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Robert L. Jerrard
University of Illinois

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Robert L. Jerrard *

Department of Mathematics
University of Illinois
1409 W. Green St.
Urbana, IL 61801

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1 introduction

Consider the energy functional

$$I_U^\epsilon(u) = \int_U E^\epsilon(u)dx,$$

for functions $u : U \to \mathbf{R}^k$, where $U \subset \mathbf{R}^k$ is a bounded, open set and the energy density $E^\epsilon(u)$ is given by

$$E^\epsilon(u) = \frac{2}{k} e^\epsilon(u)^{\frac{k}{2}},$$

$$e^\epsilon(u) = \frac{|Du|^2}{2} + \frac{1}{4\epsilon^2}(|u|^2 - 1)^2.$$

When $k = 2$ this is a well-known Ginzburg-Landau functional; associated elliptic, parabolic, and Schroedinger equations occur in a variety of contexts, including simplified models of superconductivity and superfluidity. For general k, various systems of partial differential equations, such as (1.1) below, associated with the functional I^ϵ seem to be natural model problems for the study of codimension k pattern formation, see for example Jerrard and Soner [6].

A key feature in all of these systems of PDEs associated with the above functionals is the emergence of structures of codimension k characterized by energy which scales like $\ln(1/\epsilon)$ as $\epsilon \to 0$ and by nonzero degree. In the study of these problems, it is therefore essential to understand the relationship between degree and energy. Such an understanding is the main focus of this paper.

Our basic estimate, found in Theorem 2.1, is a lower bound

$$\int_{\partial B_r} E^1(u)dH^{k-1} \geq K(k)\hat{d}^{\frac{k}{k-1}} r^{-1} - Cr^{\frac{k-1}{k-2}}$$

for a Lipschitz function u with Brouwer degree $\deg(u; \partial B_r) = \hat{d}$. After integration, this gives a lower bound for the energy on an annulus, which generalizes a similar estimate for the case $k = 2$, established by Brezis, Merle, and Riviere [2].

Theorem 2.1 readily implies some specific instances of a general informal principle:

For a set U containing an isolated vortex of u, $I^\epsilon(u; U) \geq C \ln(1/\epsilon)$.

Here we use the term "vortex" to designate a zero of a function $u : \mathbf{R}^k \to \mathbf{R}^k$ about which u has nonzero degree. (The word "vortex", although widely used in problems of this sort, is probably misleading in this context, since the objects in question have little in common, in general, with fluid dynamical vortices.)

In the above statement, the condition that a vortex be in some sense isolated is essential, as a pair consisting of two sufficiently nearby vortices of equal and opposite degree need only have $O(1)$ energy. One of our goals in this paper is to develop a weak formulation of the condition that a vortex be "isolated", under which the lower bounds of the desired sort are still valid. Thus, for example, we show in Theorem 4.2 that the $\ln(1/\epsilon)$ lower bound holds for a function u on a ball B_R if the set of radii $r \leq R$ for which $\deg(u; \partial B_r) = 1$ is large. This is accomplished via a covering argument which is presented in Theorem 3.1, and which yields bounds of the desired type as easy corollaries.

A refinement of the covering argument allows us under certain circumstances not only to assert that a set contains a certain amount of energy, but also to find a point in the set around which the energy is concentrated. This topic is addressed in Section 5. These results and certain other of our lower bounds are used in Jerrard and Soner [5] in an analysis of the asymptotic dynamics of point vortices in $\mathbf{R}^2 \times [0,T)$, a problem inspired by the formal computations of Neu [9] and others.

Finally, we derive several dynamic lower bounds. For example, in Theorem 4.4 we consider a function $u^\epsilon \in C([0,T], W^{1,\infty}(U; \mathbf{R}^k))$ which initially has an isolated vortex at the origin, and we assume that the energy is uniformly bounded in an annulus surrounding the origin. Under these circumstances, the lower bound $I^\epsilon(u^\epsilon(\cdot,t)) \leq K\ln(1/\epsilon) - C$ holds for all $t \in [0,T]$, and moreover u^ϵ continues to have an isolated vortex at the origin, in an appropriate sense.

These kinds of dynamic lower bounds are extremely easy to derive from our basic estimate if one assumes, for example, that $|u^\epsilon(\cdot,t)| > 1/2$ in an annulus surrounding the origin, for all $t \in [0,T]$. The point here is that they are still valid under weaker regularity assumptions.

Section 6 is devoted to establishing a dynamic lower bound in the case where $U \subset \mathbf{R}^d$, with $d > k$, and $u^\epsilon(\cdot,0)$ has a $(d-k)$- dimensional manifold along which each cross-section has a vortex-like structure.

This general dynamic lower bound, Theorem 6.1 is used in Jerrard and Soner [6] as part of a proof that smooth codimension k mean curvature flow is approximated in a certain sense by smooth solutions $u^\epsilon : \mathbf{R}^d \to \mathbf{R}^k$ of the parabolic system

$$u_t^\epsilon - \Delta u^\epsilon - \frac{k-2}{2e_\epsilon(u^\epsilon)} De_\epsilon(u^\epsilon) \cdot Du^\epsilon + \frac{1}{\epsilon^2}(|u^\epsilon|^2 - 1)u^\epsilon = 0. \tag{1.1}$$

In this context, the equation easily implies certain energy bounds, but pointwise estimates showing that $|u^\epsilon| > 1/2$ away from the singular set seem not to be available. Thus the results of this paper are exactly what is needed to show that a vortex-like structure persists, despite the fact that one does not have enough regularity to make strong statements about the degree around the singular set.

The estimate in Brezis, Merle, and Riviere [2] has played a fundamental role in all subsequent analysis of Ginzburg-Landau vortices in 2 dimensions, as in Bethuel, Brezis, and Hélein [1], Struwe [11], and Lin [7], [8]. Our basic estimate should similarly prove useful for the study of related questions in higher dimensions. For example, the methods of Bethuel, Brezis, and Hélein [1] and

Struwe [11] can be used in combination with our estimate to prove that minimizers u^ϵ of I^ϵ with fixed smooth boundary data of degree $\hat{d} > 0$ converge along a subsequence in $W^{1,k}_{loc}(U \setminus \{a_1, \ldots, a_{\hat{d}}\})$ as $\epsilon \to 0$ to a k-harmonic map $\bar{u} \in W^{1,k}_{loc}(U \setminus \{a_1, \ldots, a_{\hat{d}}\})$. Our results here do not give much insight into the harder problem of finding a "renormalized potential" in general dimensions which would enable one to describe possible configurations of the limiting singular points $a_1, \ldots, a_{\hat{d}}$, as is done in Bethuel, Brezis, and Hélein [1] for the two-dimensional case.

We will generally use k to denote the dimension of the space in which we work. In the final section, when we consider mappings between spaces of different dimension, d will denote the dimension of the domain and k the dimension of the range. We use the notation $B_r(y) = \{x \in \mathbf{R}^k \,|\, |x - y| \le r\}$ and $B_r = B_r(0)$. Also, we write

$$A_{r_0,r_1}(y) := B_{r_1}(y) \setminus B_{r_0}(y) \qquad \text{for } r_0 < r_1.$$

The volume of the unit ball in $\mathbf{R}^k$ will be written $\alpha(k)$, so that $H^{k-1}(\partial B_1) = k\alpha(k)$.

All of our results relate to the behavior of the scaled energy density $E^\epsilon(u)$ defined above. We will not explicitly indicate the dependence of E^ϵ on u, when no confusion can result. We also use the shorthand E to mean E^1, the unscaled energy density.

We define measures

$$d\mu^\epsilon = E^\epsilon(x)dx, \qquad d\mu = E(x)dx.$$

Again, the dependence of these measures on the function u is not reflected in our notation.

We note that all of our results remain valid, with obvious modifications in the values of certain constants, for a wide class of energy densities, including for example

$$\tilde{E}^\epsilon = \frac{1}{k}|Du|^k + \frac{1}{4\epsilon^2}(|u|^2 - 1)^2.$$

We state our results for the density E^ϵ defined above because it has certain analytic properties, as shown in Jerrard and Soner [6], which make it seem a more natural object of study than, for example $\tilde{E}^\epsilon$. However, we do not use any of these properties in this paper.

We will use a definition of the Brouwer degree of a function which is equivalent to other definitions, though perhaps not as well-known. Given a function $u \in W^{1,\infty}(U; \mathbf{R}^k)$ such that $|u| > \alpha > 0$ on ∂U, we select a smooth nonnegative function $\eta : \mathbf{R}^k \to \mathbf{R}$ supported in B_α and satisfying

$$\int \eta dx = 1.$$

Then the degree of u is defined by

$$\deg(u;\partial U)=\int_U \eta(u(x))\det Du(x)dx.$$

One easily verifies that $\deg(u;\partial U)$ is independent of the specific choice of the function η. If $u=0$ at some point in ∂U then $\deg(u;\partial U)$ is undefined.

For more information on degree and related topics, see for example Fonseca and Gangbo [4].

One property of the degree which is immediately evident from our definition is that it is additive in the following sense: suppose that u is a function as described above, and that the set U is partitioned into a collection of subsets $U_1,\ldots,U_n$, such that $|u|$ is bounded away from zero on each ∂U_i. Then

$$deg(u;\partial U)=\sum_{i=1}^{n}\deg(u;\partial U_i).$$

Also, it is well known that u must have a zero in any set V in which $\deg(u;\partial V)$ is well-defined and nonzero. Thus we may refine the above identity as follows:

$$deg(u;\partial U)=\sum_{\{i|u \text{ has a zero in } U_i\}}\deg(u;\partial U_i). \tag{1.2}$$

Acknowledgments

While working on this project, I discussed these questions and many others with H.M. Soner.

2 The basic estimate

Theorem 2.1 *Suppose that $R > 1$ and that $u \in W^{1,\infty}(B_R; B_1)$.*
If $\deg(u; \partial B_r) = \hat{d}$ for some $r \in [1, R]$, then

$$\int_{\partial B_r} E\, dH^{k-1} \geq \lambda(r; k, \hat{d}).$$

where

$$\begin{aligned}\lambda(r; k, \hat{d}) &= \min_{m \in [0,1]} \left[m^k \frac{K(k)\hat{d})^{\frac{k}{k-1}}}{r} + C^{-1}(1-m)^{2k-1} \right] \\ &\geq K(k)\hat{d}^{\frac{k}{k-1}} r^{-1} - C r^{-1-\frac{1}{2k-2}}.\end{aligned}$$

and

$$K(k) = 2\alpha(k) \left(\frac{k-1}{2} \right)^{k/2} = \int_{\partial B_1} E(v) dH^{k-1}$$

for $v(x) = x/|x|$. The constant C depends only on k and $||Du||_\infty$.

Remarks.

1. We estimate $|Du|$ from below by $|D_\tau u|$, where $D_\tau u$ is tangential part of Du. Since $u|_{\partial B_r}$, the restriction of u to ∂B_r, is Lipschitz, the tangential gradient is defined H^{k-1} a.e., and $|D_\tau u| \in L^\infty(\partial B_r)$.

2. In fact, C depends only on the modulus of continuity of u. Thus the estimate still holds, with changes in C, if for example $u \in W^{1,k}(B_R) \cap C^\alpha(B_R)$ for some $\alpha > 0$.

Proof. 1. Let $m = \min_{x \in \partial B_r} |u(x)|$, and suppose that $\overline{x}$ is a point at which the minimum is attained. If $m < 1$, then

$$|u(x)| \leq \frac{1+m}{2} \qquad \text{for all } x \in B_\sigma(\overline{x}),$$

where $\sigma = \frac{1-m}{2||Du||_{L^\infty}}$. Since we have assumed that $r \geq 1$, it is clear that

$$H^{k-1}(\partial B_r \cap B_\sigma) \geq C^{-1}(1-m)^{k-1}.$$

Thus we easily estimate

$$\begin{aligned}\int_{\partial B_r} (1-|u|^2)^k dH^{k-1} &\geq C^{-1}(1-m)^k H^{k-1}(\partial B_r \cap B_\sigma) \\ &\geq C^{-1}(1-m)^{2k-1}. \qquad (2.1)\end{aligned}$$

2. In the next several steps, we show that

$$\int_{\partial B_r} |Du|^k dH^{k-1} \geq \frac{m^k}{r} K(k) \hat{d}^{\frac{k}{k-1}}.$$

By rescaling, it suffices to show that

$$\int_{\partial B_1} |Du|^k dH^{k-1} \geq m^k K(k) \hat{d}^{\frac{k}{k-1}} \tag{2.2}$$

for all $u \in W^{1,\infty}(B_1, \mathbf{R}^k)$ such that $|u| \geq m$ on ∂B_1 and $\deg(u; \partial B_1) = \hat{d}$.

In assuming that the $\deg(u; \partial B_1)$ is well-defined, we have implicitly assumed that $m > 0$. Near ∂B_1 we may thus define

$$\rho = |u|, \qquad v = \frac{u}{|u|}.$$

Clearly $\rho \geq m$ on ∂B_1, and $|Du|^2 = \rho^2 |Dv|^2 + |D\rho|^2 |v|^2$, so we have

$$|Du|^k \geq m^k |Dv|^k \qquad \text{on } \partial B_1. \tag{2.3}$$

3. We further define a function $\hat{v} : \mathbf{R}^k \to B^1$ by

$$\hat{v}(x) = |x| v(\frac{x}{|x|}).$$

We now claim that $\deg(\hat{v}; \partial B_1) = \deg(u; \partial B_1) = \hat{d}$.

Indeed, this follows from the basic properties of degree. In particular, $\deg(\cdot; \partial U)$ is constant under homotopies $H : U \times [0,1] \to \mathbf{R}^k$, as long as $0 \notin H(\partial U, s)$ for every $s \in [0,1]$. Let

$$H(x,s) = (1-s)u(x) + s\hat{v}(x).$$

Then for $x \in \partial B_1$, $s \in [0,1]$ we have

$$H(x,s) = (1-s)u(x) + \frac{s}{|u(x)|} u(x),$$

so that $|H(x,s)| = |u(x)|(1-s) + s \geq m > 0$. It follows that $\deg(H(\cdot,s); \partial B_1)$ is constant for $s \in [0,1]$. This verifies the claim.

4. We now fix a smooth positive function η with integral 1, supported in $B_{1/2}$. We may assume that $\eta(x) = \eta(|x|)$. We have

$$|\hat{v}(x)| = |x|, \qquad D\hat{v}(x) = D\hat{v}(\frac{x}{|x|})$$

for a.e. $x \in B_1$. Thus

$$\begin{aligned}
\hat{d} &= \int_{B_1} \det D\hat{v}(x)\eta(\hat{v}(x))dx \\
&= \int_{B_1} \det D\hat{v}(\frac{x}{|x|})\eta(|x|)dx \\
&= \int_0^1 \eta(r)\int_{\partial B_r} \det D\hat{v}(\frac{x}{r})dH^{k-1}(x)dr \\
&= \int_0^1 \eta(r)r^{k-1}\int_{\partial B_1} \det D\hat{v}(x)dH^{k-1}(x)dr \\
&= \frac{1}{k\alpha(k)}\int_{\partial B_1} \det D\hat{v}(x)dH^{k-1}(x).
\end{aligned}$$

5. Let $J\hat{v}$ denote the Jacobian of $\hat{v}$, that is, $J\hat{v} = (\det D\hat{v}\ D\hat{v}^T)^{1/2}$.

Also, for $x \in \partial B_1$ let $\nu(x) = x/|x|$, the outward unit normal at x, and let the matrix $P(x)$ be the projection onto the tangent space at x, i.e. $P = \mathrm{id} - \nu \otimes \nu$.

We may now define the tangential gradient of $\hat{v}$, $D_\tau\hat{v} = PD\hat{v}P$.

Let $\lambda_1(x), \dots \lambda_{k-1}(x)$ be the eigenvalues of $D_\tau\hat{v}(x)$ corresponding to eigenvectors in the plane $\{y|y \cdot \nu(x) = 0\}$. Then the Jacobian of $\hat{v}$ on the manifold ∂B_1 is given by $J_\tau\hat{v} = |\lambda_1 \dots \lambda_{k-1}|$ (see for example Simon [10], Chapter 2).

We claim that

$$\det D\hat{v} \leq J\hat{v} = J_\tau\hat{v} \leq (k-1)^{-\frac{k-1}{2}}|D_\tau v|^{k-1}$$

on ∂B_1.

The first inequality is obvious. To see the second, we compute the gradient of $\hat{v}$ at a point $x \in \partial B_1$ to find that

$$D\hat{v}(x) = \nu(x) \otimes \hat{v}(x) + P(x)Dv(x).$$

Since $|v| = 1$, we have

$$D\hat{v}D\hat{v}^T = \nu \otimes \nu + PDvDv^TP.$$

Because $P\nu = 0$, we see that ν is an eigenvector of $D\hat{v}D\hat{v}^T$ with eigenvalue 1. The remaining $k-1$ eigenvalues are precisely $\lambda_1^2, \dots \lambda_{k-1}^2$, where the $\{\lambda_i\}_{i=1}^{k-1}$ are the eigenvalues of $D_\tau\hat{v}$ identified above. Thus

$$(J\hat{v})^2 = \det(D\hat{v}D\hat{v}^T) = \lambda_1^2 \times \dots \times \lambda_{k-1}^2 = (J_\tau\hat{v})^2.$$

Since $\hat{v} = v$ on ∂B_1, the final inequality in our claim is just the inequality of the arithmetic and geometric means:

$$(\lambda_1^2 \times \dots \times \lambda_{k-1}^2)^{1/k-1} \leq \frac{1}{k-1}(\lambda_1^2 + \dots + \lambda_{k-1}^2).$$

6. Combining Steps 4 and 5 and using Holder's inequality we obtain

$$\begin{aligned} k\alpha(k)(k-1)^{\frac{k-1}{2}}\hat{d} &\le \int_{\partial B_1} |D_\tau v|^{(k-1)} dH^{k-1} \\ &\le \left(\int_{\partial B_1} |D_\tau v|^k dH^{k-1}\right)^{frack-1k} (k\alpha(k))^{\frac{1}{k}}. \end{aligned}$$

This inequality together with (2.3) establishes the estimate (2.2), after rescaling to a ball of radius r.

7. Combining estimates (2.1) and (2.2) we see that

$$\begin{aligned} \int_{\partial B_r} E(u) dH^{k-1} &\ge \int_{\partial B_r} 2^{1-\frac{k}{2}} \frac{|Du|^k}{k} + \frac{1}{2^k}(|u|^2-1)^k dH^{k-1} \\ &\ge \min_{m\in[0,1]} \left[m^k \frac{K(k)\hat{d}^{\frac{k}{k-1}}}{r} + C^{-1}(1-m)^{2k-1} \right] \\ &:= \lambda(r;k,\hat{d}). \end{aligned}$$

where the constant C depends only on the dimension and on $||Du||_\infty$. One easily verifies by calculus that

$$\lambda(r;k,\hat{d}) \ge K(k)\hat{d}^{\frac{k}{k-1}} r^{-1} - Cr^{-1-\frac{1}{2k-2}}.$$

□

We deduce the following

Corollary 2.1 *Suppose that $u^\epsilon \in W^{1,\infty}(B_R;B_1)$ with $|Du^\epsilon| \le \kappa/\epsilon$. If $|u^\epsilon| > 0$ in the annulus $A_{\epsilon,R}$ and $\deg(u^\epsilon;\partial B_R) = \hat{d}$, then*

$$\mu^\epsilon(B_R) \ge K(k)\hat{d}^{\frac{k}{k-1}} \ln(R/\epsilon) - C(\kappa,k).$$

Proof. We define $\tilde{u}(x) := u^\epsilon(\epsilon x)$ on $B_{R/\epsilon}$, so that $\deg(\tilde{u};\partial B_r) = \hat{d}$ for all $r \in [1, R/\epsilon]$ and $|D\tilde{u}| \le \kappa$. We may then employ a change of variables and Theorem 2.1 to deduce that

$$\begin{aligned} \int_{B_R} E^\epsilon(u^\epsilon) dx &= \int_{B_{R/\epsilon}} E(\tilde{u}) dx \\ &\ge \int_1^{R/\epsilon} \int_{\partial B_r} E(\tilde{u}) dH^{k-1} dr \\ &\ge K(k)\hat{d}^{k-1/k} \ln(\frac{R}{\epsilon}) - C(k,\kappa). \end{aligned}$$

□

It is clear that the argument given above will yield analagous bounds for a variety of functionals with arbitrary growth rates $p \geq k-1$. (If $p < k-1$ then we can no longer use Holder's inequality as in Step 6 of the proof.) We record two of these results here, as they might be of some interest.

First, if $\deg(u;\partial B_r) = \hat{d}$ and $||Du||_\infty \leq \overline{C}$, then

$$\int_{\partial B_r} \frac{2}{p} e(u)^{p/2} dH^{k-1} \geq \lambda(r;k,\hat{d},p),$$

where

$$\begin{aligned}\lambda(r;k,\hat{d},p) &= \inf_{m\in[0,1]} \left[m^p \frac{K(k,p)\hat{d}^{\frac{p}{k-1}}}{r^{p+1-k}} + C^{-1}(1-m)^{k+p-1} \right] \\ &\geq K(k,p)\hat{d}^{\frac{p}{k-1}} r^{k-p-1} - C r^{(k-p-1)\frac{k+p-1}{k+p-2}},\end{aligned}$$

Secondly, note that our arguments apply also to functions which are constrained to take values in the unit sphere. In this case, in fact, the estimates are a little easier. So for example, if $u \in W^{1,p}(U;S^{k-1})$, where U is a neighborhood of ∂B_r, and $\deg(u;\partial B_r) = \hat{d}$, then

$$\int_{\partial B_r} \frac{|Du|^p}{p} dH^{k-1} \geq \tilde{K}(k,p)\hat{d}^{\frac{p}{k-1}} r^{k-p-1}.$$

Both $K(k,p)$ and $\tilde{K}(k,p)$ above are exactly given by

$$\int_{\partial B_1} E(v) dH^{k-1}$$

where E is the appropriate energy density and $v(x) = x/|x|$. (Thus in particular the estimates are always sharp if $\hat{d} = 1$.)

3 A covering argument

In this section we present a covering argument which can be used to deduce powerful and flexible lower bounds directly from the results of the previous section. We start with a fact which is both elementary and obvious, but which we state as a separate lemma, as we will refer to it several times.

Lemma 3.1 (Amalgamation) *Given any finite collection of closed balls in* $\mathbf{R}^k$, *say* $\{B_i\}_{i=1}^N$, *we can find a collection* $\{\tilde{B}_i\}_{i=1}^{\tilde{N}}$ *of pairwise disjoint balls such that*

$$\bigcup_{i=1}^{N} B_i \subset \bigcup_{i=1}^{\tilde{N}} \tilde{B}_i,$$

$$\sum_{B_j \subset \tilde{B}_i} \mathit{diam} B_j = \mathit{diam} \tilde{B}_i, \qquad \text{and} \tag{3.1}$$

$$\tilde{N} \leq N, \quad \text{with strict inequality unless } \{B_i\}_{i=1}^N \text{ is pairwise disjoint.}$$

Proof. Replace pairs of intersecting balls B_i, B_j by larger single balls $\tilde{B}$ such that $\text{diam}\tilde{B} = \text{diam}B_1 + \text{diam}B_2$, continuing until a pairwise disjoint collection is reached. This collection has the stated properties. □

In order to state the following theorem, we introduce some notation. Fix a constant $C^* = C^*(||Du||_\infty)$ such that

$$\mu(B_1(x)) \geq 4C^* \qquad \text{whenever } u(x) = 0. \tag{3.2}$$

Such a constant exists, because if $|u(x)| = 0$ then $|u(x)| \leq 1/2$ on a ball of radius $1/(2||Du||_\infty)$. We now define

$$\Lambda(r) = \int_0^r C^* \wedge \lambda(s; k, 1) ds.$$

From the definition we immediately see that $\lambda(\cdot; k, 1)$ is positive and nonincreasing, so the same clearly holds for $C^* \wedge \lambda$. If follows that

$$\Lambda(\cdot) \quad \text{is subadditive and increasing.}$$

It is also clear that

$$\Lambda(r) \geq K(k) \ln r - C(||Du||_\infty, d). \tag{3.3}$$

By integrating the conclusion of Theorem 2.1 as in Corollary 2.1 we see that

Lemma 3.2 *Suppose that* $1 \leq R_0 \leq R_1$*, and that* $u \in W^{1,\infty}(B_{R_1}; B_1)$.
If $|\deg(u; \partial B_r)| \geq 1$ *for all* $r \in [R_0, R_1]$ *then*

$$\mu(A_{R_0,R_1}) \geq \Lambda(R_1) - \Lambda(R_0).$$

□

The covering argument given here will immediately yield a variety of lower energy bounds.

Theorem 3.1 *Let* $U \subset \mathbf{R}^k$*. Suppose that* $u \in W^{1,\infty}(U; \mathbf{R}^k)$*. Then we can find a collection of balls* $\{B_i\}_{i=1}^M$ *with radii* $r_i \geq 1$ *such that*

$$\{x \in U | u(x) = 0\} \subset\subset \bigcup_{i=1}^{M} B_i, \tag{3.4}$$

$$\mu(B_i \cap U) \geq \Lambda(r_i); \tag{3.5}$$

and

$$\deg(u; \partial B_i) = 0 \qquad \text{for all } i \text{ such that } B_i \subset U. \tag{3.6}$$

Also, the interiors of the balls are pairwise disjoint.

Remarks.

1. Note that $\deg(u; \partial(B_i))$ is well-defined for each i such that $B_i \subset\subset U$ as a consequence of (3.4).
2. The definition of Λ contains constants which depend on k, $||Du||_\infty$, so the lower energy estimate in (3.5) depends on these quantities.
3. By rescaling as in the proof of Corollary 2.1, we find that if u^ϵ is a function satisfying $||Du^\epsilon||_\infty \leq \kappa/\epsilon$, and if the above hypotheses are otherwise unchanged, then we may find a collection of balls $\{B_i\}$ with pairwise disjoint interiors, each having radius $r_i \geq \epsilon$, satisfying (3.4), (3.6), and

$$\mu^\epsilon(B_i \cap U) \geq \Lambda^\epsilon(r_i) \geq K(k) \ln(r_i/\epsilon) - C(\kappa, k).$$

Here $\Lambda^\epsilon(r) := \Lambda(r/\epsilon)$. We will typically apply Theorem 3.1 in this scaled form.

Proof. If $u(x) \neq 0$ in U, then the empty collection of balls has the stated properties. So we assume that $u(x) = 0$ somewhere in U.

1. Let $Z = \bigcup_{\{x \in U | u(x)=0\}} B_1(x)$. Each component of Z has diameter at least 2, so Z has a finite number of components, say $Z_1, \ldots, Z_{M_0}$. For each $i = 1, \ldots, M_0$, let $\rho_i = (1/2)\text{diam} Z_i$.

We now claim that

$$\mu(Z_i) \geq \Lambda(2\rho_i).$$

Let $N = [\rho_i/2] + 1 \geq \rho_i/2$.

Then, since Z_i is connected and $\text{diam} Z_i = 2\rho_i \geq 4(N-1)$, we can find points $\hat{x}_1, \ldots, \hat{x}_N$ in Z_i such that $|\hat{x}_j - \hat{x}_k| \geq 4$ if $j \neq k$.

For each $j = 1, \ldots, N$, let y_j be a point such that

$$u(y_j) = 0, \qquad \hat{x}_j \in B_1(y_j) \subset Z_i.$$

Such points exist by the definition of Z_i.

If z is a point such that

$$|z - y_j| < 1 \quad \text{and} \quad |z - y_k| \leq 1$$

for some $k \neq j$, then the triangle inequality implies that $|\hat{x}_j - \hat{x}_k| < 4$, which is impossible. Thus $B_1(y_j) \cap B_1(y_k)$ consists of at most one point, and so

$$\begin{aligned} \mu(Z_i) &\geq \sum_{j=1}^{N} \mu(B_1(y_j)) \\ &\geq 4NC^* && \text{by (3.2),} \\ &\geq 2\rho_i C^* && \text{by the definition of N.} \end{aligned}$$

2. For $i = 1, \ldots, M_0$, let B_i^0 be the smallest ball containing Z_i. The radius r_i^0 of B_i^0 evidently satisfies

$$r_i^0 \leq \text{diam} Z_i = 2\rho_i \leq \mu(Z_i)/C^*.$$

Applying the amalgamation lemma to this collection, we obtain a pairwise disjoint collection $\{B_i^1\}_{i=1}^{M_1}$ such that

$$Z \subset \bigcup_{i=1}^{M_1} B_i^1$$

and for each $j \in \{1, \ldots, M_1\}$,

$$\sum_{B_i^0 \subset B_j^1} r_i^0 = r_j^1,$$

where r_j^1 is the radius of the ball B_j^1. Thus

$$\begin{aligned} \mu(B_j^1 \cap U) &\geq \sum_{Z_i \subset B_j^1} \mu(Z_i) \\ &\geq \sum_{Z_i \subset B_j^1} C^* r_i^0 \\ &= C^* r_j^1. \end{aligned} \tag{3.7}$$

Thus (3.5) holds, by the definition of Λ.

Clearly the collection $\{B_i^1\}$ satisfies (3.4). If it also satisfies (3.6) then we are finished.

3. If not, there is at least one ball, say $B_1^1 = B_{r_1^1}(x_1^1)$, such that $B_1^1 \subset U$ and $|\deg(u; \partial B_1^1)| \geq 1$. Let

$$\tilde{r} = \text{dist}(x_1^1, \partial U) \wedge \text{dist}(x_1^1, B_2^1) \wedge \ldots \wedge \text{dist}(x_1^1, B_{M_1}^1).$$

Then $|u| > 0$ in the interior of the annulus $A_{r_1^1, \tilde{r}}(x_1^1)$, so that by Lemma 3.2 we deduce that

$$\begin{aligned} \mu(B_{\tilde{r}}(x_1^1)) &= \mu(A_{r_1^1, \tilde{r}}) + \mu(B_1^1) \\ &\geq \Lambda(\tilde{r}) - \Lambda(r_1^1) + \Lambda(r_1^1) \quad = \quad \Lambda(\tilde{r}). \end{aligned}$$

Thus the collection

$$\{B_i^2\}_{i=1}^{M_1} \equiv \{B_{\tilde{r}}(x_1^1), B_2^1, \ldots, B_{M_1}^1\}$$

satisfies (3.5). It is clear that it also satisfies (3.4).

By our choice of $\tilde{r}$, $B_{\tilde{r}}(x_1^1)$ intersects either ∂U or ∂B_j^1 for some j.

If the former holds and (3.6) is satisfied, we are finished. If (3.6) does not hold but $B_{\tilde{r}}(x_1^1) \cap \partial U \neq \emptyset$, we may find another ball, say $B_2^1 \subset U$, on which u has nonzero degree, and repeat the above step.

4. Otherwise, we apply the amalgamation lemma to the collection $\{B_i^2\}_{i=1}^{M_1}$ to find a pairwise disjoint collection of balls $\{B_i^3\}_{i=1}^{M_3}$ with $M_3 < M_1$, satisfying (3.4) and (3.5). We then have

$$\begin{aligned} \mu(B_j^3 \cap U) &\geq \sum_{B_i^2 \subset B_j^3} \mu(B_i^2 \cap U) \\ &\geq \sum_{B_i^2 \subset B_j^3} \Lambda(r_i^2) \quad \geq \quad \Lambda(r_j^3), \end{aligned}$$

where we have used (3.1) and the subadditivity of Λ in the final inequality.

5. We continue in this fashion, expanding and amalgamating the balls as necessary. All of the balls obtained, both after an expansion step and an amalgamation step, satisfy (3.4) and (3.5). The process can continue as long as (3.6) remains unfulfilled. On the other hand, the process must eventually stop, as each expansion step either decreases the number of balls violating (3.6) or else leads to an amalgamation step, which then decreases the total number of balls. Since we started with a finite number of balls, (3.6) must eventually be satisfied. □

4 Some lower energy bounds

The estimates presented in this section are for the most part easy consequences of the results of the prevous section. It is convenient for applications to state these in the scaled form, as discussed in Remark 3 following the statement of Theorem 3.1.

Theorem 4.1 *Let $U \subset \mathbf{R}^k$. Suppose that $u^\epsilon \in W^{1,\infty}(U;\mathbf{R}^k)$ satisfies*

$$\deg(u^\epsilon;\partial U) \neq 0, \qquad ||Du^\epsilon|| \leq \kappa/\epsilon,$$

and

$$|u^\epsilon| > 0 \quad in \ \{x \in U | dist(x, \partial U) \leq R\}. \tag{4.1}$$

Then there is a constant $C = C(\kappa, k)$ such that

$$\mu^\epsilon(U) \geq K(k) \ln\left(\frac{R}{\epsilon}\right) - C(\kappa, k).$$

Proof. Let $\{B_i\}_{i=1}^M$ be the collection of balls found in Theorem 3.1. Since the degree of u^ϵ is nonzero, the set $\{u^\epsilon = 0\}$ must be nonempty, and hence the collection of balls is nonempty. Moreover, it is evident that at least one ball, say B_1, must have nonzero degree and must therefore intersect ∂U. Because $\deg(u^\epsilon; \partial(B_1 \cap U)) \neq 0$, u^ϵ must have a zero in $B_1 \cap U$. Now (4.1) guarantees that B_1 has radius at least $R/2$.

We immediately conclude that

$$\mu^\epsilon(U) \geq \Lambda(R/2\epsilon) \geq K(k) \ln\left(\frac{R}{\epsilon}\right) - C(\kappa, k).$$

□

The above theorem tells us nothing about a situation in which the function u^ϵ has a single isolated vortex away from ∂U but also has zeroes near ∂U. Such a situation is covered in the following

Theorem 4.2 *Suppose that $u^\epsilon \in W^{1,\infty}(B_{R_1};\mathbf{R}^k)$ with $||Du^\epsilon||_\infty \leq \kappa/\epsilon$, and that*

$$leb^1(\{r \in [0, R_1] | \deg(u^\epsilon; \partial B_r) = 1\}) \geq R$$

for some $R < R_1$, where leb^1 denotes 1-dimensional lebesgue measure. Then

$$\mu^\epsilon(B_{R_1}) \geq K(k) \ln R/\epsilon - C(\kappa, k).$$

Proof. 1. Let $\{B_i\}_{i=1}^M$ be the collection of balls found in Theorem 3.1, which as above must be nonempty, and let r_i denote the radius of B_i.

We claim first that

$$\{r \in [0, R_1] | \deg(u^\epsilon; \partial B_r) = 1\} \subset \bigcup_i \{r \in [0, R_1] | \partial B_r \cap B_i \neq \emptyset\}$$

Indeed, let $\rho \in [0, R_1]$ be a radius for which $\deg(u^\epsilon; \partial B_\rho) = 1$. We need to show that

$$B_i \cap \partial B_\rho \neq \emptyset$$

for some $i \in \{1, \ldots, M\}$. Indeed, if not then (3.4) implies that $|u^\epsilon| > 0$ on ∂B_ρ, and

$$\deg(u^\epsilon; \partial B_\rho) = \sum_{B_i \subset B_\rho} \deg(u^\epsilon; \partial B_i) = 0,$$

again using (3.4) and (1.2). As this is impossible, our claim is verified.

2. We now have

$$\begin{aligned} R &\leq \text{leb}^1(\{r \in [0, R_1] | \deg(u^\epsilon; \partial B_r) = 1\}) \\ &\leq \text{leb}^1(\bigcup_i \{r \in [0, R_1] | \partial B_r \cap B_i \neq \emptyset\}) \\ &\leq \sum_i \text{leb}^1(\{r \in [0, R_1] | \partial B_r \cap B_i \neq \emptyset\}) \\ &\leq \sum_i 2r_i. \end{aligned}$$

Thus, using as usual the subadditivity of Λ^ϵ,

$$\mu^\epsilon(U) \geq \sum_1^M \mu^\epsilon(B_i \cap U) \geq \sum_1^M \Lambda^\epsilon(r_i) \geq \Lambda^\epsilon(R/2) = \Lambda(R/2\epsilon).$$

□

The next estimate is in very much the same spirit.

Theorem 4.3 *Suppose that* $u^\epsilon \in W^{1,\infty}(B_{R_1}; \mathbf{R}^k)$ *with* $\|Du^\epsilon\|_\infty \leq \kappa/\epsilon$, *and that for some* $R_0 = R_1 - R < R_1$ *and* $\alpha > 0$,

$$leb^1(\{r \in [R_0, R_1] | \deg(u^\epsilon; \partial B_r) = m\}) \leq (1 - \alpha)R$$

holds for every integer m. *Then*

$$\mu^\epsilon(A_{R_0,R_1}) \geq K(k) \ln\left(\frac{\alpha R}{\epsilon}\right) - C(\kappa, k).$$

Proof. Apply Theorem 3.1 to the annulus A_{R_0,R_1} to find a collection of balls $\{B_i\}_{i=1}^M$ with radii r_i.

1. We first claim that $\deg(u^\epsilon;\partial B_r)$ is well-defined and constant on the set

$$S \equiv \{r \in [R_0, R_1] | \partial B_r \cap B_i = \emptyset \quad \text{for all } i = 1, \dots, M\}.$$

It is clear that the degree is well-defined in this set, since u^ϵ is positive outside of the balls $\{B_i\}$.

Select two radii $r_1 < r_2$ in set S. Using as before the additivity of the degree and (3.6), we have

$$\begin{aligned}\deg(u^\epsilon;\partial B_{r_1}) - \deg(u^\epsilon;\partial B_{r_0}) &= \deg(u^\epsilon;\partial A_{r_0,r_1}) \\ &= \sum_{B_i \subset A_{r_0,r_1}} \deg(u^\epsilon;\partial B_i) \\ &= 0.\end{aligned}$$

This is our claim.

2. From Step 1 and the hypothesis it follows that

$$\text{leb}^1(S) \le (1-\alpha)R.$$

We may then argue as before that

$$\begin{aligned}\alpha R &\le \text{leb}^1([R_0, R_1] \setminus S) \\ &\le \sum_i \text{leb}^1(\{r \in [R_0, R_1] | \partial B_r \cap B_i \ne \emptyset\}) \\ &\le \sum_i 2r_i.\end{aligned}$$

Thus

$$\mu^\epsilon(A_{R_0,R_1}) \ge \sum_i \Lambda^\epsilon(r_i) \ge \Lambda^\epsilon(\alpha R) = \Lambda(\alpha R/2\epsilon).$$

□

Finally we present a dynamic lower bound. This proof is made very easy by the availability of a measure-theoretic notion of an "isolated vortex" from Theorems 4.2 and 4.3.

Let $B_1 = \{x \in \mathbf{R}^k | |x| \le 1\}$ as usual, and let $Q = B_1 \times [0, T]$.

Theorem 4.4 *Let $u^\epsilon \in C([0,T]; W^{1,\infty}(B_1))$ satisfy*

$$||Du^\epsilon||_{L^\infty(Q)} \le \kappa/\epsilon,$$

$$\deg(u^\epsilon(\cdot;0);\partial B_r) = 1 \qquad \textit{for all } r \in [1/2, 1],$$

$$\mu_t^\epsilon(A_{1/2,1}) \le (1-\alpha)K(k)\ln(1/\epsilon) + C$$

for some $\alpha > 0$ *and all* $t \in [0, T]$. *Then for all* ϵ *sufficiently small and for all* $t \in [0, T]$ *we have*

$$leb^1(\{r \in [1/2, 1] | \deg(u^\epsilon(\cdot, t); \partial B_r) = 1\}) \geq 1/4 \tag{4.2}$$

and

$$\mu_t^\epsilon(B_1) \geq K(k)\ln(1/\epsilon) - C(\kappa, k).$$

Proof. By Corollary 4.2, the estimate (4.2) implies the other conclusion of the theorem, so it suffices to prove (4.2).

Suppose that (4.2) is false, and let $t_0 \in (0, 1]$ be the infimum of the set of times $t \in [0, T]$ for which

$$\text{leb}^1(\{r \in [1/2, 1] | \deg(u^\epsilon(\cdot, t); \partial B_r) = 1\}) < 1/4.$$

We claim that

$$\text{leb}^1(\{r \in [1/2, 1] | \deg(u^\epsilon(\cdot, t_0); \partial B_r) = i\}) < 1/4,$$

for every integer i. Indeed, fix a radius $r \in [1/2, 1]$ for which $|u^\epsilon(\cdot, t_0)| > 0$ on ∂B_r, so that $\deg(u^\epsilon(\cdot, t_0); \partial B_r)$ is defined. By continuity it follows that $|u^\epsilon(\cdot, t)| > 0$ on ∂B_r for all t close to t_0, and hence that $\deg(u^\epsilon(\cdot, t_0); \partial B_r)$ is constant for t near t_0. Thus for every integer i we have

$$\{r | \deg(u^\epsilon(\cdot, t_0); \partial B_r) = i\} \subset \bigcap_{\delta > 0} \bigcup_{|t - t_0| < \delta} \{r | \deg(u^\epsilon(\cdot, t); \partial B_r) = i\}.$$

This readily implies that the function

$$t \mapsto \text{leb}^1(\{r \in [1/2, 1] | \deg(u^\epsilon(\cdot, t); \partial B_r) = i\}) \quad \text{is lower semicontinuous.} \tag{4.3}$$

With the definition of t_0, this implies the claim.

From the claim and Corollary 4.3 we deduce that

$$\mu_t^\epsilon(A_{1/2,1}) \geq K(k)\ln(1/\epsilon) - C,$$

which contradicts the hypothesized upper bound. Thus (4.2) holds. □

5 Localization

In this section we prove a refinement of our covering theorem, which can be used to show, not only that a set contains a certain amount of energy, but also that one can find small sets on which that energy is concentrated.

Lemma 5.1 *Let $U \subset \mathbf{R}^k$ and suppose that $u \in W^{1,\infty}(U;\mathbf{R}^k)$. Let $\{B_i\}_{i=1}^M$ be the collection of balls found in Theorem 3.1, and let $\tilde{r} = \max_i r_i$.*

Then we can find a point $x^* \in U$ such that $u(x^*) = 0$ and

$$\mu(B_\sigma(x^*) \cap U) \geq \Lambda(\sigma/16)$$

for every $\sigma \in [1, 4\tilde{r}]$.

Before we give the proof of the lemma in the proof of Theorem 3.1, we illustrate its utility by noting some consequences. As before, we state these estimates in the scaled form.

Theorem 5.1 *Under the hypotheses of Theorem 4.1, there exists a point $x^* \in \{x \in U | dist(u, \partial U \geq R)\}$ such that $u^\epsilon(x^*) = 0$ and for every $\sigma \in [\epsilon, R]$,*

$$\mu^\epsilon(B_\sigma(x^*)) \geq K(k;1) \ln\left(\frac{\sigma}{\epsilon}\right) - C(\kappa, k). \tag{5.1}$$

Proof. Let $\{B_i\}_{i=1}^M$ be the collection of balls found in Theorem 3.1. We have shown in the proof of Theorem 4.1 that at least one ball, say B_1, has radius at least $R/2$.

Now using Lemma 5.1 we find a point $x^* \in U$ satisfying (5.1), and such that $u^\epsilon(x^*) = 0$. The latter identity and (4.1) imply that $\text{dist}(x^*, \partial U) \geq R$. □

Theorem 5.2 *Suppose that u^ϵ satisfies the hypotheses of Theorem 4.2 and that, in addition,*

$$\mu^\epsilon(B_{R_1}) \leq K(k) \ln(\frac{R}{\epsilon}) + C.$$

Then we may find a point $x^* \in B_{R_1}$ such that $u(x^*) = 0$ and

$$\mu^\epsilon(B_\sigma(x^*) \cap U) \geq \ln\left(\frac{\sigma}{\epsilon}\right) - C(\kappa, k)$$

for every $\sigma \in [\epsilon, R/2]$.

Proof. 1. Let $\{B_i\}_{i=1}^M$ be the collection of balls found in Theorem 3.1, and let r_i denote the radius of B_i. We have shown in the proof of Theorem 4.2 that $\sum_i r_i \geq R/2$.

We claim that the new hypothesis implies that $r_i \geq R/8$ for some i. If not, we may assume that $r_1 + \ldots + r_m \geq R/8$ and $r_{m+1} + \ldots + r_M \geq R/8$ for some $m \in \{1, \ldots, M\}$. Then

$$\begin{aligned} \mu^\epsilon(U) \geq \sum_1^m \Lambda^\epsilon(r_i) + \sum_{m+1}^M \Lambda^\epsilon(r_i) &\geq 2\Lambda^\epsilon(R/4) \\ &\geq 2K(k;1)\ln(R/4\epsilon) - C, \end{aligned}$$

which is impossible if ϵ is small enough.

Now the desired conclusion follows from Lemma 5.1. □

Remark. It is clear that we can similarly strengthen the other lower bounds which are based on Theorem 3.1.

Proof of Lemma 5.1. We use the notation from the proof of Theorem 3.1, so that r_j^i and x_j^i will denote the radius and the center, respectively, of the ball B_j^i.

1. First we establish the following

Claim: for any $\sigma \in [0, \bar{r}]$, we can find a point $x(\sigma)$ such that

$$\mu(B_\sigma(x(\sigma))\,) \geq \Lambda(\sigma/2). \tag{5.2}$$

and

$$u(x) = 0 \qquad \text{for some } x \in B_\sigma(x(\sigma)). \tag{5.3}$$

As in Step 1 of the proof of Theorem 3.1, we start by constructing the set

$$Z = \bigcup_{\{x \in U | u(x) = 0\}} B_1(x).$$

with connected components $\{Z_i\}_{i=1}^M$,

If diam$Z_i \geq \sigma$ for some i, take $x(\sigma)$ such that diam$(B_\sigma(x(\sigma)) \cap Z_i) \geq \sigma$. Exactly as in Step 1 of the proof of Theorem 3.1, we may estimate $\mu(B_\sigma(x(\sigma)) \cap Z_i$ by finding a sufficiently large number of disjoint balls of radius 1 contained in the given set. This gives

$$\begin{aligned} \mu(B_\sigma(x(\sigma))) &\geq \mu(B_\sigma(x(\sigma)) \cap Z_i) \\ &\geq C^* \mathrm{diam}(B_\sigma(x(\sigma)) \cap Z_i) \quad \geq \quad \Lambda(\sigma/2), \end{aligned}$$

which is (5.2). It is clear that in this case (5.3) holds.

2. If diam$Z_i < \sigma$ for all i, let B_i^0 be the smallest ball containing Z_i. We now proceed as in the proof of the previous theorem, alternately amalgamating and expanding balls to obtain collections which at each level satisfy (3.4) and (3.5). (In all of the following cases, the ball $B_\sigma(x(\sigma))$ that we construct must contain

some ball B_j^0, so that (5.3) will hold when we are finished.) Since by hypothesis this procedure eventually yields a ball of radius greater than σ, we may define

$$\bar{i} = \inf\{i \geq 1 | r_j^i \geq \sigma/2 \text{ for some } j = 1, \dots, M_i\}.$$

If $r_j^{\bar{i}} \in [\sigma/2, \sigma]$ for some j, then for $x(\sigma) = x_j^{\bar{i}}$ we have

$$\mu(B_\sigma(x(\sigma)) \geq \mu(B_j^{\bar{i}}) \geq \Lambda(r_j^{\bar{i}}) \geq \Lambda(\sigma/2).$$

3. Otherwise, we consider two possibilities.

Case 1: The collection $\{B_j^{\bar{i}}\}$ was obtained from $\{B_j^{\bar{i}-1}\}$ by expansion;

Case 2: The collection $\{B_j^{\bar{i}}\}$ was obtained from $\{B_j^{\bar{i}-1}\}$ by amalgamation.

If Case 1 holds, then (recalling the proof of Theorem 3.1) there is some ball, say $B_1^{\bar{i}}$, such that

$$B_1^{\bar{i}} = B_1^{\bar{i}-1} \cup A_{r_1^{\bar{i}-1}, r_1^{\bar{i}}}(x_1^{\bar{i}}),$$

with u satisfying the hypotheses of Lemma 3.2 on the annulus $A_{r_1^{\bar{i}-1}, r_1^{\bar{i}}}(x_1^{\bar{i}})$ and hence also on the smaller annulus $A_{r_1^{\bar{i}-1}, \sigma}(x_1^{\bar{i}})$. Thus

$$\mu(B_\sigma(x_1^{\bar{i}})) = \mu(B_1^{\bar{i}-1}) + \mu(A_{r_1^{\bar{i}-1}, \sigma}(x_1^{\bar{i}})) \geq \Lambda(\sigma).$$

So the claim holds with $x(\sigma) = x_1^{\bar{i}}$.

4. We now consider Case 2. In this case we can find a ball, say $B_1^{\bar{i}}$, which was formed by amalgamating balls, say $B_j^{\bar{i}-1}$, $j = 1, \dots, m$ for some integer m. We recall how the amalgamation procedure works: we may assume that

$$B_1^{\bar{i}-1} \cap B_2^{\bar{i}-1} \neq \emptyset.$$

The first Step in the procedure is to select a ball $\hat{B}_2$ containing $B_1^{\bar{i}-1} \cup B_2^{\bar{i}-1}$ and with radius $\hat{r}_2 = r_1^{\bar{i}-1} + r_2^{\bar{i}-1}$. Without loss of generality,

$$\hat{B}_2 \cap B_3^{\bar{i}-1} \neq \emptyset,$$

and so we select a ball $\hat{B}^3 \supseteq \hat{B}_2 \cup B_3^{\bar{i}-1}$ with the appropriate radius. This process continues until we arrive at $\hat{B}_m = B_1^{\bar{i}}$, which is finally disjoint from the balls in the collection $\{B_j^{\bar{i}-1}\}_{j=m+1}^{M_{\bar{i}-1}}$.

Note that $r_j^{\bar{i}-1} < \sigma/2$ for all j, so clearly we may find some integer $\bar{j}$ such that the ball $\hat{B}_{\bar{j}}$ has radius $\hat{r}_{\bar{j}} \in [\sigma/2, \sigma]$. Let $x(\sigma)$ be any point such that $\hat{B}_{\bar{j}} \subset B_\sigma(x(\sigma))$. Then, using as before the subadditivity of Λ,

$$\begin{aligned}
\mu(B_\sigma(x(\sigma))) &\geq \sum_{i=1}^{\bar{j}} \mu(B_i^{\bar{i}-1}) \\
&\geq \Lambda(\hat{r}_{\bar{j}}) \\
&\geq \Lambda(\sigma/2).
\end{aligned}$$

(If $\tilde{i} = 1$, then the first inequality above has to be replaced by $\mu(B_\sigma(x(\sigma))) \geq \sum_{i=1}^{\tilde{j}} \mu(Z_i)$, since the balls $\{B_i^0\}$ need not be disjoint. Everything else however works without change.) This establishes the claim from Step 1.

5. Now using the claim we may find a sequence of points $x_i = x(\tilde{r}/2^i)$, for $i = 1, \ldots, j$ such that $1 \leq \tilde{r}/2^j < 2$, and

$$\mu(B_{\tilde{r}/2^i}(x_i)) \geq \Lambda(\tilde{r}/2^{i+1})$$

for each $i = 1, \ldots, j$. It is not hard to see that we may take $x_{i+1} \in B_{\tilde{r}/2^i}(x_i)$. Let x^* be a point in $B_{\tilde{r}/2^j}$ such that $u(x^*) = 0$. Then we have

$$|x_i - x^*| \leq \tilde{r}/2^{i-1}$$

for every i, so that $B_{\tilde{r}/2^i}(x_i) \subset B_{\tilde{r}/2^{i-2}}(x^*)$. Given $\sigma \in [1, 4\tilde{r}]$ we may find $i \leq j$ such that

$$2^{-i+2}\tilde{r} \leq \sigma < 2^{-i+3}\tilde{r}.$$

Then $B_{\tilde{r}/2^i}(x_i) \subset B_\sigma(x^*)$, so

$$\mu(B_\sigma(x^*)) \geq \Lambda(\tilde{r}/2^{i+1}) \geq \Lambda(\sigma/16).$$

□

6 Containment of vortex submanifolds

In this section, we establish a containment result analagous to that of the previous section, except that now we are interested in functions $u^\epsilon : \mathbf{R}^d \times [0,T] \to \mathbf{R}^k$, with $d > k$. We will assume that u^ϵ initially has a $(d-k)$ dimensional manifold along which each k-dimensional cross-section has a vortex-like structure. Then we show as above that this structure persists if we assume appropriate bounds on the energy μ^ϵ in a tube surrounding the vortex submanifold.

In order to make use of the notion of degree, we need to consider maps between spaces of equal dimension. Thus the proof works by filling up $U \subset \mathbf{R}^d$ in some appropriate way with k-dimensional submanifolds on which the degree of u^ϵ can be defined and our earlier estimates can be used.

We consider the simplest possible geometry: our submanifold will be the unit cube in R^{d-k}. More general situations can be reduced to this one by a change of coordinates.

As usual, B_ρ denotes a ball of radius ρ in $\mathbf{R}^k$. It is also convenient to define $K = [0,1]^{d-k}$, and $U = B_{4\sigma} \times K \subset \mathbf{R}^d$ for some arbitrary $\sigma > 0$. We let $T_{\sigma,4\sigma}$ denote the tube $A_{\sigma,4\sigma} \times K$.

We denote typical points in $B_{4\sigma}$ and K as x and y respectively.

We will use the notation

$$m^\epsilon(y,t) = \text{leb}^1(\{r \in [\sigma, 2\sigma] | \deg(u^\epsilon(\cdot,y,t); \partial B_r) = 1\}),$$

$$V_t^\epsilon = \{y \in K | m^\epsilon(y,t) > \sigma/2\}.$$

We may think of V_t^ϵ as the subset of points in K at which the cross-section at time t exhibits an isolated vortex, in a weak sense.

Theorem 6.1 *Let* $u^\epsilon \in C([0,T]; W^{1,\infty}(U;\mathbf{R}^k))$ *with* $||Du^\epsilon|| \le \kappa/\epsilon$, *and assume that*

$$\deg(u^\epsilon(\cdot,y,0); \partial B_r) = 1$$

for all $y \in K$, $r \in [\sigma, 4\sigma]$, *and that*

$$\mu_t^\epsilon(T_{\sigma,4\sigma}) \le \kappa \tag{6.1}$$

for all $t \in [0,T]$. *Then*

$$\lim_{\epsilon \to 0, t \in [0,T]} H^{d-k}(V_t^\epsilon) = H^{d-k}(K) = 1, \tag{6.2}$$

and

$$\liminf_{\epsilon \to 0} |\ln \epsilon|^{-1} \mu_t^\epsilon(B_{2\sigma} \times K) \ge K(k). \tag{6.3}$$

Proof. 1. First note that (6.3) follows from (6.2), Theorem 4.2, and Fubini's theorem.

We assume, towards an eventual contradiction, that (6.2) does not hold, i.e. that there exist sequences $\epsilon_n \to 0$, $s_n \in [0,T]$ such that

$$H^{d-k}(V_{s_n}^{\epsilon_n}) \leq 1-\alpha < 1.$$

We may assume that $\alpha < 1/2$.

Suppose that y is a point in V_t^ϵ, so that $m^\epsilon(y,t) > \sigma/2$. We have seen in (4.3) that $m^\epsilon(y,\cdot)$ is lower semicontinuous. Consequently, it is clear that $y \in V_s^\epsilon$ for all s sufficiently close to t. Hence the function

$$t \mapsto H^{d-k}(V_t^\epsilon)$$

is lower semicontinuous. It follows that if we define

$$t_n = \inf\{t \in [0,T] | H^{d-k}(V_t^{\epsilon_n}) \leq 1-\alpha\},$$

then $H^{d-k}(V_{t_n}^{\epsilon_n}) \leq 1-\alpha$.

2. We also claim that for n large enough,

$$H^{d-k}(V_{t_n}^{\epsilon_n}) \geq \alpha.$$

Indeed, consider the set

$$W := \bigcap_{s<t_n} \bigcup_{s\leq t<t_n} V_t^{\epsilon_n}.$$

The definition of t_n implies that $H^{d-k}(W) \geq 1-\alpha$. Also, for any $y \in W$, there is a sequence $s_k \to t_n$ such that $y \in V_{s_k}^{\epsilon_n}$, and so we deduce that for any integer $j \neq 1$,

$$\liminf_{t \nearrow t_n} \text{leb}^1(\{r \in [\sigma, 2\sigma] | \deg(u^{\epsilon_n}(\cdot, y, t); \partial B_r) = j\}) \leq \sigma/2.$$

Again using lower semicontinuity, this implies that

$$\text{leb}^1(\{r \in [\sigma, 2\sigma] | \deg(u^{\epsilon_n}(\cdot, y, t_n); \partial B_r) = j\}) \leq \sigma/2,$$

for $y \in W, j \neq 1$. If in addition $y \notin V_{t_n}^{\epsilon_n}$, then the same inequality holds for $j = 1$, and Theorem 4.3 implies that

$$\int_{A_{\sigma,2\sigma}} E^{\epsilon_n}(x,y,t)dH^k(x) \geq K(k)\ln(\sigma/2\epsilon) - C.$$

Thus by (6.1) and Fubini's theorem,

$$\begin{aligned} C &\geq \int_{W\setminus V_{t_n}^{\epsilon_n}} \int_{A_{\sigma,2\sigma}} E^{\epsilon_n}(x,y,t)dH^k(x)dH^{d-k}(y) \\ &\geq H^{d-k}(W \setminus V_{t_n}^{\epsilon_n})(K(k)\ln(1/\epsilon) - C). \end{aligned}$$

Therefore

$$\lim_{n\to\infty} H^{d-k}(V_{t_n}^{\epsilon_n}) \geq H^{d-k}(W) \geq 1-\alpha.$$

Since $\alpha < 1/2$, this implies our claim.

3. We have shown so far that

$$H^{d-k}(V^{\epsilon_n}) \in [\alpha, 1-\alpha].$$

According to Lemma 6.1, to be proven below, this implies that

$$\mu^{\epsilon_n}(T_{\sigma,4\sigma}) \geq C^{-1}\ln 1/\epsilon - C,$$

an obvious contradiction to (6.1) for ϵ sufficiently small. Thus we have established the theorem. □

We now prove the lemma used above. The "time" variable does not play any role here, so we may work on $U = B_{4\sigma} \times K$. We accordingly modify our earlier notation in the obvious way:

$$m^\epsilon(y) = \mathrm{leb}^1(\{r \in [\sigma, 2\sigma] | \deg(u^\epsilon(\cdot, y); \partial B_r) = 1\}),$$

$$V^\epsilon = \{y \in K | m^\epsilon(y) > \sigma/2\}.$$

We also define $W^\epsilon = K \setminus V^\epsilon$.

As remarked before, $m^\epsilon(\cdot)$ is lower semicontinuous, so V^ϵ is open.

Lemma 6.1 *Let $u^\epsilon \in W^{1,\infty}(U; \mathbf{R}^k)$ with $\|Du^\epsilon\| \leq \kappa/\epsilon$. If*

$$\alpha \leq H^{d-k}(V^\epsilon) \leq 1-\alpha \tag{6.4}$$

for some $0 < \alpha \leq 1/2$, then

$$\mu^\epsilon(T_{\sigma,4\sigma}) \geq C^{-1}\ln 1/\epsilon - C, \tag{6.5}$$

1. We first establish (6.5) under the assumption that $d = k+1$, so that $K = [0,1] \subset \mathbf{R}$. Later we will use this to establish the general case.

For $y \in [0,1]$, let $\chi^\epsilon(y) = 1$ if $y \in V^\epsilon$ and 0 otherwise. Condition (6.4) implies that we may find a point $\overline{y} \in [0,1]$ such that

$$\int_0^{\overline{y}} \chi^\epsilon dH^1 = \alpha/2.$$

We may assume without any loss that $\overline{y} \leq 1/2$. (Otherwise we may achieve this by the change of variables $x \mapsto (1/2 - x)$.) Then

$$\int_{\overline{y}}^1 (1-\chi^\epsilon) dH^1 = 1 - \overline{y} - \alpha/2 \geq \alpha/2.$$

2. Let $\gamma = \alpha/2 \wedge 2\sigma$, and for $y \le \overline{y}$ define

$$\rho(y) = \int_y^{\overline{y}} \chi^\epsilon dH^1. \tag{6.6}$$

Note that $\rho(0) \ge \gamma, \rho(\overline{y}) = 0$, and $\rho(\cdot)$ is nonincreasing. It follows that $\rho(\cdot)$ has a unique right continuous inverse $a : [0, \gamma] \to [0, \overline{y}]$. (We specify that $a(\cdot)$ is right continuous only to guarantee that it is uniquely defined; we will not make any further use of this property.)

Similarly, there is a unique right continuous function $b : [0, \gamma] \to [\overline{y}, 1]$ satisfying the identity

$$\rho = \int_{\overline{y}}^{b(\rho)} (1 - \chi^\epsilon) dH^1 \qquad \text{for all } \rho \in [0, \gamma]. \tag{6.7}$$

Since W^ϵ is closed, it has at most countably many components, and the function $\rho(\cdot)$ defined in (6.6) is constant on each component of W^ϵ. Thus the set $\{\rho \in [0, \gamma] | a(\rho) \in W^\epsilon\}$ is at most countable. Similar reasoning applies to the set $\{\rho \in [0, \gamma] | b(\rho) \in V^\epsilon\}$, so we may find a set $R = \{r_1, r_2, \ldots\} \subset [0, \gamma]$ which is at most countable, such that

$$a(\rho) \in V^\epsilon, \qquad b(\rho) \in W^\epsilon \qquad \text{for all } \rho \in [0, \gamma] \setminus R.$$

In particular, it is clear that $a(\cdot)$ and $b(\cdot)$ are continuous away from R.

We may also assume that $0 = r_0 \le r_1 < r_2 < \ldots$.

3. Suppose that $r_i < \overline{\rho} < r_{i+1}$ for some i, so that $b(\rho) \in W^\epsilon$ for all ρ near $\overline{\rho}$. Using (6.7) we have

$$\frac{1}{h}[b(\overline{\rho} + h) - b(\overline{\rho})] = \frac{1}{h} \int_{b(\overline{\rho})}^{b(\overline{\rho}+h)} \chi^\epsilon dH^1 = 1$$

for all h sufficiently small. Thus $b' \equiv 1$, and similarly $a' \equiv -1$, on $[0, \gamma] \setminus R$.

For each $\rho \in [0, \gamma]$ we now define the set

$$S^\rho := B_{2\sigma} \times [a(\rho) + \rho, b(\rho) - \rho].$$

From the definitions of $a(\cdot)$ and $b(\cdot)$ we readily deduce that if $\rho_1 > \rho_2$, then

$$a(\rho_2) - a(\rho_1) \ge \rho_1 - \rho_2 \qquad \text{and} \qquad b(\rho_1) - b(\rho_2) \ge \rho_1 - \rho_2.$$

This immediately implies that $S^\rho \ne \emptyset$ and that

$$S^{\rho_1} \subset S^{\rho_2} \qquad \text{if } \rho_1 < \rho_2.$$

We moreover claim that S^ρ is constant in each interval (r_i, r_{i+1}). Indeed, this is clear from the fact that within any such interval, $a' \equiv -1$ and $b' \equiv 1$. Thus the set $S^i := S^\rho$ for all $\rho \in (r_i, r_{i+1})$ is well-defined.

In fact, because we have taken a and b to be right continuous, $S^i = S^\rho$ for all $\rho \in [r_i, r_{i+1})$.

4. We define functions $f_i : U \to \mathbf{R}$ by $f_i(x,y) = \text{dist}((x,y), S^i)$, and for each $\rho \in [0, \gamma]$ we define a k-dimensional manifold

$$\begin{aligned} M^\rho &= \{(x,y) \in \mathbf{R}^{k+1} | \text{dist}((x,y), S^\rho) = \rho\} \\ &= f_i^{-1}(\rho), \qquad\qquad \rho \in [r_i, r_{i+1}). \end{aligned}$$

M^ρ is a smooth cylinder which contains the disks $B_{2\sigma} \times \{a(\rho)\}$ and $B_{2\sigma} \times \{b(\rho)\}$ on its two ends. Since $\gamma \leq 2\sigma$, $M^\rho \subset U$ for all $\rho \leq \gamma$. The fact that the sets S^ρ are increasing implies that

$$f_i^{-1}(\rho_i) \cap f_j^{-1}(\rho_j) = \emptyset \text{ whenever } i \geq j \text{ and } \rho_i > \rho_i.$$

In particular, M^{ρ_1} and M^{ρ_2} are disjoint whenever $\rho_1 \neq \rho_2$.

The essential point, informally, is that the cross-section at $a(\rho)$ contains a vortex, whereas that at $b(\rho)$ does not. As a consequence of this fact, we have the following:

$$\int_{M^\rho \cap T_{\sigma,4\sigma}} E^\epsilon(x) dH^k(x) \geq C^{-1} \ln \frac{1}{\epsilon} - C. \tag{6.8}$$

We will establish this estimate in Lemma 6.2.

5. In the following computation we apply Federer's coarea formula (see for example Chapter 3 of Evans and Gariepy [3]) to integrate over the manifolds M^ρ, which are precisely level sets of the functions f_i. Recall that the Jacobian Jf of a scalar function f is simply given by $Jf = |Df|$. In particular, $Jf_i \equiv 1$ for all i, and so we may integrate over level sets of f_i without introducing any correction. Thus

$$\begin{aligned} \mu^\epsilon(T_{\sigma,4\sigma}) &\geq \sum_i \mu^\epsilon\left(T_{\sigma,4\sigma} \cap f_i^{-1}([r_i, r_{i+1}))\right) \\ &= \sum_i \int_{r_i}^{r_{i+1}} \int_{T_{\sigma,4\sigma} \cap f_i^{-1}(\rho)} E^\epsilon dH^k d\rho \\ &\geq \gamma \inf_{\rho \in [0,\gamma]} \int_{T_{\sigma,4\sigma} \cap M^\rho} E^\epsilon dH^k \\ &\geq C^{-1} \ln \frac{1}{\epsilon} - C, \end{aligned}$$

by (6.8). This is (6.5) in the case $d = k+1$.

6. Now suppose that $d > k$ is arbitrary and that (6.4) is satisfied. We will establish (6.5) by reducing this situation to the case $d = k+1$. In order to do so, we are forced to introduce more notation.

For $i = 1, \ldots, d-k$, let

$$\pi_i K = \{y \in K | y_i = 0\},$$

where y_i is the ith component of y. Similarly, for any $f \in L^1(K)$, define $\pi_i f \in L^1(\pi_i K)$ by

$$\pi_i f(y) = \int_0^1 f(y + se_i) dH^1(s).$$

For any integer $i \in \{1, \dots, d-k\}$ and $\beta \in (0, 1/2)$ we define the set

$$S_{i,\beta}^\epsilon = \{y \in \pi_i K | \beta \le \pi_i \chi^\epsilon \le 1 - \beta\}.$$

We will prove below in Lemma 6.3 that (6.4) implies that there exists some $\beta = \beta(\alpha) > 0$ such that

$$H^{d-k-1}(S_{i,\beta}^\epsilon) \ge \beta \tag{6.9}$$

for some $i \in \{1, \dots, d-k\}$. Note that if $y \in S_{i,\beta}^\epsilon$ then u^ϵ (restricted to the set $B_{4\sigma} \times \pi_i^{-1}(y)$) satisfies (6.4) with $d = k+1$ and α replaced by β. Thus we may use (6.5) to conclude that

$$\int_{A_{\sigma,4\sigma} \times \pi_i^{-1}(y)} E^{\epsilon_n} dH^{k+1} \ge C^{-1} \ln \frac{1}{\epsilon} - C$$

if $y \in S_{i,\beta}^\epsilon$. Using (6.9) and Fubini's theorem, we compute

$$\begin{aligned} \int_{T_{\sigma,4\sigma}} E^\epsilon dx &\ge \int_{S_{i,\beta}^\epsilon} \left(\int_{A_{\sigma,4\sigma} \times \pi_i^{-1}(y)} E^\epsilon dH^{k+1} \right) dH^{d-k-1}(y) \\ &\ge C^{-1} \ln \frac{1}{\epsilon} - C. \end{aligned}$$

Thus (6.5) holds for general $d > k$. □

Finally we turn our attention to the two lemmas used in the proof of Lemma 6.1.

Lemma 6.2 *Estimate (6.8) holds.*

Proof. 1. We will use the notation $M_{\sigma,4\sigma}^\rho = M^\rho \cap T_{\sigma,4\sigma}$.

We define a diffeomorphism Ψ between $M_{\sigma,4\sigma}^\rho$ and an annulus in $\mathbf{R}^k$:

First, for $(x, y) \in M^\rho$ let $\tilde{d}(x, y)$ denote the geodesic distance in M^ρ between (x, y) and the point $(0, a(\rho))$. Then for (x, y), we define $\Psi(x, y) = \tilde{d}(x, y)x/|x|$.

Note that Ψ maps circles in $M_{\sigma,4\sigma}^\rho$ of the form $\partial B_s \times \{y\}$ onto circles in $\mathbf{R}^k$ in a way that preserves distances in the direction perpendicular to these circles.

The definition of Ψ makes sense at all points in M^ρ except for $(0, b(\rho))$. It is clear that for $(x, a(\rho)) \in B_{2\sigma} \times \{a(\rho)\}$ we have

$$\Psi(x, a(\rho)) = x.$$

It is easy to see that $\tilde{d}(x,y) \leq 1 + 8\sigma$ for all $(x,y) \in M^\rho$ and all M^ρ of the form constructed above, and $|x| \geq \sigma$ on $M^\rho_{\sigma,4\sigma}$. Thus all functions Ψ of the given form are Lipschitz on $M^\rho_{\sigma,4\sigma} \cup (B_\sigma \times \{a(\rho)\})$, with a Lipschitz constant depending only on σ.

2. Let $R = \tilde{d}(x,y)$ for $(x,y) \in \partial B_\sigma \times \{b(\rho)\}$. Ψ is then a diffeomorphism of $M^\rho_{\sigma,4\sigma} \cup (B_\sigma \times \{a(\rho)\})$ onto B_R.

We may thus define $v^\epsilon : B_R \to \mathbf{R}^k$ by $v^\epsilon = u^\epsilon \circ \Psi^{-1}$.

Define

$$S_a = \{r \in [\sigma, 2\sigma] | \deg(u^\epsilon(\cdot, a(\rho)); \partial B_r) = 1\}.$$

Recall that $a(\rho) \in V^\epsilon$ by construction, so $\mathrm{leb}^1(S_a) \geq \sigma/2$. Moreover, for $r \in S_a$ and $x \in B_r$ we have $v^\epsilon(x) = u^\epsilon(\Psi^{-1}(x)) = u^\epsilon(x, a(\rho))$ and hence

$$\deg(v^\epsilon; \partial B_r) = \deg(u^\epsilon(\cdot, a(\rho)); \partial B_r) = 1.$$

Thus

$$\mathrm{leb}^1\left(\{r \in [\sigma, R] | \deg(v^\epsilon; \partial B_r) = 1\}\right) \geq \sigma/2.$$

3. We claim in addition that

$$\mathrm{leb}^1\left(\{r \in [\sigma, R] | \deg(v^\epsilon; \partial B_r) \neq 1\}\right) \geq \sigma/2.$$

To see this, let

$$S_b = \{r \in [\sigma, 2\sigma] | \deg(u^\epsilon(\cdot, b(\rho)); \partial B_r) \neq 1\}.$$

We will also abuse notation to write

$$\Psi(S_b) = \{s \in [\sigma, R] | \partial B_s = \Psi(\partial B_r \times \{b(\rho)\}) \text{ for some } r \in S_b\}.$$

Because $b(\rho) \notin V^\epsilon$ and Ψ preserves distances in the radial direction, we have $\sigma/2 \leq \mathrm{leb}^1(S_b) = \mathrm{leb}^1(\Psi(S_b))$.

Fix $s \in \Psi(S_b)$, with $r \in S_b$ such that $\partial B_s = \Psi(\partial B_r \times \{b(\rho)\})$. From the definitions of v^ϵ and Ψ we see that for $x \in \partial B_s$,

$$v^\epsilon(x) = v^\epsilon(s\frac{x}{|x|}) = u^\epsilon(\Psi^{-1}(s\frac{x}{|x|})) = u^\epsilon(r\frac{x}{|x|}, b(\rho)).$$

Because the degree of a function depends only on its boundary values, we deduce that

$$\deg(v^\epsilon; \partial B_s) = \deg(u^\epsilon(\cdot, b(\rho)); \partial B_r) \neq 1.$$

This proves the claim.

4. Since $|D\Psi| \leq C(\sigma)$, we see that $C^{-1}||Du^\epsilon||_\infty \leq ||Dv^\epsilon||_\infty \leq C||Du^\epsilon||_\infty$. Thus we may use Theorem 4.3 to deduce that

$$\int_{A_{\sigma,R}} E^\epsilon(v^\epsilon) dx \geq K(k) \ln(\frac{\sigma}{\epsilon}) - C.$$

Again using the fact that Ψ is Lipschitz, we immediately deduce that

$$\int_{M^{\rho}_{\sigma,4\sigma}} E^{\epsilon}(u^{\epsilon})dH^k \geq C^{-1}\ln(\frac{1}{\epsilon}) - C.$$

□

One last lemma remains. The proof is not hard, but the notation is extremely awkward.

Lemma 6.3 *Let $f \in L^{\infty}(K)$ with $0 \leq f \leq 1$ a.e., and suppose that*

$$\vartheta := \int_K f dy \in [\alpha, 1-\alpha]$$

for some $\alpha \leq 1/2$. Then there exists $\beta = \beta(\alpha, d-k) > 0$ such that

$$H^{d-k-1}(\{x \in \pi_i K | \beta \leq \pi_i f \leq 1-\beta\}) \geq \beta.$$

for some $i = 1, \ldots, d-k$.

Remark. Applying this lemma to $f = \chi^{\epsilon}$ yields (6.9).

Proof. 1. We write $d-k=n$, so that $K = [0,1]^n$, and we prove the lemma by induction on n.

If $n = 1$ the conclusion is obvious, with $\beta(\alpha, 1) = \alpha$.

For the induction step we will need to assume that the result holds for $n = 2$, so we establish this case independently. For $\beta = \beta(\alpha, 2)$ to be chosen, define

$$S_i = \{y \in \pi_i K | \beta \leq \pi_i f \leq 1-\beta\},$$

$i = 1, 2$. We assume that $H^1(S_1) < \beta$, and we must show that $H^1(S_2) \geq \beta$.

2. We further define sets

$$V_1 = \{y \in \pi_i K | \pi_1 f > 1-\beta\},$$

$$W_1 = \{y \in \pi_i K | \pi_1 f < \beta\},$$

As $\pi_1 K = W_1 \cup S_1 \cup V_1$, we have

$$\vartheta = \int_{W_1} \pi_1 f dH^1 + \int_{S_1} \pi_1 f dH^1 + \int_{V_1} \pi_1 f dH^1.$$

Also, from the definitions we immediately see that

$$0 \leq \int_{W_1} \pi_1 f dH^1 \leq \beta H^1(W_1),$$

$$0 \leq \int_{S_1} \pi_1 f dH^1 \leq (1-\beta)H^1(S_1) \leq \beta(1-\beta),$$

$$(1-\beta)H^1(V_1) \leq \int_{V_1} \pi_1 f dH^1 \leq H^1(V_1).$$

These imply that

$$\vartheta - 2\beta \leq H^1(V_1) \leq \frac{\vartheta}{1-\beta} \leq \vartheta + \beta,$$

where the final inequality holds whenever $\beta \leq 1 - \vartheta$, which we may take to be the case.

3. For any $A_1 \subset \pi_1 K$ and $y \in \pi_2 K$, we let

$$\nu_y^2(A) := \int_{\pi_1^{-1}(A)\cap\pi_2^{-1}(y)} f dH^1.$$

Then by Fubini's Theorem,

$$\int_{\pi_2 K} \nu_y^2(V_1) dH^1(y) = \int_{V_1} \pi_1 f dH^1 \geq \vartheta - 2\beta.$$

Also, for each $y \in \pi_2 K$, $\nu_y^2(V_1) \leq H^1(V_1) \leq \vartheta + \beta$. Define the set

$$G_V = \{y \in \pi_2 K | \vartheta - 8\beta \leq \nu_y^2(V_1) \leq \vartheta + \beta\}.$$

An obvious estimate gives

$$\int_{\pi_2 K} \nu_y^2(V_1) dH^1(y) \leq (\vartheta + \beta)H^1(G_V) + (\vartheta - 8\beta)(1 - H^1(G_V)).$$

With the above estimates, this implies that

$$H^1(G_V) \geq 2/3.$$

4. It is clear that

$$0 \leq \nu_y^2(W_1) \leq H^1(W_1)$$

for all $y \in \pi_2 K$, and that

$$\int_{\pi_2 K} \nu_y^2(W_1) dH^1 = \int_{W_1} \pi_1 f dH^1 \leq \beta.$$

These two inequalities imply that $H^1(G_W) \geq 2/3$, where

$$G_W := \{y \in \pi_2 K | 0 \leq \nu_y^2(W_1) \leq 3\beta\}.$$

5. Since $H^1(S_1) \leq \beta$ by assumption, it is obvious that $0 \leq \nu_y^2(S_1) \leq \beta$. Note also that

$$\pi_2 f(y) = \nu_y^2(V_1) + \nu_y^2(S_1) + \nu_y^2(W_1).$$

Thus for $y \in G_V \cap G_W$ we have

$$\vartheta - 8\beta \leq \pi_2 f(y) \leq \vartheta + 5\beta.$$

Moreover, Steps 4 and 5 imply that $H^1(G_V \cap G_W) \geq 1/3$. If we now take β small enough that

$$\beta \leq \vartheta - 8\beta, \qquad \vartheta + 5\beta \leq 1 - \beta, \qquad \beta \leq 1/3,$$

then the lemma holds for $n = 2$. Note that β can be chosen uniformly for all $\vartheta \in [\alpha, 1-\alpha]$, so we may write $\beta = \beta(\alpha, 2)$.

6. We now suppose that the lemma holds for $1, 2, \ldots, n-1$, with $n \geq 3$, and we set $K = [0,1]^n$.

Given $f : K \to [0,1]$ we may apply the induction hypothesis to $\pi_1 f : \pi_1 K \to [0,1]$ to find some $i \in \{2, \ldots, n\}$ for which

$$H^{n-2}(S_{i1}) \geq \beta(\alpha, n-1) := \tilde{\beta},$$

where $S_{i1} = \{y \in \pi_i \pi_1 K | \tilde{\beta} \leq \pi_i \pi_1 f \leq 1 - \tilde{\beta}\}$. We may assume that $i = 2$.

For $y = (0, 0, y_3, \ldots, y_n) \in S_{21}$, define $f_y : [0,1]^2 \to [0,1]$ by $f_y(x_1, x_2) = f(x_1, x_2, y_3, \ldots, y_n)$. Then

$$\int_{[0,1]^2} f_y(x_1, x_2)\, dx_1\, dx_2 := \vartheta(y) \in [\tilde{\beta}, 1 - \tilde{\beta}]$$

by definition of S_{21}. Applying the induction hypothesis to f_y for each $y \in S_{21}$, we have

$$H^1(\mathcal{S}_{i(y)}(y)) \geq \beta(\tilde{\beta}, 2) := \overline{\beta}, \tag{6.10}$$

where $i(y) = 1$ or 2 and

$$\mathcal{S}_{i(y)}(y) = \{x \in \pi_{i(y)}[0,1]^2 | \overline{\beta} \leq \pi_{i(y)} f_y \leq 1 - \overline{\beta}\}. \tag{6.11}$$

7. For $j = 1, 2$, define $S_{21}^j = \{y \in S_{21} | i(y) = j\}$. Since $H^{n-2}(S_{21}) \geq \tilde{\beta}$ and $S_{21} = S_{21}^1 \cup S_{21}^2$, we must have

$$H^{n-2}(S_{21}^j) \geq \tilde{\beta}/2 \tag{6.12}$$

for $j = 1$ or $j = 2$. We may as well assume that the former holds.

Let

$$\Sigma = \{y = (0, y_2, \ldots, y_n) \in \pi_1 K | \pi_2 y \in S_{21}^1, \quad (0, y_2) \in \mathcal{S}_1(\pi_2(y))\}.$$

We deduce from (6.10) and (6.12) that

$$\begin{aligned} H^{n-1}(\Sigma) &= \int_{y \in S_{21}^1} H^1(\{\hat{x}_2 \in [0,1] | (0, x_2) \in \mathcal{S}_1(y)\}) dH^1(x_2) dH^{n-2}(y) \\ &\geq \overline{\beta}\tilde{\beta}/2. \end{aligned}$$

Moreover, for $y \in \Sigma$, $\pi_2 y$ has the form $(0, 0, y_3, \ldots, y_n)$, and so $f_{\pi_2 y}$ is well-defined. For any $s \in [0, 1]$, by definition $f(s, y_2, \ldots, y_n) = f_{\pi_2 y}(s, y_2)$, so (6.11) gives

$$\pi_1 f(y) = \pi_1 f_{\pi_2 y}(0, y_2) \in [\overline{\beta}, 1 - \overline{\beta}]$$

for $y \in \Sigma$. The last two facts immediately imply that the conculsion of the lemma holds for $\beta(\alpha, n) = \overline{\beta}\tilde{\beta}/2$. □

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