - z_2|$ we conclude that

$$\begin{aligned}(1-\theta)F(z_1)+&\theta F(z_2)\\ &\geq F(z_\theta)+\nu\lim_{n\to\infty}\int_{Q_n}(\mu^2+|z_\theta|^2+|z_1-z_2|^2)^{\frac{p-2}{2}}|Du_n|^2\,dx\\ &= F(z_\theta)+c(p)\nu(\mu^2+|z_\theta|^2+|z_1-z_2|)^{\frac{p-2}{2}}\theta(1-\theta)|z_1-z_2|^2\\ &> F(z_\theta).\end{aligned}$$

□

We are now in position to give the proof of Theorem 2.2.

Proof of Theorem 2.2. Fix $B_R(x_0)$ and for any h denote by u_h the solution of the problem

$$\min\left\{\int_{B_R(x_0)}G_h(Dv)\,dx : v \in u + W_0^{1,p}(B_R(x_0))\right\},$$

where $\{G_h\}$ is the approximating sequence of C^2 convex functions provided by Lemma 2.4. From Lemma 2.3 we have that the sequence $\{u_h\}$ is bounded in $W^{1,p}(B_R)$, and is locally bounded in $W^{1,\infty}(B_R)$. Hence, we may suppose, passing possibly to a subsequence, that $u_h \rightharpoonup u_\infty$ in $W^{1,\infty} - w*$ locally in B_R. Then, using the fact $G_h \to F$ uniformly on compact sets, the convexity of F, and the minimality of u_h, we deduce that, for every $0 < \rho < R$,

$$\begin{aligned}\int_{B_\rho} F(Du_\infty)\,dx &\le \liminf_h \int_{B_\rho} F(Du_h)\,dx \\ &= \liminf_h \int_{B_\rho} G_h(Du_h)\,dx \\ &\le \liminf_h \int_{B_R} G_h(Du)\,dx \\ &= \int_{B_R} F(Du)\,dx.\end{aligned}$$

Letting $\rho \uparrow R$, since u is a local minimizer for $\mathcal{F}$, and $u = u_\infty$ on ∂B_R, we conclude that

$$\int_{B_R} F(Du_\infty)\,dx = \int_{B_R} F(Du)\,dx.$$

We claim that $u = u_\infty$. Indeed, if $u \neq u_\infty$ choose $\theta \in (0,1)$ and set $v = \theta u_\infty + (1-\theta)u$, so that by Proposition 2.5 we have

$$\begin{aligned}\int_{B_R} F(Dv)\,dx &< \theta \int_{B_R} F(Du_\infty) + (1-\theta)\int_{B_R} F(Du)\,dx \\ &= \int_{B_R} F(Du)\,dx,\end{aligned}$$

contradicting the minimality of u. Applying Lemma 2.3 to u_h, using the minimality of u_h, the growth assumption on F, and the growth estimates on G_h, we have

$$\begin{aligned}\sup_{B_{R/2}} (\mu^2 + |Du|^2)^{p/2} &\le \liminf_h \sup_{B_{R/2}} (\mu^2 + |Du_h|^2)^{p/2} \\ &\le c \liminf_h \fint_{B_R} \left(\mu^2 + \frac{1}{h^2} + |Du_h|^2\right)^{p/2} dx \\ &\le c \liminf_h \fint_{B_R} G_h(Du_h)\,dx \\ &\le c \liminf_h \fint_{B_R} G_h(Du)\,dx \\ &= c \fint_{B_R} F(Du) \\ &\le c \fint_{B_R} (\mu^2 + |Du|^2)^{p/2} dx,\end{aligned}$$

and this completes the proof of the theorem. □

Definition 2.6. *The* p-recession function *of a function* $F : \mathbb{R}^N \to [0, +\infty)$, $p \ge 1$, *is defined by*

$$F_p(z) := \limsup_{t \to +\infty} \frac{F(tz)}{t^p}$$

for $z \in \mathbb{R}^N$.

Remark 2.7. It is clear that F_p is positively homogeneous of degree p, and if F is convex, then F_p is also convex. Moreover, if

$$\frac{1}{L}|z|^p - L \leq F(z) \leq L(1+|z|^p),$$

then

$$\frac{1}{L}|z|^p \leq F_p(z) \leq L|z|^p.$$

The next lemma establishes strict quasiconvexity of F_p, provided F is strictly quasiconvex and verifies appropriate growth conditions.

Lemma 2.8. *Let $F : \mathbb{R}^N \to [0, +\infty)$ be a continuous function satisfying, for $p > 1$,*

$$0 \leq F(z) \leq L(1+|z|^p),$$

and

$$\int_{Q_1} F(z + D\varphi)\, dx \geq \int_{Q_1} \left[F(z) + \nu(\mu^2 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\right] dx$$

for every $z \in \mathbb{R}^N$, $\varphi \in C_0^1(Q_1)$ and some $0 < \nu \leq 1$ and $\mu \geq 0$. In addition, assume that there exist $t_0 > 0$ and $0 < m < p$ such that

$$\left|F_p(z) - \frac{F(tz)}{t^p}\right| \leq \frac{c_0}{t^m}$$

for every $t > t_0$ and all $z \in S^{N-1}$. Then

$$\int_{Q_1} F_p(z + D\varphi)\, dx \geq \int_{Q_1} \left[F_p(z) + \nu(|z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\right] dx \tag{2.9}$$

for every $z \in \mathbb{R}^N$ and all $\varphi \in C_0^1(Q_1)$.

Proof. Fix $\lambda > 1$ and notice that for $t > t_0\lambda$ and z such that $\lambda^{-1} < |z| < \lambda$, we have

$$\left|F_p(z) - \frac{F(tz)}{t^p}\right| \leq \frac{c_0\lambda^{p-m}}{t^m}. \tag{2.10}$$

In fact,

$$\begin{aligned}\left|F_p(z) - \frac{F(tz)}{t^p}\right| &= |z|^p \left|F_p\left(\frac{z}{|z|}\right) - F\left(t|z|\frac{z}{|z|}\right) \Big/ (t|z|)^p\right| \\ &\leq \frac{c_0|z|^p}{(t|z|)^m} \\ &\leq \frac{c_0\lambda^{p-m}}{t^m}.\end{aligned}$$

To prove (2.9), fix $z \in \mathbb{R}^N$, $\varphi \in C_0^1(Q_1)$, and take a sequence $t_h \uparrow \infty$ such that $F_p(z) = \lim_h \frac{F(t_h z)}{t_h^p}$. Fix $\lambda > \max\{1, |z| + ||D\varphi||_\infty\}$. Then, if $t_h > t_0\lambda$, from (2.10) and by virtue of the strict quasiconvexity of F, we have

$$\begin{aligned}
\int_{Q_1} F_p(z + D\varphi)\, dx &\geq \int_{Q_1 \cap \{\lambda^{-1} < |z+D\varphi|\}} F_p(z+D\varphi)\, dx \\
&\geq \frac{1}{t_h^p} \int_{Q_1 \cap \{\lambda^{-1} < |z+D\varphi|\}} F(t_h z + t_h D\varphi)\, dx - \frac{c_0 \lambda^{p-m}}{t_h^m} \\
&\geq \frac{1}{t_h^p} \int_{Q_1} F(t_h z + t_h D\varphi)\, dx - \frac{c_0 \lambda^{p-m}}{t_h^m} \\
&\quad - \frac{L}{t_h^p} \int_{Q_1 \cap \{|z+D\varphi| \leq \lambda^{-1}\}} (1 + |t_h z + t_h D\varphi|^p)\, dx \\
&\geq \int_{Q_1} \left[\frac{F(t_h z)}{t_h^p} + \nu \left(\frac{\mu^2}{t_h^2} + |z|^2 + |D\varphi|^2 \right)^{\frac{p-2}{2}} |D\varphi|^2 \right] dx - \frac{c_0 \lambda^{p-m}}{t_h^m} \\
&\quad - \frac{L}{t_h^p} \left(1 + \frac{t_h^p}{\lambda^p} \right).
\end{aligned}$$

The result follows by letting h go to $+\infty$ and then λ go to $+\infty$. □

3. Regularity Results - The Scalar Case

In order to state the main regularity result of this paper, Theorem 3.5, we recall some notations and properties of BV and SBV functions that will be of later use.

Given a set $E \subset \mathbb{R}^N$, we denote by $H^{N-1}(E)$ its $(N-1)$-dimensional Hausdorff measure. If $u : \Omega \to \mathbb{R}$ is a Borel function, $x \in \Omega$, we say that $\tilde{u}(x) \in \mathbb{R} \cup \{\infty\}$ is the *approximate limit* of u at x, $\tilde{u}(x) = \operatorname{aplim}_{y \to x} u(y)$, if

$$g(\tilde{u}(x)) = \lim_{\rho \to 0} \fint_{B_\rho(x)} g(u(y))\, dy$$

for every function $g \in C(\mathbb{R} \cup \{\infty\})$. With this definition, the set

$$S_u := \{x \in \Omega : \operatorname*{aplim}_{y \to x} u(y) \quad \text{does not exist}\}$$

is a Borel set with zero Lebesgue measure. $BV(\Omega)$ stands for the space of functions with bounded variation in Ω, and if $u \in BV(\Omega)$ one can show that the jump set S_u is countably $(N-1)$- rectifiable (see [DG] or [F]). Moreover, $H^{N-1}(\{x \in \Omega : \tilde{u}(x) = \infty\}) = 0$ (see [F]). It is also well known that if $u \in BV(\Omega)$ then the distributional derivative Du can be decomposed as $Du = \nabla u \mathcal{L}^N + D_s u$, where ∇u is the density of Du with respect to the Lebesgue N-dimensional measure $\mathcal{L}^N$, and $D_s u$ is the singular part of Du with respect to $\mathcal{L}^N$. Finally, we recall that the space of *special functions of bounded variation*,

$SBV(\Omega)$, introduced in [DGA], consists of all functions in $BV(\Omega)$ such that $D_s u$ is supported in S_u, i.e.

$$|D_s u|(\Omega \setminus S_u) = 0.$$

For the study of the main properties of SBV functions, we refer to [A1], [A2], [DGA], and we select the following SBV compactness theorem (see [A1]).

Theorem 3.1. *Let $f : [0,\infty) \to \mathbb{R}$ and $\varphi : (0,\infty] \to \mathbb{R}$ be convex and concave respectively, nondecreasing, and satisfying*

$$\lim_{t\to\infty} \frac{f(t)}{t} = \infty, \qquad \varphi(\infty) = \lim_{t\to\infty} \varphi(t), \qquad \lim_{t\to 0^+} \frac{\varphi(t)}{t} = \infty.$$

Let $\{u_n\}$ be a sequence of functions in $SBV(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ such that $\sup_n \|u_n\|_\infty < +\infty$ and

$$\sup_n \left\{ \int_\Omega f(|\nabla u_n|)\, dx + \int_{S_{u_n}} \varphi(|u_n^+ - u_n^-|)\, dH^{N-1} \right\} < \infty.$$

Then there exists a subsequence u_{n_k} and a function $u \in SBV(\Omega, \mathbb{R}^d)$ such that

$$u_{n_k} \to u \ \text{ in } L^1,\ \nabla u_{n_k} \rightharpoonup \nabla u \ \text{ in} L^1 \ \text{ and } H^{N-1}(S_u) \le \liminf_{k\to\infty} H^{N-1}(S_{u_{n_k}}).$$

Notice that if $u \in SBV(\Omega)$, then clearly $u \in W^{1,1}(\Omega \setminus \overline{S_u})$. Conversely, it follows from a modified version of Lemma 2.3 in [DGCL] that

Lemma 3.2. *If $u \in L^\infty(\Omega)$, and if $K \subset \mathbb{R}^N$ is a closed set such that $H^{N-1}(\Omega \cap K) < \infty$ and $u \in W^{1,1}(\Omega \setminus K)$, then $u \in SBV(\Omega)$ and $\overline{S_u} \subset K$.*

Density properties of $u \in BV$ at points $x \in S_u$ have been obtained in [DGCL]. In particular, the following result follows from Lemma 2.6 and Theorem 3.6 in [DGCL].

Theorem 3.3. *Let $u \in SBV(\Omega)$ be such that*

$$\int_\Omega |\nabla u|^p\, dx + H^{N-1}(S_u \cap \Omega) < +\infty$$

for some $p > 1$. Then

(i)
$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left[\int_{B_\varepsilon(x)} |\nabla u|^p\, dx + H^{N-1}(S_u \cap B_\varepsilon(x)) \right] = 0$$

for H^{N-1}a.e. $x \in \Omega \setminus S_u$;
(ii) if

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left[\int_{B_\varepsilon(x)} |\nabla u|^p\, dx + H^{N-1}(S_u \cap B_\varepsilon(x)) \right] = 0$$

then $x \notin S_u$.

The next theorem can be found in [CL], Theorem 2.6 (see also [DGCL], Remark 3.2 and Theorem 3.5) in a slightly different form, and it is proven by means of a suitable version of Sobolev-Poincaré inequality for SBV functions.

Theorem 3.4. *If* $\{u_h\} \subset SBV(B_R)$, $p > 1$, *and if*

$$\sup_h \int_{B_R} |\nabla u_h|^p dx < \infty, \qquad \lim_h H^{N-1}(S_{u_h} \cap B_R) = 0,$$

then there exist a subsequence $\{u_{h_k}\}$, *a sequence* $\{m_k\} \subset \mathbb{R}$, *and a function* $u_\infty \in W^{1,p}(B_R)$ *such that*

$$u_{h_k}(x) - m_k \to u_\infty(x) \qquad a.e. \text{ in } B_R$$

and

$$\int_{B_R} G(\nabla u_\infty)\, dx \le \liminf_{k \to \infty} \int_{B_R} G(\nabla u_{h_k})\, dx$$

for every nonnegative convex function G, *with* $G(0) = 0$. *In addition, there exist constants* α_k, β_k *such that, setting*

$$\bar{u}_{h_k} := \max\{\min\{u_{h_k}, \alpha_k\}, \beta_k\},$$

then

$$\bar{u}_{h_k} - m_k \to u_\infty \quad in \quad L^p \tag{3.1}$$

and

$$\mathcal{L}^N(\{u_{h_k} \neq \bar{u}_{h_k}\} \cap B_R) \le C(N)\left[H^{N-1}(S(u_{h_k}) \cap B_R)\right]^{\frac{N}{N-1}}. \tag{3.2}$$

In the sequel we denote by F a convex function on $\mathbb{R}^N$ satisfying the following assumptions:

$$(H1) \qquad |z|^p \le F(z) \le L(1 + |z|^p), \quad p > 1,$$

$$(H2) \qquad \int_{Q_1} F(z + D\varphi)\, dx \ge \int_{Q_1} \left[F(z) + \nu(\mu^2 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\right] dx$$

for every $z \in \mathbb{R}^N$, for all $\varphi \in C_0^1(Q_1)$, and for some $\nu > 0$, $\mu \ge 0$. Moreover we will assume that

$$(H3) \qquad \left| F_p(z) - \frac{F(tz)}{t^p} \right| \le \frac{c_0}{t^m}$$

for every $t > t_0 > 0$, for all $z \in S^{N-1}$, and some $0 < m < p$, where F_p is the p-recession function of F (see Definition 2.6). Our main goal, Theorem 3.5 below, is to show the existence of a minimizing pair (K, u) for the functional

$$\mathcal{G}(K,v) := \int_{\Omega\setminus K} F(Dv)\,dx + \alpha\int_{\Omega\setminus K} |v-g|^q\,dx + \beta H^{N-1}(K\cap\Omega)$$

where $q \geq 1$, $\alpha, \beta > 0$, K is a closed set and $v \in W^{1,p}(\Omega\setminus K)$, $g \in L^\infty(\Omega)$. In order to obtain this result, and following [DGCL], we introduce the functional

$$\bar{\mathcal{G}}(v) := \int_\Omega F(\nabla v)\,dx + \alpha\int_\Omega |v-g|^q\,dx + \beta H^{N-1}(S_v\cap\Omega)$$

defined for $v \in SBV(\Omega)$.

Now we state our regularity and existence theorem.

Theorem 3.5. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a convex function such that $F(0) = \min F(z)$ and verifying (H1),(H2),(H3), $1 \leq p < \infty$, $g \in L^\infty$, $\alpha, \beta > 0$. There exists a minimizer of $\bar{\mathcal{G}}(v)$, $u \in SBV(\Omega)$, such that $(\overline{S_u}, u)$ is a minimizer of $\mathcal{G}(K,v)$ among all pairs (K,v), where K is a closed set and $v \in W^{1,p}(\Omega\setminus K)$. Moreover*

$$\bar{\mathcal{G}}(u) = \mathcal{G}(\overline{S_u}, u)$$

and

$$H^{N-1}((\overline{S_u}\setminus S_u)\cap\Omega) = 0.$$

The existence of minimizers for $\bar{\mathcal{G}}(v)$ follows from compactness and lower semicontinuity results of [A1]. Indeed, the hypothesis $F(0) = \min F(z)$ allows us to truncate minimizing sequences in order to apply Theorem 3.1, yielding the following result.

Theorem 3.6. *If $F : \mathbb{R}^N \to \mathbb{R}$ is a convex function such that $F(0) = \min F(z)$ and verifying (H1), $1 \leq p < \infty$, $g \in L^\infty$, $\alpha, \beta > 0$, then there exists a minimizer u in $SBV(\Omega) \cap L^\infty(\Omega)$ for the functional $\bar{\mathcal{G}}(v)$.*

In order to prove Theorem 3.5 we must show that the pair $(\overline{S_u}, u)$, where u is a minimizer provided by Theorem 3.6, is indeed a minimizer for the functional $\mathcal{G}(K,v)$. Following [DGCL], we introduce some useful quantities.

Definition 3.7. *Let F_p be the p-recession function of F, $u \in SBV(\Omega)$, $c > 0$, and let $A \subset\subset \Omega$ be an open, strongly Lipschitz domain.. We define*

$$\mathcal{F}(u,c,A) := \int_A F(\nabla u)\,dx + cH^{N-1}(S_u\cap\overline{A}),$$

$$\mathcal{F}_p(u,c,A) := \int_A F_p(\nabla u)\,dx + cH^{N-1}(S_u\cap\overline{A}),$$

$$\Phi_p(u,c,A) := \inf\left\{\mathcal{F}_p(v,c,A) : v \in SBV(\Omega), v = u \,in\, \Omega\setminus\overline{A}\right\},$$

and, if $\Phi_p(u,c,A) < \infty$, we set

$$\Psi_p(u,c,A) := \mathcal{F}_p(u,c,A) - \Phi_p(u,c,A).$$

Remark 3.8. If $u \in SBV(\Omega) \cap L^\infty(\Omega)$ then, by Theorem 3.1, Φ_p is always attained.

Notice that if in the definition of $\mathcal{F}_p$ we take $H^{N-1}(S_u \cap A)$ instead of $H^{N-1}(S_u \cap \overline{A})$ then we get Φ_p identically equal to zero.

Also, if $u \in SBV(B_R(x_0))$, $\rho < R$, we set $u_\rho(y) := \rho^{-\frac{p-1}{p}} u(x_0 + \rho y)$ for $y \in B_{R/\rho}(0)$, to obtain $u_\rho \in SBV(B_{R/\rho})$, and

$$\mathcal{F}_p(u_\rho, c, B_1) = \rho^{1-N} \mathcal{F}_p(u, c, B_\rho(x_0)),$$

$$\Phi_p(u_\rho, c, B_1) = \rho^{1-N} \Phi_p(u, c, B_\rho(x_0)).$$

The next lemma is proved exactly as Lemma 4.6 in [DGCL].

Lemma 3.9. *Let $u, v \in SBV(B_R)$, $c > 0$ and $0 < \rho < R$. If $H^{N-1}(S_u \cap \partial B_\rho) = 0 = H^{N-1}(S_v \cap \partial B_\rho)$, then*

$$|\Phi_p(u, c, B_\rho) - \Phi_p(v, c, B_\rho)| \le c H^{N-1}(\{\tilde{u} \ne \tilde{v}\} \cap \partial B_\rho).$$

The following two results are straightforward generalizations of Lemma 4.7 and Theorem 4.8 in [DGCL]. For completeness we include their proofs.

Lemma 3.10. *Let $u, v \in SBV(B_R)$, $0 < \rho < \rho' < R$, $c > 0$. Then*

$$\begin{aligned}\Phi_p(u, c, B_{\rho'}) \le\ & \Phi_p(v, c, B_\rho) + c(L,p)\mathcal{F}_p(u, c, B_{\rho'} \setminus \overline{B_\rho}) \\ & + c(L,p)\mathcal{F}_p(v, c, B_{\rho'} \setminus \overline{B_\rho}) + \frac{c(L,p)}{(\rho' - \rho)^p} \int_{B_{\rho'} \setminus B_\rho} |u - v|^p \, dx.\end{aligned}$$

Proof. Define

$$\varphi(x) := \begin{cases} 0 & |x| \le \rho \\ \frac{|x| - \rho}{\rho' - \rho} & \rho < |x| < \rho' \\ 1 & \rho' \le R. \end{cases}$$

Fix $\varepsilon > 0$ and consider $w \in SBV(B_R)$ such that $w = v$ on $B_R \setminus \overline{B_\rho}$ and

$$\mathcal{F}_p(w, c, B_\rho) \le \Phi_p(v, c, B_\rho) + \varepsilon.$$

Set $z := \varphi u + (1 - \varphi) w$. By Remark 2.7 we have

$$
\begin{aligned}
&\Phi_p(u,c,B_{\rho'}) \le \mathcal{F}_p(z,c,B_{\rho'}) \\
&\le \int_{B_\rho} F_p(\nabla w)\,dx + c(L,p)\int_{B_{\rho'}\setminus B_\rho} |\nabla u|^p\,dx + c(L,p)\int_{B_{\rho'}\setminus B_\rho} |\nabla v|^p\,dx \\
&\quad + \frac{c(L,p)}{(\rho'-\rho)^p}\int_{B_{\rho'}\setminus B_\rho} |u-v|^p\,dx + cH^{N-1}(S_w \cap \overline{B_\rho}) \\
&\quad + cH^{N-1}(S_u \cap (\overline{B_{\rho'}}\setminus B_\rho)) + cH^{N-1}(S_v \cap (\overline{B_{\rho'}}\setminus B_\rho)) \\
&\le \Phi_p(v,c,B_\rho) + \varepsilon + c(L,p)\mathcal{F}_p(u,c,B_{\rho'}\setminus \overline{B_\rho}) + c(L,p)\mathcal{F}_p(v,c,B_{\rho'}\setminus \overline{B_\rho}) \\
&\quad + \frac{c(L,p)}{(\rho'-\rho)^p}\int_{B_{\rho'}\setminus B_\rho} |u-v|^p\,dx.
\end{aligned}
$$

The conclusion follows letting $\varepsilon \to 0$. □

In the following theorem we use the notation introduced in Theorem 3.4.

Theorem 3.11. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a convex function satisfying (H1), $\{u_h\} \subset SBV(\Omega)$, $c_h > 0$, $u_\infty \in W^{1,p}_{loc}(\Omega)$, $B_R(x) \subset\subset \Omega$,*

$$\lim_h c_h = +\infty,$$

$$\lim_h \mathcal{F}_p(u_h, c_h, B_\rho(x)) = \lim_h \Phi_p(u_h, c_h, B_\rho(x)) =: \alpha(\rho) < +\infty \qquad \text{for a.e. } \rho < R,$$

and

$$\lim_h [u_h - m_h] = u_\infty \qquad \text{a.e. in } B_R(x).$$

Then the function u_∞ is a local minimizer of the functional

$$v \mapsto \int_{B_R(x)} F_p(\nabla v)\,dx$$

and for $\mathcal{L}^1$ a.e. $\rho < R$

$$\alpha(\rho) = \int_{B_\rho(x)} F_p(\nabla u_\infty)\,dx.$$

Proof. Since $c_h \to +\infty$ we have

$$\sup_h \int_{B_\rho} |\nabla u_h|^p\,dx < +\infty \qquad \text{and} \qquad H^{N-1}(S_{u_h} \cap B_\rho) \to 0$$

for $\mathcal{L}^1$ a.e. $\rho < R$. Hence, by Theorem 3.4 we may find a subsequence (not relabelled), a function $u_\infty \in W^{1,p}_{loc}(B_R)$, and constants m_h such that

$$\lim_h [u_h - m_h] = u_\infty \qquad \text{a.e. in } B_R(x).$$

In addition, since F_p is a convex function (see Remark 2.7) and $F_p(0) = 0$, by Theorem 3.4 we have

$$\int_{B_\rho} F_p(\nabla u_\infty)\,dx \le \liminf_h \int_{B_\rho} F_p(\nabla u_h)\,dx$$
$$\le \liminf_h \mathcal{F}_p(u_h, c_h, B_\rho) = \alpha(\rho)$$

for $\mathcal{L}^1$ a.e. $\rho < R$. It suffices to prove that, for $\mathcal{L}^1$ a.e. $\rho < R$ and for all $v \in W^{1,p}_{loc}(B_R)$ such that $v = u_\infty$ on $B_R \setminus \overline{B_\rho}$, one has

$$\int_{B_\rho} F_p(\nabla v)\,dx \ge \alpha(\rho).$$

Consider the bounded sequence of finite Radon measures $\mu_h := |\nabla u_h|^p \mathcal{L}^N + c_h H^{N-1} \lfloor S_{u_h}$. After extracting a (not relabelled) subsequence, we may suppose that $\mu_h \xrightarrow{*} \mu$, for some finite Radon measure μ. Fix $0 < \rho < R$. By (3.2), using the facts that the sequence $\{c_h H^{N-1}(S_{u_h} \cap B_r)\}$ is bounded for a.e. $r < R$, and that $c_h \to +\infty$, we obtain

$$\begin{aligned} c_h \mathcal{L}^N(\{u_h \ne \bar{u}_h\} \cap B_R) &= \int_0^1 c_h H^{N-1}(\{\tilde{u}_h \ne \tilde{\bar{u}}_h\} \cap \partial B_\rho)\,d\rho \\ &\le C(N) c_h \left[H^{N-1}(S_{u_h} \cap B_R)\right]^{\frac{N}{N-1}} \\ &= C(N) \left[c_h H^{N-1}(S_{u_h} \cap B_R)\right]^{\frac{N}{N-1}} c_h^{\frac{-1}{N-1}} \to 0; \end{aligned}$$

hence, and after extracting a (not relabelled) subsequence, we conclude that

$$c_h H^{N-1}(\{\tilde{u}_h \ne \tilde{\bar{u}}_h\} \cap \partial B_\rho) \to 0 \tag{3.3}$$

for $\mathcal{L}^1$ a.e. $\rho < R$.

Suppose that there exist $0 < \rho < R$ for which (3.3) holds, $\varepsilon > 0$, $v \in W^{1,p}(B_\rho)$, $v = u_\infty$ on ∂B_ρ, such that $H^{N-1}(S_{u_h} \cap \partial B_\rho) = 0$ for all h, $\mu(\partial B_\rho) = 0$ and

$$\int_{B_\rho} F_p(\nabla v)\,dx < \alpha(\rho) - \varepsilon. \tag{3.4}$$

Fixing $\rho' > 0$ such that $0 < \rho < \rho' < R$, by virtue of Lemma 3.10 we have

$$\begin{aligned} \Phi_p(\bar{u}_h, c_h, B_{\rho'}) &\le \Phi_p(u_\infty, c_h, B_\rho) + c(L,p)\mathcal{F}_p(\bar{u}_h, c_h, B_{\rho'} \setminus \overline{B_\rho}) \\ &\quad + c(L,p) \int_{B_{\rho'} \setminus B_\rho} F_p(\nabla u_\infty)\,dx \\ &\quad + \frac{c(L,p)}{(\rho' - \rho)^p} \int_{B_{\rho'} \setminus B_\rho} |\bar{u}_h - m_h - u_\infty|^p\,dx \\ &\le \int_{B_\rho} F_p(\nabla v)\,dx + c(L,p)\mu_h(\overline{B_{\rho'}} \setminus B_\rho) + c(L,p) \int_{B_{\rho'} \setminus B_\rho} F_p(\nabla u_\infty)\,dx \\ &\quad + \frac{c(L,p)}{(\rho' - \rho)^p} \int_{B_{\rho'} \setminus B_\rho} |\bar{u}_h - m_h - u_\infty|^p\,dx. \end{aligned}$$

Letting $h \to +\infty$, and using (3.1) and (3.4), we obtain

$$\limsup_h \Phi_p(\bar u_h, c_h, B_{\rho'}) \le \alpha(\rho) - \varepsilon + c(L,p)\mu(\overline{B_{\rho'}} \setminus B_\rho)$$
$$+ c(L,p) \int_{B_{\rho'} \setminus B_\rho} F_p(\nabla u_\infty)\, dx,$$

and letting $\rho' \to \rho^+$ we conclude that

$$\limsup_{\rho' \to \rho^+} \limsup_h \Phi_p(\bar u_h, c_h, B_{\rho'}) \le \alpha(\rho) - \varepsilon. \tag{3.5}$$

On the other hand, (3.3) and Lemma 3.9 yield

$$\lim_h \Phi_p(\bar u_h, c_h, B_{\rho'}) = \alpha(\rho')$$

for $\mathcal{L}^1 a.e. \rho' > \rho$ which, given that $\alpha(\cdot)$ is an increasing function, is in contradiction with (3.5). □

At this point, and as in [DGCL], using the regularity result provided by Theorem 2.2 we can prove an energy decay estimate for the minimizers of the functional $\mathcal{F}_p(u, c, B_R)$.

Lemma 3.12. **[Decay Lemma]** *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a convex function satisfying (H1), (H2) and (H3). There exist $C_1 \equiv C_1(N,p,L,\nu), R_1 \equiv R_1(N,p,L,\nu)$, such that for every $c > 0$, $R < R_1$, $0 < \tau \le \frac{1}{2}$, there exist $\varepsilon \equiv \varepsilon(c,\tau)$, $\theta \equiv \theta(c,\tau)$, such that if $u \in SBV(\Omega)$, $B_R \subset\subset \Omega$ and*

$$\mathcal{F}_p(u,c,B_R) \le \varepsilon R^{N-1} \quad , \quad \Psi_p(u,c,B_R) \le \theta \mathcal{F}_p(u,c,B_R),$$

then

$$\mathcal{F}_p(u,c,B_{\tau R}) \le C_1 \tau^N \mathcal{F}_p(u,c,B_R).$$

Proof. We argue by contradiction. Suppose the result is not true; then there exist two sequences $\{\varepsilon_h\}$, $\{\theta_h\}$, with $\lim_h \varepsilon_h = \lim_h \theta_h = 0$, a sequence $\{u_h\} \subset SBV(\Omega)$, and a sequence of balls $B_{R_h}(x_h) \subset\subset \Omega$ such that

$$\mathcal{F}_p(u_h,c,B_{R_h}(x_h)) = \varepsilon_h R_h^{N-1} \quad , \quad \Psi_p(u_h,c,B_{R_h}(x_h)) = \theta_h \mathcal{F}_p(u_h,c,B_{R_h}(x_h)),$$

and

$$\mathcal{F}_p(u_h,c,B_{\tau R_h}(x_h)) > C_1 \tau^N \mathcal{F}_p(u_h,c,B_{R_h}(x_h)),$$

where C_1 is a constant to be chosen later. Rescaling, we set for every h

$$v_h(y) := R_h^{-\frac{(p-1)}{p}} \varepsilon_h^{-\frac{1}{p}} u_h(x_h + R_h y), \qquad y \in B_1(0).$$

From Remark 3.8 we obtain immediately

$$\mathcal{F}_p(v_h, c/\varepsilon_h, B_1) = 1 \quad , \quad \Psi_p(v_h, c/\varepsilon_h, B_1) = \theta_h \tag{3.6}$$

and

$$\mathcal{F}_p(v_h, c/\varepsilon_h, B_\tau) > C_1 \tau^N. \tag{3.7}$$

Since $\lim_h c/\varepsilon_h = +\infty$, then $\lim_h H^{N-1}(S_{v_h} \cap B_1) = 0$, and so by Theorem 3.4, passing possibly to a subsequence still denoted by $\{v_h\}$, there exist a sequence $\{m_h\} \subset \mathbb{R}$ and a function $v_\infty \in W^{1,p}(B_1)$ such that

$$v_h - m_h \to v_\infty \text{ a.e. in } B_1, \text{ and } \int_{B_1} F_p(\nabla v_\infty)\, dx \le \liminf_h \int_{B_1} F_p(\nabla v_h)\, dx.$$

Notice that for any h the functions $\rho \to \mathcal{F}_p(v_h, c/\varepsilon_h, B_\rho)$ are increasing, and from (3.6) we have also that $\mathcal{F}_p(v_h, c/\varepsilon_h, B_\rho) \le 1$ for every $0 < \rho \le 1$. Therefore, upon extracting another subsequence, we may suppose that

$$\lim_h \mathcal{F}_p(v_h, c/\varepsilon_h, B_\rho) \le 1 \qquad \text{for a.e. } 0 < \rho \le 1,$$

and since the functions $\rho \to \Psi_p(v_h, c/\varepsilon_h, B_\rho)$ are increasing, from (3.6) we have

$$\lim_h \Psi_p(v_h, c/\varepsilon_h, B_\rho) = 0 \quad \text{for all } 0 < \rho \le 1.$$

Now Theorem 3.11 implies that v_∞ is a local minimizer of the functional

$$w \mapsto \int_{B_1} F_p(\nabla w)\, dx,$$

and

$$\lim_h \mathcal{F}_p(v_h, c/\varepsilon_h, B_\rho) = \int_{B_\rho(x)} F_p(\nabla v_\infty) dx \qquad \text{for a.e. } \rho.$$

Using Remark 2.7 and Lemma 2.8, we may apply Theorem 2.2 to the function F_p to conclude that there exists a constant $C_2 \equiv C_2(N, p, L, \nu)$ such that

$$\sup_{B_{1/2}} |Dv_\infty|^p \le C_2 \int_{B_1} |Dv_\infty|^p dx \le C_2\, L.$$

Therefore

$$\begin{aligned}\lim_h \mathcal{F}_p(v_h, c/\varepsilon_h, B_\tau) &= \int_{B_\tau} F_p(Dv_\infty)\, dx \\ &\le L \int_{B_\tau} |Dv_\infty|^p\, dx \\ &\le L\omega_N \tau^N \sup_{B_{1/2}} |Dv_\infty|^p \\ &\le C_2\, L^2\, \omega_N\, \tau^N\end{aligned}$$

which contradicts (3.7) if we choose $C_1 = C_2 L\omega_N + 1$. □

From the latter lemma we proceed to obtaining a lower density estimate for points on S_u, whenever u is a local minimizer of the functional $\bar{\mathcal{G}}(v)$, and more generally, when u is a *quasi-minimizer.*

Definition 3.13. *We say that $u \in SBV_{loc}(\Omega)$ is a* quasi-minimizer *for $\mathcal{F}(\cdot, c, \cdot)$ if there exists a nondecreasing function $\omega : (0, +\infty) \to [0, +\infty[$ such that $\omega(t) \to 0$ as $t \to 0$ and*

$$\mathcal{F}(u,c,B_\rho) \le \mathcal{F}(v,c,B_\rho) + \rho^{N-1}\omega(\rho)$$

whenever $B_\rho \subset\subset \Omega$ *and* $v \in SBV_{loc}(\Omega), v = u$ *in* $\Omega \setminus \overline{B_\rho}$.

Lemma 3.14. [Density lower bound] *Let F satisfy (H1), (H2) and (H3). If $u \in SBV(\Omega) \cap L^\infty(\Omega)$ is a quasi-minimizer of $\mathcal{F}(\cdot,\beta,\cdot)$, then there exist θ_0, R_0, depending only on $N, p, L, \nu, c_0, m, \beta$, such that if $0 < \rho < R_0$, $x \in \overline{S_u}$, $B_\rho(x) \subset\subset \Omega$, then*

$$\int_{B_\rho(x)} |\nabla u|^p + H^{N-1}(S_u \cap B_\rho) \ge \theta_0 \rho^{N-1} \tag{3.8}$$

Moreover

$$H^{N-1}((\overline{S_u} \setminus S_u) \cap \Omega) = 0. \tag{3.9}$$

Proof. Considering $\max\{\rho, \omega(\rho)\}$, it is clear that we may assume, without loss of generality, that $\omega(\rho) \ge \rho$.

Step 1. Let us fix $0 < \tau < \frac{1}{2}$ such that $C_1\tau^N \le \tau^{N-\frac{1}{2}}$, where C_1 is the constant appearing in Lemma 3.12, and $\tau^{\frac{1}{2}}\omega(1) < 1$. We want to show that there exist ε_0 and $R_1 < \tau^4$ such that if $0 < \rho < R_1$ and

$$\mathcal{F}_p(u,\beta,B_\rho) \le \varepsilon_0 \rho^{N-1} \tag{3.10}$$

then either

$$\mathcal{F}_p(u,\beta,B_{\tau\rho}) \le \tau^{N-\frac{1}{2}} \mathcal{F}(u,\beta,B_\rho) \tag{3.11}$$

or

$$\mathcal{F}_p(u,\beta,B_{\tau\rho}) \le \tau \rho^{N-1} \omega^{\frac{1}{4}}(\rho). \tag{3.12}$$

If

$$\mathcal{F}_p(u,\beta,B_{\frac{\rho}{2}}) < \rho^{N-1}\omega^{\frac{1}{2}}(\rho) \tag{3.13}$$

then

$$\begin{aligned}\mathcal{F}_p(u,\beta,B_{\tau\rho}) &\le \mathcal{F}_p(u,\beta,B_{\frac{\rho}{2}}) \\ &< \rho^{N-1}\omega^{\frac{1}{2}}(\rho) \\ &\le \tau\rho^{N-1}\omega^{\frac{1}{4}}(\rho)\end{aligned}$$

for $0 < \rho << 1$, provided

$$\omega^{\frac{1}{4}}(\rho) < \tau. \tag{3.14}$$

Suppose now that (3.13) fails, and, by virtue of Theorem 3.6 let $\bar{u} \in SBV(\Omega)$ be such that $\bar{u} = u$ in $\Omega \setminus \overline{B_\rho}$, $||\bar{u}||_\infty \le ||u||_\infty$,

$$\mathcal{F}_p(\bar{u},\beta,B_\rho) = \Phi_p(u,\beta,B_\rho).$$

Given $\sigma > 0$, using (H1), (H3), Hölder and Young inequalities, and the fact that u is a quasi-minimizer, we have

$$\begin{aligned}
\mathcal{F}_p(u,\beta,B_\rho) &\le \int_{B_\rho\cap\{|\nabla u|\ge t_0\}} F_p(\nabla u)\,dx + \beta H^{N-1}(S_u\cap\overline{B_\rho}) + C\rho^N \\
&\le \int_{B_\rho} F(\nabla u)\,dx + C\int_{B_\rho} |\nabla u|^{p-m}\,dx + \beta H^{N-1}(S_u\cap\overline{B_\rho}) + C\rho^N \\
&\le (1+\sigma)\int_{B_\rho} F(\nabla u)\,dx + \beta H^{N-1}(S_u\cap\overline{B_\rho}) + C(\sigma)\rho^N \\
&= (1+\sigma)\mathcal{F}(u,\beta,B_\rho) + C(\sigma)\rho^N \\
&\le (1+\sigma)\mathcal{F}(\bar u,\beta,B_\rho) + C(\sigma)\rho^N + (1+\sigma)\rho^{N-1}\omega(\rho) \\
&= (1+\sigma)\mathcal{F}_p(\bar u,\beta,B_\rho) + (1+\sigma)\int_{B_\rho}[F(\nabla\bar u) - F_p(\nabla\bar u)]\,dx \\
&\quad + C(\sigma)\rho^N + (1+\sigma)\rho^{N-1}\omega(\rho) \\
&\le (1+\sigma)\mathcal{F}_p(\bar u,\beta,B_\rho) + C(\sigma)\rho^N + (1+\sigma)\rho^{N-1}\omega(\rho) \\
&\quad + (1+\sigma)\int_{B_\rho\cap\{|\nabla\bar u|<t_0\}} C(1+t_0^p)\,dx \\
&\quad + (1+\sigma)\int_{B_\rho\cap\{|\nabla\bar u|\ge t_0\}} C|\nabla\bar u|^{p-m}\,dx \\
&\le (1+2\sigma)\mathcal{F}_p(\bar u,\beta,B_\rho) + C(\sigma)\rho^N + (1+\sigma)\rho^{N-1}\omega(\rho) \\
&= (1+2\sigma)\Phi_p(u,\beta,B_\rho) + \rho^{N-1}\omega^{\frac12}(\rho)\left[C(\sigma)\frac{\rho}{\omega^{\frac12}(\rho)} + (1+\sigma)\omega^{\frac12}(\rho)\right].
\end{aligned}$$

Using the failure of (3.13) and the fact that $\omega(\rho)\ge\rho$, we deduce that

$$\mathcal{F}_p(u,\beta,B_\rho) \le (1+2\sigma)\Phi_p(u,\beta,B_\rho) + \mathcal{F}_p(u,\beta,B_\rho)[C(\sigma)\rho^{\frac12} + (1+\sigma)\omega^{\frac12}(\rho)].$$

Thus

$$\frac{1-[C(\sigma)\rho^{\frac12} + (1+\sigma)\omega^{\frac12}(\rho)]}{(1+2\sigma)}\mathcal{F}_p(u,\beta,B_\rho) \le \Phi_p(u,\beta,B_\rho)$$

with $0<\rho<<1$ such that

$$C(\sigma)\rho^{\frac12} + (1+\sigma)\omega^{\frac12}(\rho) < 1. \tag{3.15}$$

Hence

$$\begin{aligned}
\Psi_p(u,\beta,B_\rho) &= \mathcal{F}_p(u,\beta,B_\rho) - \Phi_p(u,\beta,B_\rho) \\
&\le \left[1 - \frac{1}{(1+2\sigma)} + \frac{C(\sigma)\rho^{\frac12} + (1+\sigma)\omega^{\frac12}(\rho)}{(1+2\sigma)}\right]\mathcal{F}_p(u,\beta,B_\rho) \\
&= \left[\frac{2\sigma}{(1+2\sigma)} + \frac{C(\sigma)\rho^{\frac12} + (1+\sigma)\omega^{\frac12}(\rho)}{(1+2\sigma)}\right]\mathcal{F}_p(u,\beta,B_\rho).
\end{aligned}$$

Let $\sigma<\theta/4$, where $\theta=\theta(\beta,\tau)$ is given by Lemma 3.12, and choose $0<\rho<<1$ such that

$$C(\sigma)\rho^{\frac12} + (1+\sigma)\omega^{\frac12}(\rho) < \frac{\theta}{2}. \tag{3.16}$$

Setting $\varepsilon_0 := \min\{\varepsilon, 1\}$, with $\varepsilon = \varepsilon(\beta, \tau)$ given by Lemma 3.12, and if R_1 is in agreement with (3.14)-(3.16), then we have

$$\mathcal{F}_p(u, \beta, B_\rho) < \varepsilon \rho^{N-1} \quad \text{and} \quad \Psi_p(u, \beta, B_\rho) \le \theta \mathcal{F}_p(u, \beta, B_\rho)$$

which, by virtue of Lemma 3.12 and because $\tau^{1/2} C_1 < 1$, yields (3.11).

Step 2. Let $0 < \rho < R_1$, and set $\rho_i := \tau^i \rho$. We claim that if $\mathcal{F}_p(u, \beta, B_\rho) \le \varepsilon_0 \rho^{N-1}$ then

$$\mathcal{F}_p(u, \beta, B_{\rho_i}) \le \varepsilon_0 \rho_i^{N-1}. \tag{3.17}$$

In fact, suppose that (3.17) holds for i. By Step 1 either (3.11) holds, in which case

$$\begin{aligned} \mathcal{F}_p(u, \beta, B_{\rho_{i+1}}) &\le \tau^{N-\frac{1}{2}} \mathcal{F}_p(u, \beta, B_{\rho_i}) \\ &< \tau^{N-\frac{1}{2}} \varepsilon_0 (\tau^i \rho)^{N-1} \\ &\le \varepsilon_0 (\tau^{i+1} \rho)^{N-1}, \end{aligned}$$

or (3.12) is satisfied, and then, using the fact that ω is decreasing,

$$\begin{aligned} \mathcal{F}_p(u, \beta, B_{\rho_{i+1}}) &\le \tau \rho_i^{N-1} \omega^{\frac{1}{4}}(\rho_i) \\ &\le (\tau^{i+1} \rho)^{N-1} \tau^{-N+2} \omega^{\frac{1}{4}}(\rho), \end{aligned}$$

and it suffices to choose $0 < \rho << 1$ so that

$$\tau^{-N+2} \omega^{\frac{1}{4}}(\rho) < \varepsilon_0. \tag{3.18}$$

Step 3. We claim that if $\mathcal{F}_p(u, \beta, B_\rho) \le \varepsilon_0 \rho^{N-1}$ then for all i

$$\mathcal{F}_p(u, \beta, B_{\rho_i}) \le \rho_i^{N-1} \omega^{\frac{1}{8}}(\tau^{\frac{i-1}{2}}).$$

By Steps 1 and 2 we have that either

$$\mathcal{F}_p(u, \beta, B_{\rho_i}) \le \tau^{N-\frac{1}{2}} \mathcal{F}_p(u, \beta, B_{\rho_{i-1}})$$

or

$$\mathcal{F}_p(u, \beta, B_{\rho_i}) \le \tau \rho_{i-1}^{N-1} \omega^{\frac{1}{4}}(\rho_{i-1}).$$

In the latter case, and using the fact that ω is decreasing, we have

$$\begin{aligned} \mathcal{F}_p(u, \beta, B_{\rho_i}) &\le \tau \rho_{i-1}^{N-1} \omega^{\frac{1}{4}}(\rho_{i-1}) \\ &= (\tau^i \rho)^{N-1} \omega^{\frac{1}{8}}(\tau^{i-1} \rho) \tau^{-N+2} \omega^{\frac{1}{8}}(\tau^{i-1} \rho) \\ &\le \rho_i^{N-1} \omega^{\frac{1}{8}} \left(\tau^{\frac{i-1}{2}}\right) \end{aligned}$$

provided $0 < \rho << 1$ is such that

$$\tau^{-N+2} \omega^{\frac{1}{8}}(\tau^{i-1} \rho) < 1. \tag{3.19}$$

In the first case, we denote by $h(i) \in \{0, 1, \ldots, i-1\}$ the smallest integer such that for all $j \in \{h(i)+1, \ldots, i\}$

$$\mathcal{F}_p(u, \beta, B_{\rho_j}) \le \tau^{N-\frac{1}{2}} \mathcal{F}_p(u, \beta, B_{\rho_{j-1}}).$$

If $h(i) = 0$, iterating this inequality yields

$$\begin{aligned}\mathcal{F}_p(u,\beta,B_{\rho_i}) &\leq \tau^{i(N-\frac{1}{2})}\,\mathcal{F}_p(u,\beta,B_\rho)\\ &\leq \tau^{i(N-\frac{1}{2})}\,\varepsilon_0\,\rho^{N-1}\\ &= (\tau^i\rho)^{N-1}\,\omega^{\frac{1}{8}}\left(\tau^{\frac{i-1}{2}}\right)\frac{\tau^{\frac{i}{2}}\varepsilon_0}{\omega^{\frac{1}{8}}(\tau^{\frac{i-1}{2}})},\end{aligned}$$

and the claim follows because, since $\tau^{\frac{1}{2}}\omega(1) \leq 1$ and $\varepsilon_0 \leq 1$,

$$\begin{aligned}\frac{\tau^{\frac{i}{2}}\varepsilon_0}{\omega^{\frac{1}{8}}(\tau^{\frac{i-1}{2}})} &= \frac{\tau^{\frac{i-1}{2}}}{\omega(\tau^{\frac{i-1}{2}})}\,\varepsilon_0\,\omega^{1-\frac{1}{8}}(\tau^{\frac{i-1}{2}})\tau^{\frac{1}{2}}\\ &\leq \varepsilon_0\,\tau^{\frac{1}{2}}\,\omega^{1-\frac{1}{8}}(\tau^{\frac{i-1}{2}})\\ &\leq 1.\end{aligned}$$

If $0 < h(i) < i$, iterating and using (3.12), we obtain

$$\begin{aligned}\mathcal{F}_p(u,\beta,B_{\rho_i}) &\leq \tau^{(i-h(i))(N-\frac{1}{2})}\,\mathcal{F}_p(u,\beta,B_{\rho_{h(i)}})\\ &\leq \tau^{(i-h(i))(N-\frac{1}{2})}\,\tau\,\rho_{h(i)-1}^{N-1}\,\omega^{\frac{1}{4}}(\rho_{h(i)-1})\\ &= (\tau^i\rho)^{N-1}\,\tau^{\frac{(i-h(i))}{2}}\,\tau^{-N+2}\omega^{\frac{1}{4}}(\rho\tau^{h(i)-1})\\ &\leq (\tau^i\rho)^{N-1}\,\tau^{\frac{(i-h(i))}{2}}\,\tau^{-N+2}\,\omega^{\frac{1}{8}}(\rho)\,\omega^{\frac{1}{8}}(\rho\tau^{h(i)-1}),\end{aligned}$$

where $0 < \rho << 1$ is such that

$$\tau^{-N+2}\,\omega^{\frac{1}{8}}(\rho) < 1. \tag{3.20}$$

Thus

$$\mathcal{F}_p(u,\beta,B_{\rho_i}) \leq \rho_i^{N-1}\,\tau^{\frac{(i-h(i))}{2}}\,\omega^{\frac{1}{8}}(\tau^{h(i)-1}).$$

If $h(i) - 1 \geq \frac{i-1}{2}$ then

$$\tau^{\frac{(i-h(i))}{2}}\,\omega^{\frac{1}{8}}(\tau^{h(i)-1}) \leq \omega^{\frac{1}{8}}(\tau^{\frac{i-1}{2}}).$$

If $h(i) - 1 < \frac{i-1}{2}$ then $i - h(i) > \frac{i-1}{2}$ and so

$$\begin{aligned}\tau^{\frac{(i-h(i))}{2}} &\leq \left(\tau^{\frac{i-1}{2}}\right)^{\frac{1}{2}}\\ &\leq \omega^{\frac{1}{2}}\left(\tau^{\frac{i-1}{2}}\right)\\ &\leq \omega^{\frac{1}{8}}\left(\tau^{\frac{i-1}{2}}\right);\end{aligned}$$

hence

$$\mathcal{F}_p(u,\beta,B_{\rho_i}) \leq \rho_i^{N-1}\,\omega^{\frac{1}{8}}(\tau^{\frac{i-1}{2}}).$$

We choose $R_0 \in (0, R_1)$ to be in agreement with (3.18)-(3.20).

Step 4. From Step 3 we deduce that if $\mathcal{F}_p(u,\beta,B_\rho) \leq \varepsilon_0\rho^{N-1}$ then

$$\lim_{i\to\infty}\frac{\mathcal{F}_p(u,\beta,B_{\rho_i})}{\rho_i^{N-1}}=\lim_{r\to 0}\frac{\mathcal{F}_p(u,\beta,B_r)}{r^{N-1}}=0.$$

Thus, if $x \in S_u$ and if $\mathcal{F}_p(u,\beta,B_\rho) < \varepsilon_0\rho^{N-1}$ for some $\rho < R_0$, we have

$$\lim_{r\to 0} r^{N-1}\Big[\int_{B_r}|\nabla u|^p\,dx + H^{N-1}(S_u\cap B_r)\Big]=0$$

which contradicts Theorem 3.3 (ii). In conclusion, if $x \in \overline{S_u}$ and $\rho < R_0$ then $\mathcal{F}_p(u,\beta,B_\rho) \geq \varepsilon_0\rho^{N-1}$ and this implies (3.8) for some $\theta_0 \equiv \theta_0(L,\beta)$. Finally, (3.9) follows immediately from Theorem 3.3 (i). □

Proof of Theorem 3.5. As mentioned earlier, the existence of a minimizer for $\bar{\mathcal{G}}(\cdot)$ is guaranteed by lower semicontinuity results of Ambrosio (see [A1]). Moreover if u is a minimizer for $\bar{\mathcal{G}}(\cdot)$ then u is a quasi-minimizer for $\mathcal{F}(\cdot,\beta,\cdot)$ with $\omega(\rho) = c(\alpha, q, \|g\|_\infty)\rho$. Then the last statement of the theorem is no more than (3.9), which yields $\bar{\mathcal{G}}(u) = \mathcal{G}(\bar{S}_u, u)$. To prove that $(\overline{S_u}, u)$ is a minimizing pair for $\mathcal{G}(K,v)$, consider an arbitrary pair (K,v) such that $\mathcal{G}(K,v) < \infty$, and notice that from Lemma 3.2 it follows that $v \in SBV(\Omega)$ and that $\overline{S_v} \subset K$. Therefore

$$\mathcal{G}(\overline{S_u},u)=\bar{\mathcal{G}}(u)\leq\bar{\mathcal{G}}(v)\leq\mathcal{G}(K,v),$$

and this concludes the proof. □

Remark 3.15. Following the arguments of Ambrosio and Pallara [AP], and Ambrosio, Fusco and Pallara [AFP], we expect that, under the assumptions of Theorem 3.5, $\overline{S_u}$ is locally a $C^{1,\alpha}$ hypersurface, except for a set of H^{N-1} zero measure (see [AP], Remark 3.4).

4. The Vectorial Case

The regularity result obtained in Theorem 3.5 can be applied in all its generality only to scalar valued functions. Carriero and Leaci [CL] have extended Theorem 3.5 to the vectorial case when $F(\xi) = |\xi|^p$, precisely, the functional to minimize is

$$\mathcal{G}_0(K,u):=\int_{\Omega\setminus K}|\nabla u|^p\,dx+\alpha\int_{\Omega\setminus K}|u-g|^q\,dx+\beta H^{N-1}(\Omega\cap K),$$

$q \geq 1, p > 1, \alpha, \beta > 0, g \in L^\infty(\Omega;\mathbb{R}^d)$. Here we show that lower order perturbations of $|\xi|^p$ are also allowed. In what follows, $\mathbb{M}^{d\times N}$ stands for the vector space of $d \times N$ real valued matrices.

Theorem 4.1. *Let $h : \mathbb{M}^{d\times N} \to [0,\infty)$ be a continuous function such that $h(\xi) \leq C(1 + |\xi|^r)$ for some $C > 0, p > r \geq 1$, and $h(\xi) \geq h(\xi')$ if*

$$\xi'_{ij} = \begin{cases} 0 & \text{if } i = i_0, j = 1, \dots, N \\ \xi_{ij} & \text{if } i \neq i_0 \end{cases} \tag{4.1}$$

for all $i_0 = 1, \dots, d$. *Define*

$$\mathcal{G}(K; u) := \int_{\Omega \setminus K} [|\nabla u|^p + h(\nabla u)]\, dx + \alpha \int_{\Omega \setminus K} |u - g|^q\, dx + \beta H^{N-1}(\Omega \cap K).$$

There exists a minimizer of $\mathcal{G}(\cdot, \cdot)$ *of the form* $(\overline{S_u}, u)$, $u \in SBV(\Omega, \mathbb{R}^d)$, *among all pairs* (K, v), $K \subset \Omega$ *closed,* $v \in W^{1,p}(\Omega \setminus K, \mathbb{R}^d)$. *Moreover,*

$$H^{N-1}((\overline{S_u} \setminus S_u) \cap \Omega) = 0.$$

As in Section 3, for $v \in SBV(\Omega; \mathbb{R}^d)$ we define

$$\bar{\mathcal{G}}_0(v) := \int_\Omega |\nabla v|^p\, dx + \alpha \int_\Omega |v - g|^q\, dx + \beta H^{N-1}(S_v \cap \Omega),$$

$$\mathcal{F}_0(v, c, A) := \int_A |\nabla v|^p\, dx + c H^{N-1}(S_v \cap \bar{A}),$$

$$\Phi_0(v, c, A) := \inf\{\mathcal{F}_0(w, c, A) : w \in SBV(\Omega; \mathbb{R}^d), w = v \, in \, \Omega \setminus \bar{A}\},$$

$$\Psi_0(v, c, A) := \mathcal{F}_0(v, c, A) - \Phi_0(v, c, A),$$

$$\bar{\mathcal{G}}(v) := \int_\Omega (|\nabla v|^p + h(\nabla v))\, dx + \alpha \int_\Omega |v - g|^q\, dx + \beta H^{N-1}(S_v \cap \Omega),$$

and

$$\mathcal{F}(v, c, A) := \int_A [|\nabla v|^p + h(\nabla v)]\, dx + c H^{N-1}(S_v \cap \bar{A}).$$

We recall that Theorem 3.5 was obtained from Theorem 3.6 and Lemma 3.14. Similarly, Theorem 4.1 will follow from the two results below.

Theorem 4.2. *Under the assumptions of Theorem 4.1, there exists a minimizer of the functional* $\bar{\mathcal{G}}(\cdot)$ *in* $L^\infty(\Omega; \mathbb{R}^d) \cap SBV(\Omega; \mathbb{R}^d)$.

Lemma 4.3. **[Density lower bound]** *Under the hypotheses of Theorem 4.1, if* $u \in SBV(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)$ *is a local minimizer of* $\bar{\mathcal{G}}(\cdot)$, *then there exist* θ_0, R_0, *depending only on* $N, p, L, \nu, q, c_0, m, ||u||_\infty, ||g||_\infty, \alpha, \beta$ *such that if* $0 < \rho < R_0$, $x \in \overline{S_u}$, $B_\rho(x) \subset\subset \Omega$, *then*

$$\int_{B_\rho(x)} |\nabla u|^p\, dx + \beta H^{N-1}(S_u \cap B_\rho) \geq \theta_0 \rho^{N-1}.$$

As in Section 3, Lemma 4.3 together will Theorem 3.3 will entail

$$H^{N-1}((\overline{S_u} \setminus S_u) \cap \Omega) = 0.$$

Also, Theorem 4.2 follows from the compactness result for SBV due to Ambrosio [A1], since (4.1) and the fact that $g \in L^\infty$ imply that there are minimizing sequences bounded in L^∞. Indeed, if $\{u_n\}$ is a minimizing sequence, then it suffices to truncate as follows:

$$(\bar{u}_n)_i := \begin{cases} (u_n)_i & \text{if } |(u_n)_i| \le ||g||_\infty \\ ||g||_\infty & \text{if } (u_n)_i > ||g||_\infty \\ -||g||_\infty & \text{if } (u_n)_i < -||g||_\infty \end{cases}$$

for $i = 1, \ldots, d$.

To prove Lemma 4.3, we will use the decay lemma obtained by Carriero and Leaci [CL], counterpart of Lemma 3.12.

Lemma 4.4. **[Decay Lemma]** *For all $\gamma \in (0,1)$ there exists $\tau_\gamma \in (0,1)$ such that for every $\tau \in (0, \tau_\gamma)$ and for every $c > 0$ there exist $\varepsilon = \varepsilon(c, \tau, N, p, \gamma)$, $\theta = \theta(c, \tau, N, p, \gamma)$, $R_0 = R_0(c, \tau, N, p, \gamma)$, such that if $0 < \rho < R_0$, and if $u \in SBV(\Omega; \mathbb{R}^d)$ is such that $\mathcal{F}_0(u, c, B_\rho) \le \varepsilon^p \rho^{N-1}$ and $\Psi_0(u, c, B_\rho) \le \theta \mathcal{F}_0(u, c, B_\rho)$, then*

$$\mathcal{F}_0(u, c, B_{\tau\rho}) \le \tau^{N-\gamma} \mathcal{F}_0(u, c, B_\rho). \tag{4.2}$$

Proof of Lemma 4.3. Fix $x \in \overline{S_u}$.

Step 1. We want to show that there exist $\varepsilon_0 > 0, R_1 > 0$ such that if $0 < \rho < R_1$ and if

$$\mathcal{F}(u, \beta, B_\rho(x)) < \varepsilon_0^p \rho^{N-1} \tag{4.3}$$

then either

$$\mathcal{F}(u, \beta, B_{\tau\rho}(x)) < \tau \rho^{N-\frac{1}{2}} \tag{4.4}$$

or

$$\mathcal{F}(u, \beta, B_{\tau\rho}(x)) < \tau^{N-\frac{1}{2}} \mathcal{F}(u, \beta, B_\rho(x)), \tag{4.5}$$

where, using the notation of Lemma 4.4,

$$\tau \in \left(0, \tau_{\frac{1}{4}}\right).$$

Let $\varepsilon_0 = \min\{\varepsilon_0(\beta, \tau, N, p, \frac{1}{4}), 1\}, \theta = \theta(\beta, \tau, N, p, \frac{1}{4}), R_0 = R_0(\beta, \tau, N, p, \frac{1}{4})$. Suppose that (4.3) holds and that

$$\mathcal{F}(u, \beta, B_{\tau\rho}(x)) < \rho^{N-\frac{1}{4}}. \tag{4.6}$$

Then (4.4) is satisfied, provided $0 < \rho << 1$ is such that

$$\rho < \tau^4. \tag{4.7}$$

Suppose now that (4.6) fails. By virtue of Theorem 4.2, let $\bar{u} \in SBV(B_\rho; \mathbb{R}^d) \cap L^\infty(B_\rho; \mathbb{R}^d)$ be such that $||\bar{u}||_\infty \le ||u||_\infty$, and $\bar{u}$ is a minimizer for $\mathcal{F}_0(\cdot, \beta, B_\rho)$ among all $v \in SBV(\Omega; \mathbb{R}^d), v = u \,\text{on}\, \Omega \setminus \overline{B_\rho}$, i.e.

$$\mathcal{F}_0(\bar{u},\beta,B_\rho)=\Phi_0(u,\beta,B_\rho).$$

Fix $\sigma\in(0,1)$. Using the failure of (4.6), Hölder and Young inequalities, and the fact that u is a local minimizer for $\bar{\mathcal{G}}(\cdot)$, we have

$$\begin{aligned}
\mathcal{F}_0(u,\beta,B_\rho) &\le \bar{\mathcal{G}}(u,\beta,B_\rho)\\
&\le \bar{\mathcal{G}}(\bar{u},\beta,B_\rho)\\
&\le \int_{B_\rho}|\nabla\bar{u}|^p\,dx+\int_{B_\rho}h(\nabla\bar{u})\,dx+c\rho^N+\beta H^{N-1}(S_{\bar{u}}\cap\overline{B_\rho})\\
&\le \int_{B_\rho}|\nabla\bar{u}|^p\,dx+\int_{B_\rho}|\nabla\bar{u}|^r\,dx+c\rho^N+\beta H^{N-1}(S_{\bar{u}}\cap\overline{B_\rho})\\
&\le \int_{B_\rho}|\nabla\bar{u}|^p\,dx+c\rho^N+\beta H^{N-1}(S_{\bar{u}}\cap\overline{B_\rho})\\
&\quad+\left(\int_{B_\rho}|\nabla\bar{u}|^p\,dx\right)^{\frac{r}{p}}\left(c\rho^N\right)^{1-\frac{r}{p}} \qquad (4.8)\\
&\le \int_{B_\rho}|\nabla\bar{u}|^p\,dx+c\rho^N+\beta H^{N-1}(S_{\bar{u}}\cap\overline{B_\rho})\\
&\quad+\sigma\int_{B_\rho}|\nabla\bar{u}|^p\,dx+c(\sigma)\rho^N\\
&\le (1+\sigma)\mathcal{F}_0(\bar{u},\beta,B_\rho)+c(\sigma)\rho^N\\
&\le (1+\sigma)\mathcal{F}_0(\bar{u},\beta,B_\rho)+c(\sigma)\rho^{\frac{1}{4}}\,\mathcal{F}(u,\beta,B_{\tau\rho})\\
&\le (1+\sigma)\mathcal{F}_0(\bar{u},\beta,B_\rho)+c(\sigma)\rho^{\frac{1}{4}}\,\mathcal{F}(u,\beta,B_\rho).
\end{aligned}$$

Now

$$\begin{aligned}
\mathcal{F}(u,\beta,B_\rho) &= \mathcal{F}_0(u,\beta,B_\rho)+\int_{B_\rho}h(\nabla u)\,dx\\
&\le \mathcal{F}_0(u,\beta,B_\rho)+c\rho^N+c\int_{B_\rho}|\nabla u|^r\,dx\\
&\le \mathcal{F}_0(u,\beta,B_\rho)+c\rho^N+\sigma\int_{B_\rho}|\nabla u|^p\,dx+c(\sigma)\rho^N\\
&\le (1+\sigma)\mathcal{F}_0(u,\beta,B_\rho)+c\rho^N\\
&\le (1+\sigma)\mathcal{F}_0(u,\beta,B_\rho)+c(\sigma)\rho^{\frac{1}{4}}\mathcal{F}(u,\beta,B_\rho),
\end{aligned}$$

from which we deduce that

$$\frac{1-c(\sigma)\rho^{\frac{1}{4}}}{1+\sigma}\mathcal{F}(u,\beta,B_\rho)\le\mathcal{F}_0(u,\beta,B_\rho).$$

This inequality, together with (4.8), yields

$$\begin{aligned}
\mathcal{F}_0(u,\beta,B_\rho) &\le (1+\sigma)\mathcal{F}_0(\bar{u},\beta,B_\rho)\\
&\quad+c(\sigma)\,\rho^{\frac{1}{4}}\,\frac{1+\sigma}{1-c(\sigma)\rho^{\frac{1}{4}}}\mathcal{F}_0(u,\beta,B_\rho);
\end{aligned}$$

hence

$$\left[\frac{1}{1+\sigma} - \frac{c(\sigma)\rho^{\frac{1}{4}}}{1-c(\sigma)\rho^{\frac{1}{4}}}\right] \mathcal{F}_0(u,\beta,B_\rho) \le \Phi_0(u,\beta,B_\rho)$$

and we conclude that

$$\begin{aligned}\Psi_0(u,\beta,B_\rho) &= \mathcal{F}_0(u,\beta,B_\rho) - \Phi_0(u,\beta,B_\rho)\\ &\le \left[1 - \frac{1}{1+\sigma} + \frac{c(\sigma)\rho^{\frac{1}{4}}}{1-c(\sigma)\rho^{\frac{1}{4}}}\right] \mathcal{F}_0(u,\beta,B_\rho)\\ &= \left[\frac{\sigma}{1+\sigma} + \frac{c(\sigma)\rho^{\frac{1}{4}}}{1-c(\sigma)\rho^{\frac{1}{4}}}\right] \mathcal{F}_0(u,\beta,B_\rho).\end{aligned} \tag{4.9}$$

Fix $\sigma \in (0,1)$ such that

$$\frac{\sigma}{1+\sigma} < \frac{\theta}{2} \quad \text{and} \quad (1+\sigma)\,\tau^{N-\frac{1}{4}} < \frac{\tau^{N-\frac{1}{2}}}{2}, \tag{4.10}$$

and (see (4.7)) choose $0 < \rho << 1$ satisfying

$$\rho < \tau^4, \quad c(\sigma)\,\rho^{\frac{1}{4}} < 1, \quad \frac{c(\sigma)\rho^{\frac{1}{4}}}{1-c(\sigma)\rho^{\frac{1}{4}}} < \frac{\theta}{2}. \tag{4.11}$$

It is clear that (4.9) reduces to

$$\Psi_0(u,\beta,B_\rho) < \theta\mathcal{F}_0(u,\beta,B_\rho),$$

and by (4.3) and Lemma 4.4 we have

$$\mathcal{F}_0(u,\beta,B_{\tau\rho}) \le \tau^{N-\frac{1}{4}}\mathcal{F}_0(u,\beta,B_\rho).$$

Finally, using Hölder and Young inequalities we have

$$\begin{aligned}\mathcal{F}(u,\beta,B_{\tau\rho}) &= \mathcal{F}_0(u,\beta,B_{\tau\rho}) + \int_{B_{\tau\rho}} h(\nabla u)\,dx\\ &\le \tau^{N-\frac{1}{4}}\mathcal{F}(u,\beta,B_\rho) + c(\tau\rho)^N + c\int_{B_{\tau\rho}} |\nabla u|^r\,dx\\ &\le \tau^{N-\frac{1}{4}}\mathcal{F}(u,\beta,B_\rho) + c(\tau\rho)^N + \sigma\int_{B_{\tau\rho}} |\nabla u|^p\,dx + c(\sigma)(\tau\rho)^N\\ &\le \tau^{N-\frac{1}{4}}\mathcal{F}(u,\beta,B_\rho) + c(\sigma)(\tau\rho)^N\\ &\le [\tau^{N-\frac{1}{4}}(1+\sigma) + c(\sigma)\,\tau^N\,\rho^{\frac{1}{4}}]\mathcal{F}(u,\beta,B_\rho)\\ &\le \tau^{N-\frac{1}{2}}\mathcal{F}(u,\beta,B_\rho),\end{aligned}$$

provided $0 < \rho << 1$ is small enough so that (4.11) holds and

$$c(\sigma)\,\tau^N\,\rho^{\frac{1}{4}} < \frac{\tau^{N-\frac{1}{2}}}{2}. \tag{4.12}$$

We choose R_1 accordingly.

Step 2. Now let $0 < \rho < R_0 := \min\{R_1, \varepsilon_0^{2p}\tau^{2N-4}, \tau^{2N-3}\}$, and for every $i = 0, 1, \ldots$, set $\rho_i := \tau^i \rho$. We claim that if $\mathcal{F}(u, \beta, B_\rho) < \varepsilon_0^p \rho^{N-1}$ then

$$\mathcal{F}(u, \beta, B_{\rho_i}) < \varepsilon_0^p \rho_i^{N-1}. \tag{4.13}$$

In fact, if (4.13) holds for ρ_i, then either (4.4) is verified, in which case

$$\begin{aligned}\mathcal{F}(u, \beta, B_{\rho_{i+1}}) &< \tau \rho_i^{N-\frac{1}{2}} \\ &< \varepsilon_0^p \left(\tau^{i+1}\rho\right)^{N-1},\end{aligned}$$

provided

$$\rho < \varepsilon_0^{2p} \tau^{2N-4}, \tag{4.14}$$

or (4.5) is satisfied. In the latter case we have

$$\begin{aligned}\mathcal{F}(u, \beta, B_{\rho_{i+1}}) &< \tau^{N-\frac{1}{2}} \mathcal{F}(u, \beta, B_{\rho_i}) \\ &\leq \varepsilon_0^p \rho_i^{N-1} \tau^{N-\frac{1}{2}} \\ &< \varepsilon_0^p \left(\tau^{i+1}\rho\right)^{N-1},\end{aligned}$$

proving (4.13).

Step 3. We show that if $\mathcal{F}(u, \beta, B_\rho) < \varepsilon_0^p \rho^{N-1}$ then

$$\mathcal{F}(u, \beta, B_{\rho_i}) \leq \tau^{i(N-\frac{1}{2})} \rho^{N-1}. \tag{4.15}$$

Indeed, by Step 1 and (4.13) we have that either

$$\begin{aligned}\mathcal{F}(u, \beta, B_{\rho_i}) &< \tau \rho_{i-1}^{N-\frac{1}{2}} \\ &< \tau^{i(N-\frac{1}{2})} \rho^{N-1}\end{aligned}$$

provided

$$\rho < \tau^{2N-3}, \tag{4.16}$$

or

$$\mathcal{F}(u, \beta, B_{\rho_i}) < \tau^{N-\frac{1}{2}} \mathcal{F}(u, \beta, B_{\rho_{i-1}}).$$

In the latter case, denote by $h(i) \in \{0, \ldots, i-1\}$ the smallest integer such that for all $j \in \{h(i)+1, \ldots, i\}$

$$\mathcal{F}(u, \beta, B_{\rho_j}) \leq \tau^{N-\frac{1}{2}} \mathcal{F}(u, \beta, B_{\rho_{j-1}}). \tag{4.17}$$

If $h(i) = 0$, iterating (4.17) yields

$$\begin{aligned}\mathcal{F}(u, \beta, B_{\rho_i}) &\leq \tau^{i(N-\frac{1}{2})} \mathcal{F}(u, \beta, B_\rho) \\ &\leq \tau^{i(N-\frac{1}{2})} \varepsilon_0^p \rho^{N-1} \\ &\leq \tau^{i(N-\frac{1}{2})} \rho^{N-1}.\end{aligned}$$

If $0 < h(i) \leq i-1$, iterating (4.17), and using (4.4) and (4.15) we have

$$\begin{aligned}\mathcal{F}(u,\beta,B_{\rho_i}) &\le \tau^{(i-h(i))(N-\frac{1}{2})}\,\mathcal{F}(u,\beta,B_{\rho_{h(i)}})\\ &\le \tau^{(i-h(i))(N-\frac{1}{2})}\,\tau\,\rho_{h(i)-1}^{N-\frac{1}{2}}\\ &< \tau^{i(N-\frac{1}{2})}\,\rho^{N-1}.\end{aligned}$$

The value of R_0 is choosen so as to satisfy (4.11), (4.12), (4.14), and (4.16).

Step 4. We claim that if $x \in S_u$ and if $0 < \rho < R_0$ then

$$\mathcal{F}_0(u,\beta,B_\rho) \ge \frac{\varepsilon_0^p}{2(1+\sigma)}\,\rho^{N-1}. \tag{4.18}$$

Indeed, if $\mathcal{F}_0(u,\beta,B_\rho) < \frac{\varepsilon_0^p}{2(1+\sigma)}\,\rho^{N-1}$, then, fixing $0<\sigma<1$,

$$\begin{aligned}\mathcal{F}(u,\beta,B_\rho) &= \mathcal{F}_0(u,\beta,B_\rho) + \int_{B_\rho} h(\nabla u)\,dx\\ &\le \mathcal{F}_0(u,\beta,B_\rho) + c\rho^N + \int_{B_\rho} |\nabla u|^r\,dx\\ &\le \mathcal{F}_0(u,\beta,B_\rho) + c(\sigma)\rho^N + \sigma\int_{B_\rho} |\nabla u|^p\,dx\\ &\le (1+\sigma)\mathcal{F}_0(u,\beta,B_\rho) + c(\sigma)\rho^N\\ &\le \rho^{N-1}\frac{\varepsilon_0^p}{2} + c(\sigma)\,\varepsilon_0^p\,\rho^{N-1}\,\rho\\ &< \varepsilon_0^p\,\rho^{N-1}\end{aligned}$$

for $\rho > 0$ small enough so that

$$c(\sigma)\rho < \frac{1}{2}.$$

In particular, by Step 3 it follows that

$$\lim_{\rho\to 0}\frac{1}{\rho^{N-1}}\mathcal{F}_0(u,\beta,B_\rho) = 0,$$

contradicting Theorem 3.3 (ii). We conclude that (4.18) holds.

Step 5. Finally, if $x \in \overline{S_u}$ then we may find $z \in S_u$ and $0<\rho<R_0$ such that $B(z,\rho/2) \subset B(x,\rho)$. Using (4.18), we obtain

$$\begin{aligned}\mathcal{F}_0(u,\beta,B_\rho(x)) &\ge \mathcal{F}_0(u,\beta,B_{\rho/2}(z))\\ &\ge \frac{\varepsilon_0^p}{2(1+\sigma)}\left(\frac{\rho}{2}\right)^{N-1}\\ &=: \theta_0\,\rho^{N-1},\end{aligned}$$

and this concludes the proof of the density lower bound. □

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