NAMT 95-017

Dynamical theories of electromagnetism and superconductivity based on gauge invariance and energy

Morton E. Gurtin
Carnegie Mellon University

Research Report No. 95-NA-017

October 1995

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

Dynamical theories of electromagnetism and superconductivity based on gauge invariance and energy

Morton E. Gurtin
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

1. INTRODUCTION

The purpose of this paper is twofold:

- to introduce a framework for classical electromagnetism in which the basic ingredients are the electromagnetic potential, gauge invariance, an appropriate version of the second law, and constitutive equations that define particular classes of materials;
- to develop a theory of superconductivity based on this framework supplemented by a complex wavefunction ψ and complex microforces that are work-conjugate to temporal changes in ψ.[1]

a. Electromagnetic theory

I frame the theory using nonstandard notions of work and energy. The energy transferred to a control volume is classically characterized by the Poynting vector $\mathbf{E}\times\mathbf{H}$, an identification based on an integral balance derived from Maxwell's equations and linear constitutive relations. I begin instead with a concept of energy-transfer based on experience with theories of deformation and diffusion in which temporal changes in the energy of a control volume P are brought about by the transport of material and by the work of forces associated with changes in state. Work and transport are typically represented by terms of the form

$$\int_{\partial P} \mathbf{T}\mathbf{n}\cdot\mathbf{u}^{\cdot}, \qquad -\int_{\partial P} \mu\mathbf{j}\cdot\mathbf{n} \qquad (\mathbf{n} = \text{outward unit normal to } \partial P) \qquad (1.1)$$

[1] Wherever possible I scale out unnecessary constants. I take the speed of light equal to one. In discussing superconductivity, the electromagnetic potential $\mathbf{\Theta} = (\mathbf{A},-\Phi)$ (and hence also the magnetic induction $\mathbf{B}$ and the electric field $\mathbf{E}$) are taken to be $e/\hbar$ times their standard counterparts, while the free velocity $\mathbf{v}$ of superconducting electrons is taken to be $m/\hbar$ times its standard counterpart, so that $\mathbf{v} = \nabla\theta - \mathbf{A}$ rather than $m\mathbf{v} = \hbar\nabla\theta - e\mathbf{A}$ represents the classical relation between $\mathbf{v}$, θ, and $\mathbf{A}$. Here e and m denote an (effective) charge and mass for superconducting electrons, while $\hbar$ is Planck's constant.

with T a stress, u a displacement, μ a chemical potential, and j a mass flux vector.

In electromagnetism it is the potential

$$\mathsf{a} = (\mathbf{A}, -\Phi) \qquad (\mathbf{A} = \text{magnetic potential}, \quad \Phi = \text{electric potential})$$

that is basic, and I believe that the proper analogies between electromagnetism and theories of deformation and diffusion are between **u** and a, as each characterizes the underlying kinematical state, and between μ and Φ, as each characterizes the transport of energy. Further, although the setting is *nonrelativistic,* I find it convenient to frame the theory in spacetime[2] $\mathbb{R}^3 \times \mathbb{R}$, and for that reason I use action rather than energy, although

what I call "action" is the negative of what is usually referred to as "action".[3] (1.2)

Guided by (1.1), I introduce two gauge-invariant fields, a 4-vector j, the charge-current, and a 4-tensor T, the electromagnetic stress;[4] and I use these fields to define the action flux

$$\int_{\partial \mathsf{P}} \{\mathsf{T}\mathsf{n}\cdot\mathsf{a}^{\cdot} - \Phi\mathsf{j}\cdot\mathsf{n}\} \qquad (\mathsf{n} = \text{outward unit normal to } \partial\mathsf{P}) \tag{1.3}$$

into arbitrary spacetime control volumes P. Here $\int_{\partial\mathsf{P}} \mathsf{T}\mathsf{n}\cdot\mathsf{a}^{\cdot}$ is viewed as work associated with temporal changes in the electromagnetic field, Φ as an (electro)chemical potential for current flow, and $-\int_{\partial\mathsf{P}} \Phi\mathsf{j}\cdot\mathsf{n}$ as energy carried into P by the charge-current.

As my main result I show that

Maxwell's equations are both necessary and sufficient for the action flux (1.3) *to be gauge invariant;*[5]

that is, (1.3) is gauge invariant if and only if T has the form

---

[2] Regarding notation: boldface lowercase sansserif letters $\mathsf{a}, \mathsf{g}, \mathsf{j}, \ldots$ denote 4-vectors; boldface uppercase sansserif letters $\mathsf{T}, \mathsf{M}, \mathsf{W}, \ldots$ denote 4-tensors as well as regions in spacetime; standard boldface letters denote 3-vectors; lightface letters denote scalars.

[3] "Action" then has the same sign as "free energy", which is convenient.

[4] T should not be confused with the stress tensors of Maxwell and Lorentz (cf. eqts. (33.13) and (33.32) of Wang [1979]).

[5] Analogous to the continuum mechanical result:

Galilean invariance of power $\leftrightarrow$ balance of forces.

$$\mathbb{T} = \begin{pmatrix} H\times & D \\ -D^T & 0 \end{pmatrix}$$

with $\mathrm{curl}\,H = D^{\cdot} + j$, $\mathrm{div}\,D = q$, $\mathbf{j} = (j, q)$.[6] I further establish consistency with classical electromagnetism by relating (1.3) to the more standard flow of energy represented by the Poynting vector.

I next introduce a scalar field, the action density Ω, and consider a mechanical version of the second law which asserts that the *temporal* change in Ω across $\mathbf{P}$ be not greater than the action flux into $\mathbf{P}$:

$$\int_{\partial\mathbf{P}} \Omega\, \mathbf{n}_{time} \le \mathcal{P}(\mathbf{P}) \tag{1.4}$$

with $\mathbf{n}_{time}$ the temporal component of $\mathbf{n}$. Note that, for $\mathbf{P} = P\times[t_0,t_1]$ with P a region in $\mathbb{R}^3$, the left side of (1.4) is the value of $\int_P \Omega$ at $t=t_1$ minus its value at $t=t_0$.

Finally, I use the second law (1.4) to develop a suitable constitutive theory.

b. Superconductivity

I extend the ideas described above to develop a theory of superconductivity that is consistent with, but more general than, the Ginzburg-Landau theory. What distinguishes the treatment presented here from other macroscopic theories[7] of superconductivity are the separation of basic physical laws from constitutive equations and the introduction of a balance law for complex microforces. Here I continue an approach[8] based on the belief that fundamental physical laws involving energy should account for the work associated with each kinematical process. In the Ginzburg-Landau theory the macroscopic manifestation of the kinematics of superelectrons is the wavefunction ψ. In accord with

[6] As for the remaining Maxwell equations: $\mathrm{div}\,j = -q^{\cdot}$ holds automatically; $\mathrm{curl}\,E = -B^{\cdot}$ and $\mathrm{div}\,B = 0$ are consequences of the definitions $E = -(A^{\cdot} + \nabla\Phi)$ and $B = \mathrm{curl}\,A$.

[7] I.e., the Ginzburg-Landau theories (cf., Cyrot [1970], Chapman, Howison, and Ockendon [1992]), which are variational, or the work of Zhou and Miya [1991], Maugin [1992], Zhou [1991], and Yeh and Chen [1993], who utilize thermodynamics, but do not postulate a microforce balance and are therefore compelled to lay down constitutive equations relating fields already endowed with constitutive prescriptions (e.g., (20) of Maugin [1992]).

[8] Begun in collaboration with Fried [1993,1994]. The introduction of ancillary force systems consistent with their own balance is apparently due to E. and F. Cosserat (1907) (cf. Truesdell and Noll [1965], §98 for the early history of such theories), although the development here is more closely allied to the work of Ericksen [1991] on liquid crystals and the more general treatment of Podio-Guidugli and Capriz [1983] and Capriz [1989].

this I assume that forces on superelectrons are characterized macroscopically by a complex microforce system that performs work when ψ undergoes changes. What is most important, I require that this system be consistent with a complex microforce balance, a hypothesis at least partially motivated by the following arguments:

(i) Equilibrium is, in part, described by an Euler-Lagrange equation corresponding to the vanishing of the free energy with respect to variations in ψ. This Euler-Lagrange equation represents a statical version of the complex microforce balance. In dynamics with general forms of dissipation there is no such variational principle; the use of a complex microforce balance is an attempt to extend to dynamics an essential feature of statical theories.

(ii) Standard forces in continua are associated with macroscopic length scales, while complex microforces here describe forces associated with microscopic configurations of electrons. The need for a separate balance for microforces seems a necessary consequence of the disparite length scales involved.

2. NOTATION

Although I work within a nonrelativistic setting,[9] it is useful to consider both euclidean **space**, identified with $\mathbb{R}^3$, and four-dimensional **spacetime**, $\mathbb{R}^3\times\mathbb{R}$, with both $\mathbb{R}^3$ and $\mathbb{R}^3\times\mathbb{R}$ endowed with the standard euclidean inner product, denoted by a "dot". To avoid ambiguities I write:

3-vector = vector in $\mathbb{R}^3$,
3-tensor = linear transformation of $\mathbb{R}^3$ into $\mathbb{R}^3$,
4-vector = vector in $\mathbb{R}^3\times\mathbb{R}$,
4-tensor = linear transformation of $\mathbb{R}^3\times\mathbb{R}$ into $\mathbb{R}^3\times\mathbb{R}$.

Let $\mathbb{a}$ be a 4-vector. Then

$$\mathbb{a} = (\mathbb{a}_{space}, \mathbb{a}_{time}), \tag{2.1}$$

with $\mathbb{a}_{space}$, a 3-vector, and $\mathbb{a}_{time}$, a scalar, the **spatial** and **temporal components** of $\mathbb{a}$. Every 4-tensor $\mathbb{M}$ admits the representation

$$\mathbb{M} = \begin{matrix} M & \mathbf{m} \\ \boldsymbol{\mu}^{T} & \mu , \end{matrix} \tag{2.2}$$

where M is a three-tensor viewed as a 3×3 matrix, $\mathbf{m}$ and $\boldsymbol{\mu}$ are 3-vectors viewed as 3×1 matrices, and μ is a scalar.[10]

If W is a skew 3-tensor, then there is a unique 3-vector $\mathbf{w}$, the **axial vector** of W, such that $W\mathbf{a} = \mathbf{w}\times\mathbf{a}$ for all 3-vectors $\mathbf{a}$; more succintly

$$W = \mathbf{w}\times \tag{2.3}$$

For $\mathbf{W}$ a skew 4-tensor there are 3-vectors $\mathbf{w}$ and $\mathbf{a}$ such that:

---

[9] For consistency, as I wish to formulate superconductivity within that framework; the extension of my treatment of electromagnetic theory to a (special) relativistic setting involves only minor changes.

[10] M^T denotes the transpose of a matrix M. For M *square*: trM is the trace of M; M is symmetric or skew according as $M = M^T$, $M = -M^T$; symM and skwM are the symmetric and skew parts of M. The inner product of n×m matrices M and P is defined by $M\cdot P = \mathrm{tr}(M^T P)$.

$$\mathbb{W} = \begin{bmatrix} \mathbf{w}\times & \mathbf{a} \\ -\mathbf{a}^{T} & 0 \end{bmatrix}. \qquad (2.4)$$

I use the following terminology and notation:

field = function on $\mathbb{R}^3\times\mathbb{R}$;
$\mathbb{X} = (\mathbf{x},t)$, an **event**, is a generic point of $\mathbb{R}^3\times\mathbb{R}$;
$(\,)^{\cdot}$ denotes the time derivative;
∇, div, curl, and Δ denote the spatial gradient, div, curl, and Laplacian;
∇_4 and div_4 denote the spacetime gradient and divergence.

For $\mathbb{a} = (\mathbf{a},\alpha)$ a 4-vector field,

$$\nabla_4\mathbb{a} = \begin{bmatrix} \nabla\mathbf{a} & \mathbf{a}^{\cdot} \\ \nabla\alpha^{T} & \alpha^{\cdot} \end{bmatrix}, \qquad (2.5)$$

$$\mathrm{div}_4\mathbb{a} = \mathrm{div}\,\mathbf{a} + \alpha^{\cdot}, \qquad (2.6)$$

and for $\mathbb{M}$ a 4-tensor field, represented as in (2.2),[11]

$$\mathrm{div}_4\mathbb{M} = (\mathrm{div}\mathbf{M} + \mathbf{m}^{\cdot}, \mathrm{div}\,\mu + \mu^{\cdot}). \qquad (2.7)$$

The following identities, for 3-vector fields **a** and **b**, will be useful:

$$\begin{aligned} \mathrm{div}(\mathbf{a}\times\mathbf{b}) &= \mathbf{b}\cdot\mathrm{curl}\,\mathbf{a} - \mathbf{a}\cdot\mathrm{curl}\,\mathbf{b}, \\ \mathrm{curl}\,\mathrm{curl}\,\mathbf{a} &= \nabla\mathrm{div}\,\mathbf{a} - \Delta\mathbf{a}, \\ \mathrm{skw}(\nabla\mathbf{a}) &= \tfrac{1}{2}(\mathrm{curl}\,\mathbf{a})\times, \\ \mathrm{div}(\mathbf{a}\times) &= -\mathrm{curl}\,\mathbf{a}. \end{aligned} \qquad (2.8)$$

A **spatial control volume** is a region P in $\mathbb{R}^3$; a **spacetime control volume** is a region $\mathbb{P}$ in $\mathbb{R}^3\times\mathbb{R}$; the outward unit normal vectors for ∂P and $\partial\mathbb{P}$ will be denoted by **n** and $\mathbb{n}$, respectively. The following identities, for α a scalar field, follow from the divergence theorem:

[11] For a 3-tensor field M, divM is the 3-vector field with components $\Sigma_j\, \partial M_{ij}/\partial x_j$, where i is the row index and j is summed from 1 to 3.

$$\int_{\partial\mathbf{P}} \alpha\, \mathbf{n}_{\text{space}} = \int_{\mathbf{P}} \nabla\alpha, \qquad \int_{\partial\mathbf{P}} \alpha\, \mathbf{n}_{\text{time}} = \int_{\mathbf{P}} \alpha^{\bullet}. \tag{2.9}$$

Analogous results hold for 3-vector and 3-tensor fields.

I will often use spacetime control volumes of the form $\mathbf{P} = P\times[t_0,t]$; in this case $\partial\mathbf{P}$ is the union of $\partial P\times(t_0,t)$, $P\times\{t_0\}$, and $P\times\{t\}$, and

$$\begin{aligned} \mathbf{n}_{\text{space}} &= \mathbf{n}, & \mathbf{n}_{\text{time}} &= 0 & &\text{on } \partial P\times(t_0,t), \\ \mathbf{n}_{\text{space}} &= 0, & \mathbf{n}_{\text{time}} &= -1 & &\text{on } P\times\{t_0\}, \\ \mathbf{n}_{\text{space}} &= 0, & \mathbf{n}_{\text{time}} &= 1 & &\text{on } P\times\{t\}. \end{aligned} \tag{2.10}$$

A. ELECTROMAGNETIC THEORY

3. ELECTROMAGNETIC POTENTIALS. ACTION. GAUGE TRANSFORMATIONS

a. The electromagnetic field. Action

A **path** in $\mathbb{R}^3\times\mathbb{R}$ is a smooth curve

$$\mathfrak{C} = \{\mathbf{X}(t) = (\mathbf{x}(t),t) : t_0 \le t \le t_1\}$$

parametrized by the time t.

The **electromagnetic potential** is a 4-vector field

$$\mathbb{A} = (\mathbf{A},-\Phi) \tag{3.1}$$

with $\mathbf{A}$ the **magnetic potential** and Φ the **electric potential**. The influence of the electromagnetic potential on a particle of charge e traversing a path $\mathfrak{C}$ is characterized by the integral

$$\mathfrak{A}(\mathbb{A},\mathfrak{C}) = e\int_{\mathfrak{C}} \mathbb{A}\cdot d\mathbf{X};$$

indeed a basic physical premise is that the actual path of the particle between given initial and terminal events renders

$$\mathfrak{A}_{tot}(\mathbb{A},\mathfrak{C}) = \mathfrak{A}(\mathbb{A},\mathfrak{C}) + \int_{\mathfrak{C}}\tfrac{1}{2}m\mathbf{v}\cdot d\mathbf{x} \tag{3.2}$$

stationary. Here $\mathbf{v}(t)\,(=\mathbf{x}^{\cdot}(t))$ is the velocity of the particle, while m is its mass. I will not use this principle explicitly; I introduce it only to motivate the notion of gauge invariance. To this end I refer to a path $\mathfrak{C}$ as *realizable* if $\mathfrak{C}$ renders $\mathfrak{A}_{tot}(\mathbb{A},\mathfrak{C})$ stationary when compared to all other paths whose initial and terminal events coincide with those of $\mathfrak{C}$.

b. Gauge invariance. Electric field and magnetic induction. Faraday's law

Given a scalar field χ, $\mathfrak{A}(\nabla_4\chi,\mathfrak{C})$ depends on $\mathfrak{C}$ only through its endpoints. The **gauge transformation** defined by

$$\mathbb{A} \to \mathbb{A} + \nabla_4\chi \qquad (\mathbf{A} \to \mathbf{A} + \nabla\chi, \qquad \Phi \to \Phi - \chi^{\cdot}) \tag{3.3}$$

therefore *leaves invariant all realizable particle paths.* Such transformations will be central to what follows.

Gauge-Invariant Fields. *The field* $\mathrm{skw}(\nabla_4\mathbb{A})$ *is* **gauge invariant** *(invariant under gauge transformations* $\mathbb{A}\to\mathbb{A}+\nabla_4\chi$ *) as are the* **electric field** $\mathbf{E}$ *and the* **magnetic induction** $\mathbf{B}$ defined by

$$\mathbf{E} = -(\dot{\mathbf{A}}+\nabla\Phi), \qquad \mathbf{B} = \mathrm{curl}\,\mathbf{A}. \tag{3.4}$$

Moreover,

$$\mathrm{skw}(\nabla_4\mathbb{A}) = \tfrac{1}{2}\begin{pmatrix} \mathbf{B}\times & -\mathbf{E} \\ \mathbf{E}^{\mathsf{T}} & 0 \end{pmatrix}. \tag{3.5}$$

Proof. Since the second gradient is symmetric, $\mathrm{skw}[\nabla_4(\mathbb{A}+\nabla_4\chi)] = \mathrm{skw}(\nabla_4\mathbb{A})$; hence $\mathrm{skw}(\nabla_4\mathbb{A})$ is gauge invariant. The gauge invariance of $\mathbf{E}$ and $\mathbf{B}$ follows from (3.3). Finally, since

$$\nabla_4\mathbb{A} = \begin{pmatrix} \nabla\mathbf{A} & \dot{\mathbf{A}} \\ -\nabla\Phi^{\mathsf{T}} & -\dot{\Phi} \end{pmatrix},$$

(3.5) is a consequence of $(2.8)_3$. □

The next step will be to show that functions of the electromagnetic potential and its gradient may be expressed in terms of $\mathbf{E}$ and $\mathbf{B}$. With a view toward proving this assertion, consider the functional relation

$$\mathbb{q}(\mathbb{x}) = \mathbb{Q}(\mathbb{A}(\mathbb{x}),\nabla_4\mathbb{A}(\mathbb{x})) \tag{3.6}$$

giving the value at any $\mathbb{x}$ of a field $\mathbb{q}$ when the electromagnetic potential $\mathbb{A}(\mathbb{x})$ and its gradient $\nabla_4\mathbb{A}(\mathbb{x})$ are known at $\mathbb{x}$. Functions $\mathbb{Q}$ of this type will be referred to as **local functions** and relations such as (3.6) will be written succintly as

$$\mathbb{q} = \mathbb{Q}(\mathbb{A},\nabla_4\mathbb{A}).$$

By (3.1), $\mathbb{Q}$ may be expressed as a local function $\mathcal{Q}$ of the electric and magnetic potentials and their derivatives,

$$\mathcal{Q}(\mathbb{A},\nabla_4\mathbb{A}) = \mathcal{G}(A,\Phi,\nabla A,\nabla\Phi,A^{\cdot},\Phi^{\cdot}).$$

Also, by definition, $\mathcal{Q}$ is **gauge invariant** if, for any electromagnetic potential $\mathbb{A}$,

$$\mathcal{Q}(\mathbb{A},\nabla_4\mathbb{A}) = \mathcal{Q}(\mathbb{A}+\nabla_4\chi,\nabla_4(\mathbb{A}+\nabla_4\chi)) \quad \text{for all scalar fields } \chi. \tag{3.7}$$

Gauge-Invariant Functions. *$\mathcal{Q}$ is gauge invariant if and only if $\mathcal{Q}(\mathbb{A},\nabla_4\mathbb{A})$ is independent of $\mathbb{A}$ and depends on $\nabla_4\mathbb{A}$ at most through its skew part; equivalently, there is a local function $\mathcal{H}$ such that*

$$\mathcal{Q}(\mathbb{A},\nabla_4\mathbb{A}) = \mathcal{H}(E,B). \tag{3.8}$$

Proof. As is clear from the theorem on gauge-invariant fields; functions of $\operatorname{skw}(\nabla_4\mathbb{A})$ are gauge invariant, as are functions of (E,B). To verify the converse assertion, let $\mathcal{Q}$ be gauge invariant. Choose a potential field $\mathbb{A}$ and an event $\mathbb{x}$, and choose χ in (3.7) with $\nabla_4\chi(\mathbb{x}) = -\mathbb{A}(\mathbb{x})$, $\nabla_4\nabla_4\chi(\mathbb{x}) = -\operatorname{sym}\{\nabla_4\mathbb{A}(\mathbb{x})\}$; then

$$\mathcal{Q}(\mathbb{A}(\mathbb{x}),\nabla_4\mathbb{A}(\mathbb{x})) = \mathcal{Q}(0,\operatorname{skw}\{\nabla_4\mathbb{A}(\mathbb{x})\}).$$

Thus $\mathcal{Q}(\mathbb{A}(\mathbb{x}),\nabla_4\mathbb{A}(\mathbb{x}))$ must reduce to a function of $\operatorname{skw}\{\nabla_4\mathbb{A}(\mathbb{x})\}$, and, by (3.5), any function of $\operatorname{skw}\{\nabla_4\mathbb{A}(\mathbb{x})\}$ may be expressed as a function of (E,B). □

Direct consequences of (3.4) are the first two Maxwell equations:

$$\operatorname{curl} E = -B^{\cdot}, \tag{3.9}$$

$$\operatorname{div} B = 0; \tag{3.10}$$

(3.9) is usually referred to as *Faraday's law.*

4. SECOND LAW. MAXWELL'S EQUATIONS AS A CONSEQUENCE OF GAUGE INVARIANCE

In this section I postulate a version of the second law that I believe appropriate for electromagnetic theory in the absence of thermal effects, and I show that the remaining Maxwell equations are consequences of the requirement that this version of the second law be invariant under gauge transformations.

a. Mechanical version of the second law in terms of action

I associate with the electromagnetic potential $\mathbf{a}$ two fields, a 4-vector field $\mathbf{j}$, the **charge-current**, and a 4-tensor field $\mathbf{T}$, the **electromagnetic stress**; and I use these fields to define the **action flux**

$$\mathcal{P}(\mathbf{P}) = \int_{\partial\mathbf{P}} \{\mathbf{T}\mathbf{n}\cdot\mathbf{a}^{\cdot} - \Phi\mathbf{j}\cdot\mathbf{n}\} \tag{4.1}$$

into arbitrary spacetime control volumes $\mathbf{P}$. In what follows I will relate (4.1) to the more standard flow of energy represented by the Poynting vector.

Throughout classical continuum mechanics "flows" of power and energy are reckoned across boundaries of spatial control volumes, but spacetime regions $\mathbf{P}$ are more natural to electromagnetic theory; in this regard note that for spacetime control volumes of the form $\mathbf{P} = P\times[t_0,t]$, (4.1) includes flows across the portions $P\times\{t_0\}$ and $P\times\{t\}$ of $\partial\mathbf{P}$. Further, the integral over $\partial\mathbf{P}$ includes an integral over the time interval $[t_0,t]$, so that, for example, the first term in (4.1) represents *work* done in distorting the field.

Next, I introduce a scalar field Ω, the **action density**,[12] and consider the **second law** to be the assertion that *the temporal change in* Ω *across spacetime control volumes* $\mathbf{P}$ *be not greater than the action flux into* $\mathbf{P}$:

$$\int_{\partial\mathbf{P}} \Omega\, \mathbf{n}_{\text{time}} \le \mathcal{P}(\mathbf{P}) \qquad \text{for all spacetime control volumes } \mathbf{P}. \tag{4.2}$$

b. Consequences of gauge invariance

I assume that the electromagnetic stress $\mathbf{T}$, the charge-current $\mathbf{j}$, and the action density Ω are gauge invariant, and I write $\mathcal{P}(\mathbf{P},\mathbf{a})$ to make explicit the dependence of the action flux on the potential field $\mathbf{a}$. Further, I use the phrase "all $\mathbf{P}$ and χ" as shorthand for "all spacetime control volumes $\mathbf{P}$ and all scalar fields χ."

[12] Cf. (1.2). Ω differs from the free energy by a Legendre transformation (cf. (5.11)-(5.15)).

Theorem. *The following assertions are equivalent:*

(i) *The second law is gauge invariant:*

$$\int_{\partial \mathbf{P}} \Omega\, \mathbf{n}_{\text{time}} \leq \mathcal{P}(\mathbf{P}, \mathbf{B} + \nabla_4 \chi) \qquad \text{for all } \mathbf{P} \text{ and } \chi. \tag{4.3}$$

(ii) *The action flux is gauge invariant:*

$$\mathcal{P}(\mathbf{P}, \mathbf{B}) = \mathcal{P}(\mathbf{P}, \mathbf{B} + \nabla_4 \chi) \qquad \text{for all } \mathbf{P} \text{ and } \chi. \tag{4.4}$$

(iii) *Gauge transformations involve no action flux:*

$$\mathcal{P}(\mathbf{P}, \nabla_4 \chi) = 0 \qquad \text{for all } \mathbf{P} \text{ and } \chi. \tag{4.5}$$

(iv) $\mathbf{T}$ *is skew and*

$$\operatorname{div}_4 \mathbf{T} + \mathbf{j} = \mathbf{0}, \qquad \operatorname{div}_4 \mathbf{j} = 0. \tag{4.6}$$

Proof. Note first that

$$\mathcal{P}(\mathbf{P}, \mathbf{B} + \nabla_4 \chi) = \mathcal{P}(\mathbf{P}, \mathbf{B}) + \mathcal{P}(\mathbf{P}, \nabla_4 \chi); \tag{4.7}$$

thus (iii)⇒(ii). Also, (ii)⇒(i) (granted (4.2), which is tacit). Further, (i)⇒(iii), for otherwise we could replace χ by $\alpha\chi$ and choose α sufficiently large and of the right sign to violate (i). Thus (i), (ii), and (iii) are equivalent.

The final step will be to show that (iii)⇔(iv). By (3.1) and (4.1), (iii) holds if and only if

$$\int_{\partial \mathbf{P}} \{\mathbf{T}\mathbf{n} \cdot \nabla_4 \chi^{\bullet} + \mathbf{j} \cdot \mathbf{n}\, \chi^{\bullet}\} = 0$$

for all $\mathbf{P}$ and χ, or equivalently, by the divergence theorem, if and only if

$$\int_{\partial \mathbf{P}} \{\mathbf{T} \cdot \nabla_4 \nabla_4 \chi^{\bullet} + [\operatorname{div}_4 \mathbf{T} + \mathbf{j}] \cdot \nabla_4 \chi^{\bullet} + [\operatorname{div}_4 \mathbf{j}] \chi^{\bullet}\} = 0$$

for all $\mathbf{P}$ and χ. Thus (iii) holds if and only if

$$\mathbf{T} \cdot \nabla_4 \nabla_4 \chi^{\bullet} + [\operatorname{div}_4 \mathbf{T} + \mathbf{j}] \cdot \nabla_4 \chi^{\bullet} + [\operatorname{div}_4 \mathbf{j}] \dot{\chi}^{\bullet} = 0 \tag{4.8}$$

for all scalar fields χ.

Since $\nabla_4\nabla_4\chi^{\cdot}$ is symmetric, (4.8) is satisfied if $\mathbf{T}$ is skew and (4.6) hold. Conversely, assume that (4.8) is satisfied. Then, given any event $\mathbf{x}$, we can always choose χ such that $\nabla_4\nabla_4\chi^{\cdot}(\mathbf{x})$ is an arbitrary *symmetric* 4-tensor, $\nabla_4\chi^{\cdot}(\mathbf{x})$ is an arbitrary 4-vector, and $\chi^{\cdot}(\mathbf{x})$ is an arbitrary scalar. Thus and by (4.8), $\mathbf{T}$ must be skew and consistent with (4.6). □

The next theorem represents the central result of this formulation of classical electromagnetic theory.

Maxwell's Equations. *A consequence of the invariance of the second law (or of the action flux) under gauge transformations is the existence of vector fields* H, D, *and* j *and a scalar field* q *such that*

$$\mathbf{j} = (j, q), \tag{4.9}$$

$$\mathbf{T} = \begin{matrix} H\times & D \\ -D^{T} & 0 \end{matrix}, \tag{4.10}$$

and

$$\operatorname{curl} H = D^{\cdot} + j, \tag{4.11}$$

$$\operatorname{div} D = q, \tag{4.12}$$

$$\operatorname{div} j = -q^{\cdot}. \tag{4.13}$$

Conversely, if $\mathbf{T}$ *and* $\mathbf{j}$ *are given by* (4.9) *and* (4.10) *with* H, D, j, *and* q *consistent with* (4.11) - (4.13), *then the second law and the action flux are invariant under gauge transformations.*

Proof. The invariance of the second law is equivalent to (iv) of the last theorem. Further, $\mathbf{T}$ is skew if and only if it has the form (4.10) (cf. (2.4)) and granted (4.9) and (4.10), (4.11) - (4.13) are equivalent to (4.6). □

Here H is the **magnetic field**, D is the **electric displacement**, j is the **current**, and q is the **charge** (density). Further, (3.9), (3.10), and (4.11) - (4.13) are **Maxwell's equations**, with (4.11) the *Ampère-Maxwell law* and (4.13) *balance of charge;* as is well known, (4.13) is implied by (4.11) and (4.12). The equations (4.6) represent a spacetime version of the Maxwell equations (4.11)-(4.13).[13]

To avoid repeated assumptions, I lay down the following

Hypothesis. *The second law is invariant under gauge transformations;* equivalently, *Maxwell's equations are satisfied.*

[13] In this context (4.6) are equivalent to the more standard formulation of (4.11)-(4.13) as differential forms in spacetime (cf. Wang [1979], pp. 272-273).

5. ALTERNATIVE FORMS FOR THE SECOND LAW AND FOR THE ACTION FLUX

a. Action flux

As a consequence of (4.1), (4.6), and the divergence theorem,

$$\mathcal{P}(\mathbf{P}) = \int_{\mathbf{P}}\{\mathbf{T}\cdot\nabla_4\mathbf{B}^{\cdot} - \mathbf{j}\cdot(\mathbf{B}^{\cdot}+\nabla_4\Phi)\}, \tag{5.1}$$

and, by (3.5), (4.10), and the identity $(\mathbf{b}\times)\cdot(\mathbf{c}\times) = 2\mathbf{b}\cdot\mathbf{c}$,

$$\mathbf{T}\cdot\nabla_4\mathbf{B}^{\cdot} = \mathrm{H}\cdot\mathrm{B}^{\cdot} - \mathrm{D}\cdot\mathrm{E}^{\cdot}. \tag{5.2}$$

Thus and by (3.1), (3.4), and (5.1),

$$\mathcal{P}(\mathbf{P}) = \int_{\mathbf{P}}\{\mathrm{H}\cdot\mathrm{B}^{\cdot} - \mathrm{D}\cdot\mathrm{E}^{\cdot} + \mathrm{j}\cdot\mathrm{E}\}. \tag{5.3}$$

The term $\mathbf{T}\cdot\nabla_4\mathbf{B}^{\cdot}$ represents power expended *internally* in distorting the field; (5.2) relates this term to more classical fields and shows that H and D are conjugate, in the sense of internal power, to $\mathrm{B}^{\cdot}$ and $-\mathrm{E}^{\cdot}$. The result (5.3) asserts that $\mathcal{P}(\mathbf{P})$ is balanced by the integral over $\mathbf{P}$ of the **Joule heating** $\mathrm{j}\cdot\mathrm{E}$ and $\mathrm{H}\cdot\mathrm{B}^{\cdot}-\mathrm{D}\cdot\mathrm{E}^{\cdot}$, the power expended, per unit volume, in temporal variations of the field.

Next, by $(2.8)_1$, (3.9), and (4.11),

$$-\mathrm{div}(\mathrm{E}\times\mathrm{H}) = \mathrm{H}\cdot\mathrm{B}^{\cdot} + \mathrm{E}\cdot\mathrm{D}^{\cdot} + \mathrm{j}\cdot\mathrm{E}; \tag{5.4}$$

therefore

$$\mathrm{H}\cdot\mathrm{B}^{\cdot} - \mathrm{D}\cdot\mathrm{E}^{\cdot} = -\mathrm{div}(\mathrm{E}\times\mathrm{H}) - (\mathrm{D}\cdot\mathrm{E})^{\cdot} - \mathrm{j}\cdot\mathrm{E} = -\mathrm{div}_4\mathbf{p} - \mathrm{j}\cdot\mathrm{E},$$

where

$$\mathbf{p} = (\mathrm{E}\times\mathrm{H}, \mathrm{D}\cdot\mathrm{E}).$$

Thus (5.3) and the divergence theorem yield the following result in which $\mathbf{n}_{\mathrm{space}}$ and $\mathbf{n}_{\mathrm{time}}$, defined in (2.1), are the spatial and temporal components of $\mathbf{n}$.

Poynting Vector. *For every spacetime control volume* $\mathbf{P}$,

$$\mathcal{P}(\mathbf{P}) = -\int_{\partial\mathbf{P}}\{(\mathrm{E}\times\mathrm{H})\cdot\mathbf{n}_{\mathrm{space}} + (\mathrm{D}\cdot\mathrm{E})\,\mathbf{n}_{\mathrm{time}}\}, \tag{5.5}$$

$$\mathcal{P}(\mathbf{P}) = -\int_{\partial\mathbf{P}} \mathbf{p}\cdot\mathbf{n}, \tag{5.6}$$

The field $\mathbf{E}\times\mathbf{H}$ is usually referred to as the **Poynting vector**. In classical electromagnetic theory the net flow of energy into a spatial control volume P is defined to be $-\int_{\partial P}(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n}$; the additional term $\mathbf{D}\cdot\mathbf{E}$ in (5.5) reflects the space-time structure and represents a temporal flow of energy.

For a spacetime control volume of the form

$$\mathbf{P} = P\times[t_0,t]$$

(2.10) may be used to express (5.5) as

$$\mathcal{P}(\mathbf{P}) = -\int_{t_0}^{t}\int_{\partial P}(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n} - \int_P(\mathbf{D}\cdot\mathbf{E})\Big|_{t_0}^{t}. \tag{5.7}$$

The derivative $\mathcal{P}(\mathbf{P})^\cdot$ of (5.7) with respect to t depends on $\mathbf{P}$ only through P, and I write

$$\mathcal{P}^\cdot(P) := \mathcal{P}(\mathbf{P})^\cdot \tag{5.8}$$

to make this dependence explicit. I will refer to $\mathcal{P}^\cdot(P)$ as the **electromagnetic power expended** on P. Differentiating (5.3) and (5.7) with respect to time yields the

Power-Expenditure Theorem. *The electromagnetic power expended on a spatial control volume* P *is given by*

$$\mathcal{P}^\cdot(P) = -\int_{\partial P}(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n} - \int_P(\mathbf{D}\cdot\mathbf{E})^\cdot, \tag{5.9}$$

$$\mathcal{P}^\cdot(P) = \int_P\{\mathbf{H}\cdot\mathbf{B}^\cdot - \mathbf{D}\cdot\mathbf{E}^\cdot + \mathbf{j}\cdot\mathbf{E}\}. \tag{5.10}$$

b. The second law

In what follows the assertions "for all spacetime $\mathbf{P}$" and "for all spatial P", respectively, will signify "for all spacetime control volumes $\mathbf{P}$" and "for all spatial control volume P".

Let

$$\Psi = \Omega + \mathbf{D}\cdot\mathbf{E}, \qquad \mathbf{s} = (\mathbf{E}\times\mathbf{H}, \Psi); \tag{5.11}$$

Ψ will be referred to as the **free energy**.

Alternative Forms for the Second Law. *The second law* (4.2) *is equivalent to each of the following assertions:*

$$\int_{\partial\mathbf{P}} \Psi\, \mathbf{n}_{time} \le -\int_{\partial\mathbf{P}} \{(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n}_{space} \qquad \text{for all spacetime } \mathbf{P}; \tag{5.12}$$

$$\int_{\partial\mathbf{P}} \mathbf{s}\cdot\mathbf{n} \le 0 \qquad \text{for all spacetime } \mathbf{P}; \tag{5.13}$$

$$\{\int_P \Omega\}^{\cdot} \le \mathcal{P}^{\cdot}(P) \qquad \text{for all spatial P}; \tag{5.14}$$

$$\{\int_P \Psi\}^{\cdot} \le -\int_{\partial P} (\mathbf{E}\times\mathbf{H})\cdot\mathbf{n} \qquad \text{for all spatial P}. \tag{5.15}$$

Proof. That (4.2)$\Leftrightarrow$(5.12)$\Leftrightarrow$(5.13) is clear from (5.5). By (5.9) and (5.11), (5.14)$\Leftrightarrow$(5.15). To complete the proof it suffices to show that (4.2)$\Leftrightarrow$(5.14). By (5.10), (4.2) is equivalent to the assertion that

$$\int_P \{\Omega^{\cdot} - \mathbf{H}\cdot\mathbf{B}^{\cdot} + \mathbf{D}\cdot\mathbf{E}^{\cdot} - \mathbf{j}\cdot\mathbf{E}\} \le 0 \qquad \text{for all spatial P}, \tag{5.16}$$

and, by $(2.9)_2$ and (5.3), (4.2) is equivalent to the assertion that

$$\int_{\mathbf{P}} \{\Omega^{\cdot} - \mathbf{H}\cdot\mathbf{B}^{\cdot} + \mathbf{D}\cdot\mathbf{E}^{\cdot} - \mathbf{j}\cdot\mathbf{E}\} \le 0 \qquad \text{for all spatial } \mathbf{P}. \tag{5.17}$$

Since (5.16)$\Leftrightarrow$(5.17), it follows that (4.2)$\Leftrightarrow$(5.14). □

Consequences of (5.15)-(5.17) are the

Local Forms of the Second Law. *Each of the following inequalities is equivalent to the second law* (4.2):

$$\Psi^{\cdot} \le -\operatorname{div}(\mathbf{E}\times\mathbf{H}), \tag{5.18}$$

$$\Psi^{\cdot} \le \mathbf{H}\cdot\mathbf{B}^{\cdot} + \mathbf{E}\cdot\mathbf{D}^{\cdot} + \mathbf{j}\cdot\mathbf{E}, \tag{5.19}$$

$$\Omega^{\cdot} \leq \mathbf{H}\cdot\mathbf{B}^{\cdot} - \mathbf{D}\cdot\mathbf{E}^{\cdot} + \mathbf{j}\cdot\mathbf{E}. \tag{5.20}$$

Standard treatments of electromagnetic theory are based on (5.15) and use the identity (5.4) to establish (5.19).

6. CONSTITUTIVE RELATIONS

a. Constitutive relations

I consider constitutive equations giving the action density Ω, the magnetic field **H**, the electric displacement **D**, and the current **j** as *gauge invariant* local functions of the electromagnetic potential and its gradient, or equivalently, by (3.8), as local functions of the electric field **E** and the magnetic induction **B**:

$$\begin{aligned} \Omega &= \hat{\Omega}(\mathbf{E},\mathbf{B}), \\ \mathbf{H} &= \hat{\mathbf{H}}(\mathbf{E},\mathbf{B}), \\ \mathbf{D} &= \hat{\mathbf{D}}(\mathbf{E},\mathbf{B}), \\ \mathbf{j} &= \hat{\mathbf{j}}(\mathbf{E},\mathbf{B}). \end{aligned} \tag{6.1}$$

b. Consequences of the second law[14]

The second law in the form (5.20) is used to restrict the constitutive relations. Precisely, **compatibility with thermodynamics** is the requirement that given any choice of the electromagnetic potential (field) $\mathbf{\Phi}$, if **E**, **B**, Ω, **H**, **D**, and **j** are determined through (3.4) and (6.1), then the resulting fields are consistent with (5.20).

Thermodynamic Restrictions. *The constitutive relations are compatible with thermodynamics if and only if:*

(i) *the action density determines the magnetic field and electric displacement through:*[15]

$$\hat{\mathbf{H}}(\mathbf{E},\mathbf{B}) = \hat{\Omega}_{\mathbf{B}}(\mathbf{E},\mathbf{B}), \qquad \hat{\mathbf{D}}(\mathbf{E},\mathbf{B}) = -\hat{\Omega}_{\mathbf{E}}(\mathbf{E},\mathbf{B}); \tag{6.2}$$

(ii) *the Joule-heating inequality*

$$\hat{\mathbf{j}}(\mathbf{E},\mathbf{B})\cdot\mathbf{E} \geq 0 \tag{6.3}$$

is satisfied.

Proof. The proof is standard. By (5.20) and (6.1), necessary and sufficient

[14] This part is standard and included only for completeness. The general method used here for restricting constitutive equations is due to Coleman and Noll [1963]. (Cf. Coleman and Dill [1971] for an application to electromagnetic materials with memory.)

[15] The partial derivative of a function $\Phi(a,b,c,\ldots,d)$ (of n scalar, vector, or tensor variables) with respect to b, say, is written $\Phi_b(a,b,c,\ldots,d)$.

that the constitutive relations be compatible with thermodynamics is that, for any choice of the field $\mathbb{B}$,

$$\{\hat{\Omega}_B(E,B) - \hat{H}(E,B)\}\cdot B^{\cdot} + \{\hat{\Omega}_E(E,B) + \hat{D}(E,B)\}\cdot E^{\cdot} - \hat{j}(E,B)\cdot E \le 0,$$

and the desired results follow; indeed, given any $(\mathbf{x},t)$, we can always choose $\mathbb{B} = (A,-\Phi)$ such that E, B, $E^{\cdot}$, and $B^{\cdot}$ have arbitrarily prescribed values at some $(\mathbf{x},t)$, say $(\mathbf{x},t)=(0,0)$. To verify this, let E_0, E_1, B_0, and B_1 denote arbitrarily chosen 3-vectors, and take

$$\mathbb{B} = (A,0), \qquad A(\mathbf{x},t) = -E_0 t - \tfrac{1}{2}E_1 t^2 + \tfrac{1}{2}(B_0 + B_1 t)\times\mathbf{x}. \tag{6.4}$$

Then

$$E = E_0, \quad E^{\cdot} = E_1, \quad B = B_0, \quad B^{\cdot} = B_1 \quad \text{at } (0,0) \qquad \square$$

Immediate consequences of (6.2) and (6.3) are the "Gibbs relations"

$$\Psi^{\cdot} = H\cdot B^{\cdot} + E\cdot D^{\cdot}, \qquad \Omega^{\cdot} = H\cdot B^{\cdot} - D\cdot E^{\cdot}, \tag{6.5}$$

and, by (5.10), strengthened versions of (5.14) and (5.15), namely

$$\{\int_P \Omega\}^{\cdot} = \mathcal{P}^{\cdot}(P) - \int_P j\cdot E, \tag{6.6}$$

$$\{\int_P \Psi\}^{\cdot} = -\int_{\partial P}(E\times H)\cdot n - \int_P j\cdot E, \tag{6.7}$$

in which the the Joule heating $j\cdot E$ is identified as the *energy dissipated*, per unit volume.

B. SUPERCONDUCTIVITY

7. INTRODUCTORY REMARKS

Superconductivity was discovered by Kamerlingh Onnes [1911],[16] who observed that the electrical resistance of certain metals is essentially zero below a critical temperature characteristic of the material. An attempt to describe these and related experiments led to the phenomenological theories of E. and F. London [1935] and Ginzburg and Landau [1950], and to the quantum mechanical theory of Bardeen, Cooper, and Schrieffer [1957].[17]

The Ginzburg-Landau theory of superconductivity is based on a complex order-parameter ψ in conjunction with a free energy, per unit volume, of the form

$$f(q_s) + \tfrac{1}{2}\alpha(\mathbf{X}\psi)\cdot(\overline{\mathbf{X}}\overline{\psi}), \qquad \mathbf{X} = \nabla - i\mathbf{A},$$

with $\alpha > 0$. Here ψ represents a wavefunction for superelectrons with $q_s = |\psi|^2 = \psi\overline{\psi}$ their charge density, while $f(q_s)$ is a coarse-grain free-energy with an absolute minimum at a critical value that defines the superconducting state. The Ginzburg-Landau theory, first limited to steady-state phenomena and later generalized to time-dependent behavior by Schmid [1966] and Gor'kov and E'liashberg [1968] (cf. Weller [1968]), leads to the complex PDE

$$\alpha\mathbf{X}\cdot\mathbf{X}\psi - 2\psi f'(|\psi|^2) = \beta T\psi, \qquad T = (\,)^{\cdot} + i\Phi, \tag{7.1}$$

where $\beta > 0$ is a kinetic modulus.

The PDE (7.1) is coupled to Maxwell's equations through the magnetic and electric potentials $\mathbf{A}$ and Φ, and through the current $\mathbf{j}$, which is the sum

$$\mathbf{j} = \mathbf{j}_n + \mathbf{j}_s \tag{7.2}$$

of a normal current $\mathbf{j}_n$, given by Ohm's law $\mathbf{j}_n = \sigma\mathbf{E}$, and a superconducting current

$$\mathbf{j}_s = -\tfrac{1}{2}\alpha i(\overline{\psi}\mathbf{X}\psi - \psi\overline{\mathbf{X}}\overline{\psi}). \tag{7.3}$$

The standard derivation of the Ginzburg-Landau equation begins with a free energy that is the sum of an energy due solely to the electromagnetic field and

[16] Cf. Cyrot [1970]; Chapman, Howison, and Ockendon [1992].

[17] Cf. Gor'kov [1959], who establishes the consistency of the Ginzburg-Landau theory with that of Bardeen, Cooper, and Schrieffer.

an energy

$$\mathcal{E}(\psi) = \int_R \{f(\psi\bar{\psi}) + \tfrac{1}{2}\alpha(\mathbf{X}\psi)\cdot(\bar{\mathbf{X}}\bar{\psi})\} \tag{7.4}$$

for the superelectrons, where R is the region of space occupied by the superconductor. The formal variation

$$\delta\mathcal{E}(\psi) = \int_R \{[2\psi f'(\psi\bar{\psi}) - \alpha\mathbf{X}\cdot\mathbf{X}\psi]\delta\bar{\psi} + [2\bar{\psi}f'(\psi\bar{\psi}) - \alpha\bar{\mathbf{X}}\cdot\bar{\mathbf{X}}\bar{\psi}]\delta\psi\} \tag{7.5}$$

with respect to fields ψ that vanish on ∂R yields the expression

$$\delta\mathcal{E}/\delta\bar{\psi} = 2\psi f'(\psi\bar{\psi}) - \alpha\mathbf{X}\cdot\mathbf{X}\psi \tag{7.6}$$

for the variational derivative with respect to $\bar{\psi}$. Steady-state behavior is characterized by the vanishing of $\delta\mathcal{E}/\delta\bar{\psi}$; the hypothesis underlying the standard derivation is that relaxation toward equilibrium be governed by a parameter $\beta > 0$ through a relation

$$\beta T\psi = -\delta\mathcal{E}/\delta\bar{\psi}, \tag{7.7}$$

where the kinetics is presumed characterized by $T\psi$ rather than by $\psi^{\cdot}$ to ensure compatibility with gauge invariance. A consequence of (7.6) and (7.7) is the Ginzburg-Landau equation (7.1).

Although this derivation of the Ginzburg-Landau equation is simple, elegant, and physically sound, I have three objections:

- the derivation limits the manner in which rate terms enter the equations;
- the derivation requires a-priori specification of constitutive equations;
- it is not clear how this derivation is to be generalized in the presence of processes such as deformation and heat transfer.

The major advances in nonlinear continuum mechanics over the past thirty years are based on the separation of balance laws, which are general and hold for large classes of materials, from constitutive equations, which delineate specific classes of material behavior. In the derivations presented above there is no such separation, and it is not clear whether or not there is an underlying balance law that can form a basis for more general theories. My view is that while derivations of the form (7.4)-(7.7) are useful and important, they should not be regarded as basic, but rather as precursors of more complete theories.

While variational derivations often point the way toward a correct statement of basic laws, to me such derivations obscure the fundamental nature of balance laws in any general framework that includes dissipation.

What distinguishes the development presented here from the Ginzburg-Landau treatment is not only the separation of balance laws from constitutive equations, but also the introduction of a balance law for complex microforces, defined operationally as forces whose working accompanies changes in ψ.

8. PRELIMINARIES

Let F(z) be a real function of the complex variable z. The "partial derivatives" F_z and $F_{\bar{z}}$, when they exist, may be uniquely defined through the chain rule

$$(d/d\tau)F(z(\tau)) = F_z(z(\tau))(dz/d\tau) + F_{\bar{z}}(z(\tau))(d\bar{z}/d\tau), \tag{8.1}$$

and it follows that

$$F_{\bar{z}} = \overline{F_z}\,, \tag{8.2}$$

since the left side of (8.1) equals its complex conjugate. The definition (8.1) yields the standard formal rules of differentiation; for example, if $F(z) = z\bar{z}$, then $F_z = \bar{z}$, $F_{\bar{z}} = z$.

Complex vectors $\mathbf{p} = \mathbf{u} + i\mathbf{v} \in \mathbb{C}^n$ $(\mathbf{u},\mathbf{v} \in \mathbb{R}^n)$ will also be considered; the **dot product** of two such vectors $\mathbf{p} = \mathbf{u} + i\mathbf{v}$ and $\mathbf{p}' = \mathbf{u}' + i\mathbf{v}'$ is defined by

$$\mathbf{p}\cdot\mathbf{p}' = (\mathbf{u} + i\mathbf{v})\cdot(\mathbf{u}' + i\mathbf{v}') = \mathbf{u}\cdot\mathbf{u}' - \mathbf{v}\cdot\mathbf{v}' + i(\mathbf{u}\cdot\mathbf{v}' + \mathbf{u}'\cdot\mathbf{v}).$$

This dot product, although *not an inner product*, has the important property: for $\mathbf{p} \in \mathbb{C}^n$,

$$\mathbf{p}\cdot\mathbf{q} + cc \geq 0 \text{ for all } \mathbf{q} \in \mathbb{C}^n \quad \Rightarrow \quad \mathbf{p} = \mathbf{0}. \tag{8.3}$$

Here and in what follows "cc" denotes the complex conjugate of the preceding terms:

$$A + cc = A + \bar{A}. \tag{8.4}$$

The partial derivatives $F_{\mathbf{p}}$ and $F_{\bar{\mathbf{p}}} = \overline{F_{\mathbf{p}}}$ of a real function $F(\mathbf{p})$ of $\mathbf{p} \in \mathbb{C}^n$ are defined as in (8.1).

Finally, I will use the terminology:

complex 3-vector = vector in $\mathbb{C}^3$,
complex 4-vector = vector in $\mathbb{C}^3 \times \mathbb{C}$.

9. THE WAVEFUNCTION

a. The wavefunction

To discuss superconductivity I introduce a complex field[18]

$$\psi = \varphi e^{i\theta}, \qquad \psi \neq 0 \tag{9.1}$$

called the **wavefunction**, where θ and φ are scalar fields with θ the **phase angle** and φ the **amplitude**, and with $\psi\bar{\psi} = \varphi^2$ generally viewed as an effective density of superelectrons (electron pairs). The field ψ and the electromagnetic potential

$$\mathfrak{A} = (A,-\Phi) \tag{3.1bis}$$

form the basis of the theory, with the gauge transformation

$$\mathfrak{A} \rightarrow \mathfrak{A} + \nabla_4\chi \qquad (A \rightarrow A + \nabla\chi, \qquad \Phi \rightarrow \Phi - \chi^{\cdot}) \tag{3.3bis}$$

augmented by[19]

$$\psi \rightarrow \psi e^{i\chi}, \tag{9.2}$$

or equivalently

$$\varphi \rightarrow \varphi, \qquad \theta \rightarrow \theta + \chi. \tag{9.3}$$

b. Quantum-mechanical transformations

The transformation (9.2) is ubiquitous to the theory; for that reason I say that a complex scalar- or vector-field U **transforms quantum mechanically**

[18] The assumption $\psi \neq 0$ is made for convenience; by continuity many of the final results are valid also for $\psi = 0$, where the phase angle θ is not defined. Sets on which $\psi = 0$ are referred to as "vortices" and represent an important physical phenomenon. Cf. Neu [1990], E [1994], Pismen and Rubinstein [1995].

[19] This transformation may be motivated as follows (cf. Feynman [1965], §21). The Lagrangian for a charged particle in an electromagnetic field is the integrand $\mathcal{L} = \mathcal{L}(v, A, \Phi)$ when (3.2) is written in the form $\int \mathcal{L}\,dt$, and this yields the generalized momentum $p = \partial\mathcal{L}/\partial v = mv + eA$. The quantum mechanical counterpart of this momentum is $p = \hbar\nabla\theta$, and so $\hbar\nabla\theta = mv + eA$; since v represents a particle velocity, v should be gauge invariant. Thus and by (3.3), θ should transform as in (9.3), granted a suitable rescaling and a choice of signs with $e > 0$. On the other hand, since φ^2 represents the density of particles, φ should be gauge invariant.

if

$$U \rightarrow Ue^{i\chi} \tag{9.4}$$

under a change in gauge.[20] Then for U and V complex scalar fields,

$$U, V \text{ transform quantum mechanically } \Rightarrow U\bar{V} \text{ is gauge invariant,} \tag{9.5}$$

and similarly for complex vector fields, but with $U\bar{V}$ replaced by $U\cdot\bar{V}$.

By (3.3), (9.3), and (9.5),

$$\varphi = (\psi\bar{\psi})^{\frac{1}{2}}, \quad \mathbf{v} = \nabla\theta - \mathbf{A}, \quad \text{and} \quad \lambda = -(\theta^{\cdot} + \Phi) \quad \text{are gauge invariant.} \tag{9.6}$$

I will refer to **v** as the **free velocity**[21] (of superelectrons), to λ as the **London potential**.[22]

The fields

$$\nabla\psi = e^{i\theta}[\nabla\varphi + i\varphi\nabla\theta], \qquad \psi^{\cdot} = e^{i\theta}[\varphi^{\cdot} + i\varphi\theta^{\cdot}] \tag{9.7}$$

are not gauge invariant, nor do they transform quantum mechanically; for that reason it is convenient to define an operator

$$\mathbf{\mathfrak{X}} = \nabla_4 - i\mathbb{A}, \tag{9.8}$$

so that

$$\mathbf{\mathfrak{X}} = (X,T), \qquad X = \nabla - iA, \qquad T = ()^{\cdot} + i\Phi, \tag{9.9}$$

and

$$\mathbf{\mathfrak{X}}\psi, \quad X\psi, \quad \text{and} \quad T\psi \quad \text{transform quantum mechanically.} \tag{9.10}$$

Further,

$$\nabla_4(T\psi) = \nabla_4(\psi^{\cdot} + i\Phi\psi) = \nabla_4\psi^{\cdot} + i\psi\nabla_4\Phi + i\Phi\nabla_4\psi,$$
$$(\mathbf{\mathfrak{X}}\psi)^{\cdot} = (\nabla_4\psi - i\mathbb{A}\psi)^{\cdot} = \nabla_4\psi^{\cdot} - i\mathbb{A}^{\cdot}\psi - i\mathbb{A}\psi^{\cdot}$$

---

[20] According to Park [1990], Pauli refers to such transformations as "gauge transformations of the first kind."

[21] I use this term as, modulo a rescaling, $\mathbf{v} = \nabla\theta - \mathbf{A}$ represents the velocity of a single electron in an electromagnetic field (cf. Footnote ??).

[22] λ is formally equivalent to a potential $-\mu$ introduced by the Londons [1935]. In the Londons' treatment j is the supercurrent (the normal current vanishes), and j and v essentially coincide. Further, eqt. (7) of the Londons is $j^{\cdot} = E - \nabla\mu$, and, by (9.6) with $j = v$, this yields $\nabla\mu = \nabla(\Phi + \theta^{\cdot})$; hence, modulo a spatially constant field, $\lambda = -\mu$.

and subtracting the second of these from the first yields, with the aid of $(2.8)_1$,

$$\nabla_4(T\psi) - (\mathfrak{X}\psi)^{\cdot} = i(\Phi\nabla_4\psi + \mathbf{B}\psi^{\cdot}) - i\psi\mathbf{E}. \tag{9.11}$$

Also

$$\mathbf{X}\psi = e^{i\theta}[\nabla\varphi + i\varphi(\nabla\theta - \mathbf{A})] = e^{i\theta}\nabla\varphi + i\psi\mathbf{v}, \tag{9.12}$$

$$T\psi = e^{i\theta}[\varphi^{\cdot} + i\varphi(\theta^{\cdot} + \Phi)] = e^{i\theta}\varphi^{\cdot} - i\psi\lambda, \tag{9.13}$$

$$\mathfrak{X}\psi = e^{i\theta}[\nabla_4\varphi + i\varphi(\nabla_4\theta - \mathbf{B})], \tag{9.14}$$

and

$$2\nabla\varphi = e^{-i\theta}\mathbf{X}\psi + e^{i\theta}\overline{\mathbf{X}}\overline{\psi}, \tag{9.15}$$

$$2\varphi^2\mathbf{v} = i[\psi\overline{\mathbf{X}}\overline{\psi} - \overline{\psi}\mathbf{X}\psi], \tag{9.16}$$

$$2\varphi^{\cdot} = e^{-i\theta}T\psi + e^{i\theta}\overline{T}\overline{\psi}, \tag{9.17}$$

$$2\varphi^2\lambda = i[\overline{\psi}T\psi - \psi\overline{T}\overline{\psi}]. \tag{9.18}$$

10. GAUGE INVARIANT FUNCTIONS. FUNCTIONS THAT TRANSFORM QUANTUM MECHANICALLY

Let Z denote a list of fields that transforms according to

$$Z \to Z_\chi$$

under a gauge transformation; e.g., for $Z = (\mathbf{a}, \nabla_4 \mathbf{a}, \psi, \nabla_4 \psi)$,

$$Z_\chi = (\mathbf{a} + \nabla_4 \chi, \nabla_4(\mathbf{a} + \nabla_4 \chi), e^{i\chi}\psi, e^{i\chi}(\nabla_4\psi + i\psi\nabla_4\chi)). \tag{10.1}$$

Then a local function F(Z) is **gauge invariant** if, for all Z and χ,

$$F(Z) = F(Z_\chi); \tag{10.2}$$

F(Z) **transforms quantum mechanically** if, for all Z and χ,

$$e^{i\chi} F(Z) = F(Z_\chi). \tag{10.3}$$

Gauge-Invariant Functions. *Let a local function* $F(\mathbf{a}, \nabla_4 \mathbf{a}, \psi, \nabla_4 \psi)$ *be given. Then the following are equivalent:*

(i) $F(\mathbf{a}, \nabla_4 \mathbf{a}, \psi, \nabla_4 \psi)$ *is gauge invariant;*

(ii) *There is a gauge-invariant local function* H *such that*

$$F(\mathbf{a}, \nabla_4 \mathbf{a}, \psi, \nabla_4 \psi) = H(E, B, \psi, \boldsymbol{\mathcal{X}}\psi).$$

(iii) *There is a local function* J *such that*

$$F(\mathbf{a}, \nabla_4 \mathbf{a}, \psi, \nabla_4 \psi) = J(E, B, \varphi, \nabla\varphi, \varphi^{\cdot}, \mathbf{v}, \lambda). \tag{10.4}$$

Proof. Since the arguments of J are gauge invariant, (iii)$\Rightarrow$(i). Assume next that (i) holds. Choose a potential field $\mathbf{a}$, a wavefunction ψ, and an event $\mathbf{x}$, and choose χ in (10.1) with $\chi(\mathbf{x}) = 0$, $\nabla_4\chi(\mathbf{x}) = -\mathbf{a}(\mathbf{x})$, $\nabla_4\nabla_4\chi(\mathbf{x}) = -\mathrm{sym}\{\nabla_4\mathbf{a}(\mathbf{x})\}$; then

$$F(\mathbf{a}(\mathbf{x}), \nabla_4\mathbf{a}(\mathbf{x}), \psi(\mathbf{x}), \nabla_4\psi(\mathbf{x})) = F(0, \mathrm{skw}\{\nabla_4\mathbf{a}(\mathbf{x})\}, \psi, \boldsymbol{\mathcal{X}}\psi).$$

Thus (ii) follows, since by (3.5) any function of $\mathrm{skw}\{\nabla_4\mathbf{a}(\mathbf{x})\}$ may be expressed as a function of (E,B). Therefore (i)$\Rightarrow$(ii). Finally, suppose that (ii) holds. Then, since $\boldsymbol{\mathcal{X}}\psi = (\mathbf{X}\psi, T\psi)$, (9.12) and (9.13) yield (iii), but with J a function that depends on θ

as well as $(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda) = \Theta$. But Θ is gauge invariant, while $\theta \rightarrow \theta + \chi$; hence J cannot depend on θ. Thus (ii)$\Rightarrow$(iii). □

By (iii) above, a gauge-invariant local function can depend on the phase angle θ at most through a dependence on the free velocity $\mathbf{v}$ and the London potential λ.

Quantum-Mechanical Functions. *Let a local function* $F(\mathbb{A},\nabla_4\mathbb{A},\psi,\nabla_4\psi)$ *be given. Then the following are equivalent:*

(i) $F(\mathbb{A},\nabla_4\mathbb{A},\psi,\nabla_4\psi)$ *transforms quantum mechanically.*

(ii) *There is a local function* H *that transforms quantum mechanically such that*

$$F(\mathbb{A},\nabla_4\mathbb{A},\psi,\nabla_4\psi) = H(\mathbf{E},\mathbf{B},\psi,\mathbb{X}\psi).$$

(iii) *There is a local function* J *such that*

$$F(\mathbb{A},\nabla_4\mathbb{A},\psi,\nabla_4\psi) = e^{i\theta} J(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda).$$

I omit the proof, which is similar to that of the theorem on gauge-invariant functions.

Let

$$\mathbb{G} = \mathbb{X}\psi = (\mathbf{g},\gamma), \qquad \mathbf{g} = \mathbf{X}\psi, \qquad \gamma = T\psi.$$

Lemma 10.1. *Let* $F(\psi,\mathbb{G})$ *be a real, gauge invariant, local function. Then:*

(i) $F_{\bar\psi}$ and $F_{\bar{\mathbb{G}}}$ *transform quantum mechanically;*

(ii) $\psi F_\psi + \mathbb{G}\cdot F_{\mathbb{G}}$ *is real.*

Proof. Assertion (i) follows upon differentiating

$$F(\psi,\mathbb{G}) = F(e^{i\chi}\psi, e^{i\chi}\mathbb{G}). \tag{10.5}$$

with respect to ψ and $\mathbb{G}$. On the other hand, differentiating (10.5) with respect to χ at $\chi = 0$ yields

$$\psi F_\psi - \bar\psi F_{\bar\psi} + \mathbb{G}\cdot F_{\mathbb{G}} - \bar{\mathbb{G}}\cdot F_{\bar{\mathbb{G}}} = 0, \tag{10.6}$$

so that $\psi F_\psi + \mathbb{G}\cdot F_{\mathbb{G}}$ equals its complex conjugate. This verifies (ii). □

When discussing constitutive equations I will consider gauge invariant local functions of the form

$$H(\mathbf{E},\mathbf{B},\psi,\boldsymbol{\mathfrak{g}}) = H(\mathbf{E},\mathbf{B},\psi,\mathbf{g},\gamma).$$

By the theorem on gauge-invariant functions, such functions can also be written as functions of $(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda)$, so that the partial derivatives $H_{\mathbf{v}}$, H_{λ}, H_{φ}, $H_{\nabla\varphi}$, and $H_{\varphi^{\cdot}}$ are well defined. In fact,

$$H_{\mathbf{v}} = i(\psi H_{\mathbf{g}} - \bar{\psi} H_{\bar{\mathbf{g}}}), \tag{10.7}$$

$$H_{\lambda} = -i(\psi H_{\gamma} - \bar{\psi} H_{\bar{\gamma}}), \tag{10.8}$$

$$\varphi H_{\varphi} = \psi[H_{\psi} + i(\nabla_4\theta - \boldsymbol{\mathfrak{A}})\cdot H_{\boldsymbol{\mathfrak{g}}}] + cc = \psi H_{\psi} + \boldsymbol{\mathfrak{g}}\cdot H_{\boldsymbol{\mathfrak{g}}} - e^{i\theta} H_{\boldsymbol{\mathfrak{g}}}\cdot\nabla_4\varphi + cc, \tag{10.9}$$

$$\varphi H_{\nabla\varphi} = \psi H_{\mathbf{g}} + \bar{\psi} H_{\bar{\mathbf{g}}}, \tag{10.10}$$

$$\varphi H_{\varphi^{\cdot}} = \psi H_{\gamma} + \bar{\psi} H_{\bar{\gamma}}, \tag{10.11}$$

To verify the first of (10.7), vary $\mathbf{v}$ by letting $\mathbf{v}=\mathbf{v}(\tau)$ in the list of variables $(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda)$, but hold the remaining variables fixed. Then, by (9.12),

$$d\mathbf{g}/d\tau = i\psi\, d\mathbf{v}/d\tau.$$

Thus, by the vector analog of (8.1), differentiating $H(\mathbf{E},\mathbf{B},\psi,\mathbf{g}(\tau),\gamma)$ with respect to τ yields

$$\begin{aligned}(H_{\mathbf{v}})\cdot(d\mathbf{v}/d\tau) &= (H_{\mathbf{g}})\cdot(d\mathbf{g}/d\tau) + (H_{\bar{\mathbf{g}}})\cdot(d\bar{\mathbf{g}}/d\tau)\\ &= (H_{\mathbf{g}})\cdot(i\psi\, d\mathbf{v}/d\tau) - (H_{\bar{\mathbf{g}}})\cdot(i\bar{\psi}\, d\mathbf{v}/d\tau),\end{aligned}$$

which implies (10.7), since $d\mathbf{v}/d\tau$ may be arbitrarily chosen. The verification of (10.8)-(10.11) is similar.

11. COMPLEX MICROFORCE BALANCE

a. Complex microforce balance

I consider a system of complex microforces consisting of a complex 4-vector field $\mathfrak{t}$, the **complex microstress**, and a complex scalar field Π, the **complex internal microforce**. Given any spacetime control volume $\mathbf{P}$:[23]

- $\mathfrak{t}\cdot\mathbf{n}$ represents complex microforces exerted across $\partial\mathbf{P}$ on the superelectrons within $\mathbf{P}$;
- Π represents complex microforces exerted on the superelectrons within $\mathbf{P}$ by the electromagnetic field, by the normal electrons, and by the lattice.

This force system is presumed consistent with the **complex microforce balance**

$$\int_{\partial\mathbf{P}} \mathfrak{t}\cdot\mathbf{n} + \int_{\mathbf{P}} \Pi = 0 \qquad (11.1)$$

for every spacetime control volume $\mathbf{P}$.

I assume that Π has the explicit form

$$\Pi = \pi + \mathbf{k}\cdot\mathbf{B} \qquad (11.2)$$

with π a complex scalar and $\mathbf{k}$ a complex 4-vector, and that

$$\mathfrak{t},\ \mathbf{k},\ \text{and}\ \pi\ \text{transform quantum mechanically.} \qquad (11.3)$$

Gauge invariance places an important restriction on the complex microforces. By (11.1)-(11.3) and (3.3), gauge invariance for the complex microforce balance is the requirement that

$$\int_{\partial\mathbf{P}} e^{i\chi}\{\mathfrak{t}\cdot\mathbf{n}\} + \int_{\mathbf{P}} e^{i\chi}\{\pi + \mathbf{k}\cdot(\mathbf{B} + \nabla_4\chi)\} = 0 \qquad \text{for all } \mathbf{P} \text{ and } \chi, \qquad (11.4)$$

or equivalently,

$$\mathrm{div}_4\mathfrak{t} + i(\nabla_4\chi)\cdot\mathfrak{t} + \pi + \mathbf{k}\cdot(\mathbf{B} + \nabla_4\chi) = 0 \qquad (11.5)$$

for all χ, so that $\mathfrak{t}$ and $\mathbf{k}$ must be related through

$$\mathbf{k} = -i\mathfrak{t}. \qquad (11.6)$$

[23] The microforces are *complex scalar* fields as they expend power over the *complex scalar* field ψ^*.

Conversely, (11.6) is sufficient that the complex microforce balance be gauge invariant.

I assume henceforth that (11.6) is satisfied. The complex microforce balance then has the local form

$$\mathrm{div}_4\boldsymbol{\xi} - i\boldsymbol{\alpha}\cdot\boldsymbol{\xi} + \pi = 0, \tag{11.7}$$

or equivalently,

$$\boldsymbol{\aleph}\cdot\boldsymbol{\xi} + \pi = 0. \tag{11.8}$$

b. Alternative form for the complex microforce balance

Since

$$\begin{aligned}\overline{\psi}\boldsymbol{\aleph}\cdot\boldsymbol{\xi} &= \overline{\psi}\,\mathrm{div}_4\boldsymbol{\xi} - i\overline{\psi}\boldsymbol{\alpha}\cdot\boldsymbol{\xi} = \mathrm{div}_4(\overline{\psi}\,\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\nabla_4\overline{\psi} - i\overline{\psi}\boldsymbol{\alpha}\cdot\boldsymbol{\xi}\\ &= \mathrm{div}_4(\overline{\psi}\,\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\overline{\boldsymbol{\aleph}}\overline{\psi},\end{aligned}$$

(11.8) can be written as

$$\mathrm{div}_4(\overline{\psi}\,\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\overline{\boldsymbol{\aleph}}\overline{\psi} + \overline{\psi}\pi = 0. \tag{11.9}$$

Hence the definitions

$$\mathbf{j}_s = i(\overline{\boldsymbol{\xi}}\psi - \boldsymbol{\xi}\overline{\psi}), \tag{11.10}$$

$$\boldsymbol{\sigma} = e^{-i\theta}\boldsymbol{\xi} + e^{i\theta}\overline{\boldsymbol{\xi}} \tag{11.11}$$

yield the following equations for the real and imaginary parts of (11.9):

$$\mathrm{div}_4(\varphi\boldsymbol{\sigma}) - \{\boldsymbol{\xi}\cdot\overline{\boldsymbol{\aleph}}\overline{\psi} - \overline{\psi}\pi + cc\} = 0, \tag{11.12}$$

$$\mathrm{div}_4\,\mathbf{j}_s + \{i(\boldsymbol{\xi}\cdot\overline{\boldsymbol{\aleph}}\overline{\psi} - \pi\overline{\psi}) + cc\} = 0. \tag{11.13}$$

Granted $\psi \neq 0$, the real equations (11.12) and (11.13) are together equivalent to the complex microforce balance (11.8). Note that

$$\mathbf{j}_s \text{ and } \boldsymbol{\sigma} \text{ are real and gauge invariant} \tag{11.14}$$

and related to $\boldsymbol{\ell}$ through

$$2\bar{\Psi}\boldsymbol{\ell} = \varphi\mathbf{s} + i\mathbf{j}_s . \tag{11.15}$$

12. SECOND LAW. EXPENDED POWER

a. Second law

To formulate a mechanical version of the second law appropriate to superconductivity I consider, in addition to the electromagnetic stress $\mathbf{T}$ and the action density Ω, a (real) 4-vector field $\mathbf{j}_n$, the **normal charge-current**, with $\mathbf{T}$, Ω, and $\mathbf{j}_n$ gauge invariant fields. (As before, $\mathbf{T}$ is a real 4-tensor field and Ω is a real scalar field.)

Given an arbitrary spacetime control volume $\mathbf{P}$, I assume that the **action flux** into $\mathbf{P}$ has the form

$$\mathcal{P}(\mathbf{P}) = \int_{\partial\mathbf{P}} \{\mathbf{T}\mathbf{n}\cdot\mathbf{e}^{\cdot} - \Phi\mathbf{j}_n\cdot\mathbf{n}\} + \int_{\partial\mathbf{P}} \{(\mathbf{t}\cdot\mathbf{n})\overline{\psi}^{\cdot} + \mathrm{cc}\}, \tag{12.1}$$

where

$$\int_{\partial\mathbf{P}} \{\mathbf{T}\mathbf{n}\cdot\mathbf{e}^{\cdot} - \Phi\mathbf{j}_n\cdot\mathbf{n}\} \tag{12.2}$$

represents both work associated with temporal changes in the electromagnetic field and energy carried into $\mathbf{P}$ by the flow of normal electrons, while

$$\int_{\partial\mathbf{P}} \{(\mathbf{t}\cdot\mathbf{n})\overline{\psi}^{\cdot} + \mathrm{cc}\} \tag{12.3}$$

represents work associated with temporal changes in the wavefunction. The complex internal microforce Π does not appear in (12.3), as it acts internally to the control volume $\mathbf{P}$. (In (12.1)-(12.3), $\mathbf{P}$ is considered as a control volume for the entire system consisting of the lattice, the field, and the superelectrons, but in (11.1) $\mathbf{P}$ is considered as a control volume for the superelectrons only.) Note that (12.2) differs from (4.1) only through the appearance of the normal charge-current $\mathbf{j}_n$ rather than the total charge-current $\mathbf{j}$.

As before, I write the **second law** in the form

$$\int_{\partial\mathbf{P}} \Omega\, \mathbf{n}_{\mathrm{time}} \le \mathcal{P}(\mathbf{P}) \qquad \text{for all spacetime control volumes } \mathbf{P}. \tag{12.4}$$

b. Consequences of gauge invariance

Modulo appropriate modifications of the definitions and arguments of Subsection 4b, gauge invariance for the second law is equivalent to gauge invariance of the action flux. I now turn to an investigation of this latter requirement.

The action flux can be written in the form

$$\mathcal{P}(\mathbf{P}) = \int_{\partial\mathbf{P}}\{\mathbf{T}\mathbf{n}\cdot\mathbf{e}^{\cdot} - \Phi[\mathbf{j}_n - i(\mathbf{t}\overline{\psi} - \overline{\mathbf{t}}\psi)]\cdot\mathbf{n}\} + \int_{\partial\mathbf{P}}\{\mathbf{t}\overline{T\psi} + \overline{\mathbf{t}}T\psi\}\cdot\mathbf{n}. \tag{12.5}$$

By (9.5), *the second integral is gauge invariant,* as is the term $\mathbf{j}_s$ defined by (11.10). Thus if we define $\mathbf{j}$ to be the gauge-invariant field

$$\mathbf{j} = \mathbf{j}_n + \mathbf{j}_s, \tag{12.6}$$

then

$$\mathcal{P}(\mathbf{P}) = \int_{\partial\mathbf{P}}\{\mathbf{T}\mathbf{n}\cdot\mathbf{e}^{\cdot} - \Phi\,\mathbf{j}\cdot\mathbf{n}\} + \int_{\partial\mathbf{P}}\{(\mathbf{t}\cdot\mathbf{n})\overline{T\psi} + \mathrm{cc}\} \tag{12.7}$$

and, since the first integral is gauge invariant, the results of Subsections 4b and 4c yield the existence of vector fields H, D, and j and a scalar field q consistent with (4.9) and (4.10) such that

Maxwell's equations (4.11)-(4.13) *are satisfied.*

This allows the identification of $\mathbf{j}$ with the **total charge-current** and $\mathbf{j}_s$ with the **super charge-current**.

c. Alternative forms of the action flux

Arguing as in Section 5, the action flux can be written in the form

$$\mathcal{P}(\mathbf{P}) = -\int_{\partial\mathbf{P}}\{(E\times H)\cdot\mathbf{n}_{\text{space}} + (D\cdot E)\,\mathbf{n}_{\text{time}}\} + \int_{\partial\mathbf{P}}\{(\mathbf{t}\cdot\mathbf{n})\overline{T\psi} + \mathrm{cc}\}, \tag{12.8}$$

Further, by (9.13), (11.10), and (11.11),

$$\mathbf{t}\overline{T\psi} + \overline{\mathbf{t}}T\psi = \mathbf{s}\varphi^{\cdot} - \lambda\mathbf{j}_s; \tag{12.9}$$

hence

$$\mathcal{P}(\mathbf{P}) = -\int_{\partial\mathbf{P}}\{(E\times H)\cdot\mathbf{n}_{\text{space}} + (D\cdot E)\,\mathbf{n}_{\text{time}}\} + \int_{\partial\mathbf{P}}\{(\mathbf{s}\cdot\mathbf{n})\varphi^{\cdot} - \lambda(\mathbf{j}_s\cdot\mathbf{n})\}, \tag{12.10}$$

an expression giving the superconductive contribution to $\mathcal{P}(\mathbf{P})$ as the working of a mechanical stress $\mathbf{s}$ conjugate to $\varphi^{\cdot}$ plus a transport of energy by the charge-current $\mathbf{j}_s$ with λ as corresponding electrochemical potential.

Also, by (12.8), for $\mathbb{P} = P\times[t_0,t]$ and $\mathcal{P}^{\cdot}(P) := \mathcal{P}(\mathbb{P})^{\cdot}$, (5.9) and (5.10) have the counterparts

$$\mathcal{P}^{\cdot}(P) = \int_P \{-\operatorname{div}(\mathbf{E}\times\mathbf{H}) - (\mathbf{D}\cdot\mathbf{E})^{\cdot} + \operatorname{div}_4(\mathfrak{k}\overline{T\psi} + cc)\}$$

$$= \int_P \{\mathbf{H}\cdot\mathbf{B}^{\cdot} - \mathbf{D}\cdot\mathbf{E}^{\cdot} + \mathbf{j}\cdot\mathbf{E} + \operatorname{div}_4(\mathfrak{k}\overline{T\psi} + cc)\}. \qquad (12.11)$$

d. Local form of the second law

For control volumes of the form $P\times[t_0,t]$ the second law (4.2) reduces to

$$\{\int_P \Omega\}^{\cdot} \le \mathcal{P}^{\cdot}(P); \qquad (12.12)$$

as I shall now show, this leads to the **local dissipation inequality**

$$\Omega^{\cdot} \le \mathbf{H}\cdot\mathbf{B}^{\cdot} - \mathbf{D}\cdot\mathbf{E}^{\cdot} + \mathbf{j}_n\cdot\mathbf{E} + \{\mathfrak{k}\cdot(\mathbb{H}\overline{\psi})^{\cdot} - \pi(\overline{T\psi}) - i\Phi\mathfrak{k}\cdot(\mathbb{H}\overline{\psi}) + cc\}, \qquad (12.13)$$

where the normal and superconducting currents and charges are defined by

$$\mathbb{j}_n = (\mathbf{j}_n, q_n), \qquad \mathbb{j}_s = (\mathbf{j}_s, q_s). \qquad (12.14)$$

The inequality (12.13) will be essential in restricting constitutive equations.

To verify (12.13), note that, by (9.11), (11.7), and (11.10),

$$\begin{aligned} \operatorname{div}_4(\mathfrak{k}\overline{T\psi} + cc) &= \mathfrak{k}\cdot\nabla_4(\overline{T\psi}) + (\operatorname{div}_4\mathfrak{k})(\overline{T\psi}) + cc \\ &= \mathfrak{k}\cdot\{(\mathbb{H}\overline{\psi})^{\cdot} - i(\Phi\nabla_4\overline{\psi} + \mathbb{B}\overline{\psi}^{\cdot}) + i\overline{\psi}\mathbf{E}\} - \pi(\overline{T\psi}) + i\mathbb{B}\cdot\mathfrak{k}(\overline{T\psi}) + cc \\ &= -\mathbb{j}_s\cdot\mathbf{E} + \{\mathfrak{k}\cdot(\mathbb{H}\overline{\psi})^{\cdot} - \pi(\overline{T\psi}) - i\Phi\mathfrak{k}\cdot(\mathbb{H}\overline{\psi}) + cc\}; \end{aligned} \qquad (12.15)$$

then, since $\mathbb{j}_s\cdot\mathbf{E} = \mathbf{j}_s\cdot\mathbf{E}$, (12.6), (12.11), (12.12), and (12.15) yield (12.13).

13. CONSTITUTIVE RELATIONS

a. Constitutive relations

I now generalize (6.1) to allow for superconductivity; in particular, guided by (12.13), I let

$$\mathfrak{g} = \mathfrak{D}\psi$$

and consider constitutive relations of the form

$$\begin{aligned}
\Omega &= \hat{\Omega}(E,B,\psi,\mathfrak{g}),\\
H &= \hat{H}(E,B,\psi,\mathfrak{g}),\\
D &= \hat{D}(E,B,\psi,\mathfrak{g}),\\
j_n &= \hat{j}_n(E,B,\psi,\mathfrak{g}),\\
\mathfrak{t} &= \hat{\mathfrak{t}}(E,B,\psi,\mathfrak{g}),\\
\pi &= \hat{\pi}(E,B,\psi,\mathfrak{g}),
\end{aligned} \tag{13.1}$$

where $\hat{\Omega}$, $\hat{H}$, $\hat{D}$, and $\hat{j}_n$ gauge-invariant local functions, while $\hat{\mathfrak{t}}$ and $\hat{\pi}$ are local functions that transform quantum mechanically.

b. Consequences of the second law

Compatibility with thermodynamics is now the requirement that given any choice of the fields $\mathfrak{a}$ and ψ, if E, B, Ω, H, D, j_n, $\mathfrak{t}$, and π are determined through (3.4) and (13.1), then the resulting fields are consistent with (12.13).

Thermodynamic Restrictions. *The constitutive relations are compatible with thermodynamics if and only if:*

(i) *the action density determines the magnetic field, the electric displacement, and the complex microstress through:*

$$\hat{H} = \hat{\Omega}_B, \qquad \hat{D} = -\hat{\Omega}_E, \qquad \hat{\mathfrak{t}} = \hat{\Omega}_{\bar{\mathfrak{g}}}; \tag{13.2}$$

(ii) *the* **residual dissipation inequality**

$$\hat{j}_n(E,B,\psi,\mathfrak{g})\cdot E \; - \; \{[\hat{\pi}(E,B,\psi,\mathfrak{g}) + \partial_{\bar{\psi}}\hat{\Omega}(E,B,\psi,\mathfrak{g})](\overline{T\psi}) + cc\} \;\geq\; 0 \tag{13.3}$$

is satisfied.

The proof of this theorem is given in Appendix A. I assume henceforth that *the constitutive relations are compatible with thermodynamics.*

The complex internal microforce π admits the decomposition

$$\pi = \pi_{eq} + \pi_{dis}$$

into *equilibrium* and *dissipative parts*

$$\pi_{eq} = -\hat{\Omega}_{\bar{\psi}}(\mathbf{E},\mathbf{B},\psi,\mathbf{g}), \tag{13.4}$$

$$\pi_{dis} = \hat{\pi}_{dis}(\mathbf{E},\mathbf{B},\psi,\mathbf{g}) = \hat{\pi}(\mathbf{E},\mathbf{B},\psi,\mathbf{g}) + \hat{\Omega}_{\bar{\psi}}(\mathbf{E},\mathbf{B},\psi,\mathbf{g}), \tag{13.5}$$

with only π_{dis} entering the residual dissipation inequality:

$$\hat{\mathbf{j}}_n(\mathbf{E},\mathbf{B},\psi,\mathbf{g})\cdot\mathbf{E} - \{\hat{\pi}_{dis}(\mathbf{E},\mathbf{B},\psi,\mathbf{g})(\overline{T\psi}) + cc\} \geq 0. \tag{13.6}$$

By $(13.2)_3$, and (13.5), the complex microforce balance (11.8) takes the form

$$\mathfrak{D}\cdot\{\hat{\Omega}_{\bar{\mathbf{g}}}(\mathbf{E},\mathbf{B},\psi,\mathbf{g})\} - \hat{\Omega}_{\bar{\psi}}(\mathbf{E},\mathbf{B},\psi,\mathbf{g}) + \hat{\pi}_{dis}(\mathbf{E},\mathbf{B},\psi,\mathbf{g}) = 0 \tag{13.7}$$

with $\hat{\pi}_{dis}$ consistent with (13.6); this is the most general PDE for superconductivity consistent with the constitutive relations (13.1) and the second law (12.13).

The definition (13.4) and the restrictions (i) yield the Gibbs relation

$$\Omega^{\cdot} = \mathbf{H}\cdot\mathbf{B}^{\cdot} - \mathbf{D}\cdot\mathbf{E}^{\cdot} + \{\mathfrak{t}\cdot(\overline{\mathfrak{D}\psi})^{\cdot} - \pi_{eq}\bar{\psi}^{\cdot} + cc\}.$$

Also, (i) and Lemma 10.1(ii) imply that

$$i\{\mathfrak{t}\cdot\bar{\mathbf{g}} - \pi_{eq}\bar{\psi}\} + cc = 0, \tag{13.8}$$

and the last two relations, $(9.9)_3$, and (12.15) yield a strengthened version of the second law:

$$\{\int_P \Omega\}^{\cdot} = \mathcal{P}^{\cdot}(P) - \mathcal{D}(P) \tag{13.9}$$

with

$$\mathcal{D}(P) = \int_P\{\mathbf{j}_n\cdot\mathbf{E} - \pi_{dis}(\overline{T\psi}) - \bar{\pi}_{dis}(T\psi)\} \geq 0. \tag{13.10}$$

$\mathcal{D}(P)$, the **total dissipation**, is here a consequence of Joule heating by normal electrons and working of dissipative internal microforces.[24]

Let $\mathbf{g}$ and γ denote the components of $\mathbb{g} = \mathbb{X}\psi$:

$$\mathbb{g} = (\mathbf{g},\gamma), \qquad \mathbf{g} = \mathbf{X}\psi, \qquad \gamma = \mathrm{T}\psi. \tag{13.11}$$

Then, since $\mathbf{j}_s = (\mathbf{j}_s, q_s)$, (11.10) and $(13.2)_3$ yield

$$\mathbf{j}_s = i(\psi\hat{\Omega}_{\mathbf{g}} - \overline{\psi}\hat{\Omega}_{\overline{\mathbf{g}}}), \qquad q_s = i(\psi\hat{\Omega}_{\gamma} - \overline{\psi}\hat{\Omega}_{\overline{\gamma}}); \tag{13.12}$$

a consequence of the second relation is the following

Theorem. *A nonvanishing charge-density for superelectrons requires a constitutive dependence of the action density* Ω *on the gauge-invariant time-derivative* $\mathrm{T}\psi$.[25]

c. Constitutive equations in terms of the free velocity and London potential

The thermodynamically restricted constitutive equations take on physically meaningful forms when expressed in terms of the partial derivatives $\hat{\Omega}_{\mathbf{v}}$, $\hat{\Omega}_{\lambda}$, $\hat{\Omega}_{\varphi}$, $\hat{\Omega}_{\nabla\varphi}$, $\hat{\Omega}_{\varphi^{\cdot}}$ that result when $(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda)$ are the indendent constitutive variables (cf (10.4)). In this case the important dependent variables are $\mathbf{j}_s$, q_s, and the 4-vector field $\mathbb{s}$, which is defined in (11.11) and which I now write in the form

$$\mathbb{s} = (\mathbf{s},-p). \tag{13.13}$$

Implied Constitutive Relations.

$$\begin{aligned} \mathbf{j}_s &= \hat{\Omega}_{\mathbf{v}}(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda), \qquad & q_s &= -\hat{\Omega}_{\lambda}(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda), \\ \mathbf{s} &= \hat{\Omega}_{\nabla\varphi}(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda), \qquad & p &= -\hat{\Omega}_{\varphi^{\cdot}}(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda). \end{aligned} \tag{13.14}$$

Proof. By (11.11) and $(13.2)_3$,

---

[24] Cf. Cyrot [1970], §7.2, who attributes to Tinkham [1964] the remark that the dissipation should include a contribution due to the relaxation of superelectrons.

[25] In the Ginzburg-Landau theory the energy is independent of $\mathrm{T}\psi$ and the resulting charge balance does not contain the contribution $(q_s)^{\cdot}$, an omission consistent with the underlying assumption of quasi-static conditions.

$$\varphi\mathfrak{S} = \psi\hat{\Omega}_{\mathfrak{g}} + cc, \tag{13.15}$$

and, since $\mathfrak{g} = (\mathbf{g},\gamma)$, the results (13.14) are consequences of (10.10) and (10.11). □

d. An additional hypothesis

To this point the wave function ψ is essentially arbitrary; the next assumption, which restricts ψ, is basic to the Ginzburg-Landau theory.

Charge hypothesis: ψ represents the superelectron charge-density; that is,

$$q_s = \psi\bar{\psi} \equiv \varphi^2. \tag{13.16}$$

I assume throughout the remainder of the paper that the charge hypothesis is satisfied.

By $(13.14)_2$ and (13.16), the constitutive equation for the action density can be written

$$\Omega = \tilde{\Omega}(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\varphi^{\bullet},\mathbf{v}) - \lambda\varphi^2. \tag{13.17}$$

If, in addition, Ω is independent of $\varphi^{\bullet}$,

$$\hat{\Omega}_{\varphi^{\bullet}} = 0, \tag{13.18}$$

then

$$\Omega = \tilde{\Omega}(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\mathbf{v}) - \lambda\varphi^2. \tag{13.19}$$

Further, by (9.13) and (13.16),

$$\tfrac{1}{2}i\psi\overline{T\psi} + cc = -\lambda q_s, \tag{13.20}$$

and, by (9.12), $\tilde{\Omega}(\mathbf{E},\mathbf{B},\varphi,\nabla\varphi,\mathbf{v})$ can be written as a gauge-invariant function (which I also denote by $\tilde{\Omega}$) of $(\mathbf{E},\mathbf{B},\psi,\mathbf{X}\psi)$. Thus

$$\Omega = \tilde{\Omega}(\mathbf{E},\mathbf{B},\psi,\mathbf{X}\psi) + \tfrac{1}{2}i(\psi\overline{T\psi} - \bar{\psi}T\psi), \tag{13.21}$$

provided (13.18) is satisfied. I will generally *not* require satisfaction of (13.18).

e. Weak coupling

Theories of superconductivity are often based on two assumptions: (1) that the free energy consist of uncoupled electromagnetic and superconductive contributions, with the former dependent only on $\mathbf{E}$ and $\mathbf{B}$, the latter only on ψ and $\mathfrak{g} = \mathfrak{X}\psi$; (2) that the normal current depend only on $\mathbf{E}$ and $\mathbf{B}$ (actually, $\mathbf{E}$). Within the current framework and granted the thermodynamic restrictions and the additional requirement that π depend only on ψ and $\mathfrak{g}$, these assumptions yield the constitutive relations

$$\begin{aligned} &\Omega = \hat{\Omega}_e(E,B) + \hat{\Omega}_s(\psi,\mathfrak{g}), \\ &H = \hat{H}(E,B), \qquad D = \hat{D}(E,B), \qquad j_n = \hat{j}_n(E,B), \\ &\mathfrak{t} = \hat{\mathfrak{t}}(\psi,\mathfrak{g}), \qquad \pi = \hat{\pi}(\psi,\mathfrak{g}), \end{aligned} \tag{13.22}$$

with[26]

$$\hat{H} = \partial_{B}\hat{\Omega}_e, \qquad \hat{D} = -\partial_{E}\hat{\Omega}_e, \qquad \hat{\mathfrak{t}} = \partial_{\bar{\mathfrak{g}}}\hat{\Omega}_s, \tag{13.23}$$

and with the residual dissipation inequality replaced by individual inequalities

$$\hat{j}_n(E,B)\cdot E \geq 0, \tag{13.24}$$

$$\hat{\pi}_{dis}(\psi,\mathfrak{g})(\overline{T\psi}) + cc \leq 0, \qquad \hat{\pi}_{dis}(\psi,\mathfrak{g}) = \hat{\pi}(\psi,\mathfrak{g}) + \partial_{\bar{\psi}}\hat{\Omega}_s(\psi,\mathfrak{g}), \tag{13.25}$$

which follow from (13.6) upon taking $T\psi = 0$ and then $\mathbf{E} = \mathbf{0}$. I will refer to the constitutive equations (13.22)-(13.24) as **weakly coupled**. For such constitutive equations the general PDE (13.7) has the simple form

$$\mathfrak{X}\cdot\{\partial_{\bar{\mathfrak{g}}}\hat{\Omega}_s(\psi,\mathfrak{g})\} - \partial_{\bar{\psi}}\hat{\Omega}_s(\psi,\mathfrak{g}) + \hat{\pi}_{dis}(\psi,\mathfrak{g}) = 0 \tag{13.26}$$

and is coupled to the electromagnetic field through the operators $\mathbf{X}$ and T. Further, the presence of the total current $\mathbf{j}$ (and hence of the supercurrent $\mathbf{j}_s$) in the field equation $\operatorname{curl}\mathbf{H} = \mathbf{D}^{\cdot} + \mathbf{j}$ couples the field to the flow of superelectrons. Finally, the relations (13.14) take the form

$$\begin{aligned} &j_s = \partial_{\mathbf{v}}\hat{\Omega}_s(\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda), \qquad q_s = -\partial_{\lambda}\hat{\Omega}_s(\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda), \\ &\mathbf{s} = \partial_{\nabla\varphi}\hat{\Omega}_s(\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda), \qquad p = -\partial_{\varphi^{\cdot}}\hat{\Omega}_s(\varphi,\nabla\varphi,\varphi^{\cdot},\mathbf{v},\lambda). \end{aligned} \tag{13.27}$$

[26]Here, to avoid repeated subscripts with conflicting meanings, I write, e.g., ∂_B to denote the partial derivative with respect to B.

14. CHARGE BALANCE FOR SUPERELECTRONS. REAL MICROFORCE BALANCE

a. Charge balance for superelectrons

By (11.13), $(13.2)_3$, (13.5), and Lemma 10.1(ii),

$$\mathrm{div}_4\,\mathbf{J}_s + i(\psi\,\overline{\pi}_{dis} - \overline{\psi}\,\pi_{dis}) = 0, \tag{14.1}$$

or equivalently, using (12.14),

$$(q_s)^{\cdot} = -\mathrm{div}\,j_s - m \tag{14.2}$$

with

$$m = i(\psi\,\overline{\pi}_{dis} - \overline{\psi}\,\pi_{dis}). \tag{14.3}$$

Equation (14.2) represents a **charge balance for superelectrons** with m a **mass supply** that characterizes the conversion of normal electrons to superelectrons.

b. Real microforce balance

By $(13.2)_3$ and (13.5), the balance (11.12) can be written as

$$\mathrm{div}_4(\varphi\mathbf{S}) - \{\mathbf{S}\cdot\hat{\Omega}_{\mathbf{S}} + \psi\,\hat{\Omega}_{\psi} - \psi\,\overline{\pi}_{dis} + cc\} = 0, \tag{14.4}$$

or equivalently, appealing to (10.9),

$$\varphi\,\mathrm{div}_4\mathbf{S} + \mathbf{S}\cdot\nabla_4\varphi - \varphi\,\hat{\Omega}_{\varphi} - \{e^{i\theta}\hat{\Omega}_{\mathbf{S}}\cdot\nabla_4\varphi - \psi\,\overline{\pi}_{dis} + cc\} = 0. \tag{14.5}$$

On the other hand, by (13.15), $e^{i\theta}\hat{\Omega}_{\mathbf{S}}\cdot\nabla_4\varphi + cc = \mathbf{S}\cdot\nabla_4\varphi$; the definitions

$$\omega = \omega_{eq} + \omega_{dis}, \qquad \omega_{eq} = -\hat{\Omega}_{\varphi}, \qquad \omega_{dis} = e^{i\theta}\,\overline{\pi}_{dis} + cc, \tag{14.6}$$

therefore yield the **real microforce balance**

$$\mathrm{div}_4\mathbf{S} + \omega = 0. \tag{14.7}$$

In terms of the components $\mathbf{S} = (s,-p)$, this balance can be written in the form

$$p^{\cdot} = \mathrm{div}\,s + \omega. \tag{14.8}$$

The fields s, p, ω, and ω_{dis} are real and gauge invariant (cf. (11.14)). I will refer to s as the **real microstress**, to ω as the **real internal microforce**, and to p as the **micromomentum**. This identification of p with momentum seems justified, not only by the conservation law (14.8), but also by the second of (13.14), which gives p as the negative of the derivative of the action with respect to $\varphi^{\bullet}$, and hence as the momentum corresponding to the "generalized coordinate" φ.[27]

Equivalence theorem. *The charge balance* (14.2) *for superelectrons and the real microforce balance* (14.7) *are together equivalent to the complex microforce balance* (11.8).

c. The second law revisited

By (12.10), (12.14), and (13.13), the second law (12.12) can be written as

$$\{\int_P \Psi\}^{\bullet} \le \int_{\partial P}\{-(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n} + (\mathbf{s}\cdot\mathbf{n})\varphi^{\bullet} - \lambda \mathbf{j}_s\cdot\mathbf{n}\} \tag{14.9}$$

with the total free energy Ψ (which includes kinetic energy) defined through

$$\Omega = \Psi - \mathbf{D}\cdot\mathbf{E} - \lambda q_s - p\varphi^{\bullet}, \tag{14.10}$$

a relation giving the action density the status of a grand canonical potential.

Also, by (14.3), (14.6), and the identity (9.13), the residual dissipation inequality (13.6) may be written as

$$\mathbf{j}_n\cdot\mathbf{E} - \omega_{dis}\varphi^{\bullet} + m\lambda \ge 0, \tag{14.11}$$

with the left sides of (13.6) and (14.11) equal. The dissipation $\mathcal{D}(P)$, given by (13.10) and here the left side of (14.9) minus the right, may therefore be written with integrand (14.11); this yields the balance

$$\{\int_P \Psi\}^{\bullet} = \int_{\partial P}\{-(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n} + (\mathbf{s}\cdot\mathbf{n})\varphi^{\bullet} - \lambda \mathbf{j}_s\cdot\mathbf{n}\} + \int_P\{\mathbf{j}_n\cdot\mathbf{E} - \omega_{dis}\varphi^{\bullet} + m\lambda\}. \tag{14.12}$$

Thus the classical electromagnetic energy-balance (6.7) is here augmented by the additional terms:[28]

[27] Cf. (1.2). I do not know if the micromomentum is important; it vanishes for action densities that are independent of $\varphi^{\bullet}$ and hence does not appear in the Ginzburg-Landau theories.

(i) $s\varphi^{\cdot}$, which represents working of the real microstress;

(ii) $-\lambda j_s$, which represents energy transported with the supercurrent;

(iii) $-\omega_{dis}\varphi^{\cdot}$, which represents energy dissipated through the working of the real internal microforces;

(iv) $m\lambda$, which represents energy dissipated in the conversion of normal electrons to superelectrons.

If the constitutive equations are *weakly coupled,* then for electromagnetic and superconductive free energies defined through

$$\Psi_e = \Omega_e + \mathbf{D}\cdot\mathbf{E}, \qquad \Psi_s = \Omega_s + \lambda q_s + p\varphi^{\cdot}, \tag{14.13}$$

with

$$\Omega_e = \hat{\Omega}_e(\mathbf{E},\mathbf{B}), \qquad \Omega_s = \hat{\Omega}_s(\psi,\mathbf{q}), \tag{14.14}$$

(13.2), (13.27), and (14.6) yield the Gibbs relations

$$\begin{aligned} \Psi_e^{\cdot} &= \mathbf{H}\cdot\mathbf{B}^{\cdot} + \mathbf{E}\cdot\mathbf{D}^{\cdot}, \\ \Psi_s^{\cdot} &= \mathbf{j}_s\cdot\mathbf{v}^{\cdot} + \lambda q_s^{\cdot} + m\lambda + \mathbf{s}\cdot\nabla\varphi^{\cdot} - \omega_{eq}\varphi^{\cdot} + \varphi^{\cdot}p^{\cdot} \end{aligned} \tag{14.15}$$

and these in turn yield the individual balances

$$\begin{aligned} \{\int_P \Psi_e\}^{\cdot} &= -\int_{\partial P}(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n} - \int_P \mathbf{j}_s\cdot\mathbf{E} - \mathcal{D}_e(P), \\ \{\int_P \Psi_s\}^{\cdot} &= \int_{\partial P}\{(\mathbf{s}\cdot\mathbf{n})\varphi^{\cdot} - \lambda\mathbf{j}_s\cdot\mathbf{n}\} + \int_P \mathbf{j}_s\cdot\mathbf{E} - \mathcal{D}_s(P) \end{aligned} \tag{14.16}$$

with

$$\mathcal{D}_e(P) = \int_P \mathbf{j}_n\cdot\mathbf{E}, \qquad \mathcal{D}_s(P) = \int_P\{-\omega_{dis}\varphi^{\cdot} + m\lambda\}. \tag{14.17}$$

The relations (14.16), which are "coupled" through the terms with integrand $\mathbf{j}_s\cdot\mathbf{E}$, have the local forms

$$\begin{aligned} \Psi_e^{\cdot} &= -\operatorname{div}(\mathbf{E}\times\mathbf{H}) - \mathbf{j}_s\cdot\mathbf{E} - \mathbf{j}_n\cdot\mathbf{E}, \\ \Psi_s^{\cdot} &= \operatorname{div}(\mathbf{s}\varphi^{\cdot} - \lambda\mathbf{j}_s) + \mathbf{j}_s\cdot\mathbf{E} - \omega_{dis}\varphi^{\cdot} + m\lambda. \end{aligned} \tag{14.18}$$

[28] My use of the term "working" is ambiguous, as the working of the complex microstress over $T\psi$ is, by (12.9), equal to the working of the real microstress plus the superconductive energy-transport.

d. Simple constitutive equations for the dissipative fields

By (14.3) and (14.6),

$$2\overline{\psi}\pi_{dis} = \varphi\omega_{dis} + im; \tag{14.19}$$

thus, as $\psi \neq 0$, assigning a constitutive equation for π_{dis} is equivalent to assigning constitutive equations for ω_{dis} and m, fields that may be amenable to physical interpretation. Guided by (14.11), a simple constitutive choice consistent with (14.11) (and with an assumption of weakly coupled constitutive relations) might involve equations of the form

$$\mathrm{j}_n = \Sigma \mathrm{E}, \qquad \omega_{dis} = -\beta\varphi^{\cdot}, \qquad m = \kappa q_s \lambda \tag{14.20}$$

with Σ a positive definite 3×3 matrix, the **conductivity tensor**, $\beta > 0$ a constant **kinetic modulus**, and $\kappa > 0$ a **supply modulus**. The coefficient of λ in the relation for m is chosen proportional to q_s, rather than constant, to ensure that the supply vanish with the charge density. The first of (14.20), which is *Ohm's law*, allows for anisotropy in the flow of normal electrons; when the material is *isotropic* this relation reduces to

$$\mathrm{j}_n = \sigma \mathrm{E} \tag{14.21}$$

with $\sigma \geq 0$.

Note that (14.6) and $(14.20)_2$ imply that

$$\omega = -\hat{\Omega}_{\varphi} - \beta\varphi^{\cdot}. \tag{14.22}$$

Further, by (9.13) and (14.19), the constitutive relations for ω_{dis} and m are, for $\beta = \kappa$, equivalent to the simple relation

$$\pi_{dis} = -\tfrac{1}{2}\beta T\psi, \tag{14.23}$$

and this, with (13.5), implies that

$$\pi = -\hat{\Omega}_{\overline{\psi}} - \tfrac{1}{2}\beta T\psi. \tag{14.24}$$

e. Configurational force balance

An identity important in the dynamical analysis of defects is the configurational force balance. In the present context and for weak coupling this balance has the form[29]

$$\operatorname{div}[\Omega_s \mathbf{1} - \nabla\varphi \otimes \mathbf{s}] + \mathbf{f} = 0 \qquad (14.25)$$

and may be considered as defining for f. For Ω_s independent of $\varphi^\cdot$, (13.27), (14.8), $(14.13)_2$, and (14.22) yield a specific form for this balance:

$$\operatorname{div}[\Psi_s \mathbf{1} - \nabla\varphi \otimes \mathbf{s}] = (\nabla \mathbf{v})^{\mathsf{T}} \mathbf{j}_s + \lambda \nabla q_s - \beta\varphi^\cdot \nabla\varphi. \qquad (14.26)$$

[29] 1 denotes the unit tensor, ⊗ the tensor product. A discussion of configurational forces is beyond the scope of this paper (cf. Gurtin [1995] and Fried and Gurtin [1995]).

15. ISOTROPIC QUASILINEAR THEORY

a. Constitutive assumptions

I now consider a theory based on constitutive equations that are isotropic and weakly coupled. I assume that the electromagnetic contribution to the action has the classical form

$$\Omega_e = \tfrac{1}{2}(\mu^{-1}|\mathbf{B}|^2 - \varepsilon|\mathbf{E}|^2), \tag{15.1}$$

but for the superconductive contribution I supplement the standard coarse-grain free-energy $f(q_s)$ with a gradient energy that represents a broad generalization of the gradient energy

$$\tfrac{1}{2}\alpha \mathbf{X}\psi\cdot\overline{\mathbf{X}\psi} = \tfrac{1}{2}\alpha(|\nabla\varphi|^2 + \varphi^2|\mathbf{v}|^2) \tag{15.2}$$

of the Ginzburg-Landau theory. Indeed, I see no compelling reason—other than the beauty of the resulting complex PDE—to have the same constant α govern the term $\tfrac{1}{2}\alpha|\nabla\varphi|^2$, which represents a gradient energy for the real "order-parameter" φ, and the term

$$K = \tfrac{1}{2}\alpha\varphi^2|\mathbf{v}|^2,$$

which is generally interpreted as kinetic energy associated with the supercurrent. Also, I see no reason to rule out the possibility of an energetic coupling between $\nabla\varphi$ and $\mathbf{v}$. I therefore replace (15.2) by

$$\tfrac{1}{2}(\alpha|\nabla\varphi|^2 + 2\tau\varphi\nabla\varphi\cdot\mathbf{v} + \nu\varphi^2|\mathbf{v}|^2) \tag{15.3}$$

with constants α, τ, ν real and such that (15.3) is positive-definite in $(\nabla\varphi, \varphi\mathbf{v})$. In addition, I allow for a microkinetic energy

$$k = \tfrac{1}{2}\delta(\varphi^{\cdot})^2$$

with $\delta \geq 0$. Thus, in view of (13.17), I consider a constitutive equation for the superconductive action-density of the form[30]

[30]The kinetic energies K and k enter the action density with opposite sign. Indeed, the momentum p corresponding to k is associated with a force balance that yields a Legendre transformation converting total energy to action. But the momentum corresponding to K is j_s, and j_s enters the theory through a charge balance with corresponding Legendre transformation based not on momentum, but instead on the potential λ (cf. (14.10)), as one

$$\Omega_s = f(\varphi^2) + \tfrac{1}{2}[\alpha|\nabla\varphi|^2 + 2\tau\varphi\nabla\varphi\cdot\mathbf{v} + \nu\varphi^2|\mathbf{v}|^2 - \delta(\varphi^{\cdot})^2] - \lambda\varphi^2. \tag{15.4}$$

By (13.2) and (13.27), consequences of the constitutive relations (15.1) and (15.4) are

$$\begin{aligned} &\mathbf{H} = \mu^{-1}\mathbf{B}, && \mathbf{D} = \varepsilon\mathbf{E}, \\ &q_s = \varphi^2, && \mathbf{j}_s = \nu\varphi^2\mathbf{v} + \tau\varphi\nabla\varphi, \\ &\mathbf{s} = \alpha\nabla\varphi + \tau\varphi\mathbf{v}, && p = \delta\varphi^{\cdot}. \end{aligned} \tag{15.5}$$

Unlike the Ginzburg-Landau theory in which $\mathbf{j}_s$ is proportional to $q_s\mathbf{v}$, $\mathbf{j}_s$ here involves a term $\tau\varphi\nabla\varphi$ and will generally not vanish with $\mathbf{v}$. I believe this to not contradict the physics: $\mathbf{v}$ represents the velocity of a single electron in an electromagnetic field, but here there is a stream of superelectrons and one might expect a correction term that contributes when the stream is nonuniform.

Also, note that by (14.13), the electromagnetic and superconductive free energies have the respective forms

$$\begin{aligned} \Psi_e &= \tfrac{1}{2}[\mu^{-1}|\mathbf{B}|^2 + \varepsilon|\mathbf{E}|^2], \\ \Psi_s &= f(\varphi^2) + \tfrac{1}{2}[\alpha|\nabla\varphi|^2 + 2\tau\varphi\nabla\varphi\cdot\mathbf{v} + \nu\varphi^2|\mathbf{v}|^2 + \delta(\varphi^{\cdot})^2]. \end{aligned} \tag{15.6}$$

Finally, I restrict attention to Ohm's law (14.21) for $\mathbf{j}_n$ and the relations $(14.20)_{2,3}$ for ω_{dis} and m,

$$\omega_{dis} = -\beta\varphi^{\cdot}, \qquad m = \kappa\varphi^2\lambda, \tag{15.7}$$

so that, by (14.22),

$$\omega = -2\varphi f'(\varphi^2) - \nu\varphi|\mathbf{v}|^2 - \tau\nabla\varphi\cdot\mathbf{v} + 2\varphi\lambda - \beta\varphi^{\cdot}. \tag{15.8}$$

For the special case

$$\alpha = \nu, \qquad \beta = \kappa, \qquad \tau = 0, \qquad \delta = 0, \tag{15.9}$$

the superconductive action-density has the simple form

$$\Omega_s = f(\psi\bar{\psi}) + \tfrac{1}{2}\alpha\mathbf{X}\psi\cdot\bar{\mathbf{X}}\bar{\psi} + \tfrac{1}{2}i(\psi\bar{T}\bar{\psi} - \bar{\psi}T\psi) \tag{15.10}$$

---

would expect when modelling transport.

(cf. (13.20)) and, by $(13.23)_3$ and (14.24),

$$\mathfrak{k} = \tfrac{1}{2}(\alpha\mathbf{X}\psi, i\psi), \qquad \pi = \tfrac{1}{2}iT\psi - \psi f'(\psi\bar{\psi}) - \tfrac{1}{2}\beta T\psi. \tag{15.11}$$

b. Basic system of PDEs

Substituting the constitutive equations (15.5)-(15.7) into the charge balance (14.2) and the real microforce balance (14.7), and choosing the gauge such that

$$\operatorname{div}\mathbf{A} = 0, \tag{15.12}$$

yields, by virtue of the definitions (9.6) of λ and $\mathbf{v}$, a hyperbolic PDE for the amplitude φ coupled to a parabolic PDE for the phase angle θ:

$$\delta\varphi^{\cdot\cdot} + \beta\varphi^{\cdot} + 2\varphi(\theta^{\cdot} + \Phi) = \alpha\Delta\varphi + \tau\varphi\Delta\theta - \nu\varphi|\nabla\theta - \mathbf{A}|^2 - 2\varphi f'(\varphi^2), \tag{15.13}$$

$$-2\varphi^{\cdot} + \kappa\varphi(\theta^{\cdot} + \Phi) = \nu\varphi\Delta\theta + \tau\Delta\varphi + 2\nu\nabla\varphi\cdot(\nabla\theta - \mathbf{A})] + \tau\varphi^{-1}|\nabla\varphi|^2. \tag{15.14}$$

These are the basic PDEs of the theory. Note that the terms $2\varphi(\theta^{\cdot} + \Phi)$ and $-2\varphi^{\cdot}$, which are consequences of the term $-\lambda\varphi^2$ in (15.4), are nondissipative, as they enter the equations skew-symmetrically.

The PDEs (15.13) and (15.14) may be written as a complex PDE; the result, which I write only for the special case $\tau = \delta = 0$, is

$$\bar{\psi}[\alpha_+\mathbf{X}\cdot\mathbf{X}\psi - 2\psi f'(\psi\bar{\psi}) + (2i - \beta_+)T\psi] + \psi[\alpha_-\bar{\mathbf{X}}\cdot\bar{\mathbf{X}}\bar{\psi} - \beta_-\bar{T}\bar{\psi}] - \tfrac{1}{2}\alpha_-[e^{-i\theta}\mathbf{X}\psi + e^{i\theta}\bar{\mathbf{X}}\bar{\psi}]^2 = 0 \tag{15.15}$$

with

$$\alpha_+ = \tfrac{1}{2}(\alpha + \nu), \qquad \alpha_- = \tfrac{1}{2}(\alpha - \nu),$$
$$\beta_+ = \tfrac{1}{2}(\beta + \kappa), \qquad \beta_- = \tfrac{1}{2}(\beta - \kappa).$$

The verification of (15.15) follows from (9.15)-(9.18), (15.5)-(15.8), and the complex microforce balance in the form

$$2\operatorname{div}_4(\bar{\psi}\,\mathfrak{k}) - \mathbf{s}\cdot\nabla_4\varphi + \varphi\omega + im = 0$$

(cf. (11.9), (11.15), (14.2), and (14.7)).

For the restricted theory based on (15.9), a direct consequence of (15.11) and the balance $\mathbf{X}\cdot\mathfrak{k} + \pi = 0$ is the complex PDE

$$(\beta - 2i)T\psi = \alpha \mathbf{X}\cdot\mathbf{X}\psi - 2\psi f'(\psi\bar{\psi}). \tag{15.16}$$

The term $-2iT\psi$ follows from the last term in (15.10); if this term is neglected, then (15.16) reduces to the complex time-dependent Ginzburg-Landau equation. Granted (15.12), (15.16) is equivalent to the *parabolic* PDEs

$$\beta\varphi^{\cdot} + 2\varphi(\theta^{\cdot} + \Phi) = \alpha[\Delta\varphi - \varphi|\nabla\theta - \mathbf{A}|^2] - 2\varphi f'(\varphi^2), \tag{15.17}$$

$$-2\varphi^{\cdot} + \beta\varphi(\theta^{\cdot} + \Phi) = \alpha[\varphi\Delta\theta + 2\nabla\varphi\cdot(\nabla\theta - \mathbf{A})], \tag{15.18}$$

which may be obtained by subjecting (15.13) and (15.14) to (15.9). If the terms $2\varphi(\theta^{\cdot} + \Phi)$ and $-2\varphi^{\cdot}$ in (15.17) and (15.18) are dropped, then these equations are the time-dependent Ginzburg-Landau equations.

Returning to the more general theory unencumbered by (15.9), the basic system of equations consists of (15.13) and (15.14), the electromagnetic equations

$$\mathbf{E} = -(\mathbf{A}^{\cdot} + \nabla\Phi), \qquad \mathbf{B} = \operatorname{curl}\mathbf{A}, \tag{3.4bis}$$

$$\operatorname{curl}\mathbf{H} = \mathbf{D}^{\cdot} + \mathbf{j}, \tag{4.11bis}$$

$$\mathbf{H} = \mu^{-1}\mathbf{B}, \qquad \mathbf{D} = \varepsilon\mathbf{E}, \tag{15.5bis}$$

and the current equations

$$\mathbf{j} = \mathbf{j}_n + \mathbf{j}_s, \tag{15.19}$$

$$\mathbf{j}_n = \sigma\mathbf{E}, \qquad \mathbf{j}_s = \nu\varphi^2(\nabla\theta - \mathbf{A}). \tag{15.20}$$

(The Maxwell equation (4.12) can be considered as defining for q.)

c. Energy identities

The relations (14.18), when combined with (3.4), (9.6), (14.21), and (15.5)-(15.7), result in an energy identity

$$\tfrac{1}{2}[\mu^{-1}|\mathbf{B}|^2 + \varepsilon|\mathbf{E}|^2]^{\cdot} = -\operatorname{div}(\mu^{-1}\mathbf{E}\times\mathbf{B}) - \sigma|\mathbf{A}^{\cdot} + \nabla\Phi|^2 - \ell \tag{15.21}$$

for the electromagnetic field and an energy identity

$$\{f(\varphi^2) + \tfrac{1}{2}[\alpha|\nabla\varphi|^2 + 2\tau\varphi\nabla\varphi\cdot(\nabla\theta - \mathbf{A}) + \nu\varphi^2|\nabla\theta - \mathbf{A}|^2 + \delta(\varphi^\cdot)^2]\}^\cdot =$$
$$\operatorname{div}\{[\alpha\nabla\varphi + \tau\varphi(\nabla\theta - \mathbf{A})]\varphi^\cdot + (\theta^\cdot + \Phi)[\nu\varphi^2(\nabla\theta - \mathbf{A}) + \tau\varphi\nabla\varphi]\} +$$
$$\beta(\varphi^\cdot)^2 + \kappa\varphi^2(\theta^\cdot + \Phi)^2 + \ell, \qquad (15.22)$$

for the superconducting electrons, coupled through the field

$$\ell = \mathbf{j}_s\cdot\mathbf{E} = -[\nu\varphi^2(\nabla\theta - \mathbf{A}) + \tau\varphi\nabla\varphi]\cdot(\mathbf{A}^\cdot + \nabla\Phi). \qquad (15.23)$$

For the standard Ginzburg-Landau theory restricted by (15.9), the second identity reduces to

$$\{f(\varphi^2) + \tfrac{1}{2}\alpha[|\nabla\varphi|^2 + \varphi^2|\nabla\theta - \mathbf{A}|^2]\}^\cdot =$$
$$\alpha\operatorname{div}[(\nabla\varphi)\varphi^\cdot + \varphi^2(\theta^\cdot + \Phi)(\nabla\theta - \mathbf{A})] + \beta[(\varphi^\cdot)^2 + \varphi^2(\theta^\cdot + \Phi)^2] + \ell \qquad (15.24)$$

with

$$\ell = -\alpha\varphi^2(\nabla\theta - \mathbf{A})\cdot(\mathbf{A}^\cdot + \nabla\Phi). \qquad (15.25)$$

Also important is the configurational force balance (14.26), which, by (15.5) and (15.6), and for the special case $\tau = \delta = 0$, has the form

$$\nabla[f(\varphi^2) + \tfrac{1}{2}\alpha|\nabla\varphi|^2] - \alpha\operatorname{div}(\nabla\varphi\otimes\nabla\varphi) =$$
$$-[\theta^\cdot + \Phi + \tfrac{1}{2}\nu|\nabla\theta - \mathbf{A}|^2]\nabla q_s - \beta\varphi^\cdot\nabla\varphi. \qquad (15.26)$$

d. Momentum identity for the supercurrent

Assume that $\tau = 0$, so that $\mathbf{j}_s = \nu q_s\mathbf{v}$. Consider the quantity

$$(q_s\mathbf{v})^\cdot + \operatorname{div}(\mathbf{j}_s\otimes\mathbf{v}). \qquad (15.27)$$

Since $\mathbf{v}\otimes\mathbf{j}_s = \mathbf{j}_s\otimes\mathbf{v}$, the charge balance (14.3) can be used to write (15.27) as

$$q_s\mathbf{v}^\cdot + \nu q_s(\nabla\mathbf{v} - \nabla\mathbf{v}^\mathsf{T})\mathbf{v} + \nu q_s(\nabla\mathbf{v})^\mathsf{T}\mathbf{v} - m\mathbf{v},$$

and hence, by $(2.8)_3$, (3.4), and $(9.6)_3$, as

$$q_s(\mathbf{E} + \nu\mathbf{v}\times\mathbf{B}) + q_s\nabla[-\lambda + \tfrac{1}{2}\nu|\nabla\theta - \mathbf{A}|^2] - m\mathbf{v}.$$

This establishes the identity

$$(q_s\mathbf{v})^{\cdot} + \operatorname{div}(\mathbf{j}_s \otimes \mathbf{v}) = q_s[\mathbf{E} + \nu\mathbf{v}\times\mathbf{B} + \nabla(-\lambda + \tfrac{1}{2}\nu|\nabla\theta - \mathbf{A}|^2)] - m\mathbf{v}. \tag{15.28}$$

The operation $(\,)^{\cdot}$ is not the time-derivative following the motion of superelectrons, but rather the derivative with respect to t holding the spatial position $\mathbf{x}$ fixed. Thus, modulo a multiplicative constant, the left side of (15.28) represents the rate of change of momentum, per unit volume, following the superelectrons. The identity (15.28) is therefore a *momentum-balance for superelectrons* with $\mathbf{E} + \nu\mathbf{v}\times\mathbf{B}$ a Lorentz-force, per unit charge, $\nabla(-\lambda + \tfrac{1}{2}\nu|\nabla\theta - \mathbf{A}|^2)$ a conservative quantum-mechanical force, per unit charge, and $-m\mathbf{v}$ a supply of momentum induced by the conversion of normal electrons to superelectrons.

It should be emphasized that (15.28) is not a separate field equation, but rather a consequence of the charge balance and the definitions of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{v}$, and λ.

Note the presence of the "potential"

$$(-\lambda + \tfrac{1}{2}\nu|\nabla\theta - \mathbf{A}|^2) = (\theta^{\cdot} + \Phi + \tfrac{1}{2}\nu|\nabla\theta - \mathbf{A}|^2)$$

in both the configurational force balance (15.26) and the momentum balance (15.28).

16. ANISOTROPIC QUASILINEAR THEORY

I now consider a theory based on constitutive equations that are weakly coupled, but anisotropic, with gradient energy of the form[31]

$$\Gamma(\mathbf{g}) = \tfrac{1}{2}\bar{\mathbf{g}}\cdot\mathbf{M}\mathbf{g}, \qquad \mathbf{g} = \mathbf{X}\psi, \tag{16.1}$$

where $\mathbf{M}$ is a complex 3×3 matrix. Because of (9.10), the energy (16.1) is gauge invariant. Further, since $\Gamma(\mathbf{g})$ is real, $\bar{\mathbf{g}}\cdot\mathbf{M}\mathbf{g} = \mathbf{g}\cdot\bar{\mathbf{M}}\bar{\mathbf{g}}$, and $\mathbf{M}$ is Hermitian:

$$\mathbf{M} = \bar{\mathbf{M}}^{\mathsf{T}}. \tag{16.2}$$

The real and imaginary parts of $\mathbf{M}$ are therefore symmetric and skew, respectively, and $\mathbf{M}$ may be written in the form

$$\mathbf{M} = \mathbf{K} + i\mathbf{k}\times \tag{16.3}$$

(cf. (2.3)) with $\mathbf{K}$ a real, symmetric 3×3 matrix and $\mathbf{k}$ a real 3-vector. Thus, by (9.12) and (13.16),

$$\Gamma(\mathbf{g}) = \tfrac{1}{2}[\nabla\varphi\cdot\mathbf{K}\nabla\varphi + q_s\mathbf{v}\cdot\mathbf{K}\mathbf{v} + \mathbf{v}\cdot(\mathbf{k}\times\nabla q_s)]. \tag{16.4}$$

For convenience, I do not allow for a microkinetic energy, and therefore write the superconductive contribution to the action density in the form

$$\Omega_s = f(\varphi^2) + \Gamma(\mathbf{g}) - \lambda\varphi^2 = f(\psi\bar{\psi}) + \tfrac{1}{2}\bar{\mathbf{X}}\bar{\psi}\cdot\mathbf{M}\mathbf{X}\psi + \tfrac{1}{2}i(\psi\bar{\mathrm{T}}\bar{\psi} - \bar{\psi}\mathrm{T}\psi). \tag{16.5}$$

By (13.27), $q_s = \varphi^2$, $p = 0$, and

$$\mathbf{j}_s = q_s\mathbf{K}\mathbf{v} + \varphi\mathbf{k}\times\nabla\varphi, \qquad \mathbf{s} = \mathbf{K}\nabla\varphi - \varphi\mathbf{k}\times\mathbf{v},$$

so that $\mathbf{j}_s$ involves a term $\mathbf{k}\times\nabla\varphi$ that is present even when $\mathbf{v}$ vanishes. This term yields a single fixed direction (that parallel to $\mathbf{k}$) in which nonuniformity in the supercharge does not affect the supercurrent. This specificity in the coupling between $\mathbf{j}_s$ and $\nabla\varphi$ is misleading, as one could easily allow for more general and possibly more appropriate forms of coupling through an anisotropic generalization of (15.3). For that reason I set

[31] I could also modify (15.3) by replacing α, τ, and ν by real matrices. This would be more general than (16.1), but (in contrast to the last section) my objective here is to give a simple and direct anisotropic generalization of the Ginzburg-Landau theory.

$$\mathbf{k} = 0,$$

so that

$$\mathrm{j}_s = q_s \mathbf{K}\mathbf{v}, \qquad \mathbf{s} = \mathbf{K}\nabla\varphi. \tag{16.6}$$

To these constitutive equations I adjoin the dissipative relations (14.20). The analysis of the last section then yields the complex PDE

$$(\beta - 2i)T\psi = \mathbf{X}\cdot(\mathbf{K}\mathbf{X}\psi) - 2\psi f'(\psi\bar{\psi}) \tag{16.7}$$

and, if the gauge is chosen so that

$$\mathrm{div}(\mathbf{K}\mathbf{A}) = 0, \tag{16.8}$$

the corresponding real PDEs

$$\beta\varphi^{\cdot} + 2\varphi(\theta^{\cdot} + \Phi) = \mathrm{div}(\mathbf{K}\nabla\varphi) - \varphi(\nabla\theta - \mathbf{A})\cdot\mathbf{K}(\nabla\theta - \mathbf{A}) - 2\varphi f'(\varphi^2), \tag{16.9}$$

$$-2\varphi^{\cdot} + \beta\varphi(\theta^{\cdot} + \Phi) = \varphi\,\mathrm{div}(\mathbf{K}\nabla\theta) + 2\nabla\varphi\cdot\mathbf{K}(\nabla\theta - \mathbf{A}). \tag{16.10}$$

As before there are associated energy identities. The identity for the superconducting electrons follows from $(14.18)_1$ supplemented by (3.4), (9.6), (14.20), (16.5), and (16.6); the result is

$$\left\{f(\varphi^2) + \tfrac{1}{2}[\nabla\varphi\cdot\mathbf{K}\nabla\varphi + \varphi^2(\nabla\theta - \mathbf{A})\cdot\mathbf{K}(\nabla\theta - \mathbf{A})]\right\}^{\cdot} =$$
$$\mathrm{div}\left\{[\varphi^{\cdot}\mathbf{K}\nabla\varphi + (\theta^{\cdot} + \Phi)\varphi^2\mathbf{K}(\nabla\theta - \mathbf{A})]\right\} + \beta[(\varphi^{\cdot})^2 + \varphi^2(\theta^{\cdot} + \Phi)^2] + \ell \tag{16.11}$$

with

$$\ell = \mathrm{j}_s\cdot\mathbf{E} = -[\varphi^2\mathbf{K}(\nabla\theta - \mathbf{A})]\cdot(\mathbf{A}^{\cdot} + \nabla\Phi). \tag{16.12}$$

I will not write the electromagnetic energy identity explicitly: granted a specification of the constitutive relation $\Omega = \hat{\Omega}_e(\mathbf{E},\mathbf{B})$, it follows directly from $(14.18)_1$, (13.2), (14.13), $(14.20)_1$, and (16.12).

Finally, the steps used to derive the configurational balance (15.26) and the momentum balance (15.28) here yield the respective identities

$$\nabla[f(\varphi^2) + \tfrac{1}{2}\nabla\varphi\cdot K\nabla\varphi] - \operatorname{div}[(K\nabla\varphi)\otimes\nabla\varphi] =$$
$$-[\theta^{\cdot} + \Phi + \tfrac{1}{2}(\nabla\theta - \mathbf{A})\cdot K(\nabla\theta - \mathbf{A})]\nabla q_s - \beta\varphi^{\cdot}\nabla\varphi. \qquad (16.13)$$

and

$$j_s^{\cdot} + \operatorname{div}(j_s\otimes\mathbf{v}_s) =$$
$$q_s K\{(\mathbf{E} + \mathbf{v}_s\times\mathbf{B}) + \nabla[-\lambda + \tfrac{1}{2}(\nabla\theta - \mathbf{A})\cdot K(\nabla\theta - \mathbf{A})]\} - m\mathbf{v}_s \qquad (16.14)$$

with $\mathbf{v}_s = K\mathbf{v}$, so that $j_s = q_s\mathbf{v}_s$.

APPENDIX A. PROOF OF THE THEOREM ON THERMODYNAMIC RESTRICTIONS

Let

$$\mathfrak{z} = (\mathbf{E},\mathrm{B},\psi,\mathfrak{d}). \tag{A1}$$

A necessary and sufficient condition for compatibility with thermodynamics (CWT) is that, for any choice of the fields $\mathfrak{a}$ and ψ,

$$\{\hat{\Omega}_{\mathbf{B}}(\mathfrak{z}) - \hat{\mathrm{H}}(\mathfrak{z})\}\cdot\mathrm{B}^{\cdot} + \{\hat{\Omega}_{\mathbf{E}}(\mathfrak{z}) + \hat{\mathrm{D}}(\mathfrak{z})\}\cdot\mathrm{E}^{\cdot} + \{[\hat{\Omega}_{\bar{\mathfrak{d}}}(\mathfrak{z}) - \hat{\mathfrak{t}}(\mathfrak{z})]\cdot\bar{\mathfrak{d}}^{\cdot} + \mathrm{cc}\} - \mathcal{F}(\mathfrak{z}) + \mathcal{G}(\mathfrak{z}) \le 0, \tag{A2}$$

where

$$\begin{aligned}\mathcal{F}(\mathfrak{z}) &= \hat{\mathrm{j}}_n(\mathfrak{z})\cdot\mathbf{E} - \{[\hat{\pi}(\mathfrak{z}) + \hat{\Omega}_{\bar{\psi}}(\mathfrak{z})](\overline{\mathrm{T}\psi}) + \mathrm{cc}\},\\ \mathcal{G}(\mathfrak{z}) &= \mathrm{i}\Phi\{\bar{\psi}\hat{\Omega}_{\bar{\psi}}(\mathfrak{z}) + \hat{\mathfrak{t}}(\mathfrak{z})\cdot\bar{\mathfrak{d}}\} + \mathrm{cc}.\end{aligned} \tag{A3}$$

The remainder of the proof will precede as a series of assertions.

Assertion 1. $\mathfrak{a}$ and ψ can always be chosen such that $\mathbf{E}$, $\mathbf{B}$, $\mathbf{E}^{\cdot}$, $\mathrm{B}^{\cdot}$, ψ, $\mathfrak{d} = \mathfrak{X}\psi$, and $\mathfrak{d}^{\cdot}$ have arbitrarily prescribed values at some point, say $\mathfrak{X} = (\mathbf{x},t) = (0,0)$.

Choose arbitrary 3-vectors $\mathbf{E}_0$, $\mathbf{E}_1$, B_0, and B_1, an arbitrary complex scalar field $\psi_0 \neq 0$, and arbitrary complex 4-vector fields $\mathfrak{d}_0$ and $\mathfrak{d}_1$. Let $\eta(\tau)$ be a real function on $\mathbb{R}$ with

$$\eta'(0) = 1, \qquad |\eta| < \tfrac{1}{2}|\psi_0|, \tag{A4}$$

let $\mathfrak{a} = (\mathbf{A},-\Phi)$ be given by (6.4), and let

$$\psi(\mathbf{x},t) = \psi_0 + \eta(\mathfrak{X}\cdot(\mathfrak{d}_0 + t\,\mathfrak{d}_1)) - \mathrm{i}\eta(t\mathbf{x}\cdot\mathbf{E}_0\psi_0).$$

Then, by (A4), $\psi \neq 0$, and, since $\mathbf{A}(0,0) = \mathbf{0}$, $\mathbf{A}^{\cdot}(0,0) = -\mathbf{E}_0$, and $\Phi \equiv 0$,

$$\mathbf{E} = \mathbf{E}_0,\ \ \mathbf{E}^{\cdot} = \mathbf{E}_1,\ \ \mathrm{B} = \mathrm{B}_0,\ \ \mathrm{B}^{\cdot} = \mathrm{B}_1,\ \ \psi = \psi_0,\ \ \mathfrak{d} = \mathfrak{d}_0,\ \ \mathfrak{d}^{\cdot} = \mathfrak{d}_1 \quad \text{at} \quad (0,0),$$

which establishes Assertion 1.

Assertion 2. (i) $\Rightarrow$ $\mathcal{G}(\mathfrak{z}) = 0$.

Assume (i). The third of (13.2) implies that

$$\mathcal{G}(\mathfrak{z}) = i\Phi\{\overline{\psi}\hat{\Omega}_{\overline{\psi}}(\mathfrak{z}) + \hat{\Omega}_{\overline{\mathfrak{A}}}(\mathfrak{z})\cdot\overline{\mathfrak{A}}\} + cc,$$

so that, by (10.6) and the fact that $\hat{\Omega}$ is gauge invariant, $\mathcal{G}(\mathfrak{z}) = 0$.

Assertion 3. (CWT) $\Rightarrow$ (i).

Assume (CWT). Since the list $\mathfrak{z}$ does not include $E^{\cdot}$, $B^{\cdot}$, and $\mathfrak{A}^{\cdot}$, and since these variables appear linearly in (A2), $\hat{H}(\mathfrak{z}) = \hat{\Omega}_B(\mathfrak{z})$, $\hat{D}(\mathfrak{z}) = -\hat{\Omega}_E(\mathfrak{z})$, and

$$\{\hat{\Omega}_{\overline{\mathfrak{A}}}(\mathfrak{z}) - \hat{\mathfrak{l}}(\mathfrak{z})\}\cdot\overline{\mathfrak{A}}^{\cdot} + cc = 0.$$

Thus and by (8.3), $\hat{\mathfrak{l}}(\mathfrak{z}) = \hat{\Omega}_{\overline{\mathfrak{A}}}(\mathfrak{z})$. Therefore (i) is satisfied.

Assertion 4. (CWT) $\Rightarrow$ (ii).

Assume (CWT). Then (A2) and (i) yield

$$\mathcal{F}(\mathfrak{z}) - \mathcal{G}(\mathfrak{z}) \geq 0 \tag{A5}$$

for all $\mathfrak{z}$. Thus and by Assertion 2, the inequality (A5) reduces to (ii).

Assertion 5. (i), (ii) $\Rightarrow$ (CWT).

This follows from (A2) and Assertion 2. □

APPENDIX B. ALTERNATIVE FORMULATION IN TERMS OF REAL FIELDS

The results established thus far should underline the importance of the London potential λ and the free velocity $\mathbf{v}$. I now sketch an alternative formulation in which these fields play a major role. The primitive concepts remain the vector potential $\mathbf{B} = (\mathbf{A},-\Phi)$, the amplitude φ, and the phase angle θ, but the requirement that φ and θ enter the theory through a complex wavefunction is dropped.

I take as basic the following forms of the charge balance, the real microforce balance, and the second law:

$$\{\int_P q_s\}^{\cdot} = -\int_{\partial P} \mathbf{j}_s\cdot\mathbf{n} - \int_P m, \qquad \{\int_P p\}^{\cdot} = \int_{\partial P} \mathbf{s}\cdot\mathbf{n} + \int_P \omega, \tag{B1}$$

$$\{\int_P \Psi\}^{\cdot} \le \int_{\partial P}\{-(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n} + (\mathbf{s}\cdot\mathbf{n})\varphi^{\cdot} - \lambda\mathbf{j}_s\cdot\mathbf{n}\} \tag{B2}$$

for all spatial control volumes P (cf. (14.2), (14.8), and (14.9)); equivalently,

$$(q_s)^{\cdot} = -\operatorname{div}\mathbf{j}_s - m, \tag{B3}$$

$$p^{\cdot} = \operatorname{div}\mathbf{s} + \omega, \tag{B4}$$

$$\Psi^{\cdot} \le \operatorname{div}\{-(\mathbf{E}\times\mathbf{H}) + \mathbf{s}\varphi^{\cdot} - \lambda\mathbf{j}_s\}. \tag{B5}$$

These are to be augmented by the auxiliary equations

$$\mathbf{E} = -(\mathbf{A}^{\cdot}+\nabla\Phi), \qquad \mathbf{B} = \operatorname{curl}\mathbf{A}, \tag{B6}$$

$$\mathbf{v} = \nabla\theta - \mathbf{A}, \qquad \lambda = -(\theta^{\cdot} + \Phi), \tag{B7}$$

which may be considered as defining (cf. (3.4), (9.6)), the current equation

$$\mathbf{j} = \mathbf{j}_n + \mathbf{j}_s \tag{B8}$$

(cf. (12.6)), and the Maxwell equation

$$\operatorname{curl}\mathbf{H} = \mathbf{D}^{\cdot} + \mathbf{j} \tag{B9}$$

(cf (4.11)). (B3)-(B9) then yield the local dissipation inequality

$$\Omega^{\cdot} \le \mathbf{H}\cdot\mathbf{B}^{\cdot} - \mathbf{D}\cdot\mathbf{E}^{\cdot} - q_s\lambda^{\cdot} + \mathbf{j}_n\cdot\mathbf{E} + \mathbf{j}_s\cdot\mathbf{v}^{\cdot} + m\lambda + \mathbf{s}\cdot\nabla\varphi^{\cdot} - \omega\varphi^{\cdot} - p\varphi^{\cdot\cdot} \tag{B10}$$

with the action density defined by

$$\Omega = \Psi - \mathbf{D}\cdot\mathbf{E} - \lambda q_s - p\varphi^{\cdot} \tag{B11}$$

(cf. (14.10)).

Within this framework constitutive equations equivalent to (13.1) give

$$\Omega,\ \mathrm{H},\ \mathbf{j}_n,\ \mathbf{j}_s,\ q_s,\ m,\ \mathbf{s},\ \mathbf{p},\ \omega \text{ as functions of } (\mathbf{E},\mathbf{B},\mathbf{v},\lambda,\varphi,\nabla\varphi,\varphi^{\cdot}). \tag{B12}$$

Compatibility with thermodynamics, defined as in Subsection 13b, but with (B10) as the underlying inequality, then yields the restrictions $(13.2)_{1,2}$, (13.27), and (14.11), and renders this formulation of the theory completely equivalent to the complex formulation.

APPENDIX C. RELATION TO THE LONDON THEORY

A gross simplification of the formulation discussed in Appendix B leads to a minor modification of the London theory. The simplified formulation is quasi-static and neglects real microforces as well as the influence of normal electrons. Thus

$$s = 0, \quad \omega = 0, \quad p = 0, \quad j_n = 0, \quad j = j_s, \quad m = 0, \quad \mathbf{D}^{\cdot} = 0, \quad (q_s)^{\cdot} = 0, \tag{C1}$$

and the second law (B5) takes the form

$$\Psi^{\cdot} \le -\operatorname{div}\{(\mathbf{E}\times\mathbf{H}) + \lambda \mathbf{j}\}. \tag{C2}$$

In addition, not all of the auxiliary equations (B6) and (B7) are needed. One simply notes that (B6) and (B7) yield the *London equations*[32]

$$\mathbf{E} = \mathbf{v}^{\cdot} + \nabla\lambda, \qquad \mathbf{B} = -\operatorname{curl}\mathbf{v}, \tag{C3}$$

which I consider as defining for **E** and **B**. These and the Maxwell equation

$$\operatorname{curl}\mathbf{H} = \mathbf{j} \tag{C4}$$

are the basic equations of the theory, the charge balance

$$\operatorname{div}\mathbf{j} = 0 \tag{C5}$$

being a consequence of (C4). By (C2)-(C5),

$$\Psi^{\cdot} \le \mathbf{H}\cdot\mathbf{B}^{\cdot} + \mathbf{j}\cdot\mathbf{v}^{\cdot}, \tag{C6}$$

which replaces (B10) as the local dissipation inequality.

I consider constitutive equations giving

$$\Psi, \mathbf{H}, \mathbf{j} \text{ as functions of } (\mathbf{E},\mathbf{B},\mathbf{v},\lambda). \tag{C7}$$

Compatibility with thermodynamics with (C6) as the underlying inequality then renders Ψ, **H**, and **j** independent of **E** and λ with

---

[32] Cf. eqts. (6) and (7) of the Londons [1935].

$$\Psi = \hat{\Psi}(\mathbf{B},\mathbf{v}), \qquad \mathbf{H} = \hat{\Psi}_{\mathbf{B}}(\mathbf{B},\mathbf{v}), \qquad \mathbf{j} = \hat{\Psi}_{\mathbf{v}}(\mathbf{B},\mathbf{v}). \tag{C8}$$

Note that, by (C6) and (C8), the dissipation inequality becomes an equality,

$$\Psi^{\cdot} = \mathbf{H}\cdot\mathbf{B}^{\cdot} + \mathbf{j}\cdot\mathbf{v}^{\cdot}, \tag{C9}$$

or equivalently,

$$\{\int_{P}\Psi\}^{\cdot} = \int_{\partial P}\{-(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n} - \lambda\mathbf{j}_s\cdot\mathbf{n}\} \tag{C10}$$

and the theory is dissipationless.

Finally, the free energy[33]

$$\Psi = \tfrac{1}{2}(\mu^{-1}|\mathbf{B}|^2 + q|\mathbf{v}|^2), \tag{C11}$$

with μ and $q=q_s$ strictly positive and constant yield

$$\mathbf{H} = \mu^{-1}\mathbf{B}, \qquad \mathbf{j} = q\mathbf{v}, \tag{C12}$$

and, by (C3) and (C4) result in the field equation

$$\Delta\mathbf{B} = q\mu\mathbf{B}. \tag{C13}$$

[33]The Londons [1935] take $\Psi = \ldots + (\text{const})j^2 + (\text{const})q^2$ and do not require that $q^{\cdot} = 0$; but since $\mathbf{j} = q\mathbf{v}$, a kinetic energy $(\text{const})j^2$ and a temporal dependence of q seem inconsistent.

APPENDIX D. VIRTUAL FORCES AND SUPPLIES

The constitutive theories developed in this paper utilize the second law to restrict constitutive relations, the premise being that for every choice of the electromagnetic potential and wave function the resulting "constitutive process"—as defined through the constitutive equations—must be consistent with the local dissipation inequality (which is (5.20) for electromagnetism and (12.13) for superconductivity). This method of restricting constitutive equations is generally referred to as the Coleman-Noll procedure (CNP).[34] One might argue that CNP, as applied here, is flawed, as the resulting constitutive processes will generally not satisfy the underlying balance laws. In fact, *the rational application of CNP requires virtual forces and virtual supplies that ensure satisfaction of the underlying balance laws in all constitutive processes.*[35] I now discuss a modification of the basic framework that includes such virtual fields, and show that the resulting constitutive restrictions are unchanged.

a. Electromagnetic theory

I allow for two virtual fields: a 4-vector field $\mathbf{f}$, a force,[36] and a scalar field r, a supply of electrons, with $\mathbf{f}$ and r gauge invariant; and I use these fields to rewrite the action flux (4.1) in the form

$$\mathcal{P}(\mathbf{P}) = \int_{\partial \mathbf{P}} \{\mathbf{T}\mathbf{n}\cdot\mathbf{a}^{\cdot} - \Phi\mathbf{j}\cdot\mathbf{n}\} + \int_{\mathbf{P}} \{\mathbf{f}\cdot\mathbf{a}^{\cdot} + \Phi r\} \tag{D1}$$

for all spacetime control volumes $\mathbf{P}$. For the second law I retain (4.2), but with this more general version of $\mathcal{P}(\mathbf{P})$.

The requirement that the second law be gauge invariant (or equivalently that the action flux be gauge invariant) is then equivalent to the requirement that $\mathbf{T}$ be skew and

$$\mathrm{div}_4\mathbf{T} + \mathbf{j} + \mathbf{f} = 0, \qquad \mathrm{div}_4\mathbf{j} = r. \tag{D2}$$

Further, since $\mathbf{T}$ is skew, $\mathrm{div}_4(\mathrm{div}_4\mathbf{T})=0$; hence

---

[34]Cf. Coleman and Noll [1963], where the application is thermoelasticity.

[35]This may seem artificial, but it is no more artificial than theories based on variational principles, as these require arbitrary variations of the fields, even though such variations are generally inconsistent with the resulting balance laws. CNP has the same goal as variational procedures: to ensure a properly invariant theory consistent with basic physical laws *under the widest possible set of circumstances.*

[36]In the sense that $\mathbf{f}$ performs work.

$$\mathrm{div}_4\mathfrak{f} + r = 0, \tag{D3}$$

which I consider as defining for r. Further, using the explicit representations (4.9) and (4.10) and writing $\mathfrak{f} = (f, g)$, (D2) may be written as the generalized Maxwell relations

$$\mathrm{curl}\,H = D^{\cdot} + j + f, \qquad \mathrm{div}\,D = q + g, \qquad \mathrm{div}\,j = -q^{\cdot} + r, \tag{D4}$$

where by (D3) the third equation is redundant.

The constitutive equations remain (6.1). The virtual field $\mathfrak{f}$ is not specified by a constitutive equation, but instead is allowed to be assignable in any way compatible with the balance (D2), just as the body forces and the heat supply are often left assignable in the more standard theories of mechanics and heat conduction. The balance law $(D2)_1$ is then nonrestrictive: it simply gives the virtual force $\mathfrak{f}$ needed to support any given constitutive process.

In localizing the second law one seeks to eliminate the virtual fields and obtain an inequality involving only constitutive quantities. Here the elimination of $\mathfrak{f}$ and r yields the original dissipation inequality (5.20), so the resulting theory remains unchanged.

b. Superconductivity

To treat superconductivity I augment the virtual force $\mathfrak{f}$ with a real scalar field r_n, which represents a virtual supply of normal electrons, and a complex scalar field $\mathfrak{z}$, which I interpret as a virtual microforce. I assume that $\mathfrak{f}$ and r are gauge invariant, while $\mathfrak{z}$ transforms quantum mechanically.

I rewrite the complex microforce balance in the form

$$\int_{\partial \mathbf{P}} \mathfrak{t}\cdot\mathbf{n} + \int_{\mathbf{P}} (\Pi + \mathfrak{z}) = 0 \tag{D5}$$

for every spacetime control volume $\mathbf{P}$, and arguing as in Section 11, arrive at the local balance

$$\mathfrak{X}\cdot\mathfrak{t} + \pi + \mathfrak{z} = 0 \tag{D6}$$

with π given by (11.2)

I consider the second law in the form (12.4), but with

$$\mathcal{P}(\mathbf{P}) = \int_{\partial\mathbf{P}}\{\mathbf{Tn}\cdot\mathbf{e}^{\cdot} - \Phi\mathbf{j}_n\cdot\mathbf{n}\} + \int_{\mathbf{P}}\{\mathbf{f}\cdot\mathbf{e}^{\cdot} + \Phi r_n\} +$$

$$\int_{\partial\mathbf{P}}\{(\mathbf{t}\cdot\mathbf{n})\overline{\psi}^{\cdot} + cc\} + \int_{\mathbf{P}}\{\zeta\overline{\psi}^{\cdot} + cc\}, \tag{D7}$$

for the action flux. Then

$$\mathcal{P}(\mathbf{P}) = \int_{\partial\mathbf{P}}\{\mathbf{Tn}\cdot\mathbf{e}^{\cdot} - \Phi\mathbf{j}\} + \int_{\partial\mathbf{P}}\{\mathbf{f}\cdot\mathbf{e}^{\cdot} + \Phi r\} +$$

$$\int_{\partial\mathbf{P}}\{(\mathbf{t}\cdot\mathbf{n})\overline{T\psi} + cc\} + \int_{\mathbf{P}}\{\zeta\overline{T\psi} + cc\}, \tag{D8}$$

with $\mathbf{j}$ given by (12.6) and

$$r = r_n + r_s, \qquad r_s = i(\zeta\overline{\psi} - \overline{\zeta}\psi),$$

Gauge invariance then yields (D2)-(D4), and, further, the local dissipation inequality retains its past form (12.13).

As before, the virtual fields $\mathbf{f}$, r_n, and ζ are not included in the list of constitutive equations, which remains (13.1). Given any constitutive process, the fields $\mathbf{f}$, r_n, and ζ are chosen to ensure satisfaction of the resulting balances, which are (D2) and (D6), and, again, the resulting theory remains unchanged.

Acknowledgment. I am grateful to the National Science Foundation and the Army Research Office for their support. D. Kinderlehrer suggested that superconductivity might be a fertile area for research; my interest was further aroused by an interesting seminar of K. Rubenstein. At that time I began a collaborative effort with E. Fried focused on modelling superconductivity within the continuum theory of mixtures. However, numerous commitments prevented Fried from joining me in this project.[37] I acknowledge the many discussions I have had with Fried and with P. Podio-Guidugli; their penetrating comments were instrumental in my better understanding the basic concepts underlying electromagnetism and superconductivity. Finally, I acknowledge valuable comments of A. DiCarlo and A. Marrucci during a series of lectures I gave on superconductivity at the Istituto per le Applicazioni del Calcolo in Rome.

[37] Also, I discovered that a more fruitful approach was to place the Ginzburg-Landau theory within a continuum mechanical framework, as this allows for the macroscopic characterization of the basic quantum mechanical features of superconductivity.

REFERENCES

[1913] Kamerlingh Onnes, H., Report on the Researches made in the Leiden Cryogenic Laboratory between the Second and Third International Congress of Refrigeration, *Commun. Kamerlingh Onnes Lab. Univ. Leiden,* Suppl. 34b, 55-70

[1935] London, F. and H. London, The electromagnetic equations of the supraconductor, *Proc. Roy. Soc. Lond.* **A149**, 72-87

[1950] Ginzburg, V. L. and L. D. Landau, On the theory of superconductivity, *Zh. Eksp. Teor. Fiz.* **20**, 1064-1082

[1957] Bardeen, J., Cooper, L. N., and J. R. Schrieffer, Theory of superconductivity, *Phys. Rev.* **108**, 1175-1204

[1959] Gor'kov, L. P., Microscopic derivation of the Ginzburg-Landau equations in the theory of superconductivity, *Sov. Phys. JETP,* **9**, 1364-1367

[1963] Coleman, B.D. and W. Noll, The thermodynamics of elastic materials with heat conduction and viscosity, *Arch. Rational Mech. Anal.* **13**, 245-261 (1963)

[1964] Tinkham, M., Viscous flow of flux in type-II superconductors, *Phys. Rev. Lett.* **13**, 804-807

[1965] Feynman, R. P., Leighton, R. B., and M. Sands, *The Feynman Lectures on Physics: Quantum Mechanics.* Addison-Wesley, Reading, Mass.

[1965] Truesdell, C. A.. & W. Noll, The non-linear field theories of mechanics, *Handbuch der Physik,* III/3 (ed. S. Flugge), Springer Verlag, Berlin (1965)

[1966] Schmid, A., A time dependent ginzburg-Landau equation and its application to the problem of resistivity in the mixed state, *Phys. Mat. Cond. (Phys. Kondens Materie)* **5**, 302-317

[1968] Gor'kov, L. P. and G. M. E'liashberg, Generalisation of the Ginzburg-Landau equations for nonstationary problems in the case of alloys with paramagnetic impurities, *Sov. Phys. JETP,* **27**, 328-334

[1968] Weller, W., Nonequilibrium thermodynamics of superconducting alloys in high magnetic fields, *Phys. Stat. Solidi,* **30**, 373-378

[1970] Cyrot, M., Ginzburg-Landau theory for superconductors, *Rep. Prog. Phys.* **36**, 103-158

[1971] Coleman, B. D. and E. H. Dill, On the thermodynamics of electromagnetic fields in materials with memory, *Arch. Rational Mech. Anal.* **41**, 132-162

[1979] Wang, C.-C., *Mathematical Principles of Mechanics and Electromagnetism. Part B: Electromagnetism and Gravitation,* Plenum Press, New York

[1983] Capriz, G. and P. Podio-Guidugli, Structured continua from a Lagrangian point of view, *Ann. Matem. Pura Appl.* **135**, 1-25

[1989] Capriz, G., *Continua with Microstructure. Springer Tracts in Natural Philosophy.* Springer Verlag

[1990] Park, F., *Classical Dynamics and its Quantum Mechanical Analogues,* Springer-Verlag, Berlin

[1990] Neu, J. C., Vortices in complex scalar fields, *Physica D,* **43**, 385-406

[1991] Ericksen, J. L., Liquid crystals with variable degree of orientation, *Arch. Rational Mech. Anal.* **113**, 97-120

[1991] Zhou, S.-A., *Electrodynamic Theory of Superconductors,* Peter Peregrinus, London

[1993] Zhou, S.-A. and K. Miya, A nonequilibrium theory of thermoelastic superconductors, *Int. J. Appl. Electromag. Mat.* **2**, 21-38

[1992] Chapman, S. J., S. D. Howison, and J. R. Ockendon, Macroscopic models for superconductivity, *SIAM Rev.* **34**, 9529-9560

[1992] Maugin, G. A., Irreversible thermodynamics of deformable superconductors, *C. R. Acad. Sci. Paris* **314**, 889-894

[1993] Fried, E. and M. E. Gurtin, Continuum theory of thermally induced phase transitions based on an order parameter, *Physica D,* **68**, 326-343

[1993] Yeh, C. S. and K. C. Chen, A phenomenological theory for elastic superconductors, *Continuum Mech. Thermodyn.* **5**, 127-144

[1994] E, Weinan, Dynamics of vortex liquids in Ginzburg-Landau theories with applications to superconductivity, *Phys. Rev. B,* **50**, 1126-1135

[1994] Fried, E. and M. E. Gurtin, Dynamic solid-solid phase transitions with phase characterized by an order parameter, *Physica D,* **72**, 287-308

[1994] Gurtin, M. E., Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance, *Physica D,* Forthcoming

[1995] Fried, E. and M. E. Gurtin, A phase field theory for solidification based on a general anisotropic sharp-interface theory with interfacial energy and entropy, *Physica D,* Forthcoming

[1995] Gurtin, M. E., The nature of configurational forces, *Arch. Rational Mech. Anal.* **131**, 67-100

[1995] Pismen, L. and J Rubinstein, Motion of vortex lines in the Ginzburg-Landau model, *Physica D,* Forthcoming