Remarks on the Homogenization Method in Optimal Design Problems

Luc Tartar
Carnegie Mellon University

Research Report No. 95-NA-021

November 1995

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

Remarks on the Homogenization Method in Optimal Design Problems

Luc TARTAR

Department of Mathematics, CARNEGIE-MELLON University
Pittsburgh, PA 15213-3890, U.S.A.

Abstract: The method of Homogenization for problems of Optimal Design, developed by F. MURAT and the author, is recalled. It is shown how to avoid the characterization of effective properties of mixtures for a general functional that does not involve gradients.

History of the subject

In the early 70s, while François MURAT was working on some academic problems of optimization that had been proposed by Jacques-Louis LIONS [1], he found that a few of them had no solution [2]. His results were quite unexpected for me, and as we were sharing an office in Jussieu in those days, we had many occasions to discuss both his original proof and the various generalizations that he had then obtained [3], and the subject was so fascinating that it marked the beginning of a long and fruitful collaboration, although many of our results have been only partially published (in the sequel I will use "we" to mean F. MURAT and myself). Essentially the initial problem was to minimize

$$J(a) = \int_0^L |u(x) - z(x)|^2\,dx, \tag{1}$$

where

$$-\frac{d}{dx}\Big(a\frac{du}{dx}\Big) + a\,u = f \text{ in } (0,L), u \in H_0^1(0,L), \tag{2}$$

$z \in L^2(0,L)$ being given, and

$$a \in \mathcal{A} = \{a \in L^\infty(0,L), \alpha \le a \le \beta \text{ a.e. in } (0,L)\}. \tag{3}$$

F. MURAT was trying to apply the direct method of the Calculus of Variations, and he noticed that if a sequence $a_n \in \mathcal{A}$ is such that $a_n \rightharpoonup a_+$ in $L^\infty(0,L)$ weak $\star$ and $\frac{1}{a_n} \rightharpoonup \frac{1}{a_-}$ in $L^\infty(0,L)$ weak $\star$, then the corresponding sequence of solutions u_n of (2) converges in $H_0^1(0,L)$ weak to the solution u_∞ of

$$-\frac{d}{dx}\Big(a_-\frac{du_\infty}{dx}\Big) + a_+\,u_\infty = f \text{ in } (0,L), u_\infty \in H_0^1(0,L), \tag{4}$$

and

$$J(a_n) \to \tilde{J}(a_-,a_+) = \int_0^L |u_\infty(x) - z(x)|^2\,dx. \tag{5}$$

He constructed then a particular sequence with $a_- < a_+$, defined $z = u_\infty$, implying then $\inf_{a\in\mathcal{A}} J(a) = 0$, and checked that it was impossible to have $u = z$ in (2) for any $a \in \mathcal{A}$, by considering (2) as a differential equation for a, which had no solution staying between α and β in the interval $(0,L)$ for the choice that he had made.

We were naturally led to characterize all the possible pairs (a_-,a_+) and more generally we proved that if a sequence $U^{(n)}$ of measurable functions from an open set $\Omega \subset R^N$ into R^p satisfies $U^{(n)}(x) \in K$ a.e. $x \in \Omega$ for a bounded set K, and $U^{(n)} \rightharpoonup U^{(\infty)}$ in $L^\infty(\Omega; R^p)$ weak $\star$, then the characterization of all the possible limits $U^{(\infty)}$ is $U^{(\infty)}(x) \in \overline{conv(K)}$, the closed convex hull of K, a.e. $x \in \Omega$. This result might not have been stated in such a simple form before, and Ivar EKELAND told me that it had been implicitely used in some work of CASTAING and was related to a classical result of LYAPUNOV valid for a set endowed with a nonnegative measure without atoms, and indeed our proof extended easily to such a general case. Our characterization appeared quite useful when we tried to understand the more general situation where u is the solution of

$$-div\Big(a\, grad(u)\Big) = f \text{ in } \Omega,\ u \in H_0^1(\Omega), \tag{6}$$

with

$$a \in \mathcal{A} = \{a \in L^\infty(\Omega), \alpha \le a \le \beta \text{ a.e. in } (\Omega)\}, \tag{7}$$

and one wants to minimize

$$J(a) = \int_\Omega g\Big(x, u(x), a(x)\Big)\, dx. \tag{8}$$

Of course, we had found that the main difficulty was to consider (6) for a sequence a_n converging only weakly, and to identify the weak limits of u_n and of $a_n\, grad(u_n)$, but at that moment we were not aware that Sergio SPAGNOLO had already studied a similar question [4,5], and we were led to rediscover most of his results by a different approach which, after an improvement of our initial method that I based on our Div-Curl lemma, appeared more powerful. Although our names are rarely quoted nowadays, it is our method that almost everybody uses now, but many do not seem to understand that our notion of H-convergence is indeed much more general that the notion of G-convergence that S. SPAGNOLO had introduced, a reminder of its relation with the convergence of GREEN kernels. At a CIME session in Varenna in 1970, I had met S. SPAGNOLO who had asked me if some of my results about nonlinear interpolation had anything to do with his own results, which he quickly mentioned, but although I could say that there was no relation because the coefficients of his equations were not regular, I did not catch much about what his results really were. I think that after obtaining our initial results, we finally became aware of what S. SPAGNOLO had done through some work of Tullio ZOLEZZI, and one of his articles indeed puzzled us for a while, as we thought that one of his theorems contradicted some of ours [6]. F. MURAT had first identified what one calls now the effective conductivity of a layered material, and his formula had told us that, for $N \ge 2$, one could not characterize the limit of u_n even if one knew the limits in $L^\infty(\Omega)$ weak $\star$ of $h(a_n)$ for all continuous functions h, an information which I described later by using the notion of parametrized measures, until John M. BALL told me that the notion had actually been introduced by Laurence C. YOUNG, and should be called the YOUNG measure associated to the sequence a_n. The puzzling theorem in T. ZOLEZZI's article stated that if a sequence a_n converges weakly to a_+ in $L^\infty(\Omega)$ then the corresponding sequence of solutions u_n converges weakly to the solution associated to a_+. F. MURAT thought that some nuance in Italian might have tricked us in mistranslating what was meant, but as we were pondering if "debolmente" could mean anything else than weakly, it suddenly appeared that our mistake had been to read correctly weakly and to interpret it incorrectly as weakly $\star$, as indeed it was the first time that we had seen any mention of the weak topology of $L^\infty(\Omega)$ in a concrete situation; we understood then why there was a reference to an article of Alexandre GROTHENDIECK, who had shown that convergence in $L^\infty(\Omega)$ weak implies strong convergence in $L^p_{loc}(\Omega)$ for every finite p.

In our initial proof we assumed that the sequence $a_n \in \mathcal{A}$ was such that $a_n \rightharpoonup a_+$ in $L^\infty(0,L)$ weak $\star$ and $\dfrac{1}{a_n} \rightharpoonup \dfrac{1}{a_-}$ in $L^\infty(0,L)$ weak $\star$, because these limits had played a role in the layered case, and that $E^{(n)} = grad(u_n) \rightharpoonup E^{(\infty)}$ in $L^2(\Omega; R^N)$ weak and $D^{(n)} = a_n\, grad(u_n) \rightharpoonup D^{(\infty)}$ in $L^2(\Omega; R^N)$ weak. Using an integration by parts, we deduced that $(E^{(n)}.D^{(n)}) \rightharpoonup (E^{(\infty)}.D^{(\infty)})$ in $\mathcal{D}'(\Omega)$ (or $\mathcal{M}(\Omega)$ weak $\star$), and we were led to identify the convex hull of the set $\Big(E, a\,E, a|E|^2, a, \dfrac{1}{a}\Big)$, parametrized by $a \in [\alpha,\beta], E \in R^N$, and the explicit description of that convex hull gave us the missing link to prove the existence (for a subsequence) of a symmetric tensor $a_\infty \in \mathcal{A}$, independent of f, such that $D^{(\infty)} = a_\infty E^{(\infty)}$, and moreover that $a_- I \le a_\infty \le a_+ I$ a.e. in Ω. Actually, our analysis provided the inequality $(D^{(\infty)} - a_+ E^{(\infty)}.D^{(\infty)} - a_- E^{(\infty)}) \le 0$ a.e. in Ω, which I will use later on.

It is useful to mention that we had assumed no periodicity hypothesis on the coefficients of our equations, although we had been aware of that framework after reading notes of Enrique SANCHEZ-PALENCIA [7,8], but his work helped us understanding something more important. Up to that point, we had been dealing with abstract mathematical questions about Partial Differential Equations in variational form, using and improving results from Functional Analysis, and we had never used any physical interpretation of our equations for the quite simple reason that we were not so confident with the knowledge of Continuum

Mechanics and Physics that we had been taught at Ecole Polytechnique. The new understanding that we obtained from reading the work of E. SANCHEZ-PALENCIA and comparing it to ours was that the weak convergence methods and the new H-convergence that we had been using (although the term was coined much later), were actually a new mathematical approach for modelling the relations between microscopic and macroscopic levels (I have learned since that the word microscopic should be replaced by mesoscopic when talking to people who think that microscopic only means the scale of atoms). In those days, relations between microscopic and macroscopic levels were only explained using a probabilistic interpretation and ensemble averages, and this is still the case in many circles. I was not writing much in those days, and the only articles that I wrote then were for the proceedings of a conference in Roma in April 1974 and another one at IRIA in June 1974 [9,10], and it was between these two conferences that we had discovered the Div-Cul lemma, in the process of reviewing all the situations that we knew where a_∞ could be explicitly computed, and [10] is the earlier reference with a hint to that new philosophy about Continuum Mechanics and Physics which I have advocated for the last twenty years.

In the same period some numerical computations about similar Optimal Design problems were performed in Nice, by Jean CÉA and his team, and we were aware of the work of Denise CHENAIS, which meant that if one imposed some kind of regularity condition on an interface between two materials then a classical optimal solution could be found, while our work suggested that if one did not impose such a condition there might be no classical *solution*, in which case one would have to use generalized solutions corresponding to mixtures. In some simple cases we could propose a new relaxed problem that seemed to have much better numerical stability properties, and if I computed necessary conditions of optimality in [10], it was partly for telling J. CÉA that there were stronger necessary conditions of optimality due to the fact that a classical optimal solution had to be better than all possible mixtures, but I could not convince him that if he refined his triangulations enough he might start seeing oscillations and that our analysis described what kind of oscillations were useful, so that it was not so important to resolve these oscillations numerically. He might have believed that the situation in his work with MALANOWSKI was general [11], while it was obviously the result of a small miracle due to the very special form of their function g, as they had $g(x,u,a) = f(x)u$. Had the computers been more powerful in those days, he might have discovered numerical oscillations in refining his triangulations, but the cost would have been prohibitive at the time and only coarse triangulations were used. The necessary conditions of optimality which I had computed considered a mixture of two isotropic materials, without constraints upon the proportions, and the necessary conditions of optimality that I had obtained were much stronger than the usual ones obtained by pushing the interface along its normal, an idea going back to HADAMARD. The classical idea, which is often only derived in a formal way (although F. MURAT & Jacques SIMON had spent some time putting it into a rigourous framework [12]), gives conditions to be satisfied on the interface, while I was obtaining necessary conditions which were valid everywhere. As J. CÉA was optimizing among all the domains obtained as unions of triangles of his triangulation, he was also obtaining necessary conditions which were valid everywhere, but they were not as strong as mine.

After some discussions with Guy CHAVENT, who was studying the related problem of identifying the permeability of an oil field from measurements at various points, I had proposed a numerical approach for solving numerically the type of optimization problem that we had been studying, but the numerical method that I had proposed did not work well at all. As we knew that the optimal mixture that we were looking for was locally obtained as a layered medium, I had chosen to parametrize the possibilities with a proportion $\theta \in [0,1]$ and an angle describing the orientation of layers (as I was considering a 2-dimensional problem), but that method appeared to be quite unstable because when θ is 0 or 1, i.e. the material is isotropic, the orientation of the layers is not determined. I did not try another numerical method, but I had learned that even when the solution of an optimization problem is on the boundary of a set, it might not be a good idea to move only along the boundary of this set in order to find the solution, and a better approach could be to cut through the set in order to arrive more quickly at the interesting points on the boundary.

At that point, the crucial difficulty that we were facing was that we did not know how to characterize the set of possible a_∞ corresponding to mixing in given proportions some (isotropic) materials. It should be said that for what concerned bounds, the method that we had used for obtaining bounds was not restricted to the case of symmetric tensors, although we had mostly concentrated our attention on the case of symmetric operators, where the computations are simpler.

During the year 1974-75, where I visited the University of Wisconsin at Madison and the Mathematics

Research Center, I found the simplifying approach that F. MURAT later called H-convergence, and although my approach is often called the energy method (without usually attributing it to me), it should better be called the method of oscillating test functions.

I also learned from Carl DEBOOR about the term "homogenization", which had been coined by Ivo BABUŠKA [13]. I. BABUŠKA had been motivated by Engineering questions with periodic structures, and I learned from him why it was not enough to describe the effective coefficients and how important it was to understand local amplifying factors: this motivated me later to prove results about correctors.

When in the Spring of 1977, I gave my PECCOT lectures at Collège de France on Homogenization in Partial Differential Equations, I described the basic theory of Compensated Compactness and the basic theory of H-convergence, which we had developped (some terminology and simplifying notation that I use here were actually introduced later [14]). In the context of a diffusion equation, which is only a simple model as all our methods are variational in nature and can be extended to many other situations in Continuum Mechanics or Physics (for equations or systems which need not be elliptic), the natural class is to choose the coefficients satisfying

$$\begin{aligned} a \in \mathcal{M}(\alpha,\beta;\Omega) = \Big\{ a \in L^\infty\Big(\Omega;\mathcal{L}(R^N,R^N)\Big), (a(x)\xi.\xi) \geq \alpha|\xi|^2, \\ (a(x)\xi.\xi) \geq \frac{1}{\beta}|a(x)\xi|^2 \text{ for all } \xi \in R^N, \text{ a.e. } x \in \Omega \Big\}, \end{aligned} \tag{9}$$

and the basic result is that $\mathcal{M}(\alpha,\beta;\Omega)$ is compact for the topology of H-convergence, i.e. from any sequence $a_n \in \mathcal{M}(\alpha,\beta;\Omega)$, one can extract a subsequence a_m which H-converges to $a_{eff} \in \mathcal{M}(\alpha,\beta;\Omega)$, i.e.

$$\begin{aligned} &\text{if i) } E^{(m)} \rightharpoonup E^{(\infty)} \text{ in } L^2(\Omega;R^N) \text{ weak} \\ &\quad \text{ii) } D^{(m)} = a_m E^{(m)} \rightharpoonup D^{(\infty)} \text{ in } L^2(\Omega;R^N) \text{ weak} \\ &\quad \text{iii) } curl E^{(m)} \text{ stays in a compact of } H^{-1}_{loc}\Big(\Omega;\mathcal{L}_a(R^N,R^N)\Big) \text{ strong} \\ &\quad \text{iv) } div D^{(m)} \text{ stays in a compact of } H^{-1}_{loc}(\Omega) \text{ strong} \\ &\text{then } D^{(\infty)} = a_{eff} E^{(\infty)} \text{ a.e. in } \Omega. \end{aligned} \tag{10}$$

There exists a sequence of correctors $P^{(m)} \in L^2(\Omega;\mathcal{L}(R^N,R^N))$ associated to the subsequence a_m, satisfying

$$\begin{aligned} &P^{(m)} \rightharpoonup I \text{ in } L^2\Big(\Omega;\mathcal{L}(R^N,R^N)\Big) \text{ weak} \\ &Q^{(m)} = a^{(m)}P^{(m)} \rightharpoonup a_{eff} \text{ in } L^2\Big(\Omega;\mathcal{L}(R^N,R^N)\Big) \text{ weak} \\ &curl(P^{(m)}\lambda) \text{ stays in a compact of } H^{-1}_{loc}\Big(\Omega;\mathcal{L}_a(R^N,R^N)\Big) \text{ strong for all } \lambda \in R^N \\ &div(Q^{(m)}\lambda) \text{ stays in a compact of } H^{-1}_{loc}(\Omega) \text{ strong for all } \lambda \in R^N, \end{aligned} \tag{11}$$

and the role of $P^{(m)}$ is to describe what oscillations must exist (at a microscopic level) in $E^{(m)}$ when one has only measured (at a macroscopic level) what the weak limit E^∞ is: if (10) holds then one has

$$P^{(m)}E^{(\infty)} - E^{(m)} \to 0 \text{ in } L^1_{loc}(\Omega;R^N) \text{ strong}, \tag{12}$$

and the convergence can be shown to hold in $L^p_{loc}(\Omega;R^N)$ strong for some $p > 1$ if one knows better integrability properties for $E^{(\infty)}$ or if one uses MEYERS's regularity theorem. If all a_m are symmetric, then a_{eff} is symmetric (but $P^{(m)}$ is not symmetric in general), and in this case one has

$$\begin{aligned} &\text{if i) } a_m \rightharpoonup a_+ \text{ in } L^\infty\Big(\Omega;\mathcal{L}_s(R^N,R^N)\Big) \text{ weak } \star \\ &\quad \text{ii) } (a_m)^{-1} \rightharpoonup (a_-)^{-1} \text{ in } L^\infty\Big(\Omega;\mathcal{L}_s(R^N,R^N)\Big) \text{ weak } \star \\ &\text{then } a_- \leq a_{eff} \leq a_+ \text{ a.e. in } \Omega. \end{aligned} \tag{13}$$

I derived a few months after a general method, although of rather difficult use, for deriving bounds on effective coefficients [15]. The underlying idea is that (11) formally says that $a^{(m)} = Q^{(m)}(P^{(m)})^{-1}$, with $P^{(m)}$ and $Q^{(m)}$ satisfying linear differential constraints, and it was natural then that I tried to apply to this situation the Compensated Compactness theory that we had developped the year before. For every real function F on $\mathcal{L}(R^N, R^N) \times \mathcal{L}(R^N, R^N)$ such that

$$\begin{aligned} &\text{if i) } P^{(n)} \rightharpoonup P^{(\infty)} \text{ in } L^2\Big(\Omega; \mathcal{L}(R^N, R^N)\Big) \text{ weak} \\ &\quad \text{ii) } Q^{(n)} \rightharpoonup Q^{(\infty)} \text{ in } L^2\Big(\Omega; \mathcal{L}(R^N, R^N)\Big) \text{ weak} \\ &\quad \text{iii) } curl(P^{(n)}\lambda) \text{ stays in a compact of } H^{-1}_{loc}\Big(\Omega; \mathcal{L}_a(R^N, R^N)\Big) \text{ strong for all } \lambda \in R^N \\ &\quad \text{iv) } div(Q^{(n)}\lambda) \text{ stays in a compact of } H^{-1}_{loc}(\Omega) \text{ strong for all } \lambda \in R^N \\ &\text{then } \liminf_{n\to\infty} \int_\Omega F(P^{(n)}, Q^{(n)})\varphi\, dx \geq \int_\Omega F(P^{(\infty)}, Q^{(\infty)})\varphi\, dx \text{ for all } \varphi \in \mathcal{D}(\Omega), \varphi \geq 0, \end{aligned} \tag{14}$$

one defines g by

$$g(a) = \sup_{P \in \mathcal{L}(R^N, R^N)} F(P, a\,P), \tag{15}$$

and one deduces that

$$\liminf_{n\to\infty} \int_\Omega g(a_n)\varphi\, dx \geq \int_\Omega g(a_{eff})\varphi\, dx \text{ for all } \varphi \in \mathcal{D}(\Omega), \varphi \geq 0. \tag{16}$$

Of course, the quadratic theorem of Compensated Compactness [16] provides an analytic characterization of those quadratic functions F which satisfy (14), i.e. $F = F_0 + affine(P, Q)$ where F_0 is an homogeneous quadratic function which satisfies

$$F_0(\eta \otimes \xi, Q) \geq 0 \text{ for all } \eta, \xi \in R^N \text{ and } Q \in \mathcal{L}(R^N, R^N) \text{ such that } Q\xi = 0. \tag{17}$$

It should be emphasized that this method for obtaining bounds requires no assumption of symmetry for a.

It was only a few years after, while I was visiting the COURANT Institute, that I tried to find which F_0 would be suitable for the case of mixing isotropic materials, restricting myself to the case where a_{eff} was also isotropic. I was then looking for functions invariant under orthonormal changes of basis, and found that the natural ones to use were $\pm trace(Q^T P), trace(P^T P) - \big(trace(P)\big)^2, (N-1)trace(Q^T Q) - \big(trace(Q)\big)^2$. After having computed the particular bounds that my method gave in the case of a mixture of two isotropic materials using given proportions, George PAPANICOLAOU suggested that I compare my bounds to the HASHIN-SHTRIKMAN bounds, and indeed they were the same, but I could not make much sense out of the derivation of their necessary conditions, although I could easily apply their idea using coated spheres in order to construct materials having the effective properties corresponding to the bounds that I had proved to hold. It was also during that visit that I taught Robert V. KOHN about our method of Homogenization for Optimal Design problems, and if he often forgot to mention our names concerning these matters, it was not because he had never heard of our work.

F. MURAT thought then that the same functions that I had used for finding the best bounds for isotropic mixtures of two (isotropic) materials could also give the best bounds for anisotropic mixtures. It was easy to check what the necessary conditions were in the anisotropic case, but despite some advice by L. Edward FRAENKEL on how to use ellipsoidal coordinates, it took us some time to construct some optimal geometries using coated confocal ellipsoids. These results were presented at a meeting at New York University in June 1981, and they gave the missing link in the method that I had partially described in [10], but when I conjectured that one could probably extend the method to the question of mixing more than two (isotropic) materials, Graeme MILTON immediately pointed out that some of his computations showed that the construction using coated spheres did not always work. It is still not clear if my method for obtaining bounds, which by the way was later described as the "translation method" by G. MILTON and is widely used now with usually no mention of my name, gives optimal bounds or not in the case of mixing more than two materials. However, this is not an important issue for the question of using Homogenization in problems of Optimal Design, as some simplifying arguments were then discovered.

In the following Spring, I taught these results in a topics course at Ecole Polytechnique, and I asked two students, Philippe BRAIDY and Didier POUILLOUX, to make a numerical study comparing the materials that we had constructed by using confocal ellipsoids and those that could be constructed by successive layerings, a method that we had already used for the results quoted in [10] and which I did not think general enough, but to my surprise they announced that the two sets seemed equal, and they quickly provided a proof using N layerings in orthogonal directions, where in each layering the normal direction was a common eigenvector for the two materials being mixed. While visiting the Mathematical Sciences Research Institute in Berkeley in the Spring of 1983, I generalized their result by deriving a formula for layering materials in arbitrary directions which had the advantage that it could be reiterated easily, and I wrote all the details in the proceedings of a meeting dedicated to Ennio DEGIORGI, held in Paris in November 1983 [17].

During the same Spring, Michael RENARDY had told me about his work with Daniel JOSEPH on POISEUILLE flows of two immiscible fluids, which in applications were either melted polymers or crude oil and water, the water being added in small quantity in a pipeline for lubricating it. For a cylinder with arbitrary cross section Ω, there are infinitely many possible POISEUILLE flows, but when Ω is a disc one only observes the flow where the less viscous fluid occupies an annular region near the boundary, and they had imagined that it was related to a maximization of dissipation. This situation was described by (6) with $f = 1$ when one seeks to maximize $\int_\Omega a|grad(u)|^2\,dx = \int_\Omega u\,dx$, and indeed there is a classical solution in the case of a circular cross section, but I told M. RENARDY that I expected no classical solution in general, contrary to what they had conjectured. To prove that there was no classical solution, I looked at the necessary conditions of optimality as I had done in [10], but using the new characterization of all mixtures of two isotropic materials in given proportion that we had obtained two years before, and to my surprise I discovered that our precise characterization could be ignored completely and that the same necessary conditions could be derived from the crude bounds that we had derived ten years earlier. A classical solution satisfying the necessary conditions of optimality must satisfy both DIRICHLET and NEUMANN conditions on the interface, which is quite unlikely in general, but the precise argument for rejecting that possibility was mentioned to me by Joel SPRUCK who reminded me of a result of James SERRIN that I had heard at a conference in Jerusalem in 1972 (and he told me about a quicker proof by Hans WEINBERGER), valid in the case of simply connected domains, and showing that for such domains only the circular one had a classical solution. Of course, this negative result when Ω is not a disc does not tell what configurations will be chosen by a mixture of fluids in a pipe, as the question of what mixtures of fluids really try to optimize must be studied further. Although the idea that turbulent flows are trying to optimize something has been suggested before, as mentioned by D. JOSEPH [18], it is not clear what it is that they could be trying to optimize, but that idea is consistent with the Homogenization approach that in order to optimize some criterion one has to create adequate microstructures. The details of our method were finally written for the purpose of a school on Homogenization at Bréau sans Nappe in July 1983 [19].

During the Spring in Berkeley, R. KOHN had mentioned that he was going to visit Konstantin LUR'IE in Leningrad in the Fall, and he had asked me if he could describe some of our unpublished work (and possibly my recent results). Of course I did not see anything wrong with that, although I later thought that there had been a few hints before which showed that R. KOHN was often forgetting to quote my ideas or was even attributing them to others. The result of that visit was that K. LUR'IE is often refered to now for our results that I had described in 1981 and that R. KOHN had explained to him in 1983, and as R. KOHN himself does systematically refer to him for things that he had heard from me, one must exercise great care before accepting any attribution of ideas mentioned by R. KOHN, K. LUR'IE and their followers for what concerns the period after the Fall of 1983.

What I know about the early work of K. LUR'IE mostly comes from what Jean-Louis ARMAND told me about it. J.-L. ARMAND had found some strange effects in the numerical approximation of some Optimal Design problems, and as he had found references to K. LUR'IE's work as a possible explanation of the difficulties that he had encountered (essentially due to the oscillations that I had warned J. CÉA against), he paid a visit to K. LUR'IE in Leningrad. As it was K. LUR'IE who had mentioned to him about [10], it was clear that K. LUR'IE knew about our work (but this reference is usually absent from all recent articles), and J.-L. ARMAND would probably not have traveled so far, had he known that there were some specialists in France who could answer his questions: after he had made contact with me, we even taught together a topics course at Ecole Polytechnique.

According to J.-L. ARMAND, K. LUR'IE was trying to extend PONTRYAGUIN's theory to partial differential equations, probably after he had found a problem of optimization without a solution because no function could satisfy the necessary conditions that he had derived (I suppose that [20] is the right reference). K. LUR'IE had devised a way to obtain necessary conditions of optimality which improved the one consisting in just perturbing the interface between two materials: he would cut out many small balls in one region and many small balls of the same volume in the other region and exchange the contents of these balls, and then he would estimate the change in the cost function to be minimized, giving a necessary condition of optimality. Up to this point one can say that this is almost PONTRYAGUIN's idea, but then he thought that there was no reason to restrict oneself to spherical shapes and he cut out many small ellipsoids and found that it was better to choose very flat ellipsoids with short axes perpendicular to some direction, and in the limit it started looking like a layered medium. I do not know the level of mathematical precision that K. LUR'IE had used for these computations, but my guess is that they were slightly formal, but he certainly had the right idea, and he was certainly glad to find in [10] a purely mathematical framework that did exactly what he had discovered in a clever way. As K. LUR'IE pointed out to J.-L. ARMAND, he had been lucky that [10] had been published in the proceedings of an optimization conference, as this had enabled him to discover more easily that reference.

The method that we had developped is adapted to minimizing functionals like (8), which are weakly continuous in $u \in H_0^1(\Omega)$, and this is an important limitation for some applications. It is useful then to consider more general functionals, like

$$J(a) = \int_\Omega G\Big(x, u, a, grad(u)\Big)\, dx, \tag{18}$$

where $grad(u)$ occurs in a nontrivial way, as some cases like

$$G\Big(x, u, a, grad(u)\Big) = \varphi(x,u)\Big(a\, grad(u).grad(u)\Big) + \Big(grad(u).a\, \psi_1(x,u) + \psi_2(x,u)\Big) + g(x,u,a), \tag{19}$$

can be handled by our method when $\varphi, \psi_1, \psi_2, g$ have natural regularity and growth properties, so that any G satisfying (19) can be said to depend upon $grad(u)$ in a fake way. Results for functionals depending upon $grad(u)$ in a nontrivial way are still very fragmentary, and I have partial results which extend those that I had described in [21], but I will not discuss them here.

Of great importance for applications are questions related to Elasticity, and the first results following our ideas were probably those of R. KOHN & Gilbert STRANG [22]. Unfortunately most works have been concerned with inadequate approximations like Linearized Elasticity and only consider functionals whose dependence upon the stress is fake. As one aspect of Optimal Design in Elasticity consists in cutting holes out of plates in order to keep enough strength but use less material, it is useful to point out that the system of Linearized Elasticity and the presence of holes creates a few technicalities in the mathematical apparatus, which are often just swept under the rug in many articles which are then incomplete mathematical papers.

It has been suggested by Owen RICHMOND [23] that dealing with plates with holes could lead to theories with higher order gradients, and although I am not entirely sure about what precise mathematical result to conjecture, it seems clear that one has better go back to the derivation of the equations for plates, starting from 3-dimensional Finite Elasticity, which is the only type of Elasticity that real materials can follow, and then study the various possible limiting behaviours as the thickness and the typical distance between holes tend to zero.

Most mathematical problems of control are idealizations, as real situations are usually much too complex to be analyzed with existing mathematical tools, and it does not make much sense then to specialize on only one particular functional, as is mostly done by those interested in Linearized Elasticity. The role of mathematicians is to put in evidence general methods for solving large classes of problems, and we had put a message in [19], namely that, after spending many years trying to characterize the best information on effective properties in terms of proportions, we had realized that we did not need such a knowledge and that a large class of Optimal Design problems, with state described by equations like (6) and cost functions described by functionals like (8), could be solved by generalized solutions corresponding to mixtures. We could have mentioned that our method could be applied as well to functionals like (18)-(19), which depend upon $grad(u)$ in a fake way, but we did not think about it. As I have mentioned, our method does not apply directly to functionals whose dependence upon $grad(u)$ is not fake. It was pointed out by R. KOHN that one needs to know a little more about the characterization of effective properties in the case where for the same a one solves (6) with various right side f_j and the cost function uses explicitly the corresponding solutions u_j.

At a meeting in Trieste in September 1993, Martin BENDSØE had described a numerical approach which avoided a precise characterization of effective coefficients in Elasticity, and this had enraged K. LUR'IE, but what M. BENDSØE was saying was consistent with our message of twelve years ago: it is important to realize that Homogenization plays a role in some problems of Optimal Design, but it is also important to realize that fortunately it is not entirely necessary to characterize which effective properties are possible for given proportions of various materials used for creating every possible mixture. The efficient method that M. BENDSØE was describing took advantage of that philosophy, and added to it the necessity of an interacting procedure: in practical applications no one gives a precise functional to minimize, and it is not always useful to spend too much time minimizing in detail a functional which is only used at a given instant of the search for an efficient design, and one might want to use the information about generalized solutions of various cost functions in order to discover a purely classical efficient design, as the technological cost of creating these mixtures has not been taken into account.

The purpose of this article is then to try to simplify the technical details of the approach that we had invented twenty years ago, and as the characterization of effective properties has not yet become a simple matter, it will have to be avoided.

Position of the problem

Assume that $u \in H_0^1(\Omega)$ is the solution of the equation

$$-div\Big(A\, grad(u)\Big) = f \text{ in } \Omega, \tag{20}$$

where $A \in \mathcal{M}(\alpha, \beta; \Omega)$ is symmetric, i.e. $\alpha I \leq A \leq \beta I$ a.e. in Ω, and A is chosen among some available materials, which might be in limited quantity. One makes the assumption that all the materials used can be rotated in arbitrary way, and for simplifying the details, one assumes that only a finite number of materials $M_1, \ldots, M_m$, are available. The admissible A are then of the form

$$A = \sum_{i=1}^{m} \chi_i R^T M_i R, \tag{21}$$

with

$$R \in L^\infty\Big(\Omega, SO(N)\Big), \tag{22}$$

and the characteristic functions $\chi_i, i = 1, \ldots, m$, must satisfy

$$\begin{aligned} &\sum_{i=1}^{m} \chi_i = 1 \text{ a.e. in } \Omega \\ &\int_\Omega \chi_i \, dx \leq \gamma_i, i = 1, \ldots, m, \end{aligned} \tag{23}$$

where γ_i denotes the given available quantity of material $\#i$, $i = 1, \ldots, m$. Of course, one assumes that

$$\sum_{i=1}^{m} \gamma_i \geq meas(\Omega), \tag{24}$$

so that there is enough material to fill Ω with, and there are characteristic functions satisfying the constraints. The problem studied is then to find if there is a classical Optimal Design A minimizing the cost function

$$J(A) = \int_\Omega \Big(\sum_{i=1}^{m} \chi_i g_i(x,u) \Big)\, dx + \sum_{i=1}^{m} h_i \Big(\int_\Omega \chi_i \, dx \Big), \tag{25}$$

to introduce a relaxed problem describing generalized solutions corresponding to mixtures of the originally available materials, to derive necessary conditions of optimality for generalized solutions (conditions which are of course valid for classical solutions), to describe stable algorithms for computing generalized solutions (and classical solutions when they exist), and more generally to learn as much as possible about how to attack more general questions. Of course, the method described applies to other functionals J, but (25) serves as a prototype, where the important features of this class of problems can be discovered and studied.

One assumes that

$$u \mapsto g_i(x,u) \text{ is continuous from } H_0^1(\Omega) \text{ weak into } L^1(\Omega) \text{ strong}, i = 1, \ldots, m, \tag{26}$$

which means that $g_i, i = 1, \ldots, m$, satisfy CARATHÉODORY conditions with suitable growth with respect to u. Such a property generalizes to functionals depending upon $grad(u)$ in a fake way, but it does not extend to general functionals depending upon $grad(u)$.

The method that we had initially introduced consisted in constructing a relaxed problem where u still solves (20) but A must satisfy

$$A(x) \in \mathcal{K}\Big(\theta_1(x), \ldots, \theta_m(x)\Big) \text{ a.e. } x \in \Omega. \tag{27}$$

where $\mathcal{K}(\theta_1, \ldots, \theta_m)$ denotes the set of all possible effective tensors associated to mixtures using the initially available materials $M_1, \ldots, M_m$, with local proportions $\theta_1, \ldots, \theta_m$, and to minimize the relaxed functional

$$J_1(A, \theta_1, \ldots, \theta_m) = \int_\Omega \Big(\sum_{i=1}^{m} \theta_i g_i(x,u) \Big)\, dx + \sum_{i=1}^{m} h_i \Big(\int_\Omega \theta_i \, dx \Big), \tag{28}$$

$\theta_1, \ldots, \theta_m$, satisfying the constraints

$$\begin{aligned} &0 \leq \theta_i \leq 1, i = 1, \ldots, m, \sum_{i=1}^{m} \theta_i = 1 \text{ a.e. in } \Omega \\ &\int_\Omega \theta_i \, dx \leq \gamma_i, i = 1, \ldots, m. \end{aligned} \tag{29}$$

The main difficulty is that the sets $\mathcal{K}(\theta_1, \ldots, \theta_m)$ are not known in general. We had characterized the case where $m = 2$ with M_1 and M_2 isotropic, but even the case $m = 1$ with M_1 anisotropic is not completely understood yet. Fortunately, the characterization of $\mathcal{K}(\theta_1, \ldots, \theta_m)$ can be avoided.

Presentation of the results

The method that I present here is based on the fact that, although $\mathcal{K}(\theta_1, \ldots, \theta_m)$ is not known, one can characterize the sets

$$\mathcal{K}(\theta_1, \ldots, \theta_m)E = \Big\{ A\,E : \; A \in \mathcal{K}(\theta_1, \ldots, \theta_m) \Big\}, E \in R^N. \tag{30}$$

In order to simplify the notation, $(\theta_1, \ldots, \theta_m)$ will be abbreviated as θ.

Proposition 1. Assume that $N \geq 2$. For any symmetric M, let $\lambda_1(M)$ denote the smallest eigenvalue of M and let $\lambda_N(M)$ denote the largest eigenvalue of M. Define $\lambda_-(\theta)$ and $\lambda_+(\theta)$ by

$$\begin{aligned} \frac{1}{\lambda_-(\theta)} &= \sum_{i=1}^{m} \frac{\theta_i}{\lambda_1(M_i)}, \\ \lambda_+(\theta) &= \sum_{i=1}^{m} \theta_i \lambda_N(M_i). \end{aligned} \tag{31}$$

Then

$$D \in \mathcal{K}(\theta)E \text{ if and only if } \Big(D - \lambda_-(\theta)E . D - \lambda_+(\theta)E\Big) \leq 0, \tag{32}$$

or equivalently

$$\mathcal{K}(\theta)E \text{ is the closed ball with diameter } [\lambda_-(\theta)E, \lambda_+(\theta)E], \tag{33}$$

or

$$\mathcal{K}(\theta)E = \Big\{B\,E : \lambda_-(\theta)I \leq B \leq \lambda_+(\theta)I\Big\}. \tag{34}$$

Using this characterization, one shows then that a relaxed problem consists in minimizing J_1 given by (28), where θ still satisfies (29) and u is still given by (20), but where A satisfies now

$$A \in \mathcal{B}(\theta) = \Big\{B : \lambda_-(\theta)I \leq B \leq \lambda_+(\theta)I\Big\}. \tag{35}$$

Once one has solved this problem, one uses Proposition 1 to replace $A \in \mathcal{B}(\theta)$ by some $A_{eff} \in \mathcal{K}(\theta)$ such that $A_{eff}\, grad(u) = A\, grad(u)$ a.e. in Ω, and one has a generalized solution of the initial problem.

Although the understanding of Homogenization has been instrumental in discovering which relaxed problem to introduce, it is useful to have a direct proof of the existence of a solution of this relaxed problem that uses as little as possible from the theory of Homogenization. It is even useful to forget that θ comes from proportions and prove the following more general result.

Proposition 2. Let Θ be a nonempty bounded weak $\star$ closed convex set of V', where V is a separable BANACH space. Let μ_-, μ_+ be two maps from Θ to $L^\infty(\Omega)$ such that

$$\begin{aligned} &\alpha \leq \mu_-(\theta) \leq \mu_+(\theta) \leq \beta, \text{ a.e. in } \Omega, \text{ for all } \theta \in \Theta, \\ &\frac{1}{\mu_-} \text{ and } \mu_+ \text{ are (sequentially) weakly } \star \text{ upper semi-continuous on } \Theta, \end{aligned} \tag{36}$$

where the usual order and weak $\star$ topology are used for $L^\infty(\Omega)$. Let A be a symmetric tensor satisfying

$$\mu_-(\theta)I \leq A \leq \mu_+(\theta)I, \text{ a.e. in } \Omega. \tag{37}$$

The set of admissible (θ, A) is then convex and (sequentially) weak $\star$ compact in $V' \times L^\infty\Big(\Omega; \mathcal{L}_s(R^N, R^N)\Big)$, and if u is solution of (20), the set of resulting (θ, u) is (sequentially) weak $\star$ compact in $V' \times H_0^1(\Omega)$, so that every functional which is (sequentially) weak $\star$ continuous on $\Theta \times H_0^1(\Omega)$ attains its minimum.

The next step is to derive (first order) necessary conditions of optimality. One assumes now that the functions $h_i, i = 1, \ldots, m$, are differentiable, and that

$$v \mapsto \frac{\partial g_i}{\partial u}(x, v) \text{ is continuous from } H_0^1(\Omega) \text{ weak into } H^{-1}(\Omega) \text{ strong}, i = 1, \ldots, m. \tag{38}$$

The purpose of that condition is to have a functional which is GÂTEAUX differentiable with a GÂTEAUX derivative which is continuous from $H_0^1(\Omega)$ weak into $H^{-1}(\Omega)$ strong, and the following necessary conditions are valid in the general framework of Proposition 2 under such hypotheses, but in order to simplify the exposition, only the functional (28) will be considered.

Proposition 3. Assume that (38) holds and that (θ^*, A^*) is an optimal solution which minimizes the functional (28) under the constraints (29), (35). Denoting the corresponding state by u^*, one defines the adjoint state $p^* \in H_0^1(\Omega)$ as the solution of

$$-div\Big(A^* grad(p^*)\Big) = \sum_{i=1}^{m} \theta_i^* \frac{\partial g_i}{\partial u}(x, u^*) \text{ in } \Omega, \tag{39}$$

and a necessary condition of optimality is that

$$\begin{aligned} &\int_\Omega \Big[\sum_{i=1}^{m} \theta_i k_i^* - \Big(A\, grad(u^*).grad(p^*)\Big)\Big]\, dx \text{ is minimum at } (\theta^*, A^*) \\ &\text{when } (\theta, A) \text{ satisfy the constraints (29), (35), and} \\ &k_i^*(x) = g_i(x, u^*) + h_i'\Big(\int_\Omega \theta_i^*\, dx\Big), \text{ a.e. in } \Omega, i = 1, \ldots, m. \end{aligned} \tag{40}$$

The only information about the constraints (29) and (35) which has been used in order to derive (40) is that the set of admissible (θ, A) is convex, and the next step is to interpret what (40) means for the precise constraints (29), (35). Let

$$\Omega_0 = \{x \in \Omega, |grad(u^*)|\, |grad(p^*)| = 0\}. \tag{41}$$

Taking $\theta = \theta^*$ and varying A in $\mathcal{B}(\theta^*)$, one deduces from (40) some information about A^* outside Ω_0. Let

$$\begin{aligned} e_u &= \frac{grad(u^*)}{|grad(u^*)|} \text{ in } \Omega \setminus \Omega_0, \\ e_p &= \frac{grad(p^*)}{|grad(p^*)|} \text{ in } \Omega \setminus \Omega_0, \\ &\cos(\varphi^*) = (e_u.e_p), \text{ with } 0 \le \varphi^* \le \pi \text{ in } \Omega \setminus \Omega_0, \end{aligned} \tag{42}$$

then (40) implies

$$\begin{aligned} A^* e_u &= \frac{\lambda_+(\theta^*) + \lambda_-(\theta^*)}{2} e_u + \frac{\lambda_+(\theta^*) - \lambda_-(\theta^*)}{2} e_p \text{ in } \Omega \setminus \Omega_0, \\ A^* e_p &= \frac{\lambda_+(\theta^*) - \lambda_-(\theta^*)}{2} e_u + \frac{\lambda_+(\theta^*) + \lambda_-(\theta^*)}{2} e_p \text{ in } \Omega \setminus \Omega_0. \end{aligned} \tag{43}$$

On the set where $e_p = e_u$, i.e. the subset of $\Omega \setminus \Omega_0$ where $\varphi^* = 0$, one can create A^* by using a fibered material with fibers parallel to e_u, where the material M_i (used with proportion θ_i^*) is turned so that e_u is an eigenvector for the eigenvalue $\lambda_N(M_i)$. On the set where $e_p = -e_u$, i.e. the subset of $\Omega \setminus \Omega_0$ where $\varphi^* = \pi$, one can create A^* by using a layered material with layers perpendicular to e_u, where the material M_i (used with proportion θ_i^*) is turned so that e_u is an eigenvector for the eigenvalue $\lambda_1(M_i)$. On the set where $e_p \neq \pm e_u$, i.e. the subset of $\Omega \setminus \Omega_0$ where $0 < \varphi^* < \pi$, one can create A^* by using a layered material with layers perpendicular to $e_u - e_p$, where the material M_i (used with proportion θ_i^*) is turned so that $e_u - e_p$ is an eigenvector for the eigenvalue $\lambda_1(M_i)$ and $e_u + e_p$ is an eigenvector for the eigenvalue $\lambda_N(M_i)$.

With the notations of (42), the necessary condition of optimality (40) becomes equivalent to

$$\begin{aligned} &\int_\Omega \Big[\sum_{i=1}^{m} \theta_i k_i^* + |grad(u^*)|\, |grad(p^*)| \Big(-\lambda_+(\theta) \cos^2\Big(\frac{\varphi^*}{2}\Big) + \lambda_-(\theta) \sin^2\Big(\frac{\varphi^*}{2}\Big)\Big)\Big]\, dx \\ &\text{is minimum at } \theta^* \text{ when } \theta \text{ satisfies the constraints (29).} \end{aligned} \tag{44}$$

As the integrand in (44) is convex in θ, the condition is equivalent to a first order condition

$$\int_\Omega \Big(\sum_{i=1}^m \theta_i K_i^*\Big)\,dx \text{ is minimum at } \theta^* \text{ when } \theta \text{ satisfies the constraints (29),}$$
$$K_i^*(x) = k_i^*(x) - |grad(u^*)|\,|grad(p^*)|\Big[\lambda_N(M_i)\cos^2\Big(\frac{\varphi^*}{2}\Big) + \frac{\lambda_1(M_i)}{\big(\lambda_-(\theta^*)\big)^2}\sin^2\Big(\frac{\varphi^*}{2}\Big)\Big]. \tag{45}$$

One can then transform the preceding condition by using LAGRANGE multipliers. It is useful to notice that up to that point, the analysis has only consisted in observing what A^* should be in terms of θ^*, and that the set of admissible (θ, A) is convex, and therefore everything that was done can be extended to the framework of Proposition 2, and after one has eliminated A one is led to a problem of minimization on Θ.

As one has not obtained information about A^* on Ω_0, one notices that on the subset where $grad(u^*) = 0$, one can change A^* without changing the solution of (20), and therefore one can choose there a particular A^* corresponding to a layered material. This argument does not work on the set where $grad(u^*) \neq 0$ and $grad(p^*) = 0$, and one adapts an argument of RAITUM [24] for changing A^* on this set. Let

$$\Omega_c = \Big\{x \in \Omega : \Big(A^*grad(u^*) - \lambda_-(\theta^*)grad(u^*).A^*grad(u^*) - \lambda_+(\theta^*)grad(u^*)\Big) < 0\Big\}, \tag{46}$$

i.e. the set where $A^*grad(u^*)$ cannot be obtained by a layered material, and the preceding analysis shows that on Ω_c one must have $grad(u^*) \neq 0$ and $grad(p^*) = 0$. Denoting $E^* = grad(u^*)$ and $D^* = A^*grad(u^*)$, one defines

$$\begin{aligned}\Theta_c = \Big\{\theta \in L^\infty(\Omega_c, R^m) : \Big(D^* - \lambda_-(\theta)E^*.D^* - \lambda_+(\theta)E^*\Big) \le 0 \text{ a.e. in } \Omega_c,\\ 0 \le \theta_i \le 1, i = 1, \ldots, m, \sum_{i=1}^m \theta_i = 1, \text{ a.e. in } \Omega_c,\\ \int_{\Omega_c} \theta_i\,dx = \int_{\Omega_c} \theta_i^*\,dx, i = 1, \ldots, m.\Big\},\end{aligned} \tag{47}$$

which is nonempty as it contains θ^*, and convex (weakly $\star$ compact) because λ_- is convex and λ_+ is concave. Then one defines the functional J_c on Θ_c by

$$J_c(\theta) = \int_{\Omega_c} \Big(\sum_{i=1}^m \theta_i g_i(x, u)\Big)\,dx,, \tag{48}$$

and J_c attains its minimum on a weakly $\star$ compact convex subset $\Theta^{opt} \subset \Theta_c$, containing θ^*, and on Ω_c one replaces θ^* by an extreme point θ^{opt} of Θ^{opt}. Using an argument which I learned from Zvi ARTSTEIN in 1975 [25], one shows that one must have $\big(D^* - \lambda_-(\theta^{opt})E^*.D^* - \lambda_+(\theta^{opt})E^*\big) = 0$, and the property of λ_- and λ_+ which is used in this argument is that if one has $\theta_i^{opt} = 1$ for some i on a subset $\omega \subset \Omega_c$, then one has $\lambda_-(\theta^{opt}) = \lambda_+(\theta^{opt})$ a.e. on ω.

Proofs

An essential ingredient in the proofs of the preceding results is the following lemma.

Lemma 4. Assume that

$$\begin{aligned}&E^{(n)} \rightharpoonup E^{(\infty)} \text{ in } L^2(\Omega; R^N) \text{ weak},\\ &D^{(n)} \rightharpoonup D^{(\infty)} \text{ in } L^2(\Omega; R^N) \text{ weak},\\ &(E^{(n)}.D^{(n)}) \rightharpoonup (E^{(\infty)}.D^{(\infty)}) \text{ in } \mathcal{M}(\Omega) \text{ weak } \star,\\ &\Big(D^{(n)} - b_n E^{(n)}.D^{(n)} - a_n E^{(n)}\Big) \le 0 \text{ a.e. in } \Omega,\end{aligned} \tag{49}$$

where

$$\begin{aligned}&\alpha \le b_n \le a_n \le \beta \text{ a.e. in } \Omega,\\&\frac{1}{b_n} \rightharpoonup \frac{1}{b_\infty} \text{ in } L^\infty(\Omega) \text{ weak } \star,\\&a_n \rightharpoonup a_\infty \text{ in } L^\infty(\Omega) \text{ weak } \star,\end{aligned} \tag{50}$$

then one has

$$\Big(D^{(\infty)} - b_\infty E^{(\infty)}.D^{(\infty)} - a_\infty E^{(\infty)}\Big) \le 0 \text{ a.e. in } \Omega. \tag{51}$$

Of course, the information on $(E^{(n)}.D^{(n)})$ will usually be deduced by integration by parts (or by applying the Div-Curl lemma) when $E^{(n)} = grad(u_n)$ with $u_n \in H_0^1(\Omega)$ satisfying an equation like (20), but it is useful to realize that if Homogenization has played an essential role in the development of this method, most of the results can be explained with few technical results from Homogenization, and one must then isolate the crucial steps in the analysis. Lemma 4 is actually a consequence of the following more general result.

Lemma 5. On the domain $a > b > 0, E, D \in R^N$

$$\frac{1}{a-b}\Big(D - b\,E.D - a\,E\Big) \text{ is a convex function of } E, D, (E.D), a, \frac{1}{b}. \tag{52}$$

More precisely

$$\frac{1}{a-b}\Big(D - b\,E.D - a\,E\Big) = \sup_{v,w\in R^N}\Big\{-(D.E) + 2(E.v) + 2(D.w) - \frac{|v|^2}{b} - 2(v.w) - a|w|^2\Big\}. \tag{53}$$

Indeed the supremum is attained when

$$\begin{aligned}&\frac{v}{b} + w = E,\\&v + a\,w = D,\end{aligned} \tag{54}$$

i.e.

$$\begin{aligned}v &= \frac{b(a\,E - D)}{a-b},\\ w &= \frac{D - b\,E}{a-b},\end{aligned} \tag{55}$$

from which (53) follows. Notice that the function can be extended to be 0 if $a = b$ and $D = a\,E$ and $+\infty$ elsewhere.

Lemma 4 follows as one has $-(D^{(n)}.E^{(n)}) + 2(E^{(n)}.v) + 2(D^{(n)}.w) - \frac{|v|^2}{b} - 2(v.w) - a|w|^2 \le 0$ a.e. in Ω for all $v, w \in R^N$, and therefore the same inequality is true with (n) replaced by (∞), and taking then the supremum in v, w gives (51).

If $D = R^T M_i R E$ for some $R \in SO(N)$, then elementary arguments of Linear Algebra show that $(D - \lambda_1(M_i)E.D - \lambda_N(M_i)E) \le 0$. Let $A^{(n)}$ be given by (21) for sequences of characteristic functions $\chi_i^{(n)}$ converging to θ_i in $L^\infty(\Omega)$ weak $\star$ for $i = 1, \ldots, m$, let $E^{(n)} = grad(u_n)$ where u_n solves (20) and $D^{(n)} = A^{(n)}E^{(n)}$. If $E^{(n)} \rightharpoonup E^{(\infty)}$ in $L^\infty(\Omega; R^N)$ weak $\star$ and $D^{(n)} \rightharpoonup A_{eff}E^{(\infty)}$, then one can apply Lemma 4 with $b_n = \sum_{i=1}^m \chi_i^n \lambda_1(M_i)$ and $a_n = \sum_{i=1}^m \chi_i^n \lambda_N(M_i)$, so that $b_\infty = \lambda_-(\theta)$ and $a_\infty = \lambda_+(\theta)$, and Lemma 4 implies the first part of (32), i.e. if $D \in \mathcal{K}(\theta)$ one has $(D - \lambda_-(\theta)E.D - \lambda_+(\theta)E) \le 0$.

Let e_1, e_2 be two orthogonal unit vectors. If one creates a laminated material with layers perpendicular to e_1 by using the material M_i with proportion θ_i and choosing the rotation such that e_1 is an eigenvector for the eigenvalue $\lambda_1(M_i)$ and e_2 is an eigenvector for the eigenvalue $\lambda_N(M_i)$, then any effective tensor obtained (as for $N > 3$ the rotation is not completely determined) must have e_1 as an eigenvector for the eigenvalue $\lambda_-(\theta)$ and e_2 as an eigenvector for the eigenvalue $\lambda_+(\theta)$. For such an effective tensor one has $(D - \lambda_-(\theta)E.D - \lambda_+(\theta)E) = 0$ if E is any combination of e_1 and e_2.

The precedings arguments have shown that the set $\{D = A\,E : A \in \mathcal{K}(\theta)\}$ is included in the closed ball of diameter $[\lambda_-(\theta)E, \lambda_+(\theta)E]$ and contains its boundary. That it also contains its interior follows from the following convexity result.

Lemma 6. For any $E \in R^N$ the set $\{A\,E : A \in \mathcal{K}(\theta)\}$ is a (closed) convex set of R^N. More generally, for any $k = 1, \dots, N-1$, and $E_1, \dots, E_k \in R^N$ the set $\{(A\,E_1, \dots, A\,E_k) : A \in \mathcal{K}(\theta)\}$ is a (closed) convex set of $(R^N)^k$.

This follows from the formula for layering two materials with layers perpendicular to the unit vector e. If one uses material A with proportion η and material B with proportion $1 - \eta$, then the effective tensor is given by

$$A_{eff} = \eta\,A + (1-\eta)B - \eta(1-\eta)(B-A)\frac{e \otimes e}{\eta(Be.e) + (1-\eta)(Ae.e)}(B-A), \quad (56)$$

and therefore $A_{eff}E = \eta\,A\,E + (1-\eta)B\,E$ if $(B-A)E$ is perpendicular to e. For $k \le N-1$, there exists $e \ne 0$ orthogonal to $(B-A)E_i$ for $i = 1, \dots, k$, showing the lemma.

For $N = 2$, one can have $\mathcal{K}(\theta) = \{A : I \le A \le 2I, det(A) = 2\}$, which is not convex, and therefore $k = N$ cannot be allowed in Lemma 6.

Proposition 2 follows easily from Lemma 4. Let θ_n be a minimizing sequence converging to θ_∞ in Θ weak $\star$, let $A^{(n)}$ satisfy (37) for θ_n, let $E^{(n)} = grad(u_n)$ and $D^{(n)} = A^{(n)}grad(u_n)$ where u_n is the solution of (20) for $A^{(n)}$. One can apply Lemma 4 with $b_n = \mu_-(\theta_n)$ and $a_n = \mu_+(\theta_n)$ and as (36) implies that one will have $\mu_-(\theta_\infty) \le b_\infty$ and $a_\infty \le \mu_+(\theta_\infty)$, one deduces that $D^{(\infty)} = A^{(\infty)}E^{(\infty)}$ with some $A^{(\infty)}$ satisfying (37) for θ_∞.

Once one has noticed that the set of (θ, A) is convex, Proposition 3 consists in writing that $J(A^*, \theta^*) \le J(A_\varepsilon, \theta_\varepsilon)$ with $\theta_\varepsilon = (1-\varepsilon)\theta^* + \varepsilon\theta$ and $A_\varepsilon = (1-\varepsilon)A^* + \varepsilon A$ for $0 \le \varepsilon \le 1$, and compute the derivative with respect to ε at $\varepsilon = 0$. As u becomes $u^* + \varepsilon\delta u + o(\varepsilon)$, δu appears in the expression of the derivative and the introduction of the adjoint state p^* has the effect of eliminating δu and have only $\theta - \theta^*$ and $A - A^*$ appear, and (40) is what one obtains by this classical procedure. The rest is interpretation of (40), using elementary Linear Algebra for the question of maximizing $(A\,grad(u^*).grad(p^*))$ for A satisfying (35) for θ^*.

As a final comment, I want to point out that if I have avoided the yet poorly understood question of characterizing effective properties of mixtures in terms of the proportions used, I have used some general knowledge about Homogenization which corresponds to what I had taught in my PECCOT lectures in 1977. The corresponding material is described in the already written part of my lecture notes [26], for which I am still writing the third part.

Acknowledgements

My research work has been supported by CARNEGIE-MELLON University, the National Science Foundation through grant DMS-94-01310, and the Army Research Office, through a grant to the Center for Nonlinear Analysis.

References

[1] LIONS J.-L., *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod - Gauthier-Villars, Paris, 1968.

[2] MURAT F., "Un contre-exemple pour le problème du contrôle dans les coefficients," C. R. Acad. Sci. Paris, Sér. A-B 273 (1971), A708-A711.

[3] MURAT F., "Théorèmes de non-existence pour des problèmes de contrôle dans les coefficients," C. R. Acad. Sci. Paris, Sér. A-B 274 (1972), A395-A398.

[4] SPAGNOLO S., "Sul limite delle soluzioni di problemi di Cauchy relativi all' equazione del calore," Ann. Scuola Norm. Sup. Pisa (3) 21 (1967), 657-699.

[5] SPAGNOLO S., "Sulla convergenza di soluzioni di equazioni paraboliche ed ellitiche," Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 571-597.
[6] ZOLEZZI T., "Teoremi d'esistensa per problemi di controllo ottimo retti da equazioni ellittiche o paraboliche," Rend. Sem. Mat. Padova, 44 (1970), 155-173.
[7] SANCHEZ-PALENCIA E., "Solutions périodiques par rapport aux variables d'espaces et applications," C. R. Acad. Sci. Paris, Sér. A-B 271 (1970), A1129-A1132.
[8] SANCHEZ-PALENCIA E., "Equations aux dérivées partielles dans un type de milieux hétérogènes," C. R. Acad. Sci. Paris, Sér. A-B 272 (1972), A395-A398.
[9] TARTAR L., "Convergence d'opérateurs différentiels," *Atti Giorni Analisi Convessa e Applicazioni* (*Roma*, 1974), pp. 101-104.
[10] TARTAR L., "Problèmes de contrôle des coefficients dans des équations aux dérivées partielles," *Control theory, numerical methods and computer systems modelling (Internat. Sympos., IRIA LABORIA, Rocquencourt,* 1974) pp. 420-426. *Lecture Notes in Econom. and Math. Systems, Vol.* 107, *Springer, Berlin,* 1975.
[11] CÉA J. & MALANOWSKI K., "An example of a max-min problem in partial differential equations," SIAM J. Control, 8 (1970), 305-316.
[12] MURAT F. & SIMON J., "Sur le contrôle par un domaine géométrique," *Publication 76015, Laboratoire d'Analyse Numérique, Université Paris 6,* 1976.
[13] BABUŠKA I., "Homogenization and its application. Mathematical and computational problems," *Numerical Solution of Partial Differential Equations-III, (SYNSPADE 1975, College Park MD, May* 1975) pp. 89-116. *Academic Press, New York,* 1976.
[14] MURAT F., "H-convergence," *Séminaire d'analyse fonctionnelle et numérique, Université d'Alger,* 1977-78.
[15] TARTAR L., "Estimations de coefficients homogénéisés," *Computing methods in applied sciences and engineering (Proc. Third Internat. Sympos., Versailles,* 1977), I, pp. 364-373. *Lecture Notes in Math.*, 704, *Springer, Berlin,* 1979.
[16] TARTAR L., "Compensated compactness and applications to partial differential equations," *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV,* pp. 136-212. *Res. Notes in Math.*, 39, *Pitman, San Francisco, Calif.*, 1979.
[17] TARTAR L., "Estimations fines de coefficients homogénéisés", *Ennio de Giorgi Colloquium,* pp. 168-187. *Research Notes in Mathematics* 125, *Pitman, London,* 1985.
[18] JOSEPH D., *Stability of fluid motions*, Springer tracts in natural philosophy, v. 3, Springer-Verlag, Berlin, New York, 1976.
[19] MURAT F. & TARTAR L., "Calcul des variations et homogénéisation," *Les Méthodes de l'Homogénéisation: Théorie et Applications en Physique,* pp. 319-369. *Coll. de la Dir. des Etudes et Recherches de Elec. de France* 57, *Eyrolles (Paris)*, 1985.
[20] LURIE K.A., "On the optimal distribution of the resistivity tensor of the working substance in a magnetohydrodynamic channel," J. Appl. Mech. (PMM) 34 (1970), 255-274.
[21] TARTAR L., "Remarks on Optimal Design Problems," *Calculus of variations, Homogenization and Continuum Mechanics, G. BUTTAZZO, G. BOUCHITTÉ, P. SUQUET eds.*, 279-296. *World Scientific, Singapore,* 1994.
[22] KOHN R. & STRANG G., "Optimal design and relaxation of variational problems, I, II, III," Comm. Pure Appl. Math. 39 (1986), 113-137, 139-182, 353-377.
[23] RICHMOND O., Personal communication, Alcoa Research Center, Fall 1987.
[24] RAITUM U.E., "The extension of extremal problems connected with a linear elliptic equation," Soviet Math. Dokl. 19, (1978), 1342-1345.
[25] ARTSTEIN Z., "Look for the extreme points", SIAM Review 22 (1980), 172-185.
[26] TARTAR L., *Homogenization, Compensated Compactness and H-Measures*, (CBMS-NSF Conference, Santa Cruz, June 1993), Lectures Notes in preparation.

Note: A forthcoming book, edited by R. KOHN and published by Birkhäuser, will contain translations into English of some of the early references on Optimal Design which were originally published in Russian or in French, namely [10, 14, 15, 19]; the translation of [10, 14] will appear as joint work with F. MURAT, as indeed they were mostly describing some joint work that we had described separately.