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On Material Frame-Indifference

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by Walter Noll, November 1995

1. Introduction.

There is a considerable amount of confusion in the literature about the meaning of material frame-indifference, even among otherwise knowledgeable people. This paper is an attempt at clarification. I first started thinking about this issue when I was a student and tried to learn the theories of linearized elasticity and of linearly viscous fluids ("Navier-Stokes fluids"). In the former, one assumes that the stress $\mathbf{T}$ at a material point is determined by the gradient of the displacement-vector field $\mathbf{u}$. In the latter, one assumes that the stress $\mathbf{T}$ at a material point is determined by the gradient of the velocity field $\mathbf{v}$ and the density. For either case, denote the function which describes the dependence on the gradient by $\hat{\mathbf{T}}$. Then the following additional assumptions are most often introduced, not necessarily in the order and in the form given here.

(1) $\hat{\mathbf{T}}(\mathbf{A})$ depends only on the symmetric part of $\mathbf{A}$, i.e. †)

$$\hat{\mathbf{T}}(\mathbf{A}) = \hat{\mathbf{T}}(\frac{1}{2}(\mathbf{A} + \mathbf{A}^{\top})) \quad \text{for all lineons } \mathbf{A}.$$

(2) The function $\hat{\mathbf{T}}$ is linear.

(3) We have $\hat{\mathbf{T}}(\mathbf{QEQ}^{\top}) = \hat{\mathbf{T}}(\mathbf{E})$ for all symmetric lineons $\mathbf{E}$ and all orthogonal lineons $\mathbf{Q}$.

Using some fairly elementary pure mathematics, it is then proved that the function $\hat{\mathbf{T}}$ must be given by the following specific formula:

$$\hat{\mathbf{T}}(\mathbf{E}) = 2\mu\mathbf{E} + (\lambda(\text{tr}\mathbf{E}) + p)\mathbf{1}_{\mathcal{V}} \quad \text{for all symmetric lineons } \mathbf{E}.$$

(In the case of elasticity, μ is the shear modulus, $\lambda + \frac{2}{3}\mu$ is the modulus of compression, and p is most often assumed to be zero. In the case of viscous fluid theory μ is the shear viscosity, $\lambda + \frac{2}{3}\mu$ is the bulk viscosity, and p is the pressure, all of which may depend on the density.)

Most of the justifications that I found in the textbooks for these assumptions were mysterious to me. I was not satisfied by the justification that the two theories have been spectacularly successful for describing many physical phenomena and for designing machines, bridges, ships, airplanes, etc.. I now know that the assumption (1) is a consequence of the principle of frame-indifference.

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†) We use the mathematical infrastructure, notation, and terminology of [FDS]. In particular, we use "lineon" as an abbreviation for "linear transformation from an inner-product space to itself". Many people use the term "tensor", but this term has also many other meanings and hence "lineon" is more specific. Given a lineon $\mathbf{A}$, we denote its transpose (called "adjoint" by some) by $\mathbf{A}^{\top}$ and its trace by tr$\mathbf{A}$.

In the case of viscous fluid theory, assumption (3) also follows from the principle of frame-indifference. In the case of elasticity, however, assumption (3) means that the material possesses a special kind of material symmetry, namely isotropy; there are many elastic materials for which (3) is not appropriate. In the case of elasticity, assumption (2) follows from the fact that the theory is obtained by linearization from the theory of finite elasticity and can be valid only approximately for small deformations from a stress-free "natural" configuration. (If large deformations are taken into account, then assumption (1) becomes inconsistent with the principle of frame-indifference.) In the case of viscous fluid theory, the only justification for assumption (2) that I know of is that it comes out as a first approximation when one applies the retardation theorem for general simple fluids as described in [CN].

In classical particle physics it is often assumed that the force that is exerted at a given instant by one particle on another depends only on the position of the two particles. Then the following additional assumptions are introduced:
(a) The force has the direction given by the straight line joining the two particles.
(b) The magnitude of the force depends only on the distance between the two particles. The textbooks do not usually give any convincing reasons for making these two assumptions beyond claiming that they are reasonable. In fact, both of them are consequences of the principle of material frame-indifference, as we will prove in Sect.4.

In recent years, the theory of liquid crystals has become a very popular subject. A large part of this theory deals only with bodies that are not subject to deformation, but only to changes involving a *director field* $\mathbf{n}$, which influences the optical and electromagnetic properties of the body. It is assumed that this director field gives rise to a free energy with a density σ per unit volume, and that this density at a given material point depends only on the values of $\mathbf{n}$ and its gradient at that point. Denote the function which describes this dependence by $\hat{\sigma}$. The following additional assumption is then introduced: The identity

$$\hat{\sigma}(\mathbf{Qu}, \mathbf{QGQ}^{\top}) = \hat{\sigma}(\mathbf{u}, \mathbf{G}) \quad \text{for all vectors } \mathbf{u} \text{ and all lineons } \mathbf{G}$$

is valid for all proper orthogonal lineons $\mathbf{Q}$ when dealing with cholesteric liquid crystals or for all orthogonal lineons $\mathbf{Q}$, proper or not, when dealing with nematic liquid crystals. Some authors have claimed that this assumption is justified by the principle of frame-indifference. (Even "Galilean Invariance" has been invoked by some as a justification.) These authors are mistaken. Rather, the assumption expresses a certain kind of material symmetry. Roughly, it states that the director-field interacts isotropically (or hemitropically in the cholesteric case) with the underlying body. In other words, the body has no implicit preferred directions in addition to the explicit one given by the director-field itself. It is quite conceivable that there are materials for which this assumption fails to be appropriate, although one should not call such materials "*liquid* crystals".

The term "principle of material frame-indifference" was introduced in 1965 by C. Truesdell and me in our contribution [NLFT] to the Encyclopedia of

Physics. Earlier, I had used the term "principle of objectivity", and some people use this term to this day. I meant "objectivity" to express independence of the "observer", but Truesdell disliked the term as being too easily misinterpreted. In fact, "observer" is also easily misinterpreted; a much better term is "frame of reference", or "frame" for short. In my doctoral thesis in 1954, I used the term "principle of isotropy of space", but I discarded it soon thereafter because I realized that there is really no such thing as (physical) "space". Of course, the principle has been applied implicitly for a long time without the use of an explicit name or formulation.

2. Frames of reference, motion.

Most people, whether they are physicists or not, use the words "position", "place", and "motion" as if they had a completely unambiguous meaning. It should be obvious, however, that it makes no sense to speak of the position or place of a material particle or point except relative to a *frame of reference*. Similarly, "motion" means change of position with time and hence, like "position" makes no sense except relative to a frame of reference. Usually, it is tacitly understood that the frame of reference to be used is the one that is determined by the background. In our daily lives, the background is most often rigidly attached to the earth (the road, the building we live or work in, the scenery, etc.). Sometimes, for example when we are inside a railroad car, ship, airplane, or spacecraft, the background is given by the interior walls of a conveyence that is in motion relative to the frame of reference provided by the earth. When our ancesters looked at the sky at night, they saw the background provided by the fixed stars und used the frame of reference determined by it to describe the motion of the planets (recall that "planet" is the Greek word for wanderer).

We all know about the trouble that Galileo had when he asserted that the earth moves around the sun rather than the sun around the earth, as church dogma had it at that time. On the face of it, neither of these assertions makes any sense because frames of reference are not specified. Both of these assertions are in fact true if "move" is understood relative to different but suitable frames of reference. I cannot understand, therefore, what the fuss was all about.

When describing a physical process, there is sometimes no obvious background that can be used to determine a frame of reference. Even if there is, the frame obtained from it may not lead to the simplest description of the process and a simpler description may be obtained by using a frame that seems, at first, artificial. The true value of the Copernican frame, although quite artificial at the time when it was proposed, is that it yields simpler motions for the planets than the frame determined by the the earth or the frame determined by the fixed stars and the condition that the center of the earth be at rest. The choice of a frame of reference is a matter of expediency, not of truth.

As stated in the beginning of this section, a frame of reference should make it possible to speak of *locations*. Mathematically, locations are points in a (genuine) *Euclidean space* (as defined precisely in Chapter 4 of [FDS]), which we will then call a **frame-space**. Such a frame-space can be constructed from a suitable rigid material system (the most important example is the earth). Mathematically, such a rigid system is a metric set, i.e., a set $\mathcal{S}$ endowed with structure by the specification of a function $d : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{P}$ †). Given any two points x and y of the rigid system, $d(x, y)$ should be interpreted as the distance from x to y as measured, for example, with a measuring tape. The observed facts of such measurements show that $\mathcal{S}$ is isometric to a suitable subset of any given 3-dimensional Euclidean space. Of course, there are infinitely many such spaces, any two of which are isomorphic. In Sect.6 I will show how one can construct, by an intrinsic mathematical construction, a particular such space, and how the given rigid system can be imbedded in that space, which we then call the *frame-space determined by the given rigid system.*

Only after such a frame-space has been constructed does it make any sense to use geometric concepts such as straight line, direction, angle, etc. and also to talk about vectors, namely as members of the translation-space of the frame-space. Hence concepts such as velocity, acceleration, and force also require the specification of a frame-space.

If one deals only with the internal properties of a body not subject to deformation, one can use the body itself as a metric set from which a frame-space can be constructed. In this case it is not natural to consider any other frame-space, and hence frame-indifference is not an issue. This is the case for the theory of liquid crystals mentioned above. However, frame-indifference does come into play in theories that deal with *deforming* liquid crystals.

Remark: Some people confuse the concept of a frame of reference with that of a coordinate system. It makes no sense to talk about a coordinate system unless a frame-space (or at least some kind of manifold) is given first. One can consider many different coordinate systems on one and the same frame-space. Using coordinate systems in conceptual considerations is an impediment to insight; they have a legitimate place only in the context of very specific situations.

3. Inertia.

In elementary science and physics courses, students are very often confronted with statements such as "a particle will move along a straight line with constant speed unless it is subject to an outside force." Later they will learn about "Newton's law" $\mathbf{f} = m\mathbf{a}$: the force acting on a particle is proportional to its acceleration, its inertial mass m being the proportionality factor. As was pointed out in the previous section, these statements aquire a meaning only after a frame of reference has been specified. They *cannot* be valid relative to ***every*** frame of reference. In fact, one can always construct frames relative to which the particle will undergo any motion prescribed at will. Newton's law is valid only in certain preferred frames, which we will call **inertial frames**. It seems that Newton dealt with this issue by postulating a particular preferred frame,

†) $\mathbb{P}$ denotes the set of all positive real numbers (including zero). We will use $\mathbb{R}$ to denote the set of *all* real numbers.

which he called "absolute space". However, there are infinitely many inertial frames, any one of which moves relative to any other in a uniform translational motion. Hence the laws of inertia remain valid under such changes of frame. This fact is often called "Galilean Invariance". It turns out that the frame of reference determined by the fixed stars and the condition that the center of the sun be at rest is, for most practical purposes, an inertial frame. This fact, together with the inverse-square law for gravitation, made it possible not only to explain the orbits of the planets and their moons, but even to make accurate predictions about the orbits of artificial satellites. Thus Newtonian mechanics became, perhaps, the first triumph of modern mathematical science. However, I believe that Newton's *absolute space* is a chimera.

Inertia plays a fundamental role in classical particle mechanics and also in the mechanics that deals with the motion of rigid bodies. However, when dealing with deformable bodies, inertia plays very often a secondary role. In some situations, it is even appropriate to neglect inertia altogether. For example, when analyzing the forces and deformations that occur when one squeezes toothpaste out of a tube, inertial forces are usually negligible. Thus, I believe that the basic concepts of mechanics in general should *not* include items such as momentum, kinetic energy, and angular momentum, because they are relevant only when inertia is important. What remains are the two fundamental balance laws:
(1) The sum of all the forces (including the inertial forces) acting on a system or any of its parts should be zero.
(2) The sum of the moments (including the moments of inertial forces) acting on a system or any of its parts should be zero.

As far as these balance laws are concerned, inertial forces should be treated on equal terms with other kinds of forces. In this context, Newton's law $\mathbf{f} = m\mathbf{a}$ should be viewed as the result of the combination of two laws. The first is the force-balance law in the form $\mathbf{f} + \mathbf{i} = \mathbf{0}$, where $\mathbf{f}$ denotes the sum or the non-inertial forces acting on the particle, while $\mathbf{i}$ denotes the inertial force acting on it. The second law is the constitutive law of inertia. It states that $\mathbf{i} = -m\mathbf{a}$ when an inertial frame of refence is used. If the frame used is arbitrary, not necessarily inertial, the constitutive law of inertia takes the form

$$\mathbf{i} = -m(\mathbf{u}^{\bullet\bullet} + 2\mathbf{A}\mathbf{u}^{\bullet} + (\mathbf{A}^{\bullet} - \mathbf{A}^2)\mathbf{u}. \qquad (1)$$

Here, the value $\mathbf{u}(t)$ of the function $\mathbf{u}$ at time t denotes the position vector of the particle relative to a reference point (often called "origin") which is at rest in some inertial frame, although not neseccarily in the frame used. The value $\mathbf{A}(t)$ of the function $\mathbf{A}$ at time t is a skew lineon; it measures the rate of rotation of the given frame relative to some inertial frame. Dots denote time-derivatives. If the refence point is at rest not only in some inertial frame but also in the frame used and if $\mathbf{A}$ is constant, i.e. if $\mathbf{A}^{\bullet}$ is zero, then the contributions to the inertial force given by the second and third term on the right of (1) are called *Coriolis* force and *centrifugal* force, respectively. For example, the frame of refence determined by the earth is approximately inertial for smallscale phenomena, but the contribution of the Coriolis force can be decisive when large-scale wind or ocean-current phenomena are analyzed.

4. Frame-Indifference.

Physical processes are usually described in a mathematical framework provided by a frame of refence with its corresponding frame-space $\mathcal{F}$. Concepts such as *vector*, *lineon*, or *tensor* become meaningful only relative to the given frame-space. For example, a vector is a member of the translation-space $\mathcal{V}$ of $\mathcal{F}$, and a lineon is a member of the space $\text{Lin}\,\mathcal{V}$ of all linear transformations of $\mathcal{V}$ into itself.

Now consider two frames of refence with corresponding frame-spaces $\mathcal{F}$ and $\mathcal{F}'$ and denote their translation spaces by $\mathcal{V}$ and $\mathcal{V}'$, repectively. If x is the position of a material point or particle at a given time in the frame-space $\mathcal{F}$, then the position of the *same* particle at the *same* time in the frame-space $\mathcal{F}'$ will be given by $x' = \alpha(x)$ where $\alpha : \mathcal{F} \longrightarrow \mathcal{F}'$ is an isometry and hence a Euclidean isomorphism (see Sect.45 of [FDS]), which may depend on time. The gradient $\mathbf{Q} := \nabla\alpha$ is an orthogonal, i.e. inner-product preserving, linear mapping from $\mathcal{V}$ onto $\mathcal{V}'$. A vector $\mathbf{u}$ and a lineon $\mathbf{L}$ relative to the frame-space $\mathcal{F}$ will appear as $\mathbf{u}' = \mathbf{Q}\mathbf{u}$ and $\mathbf{L}' = \mathbf{Q}\mathbf{L}\mathbf{Q}^{\top}$, repectively, relative to the frame-space $\mathcal{F}'$. Now, if we want to compare the description of a physical process as seen in the two frame-spaces $\mathcal{F}$ and $\mathcal{F}'$ and if we want the comparison to take place in the frame-space $\mathcal{F}$, say, we must return from $\mathcal{F}'$ to $\mathcal{F}$ via some fixed Euclidean isomorphism. We replace the time-dependent isomorphism α above with its composite with this fixed isomorphism, and denote the resulting automorphism of $\mathcal{F}$ again by α. Then the fomulas for a change of frame given above remain valid with $\mathcal{F}'$ and $\mathcal{V}'$ replaced by $\mathcal{F}$ and $\mathcal{V}$, repectively.

Consider now a physical system. The **principle of material frame-indifference**, as applied to this system, can then be formulated as follows:

The constitutive laws governing the internal interactions between the parts of the system should not depend on whatever external frame of reference is used to describe them.

The principle should apply to interactions of any kind, be they mechanical, thermodynamical, optical, electromagnetic, or whatever.

It is important to note that the principle applies only to *internal* interactions, not to actions of the environment on the system and its parts, because usually the frame of reference employed is actively connected with the environment. For example, if one considers the motion of a fluid in a container, one usually uses the frame of reference determined by the container, which certainly affects the fluid. Inertia should always be considered as an action of the environment on the given system and its parts, and hence its description does depend on the frame of reference used. As we saw in Sect.3, the inertial force has a simple description only when an inertial frame is used. It is also important to note that the principle applies only to *external* frames of refence, not to frames that are constructed from the system itself, as is the case in the theory of liquid crystals not subject to deformation described in Sect.1.

To illustrate how the principle of material frame-indifference is applied, we consider the simple example already mentioned in Sect.1, namely a system consisting of only two particles and the force-interaction between them.†) We assume that the force $\mathbf{f}$ exerted at a given time on the first particle by the second depends only on the positions of the two particles at that time. Denote the function that describes this dependence by $\hat{\mathbf{f}} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{V}$, so that

$$\mathbf{f} = \hat{\mathbf{f}}(x, y) \tag{2}$$

is this force when the particles are located at x, and y, respectively. Now, after a change of frame given by the Euclidean automorphism $\alpha : \mathcal{F} \longrightarrow \mathcal{F}$, the particles appear at the locations $x' = \alpha(x)$ and $y' = \alpha(y)$ and the force appears to be $\mathbf{f}' = \mathbf{Q}\mathbf{f}$, where $\mathbf{Q} := \nabla\alpha$. The principle of material frame-indifference states that the function $\hat{\mathbf{f}}$ should also describe the dependence of the force on the locations after the change of frame, so that

$$\mathbf{Q}\mathbf{f} = \mathbf{f}' = \hat{\mathbf{f}}(x', y') = \hat{\mathbf{f}}(\alpha(x), \alpha(y)). \tag{3}$$

Combining (2) and (3), we find that the function $\hat{\mathbf{f}}$ must satisfy

$$\mathbf{Q}\hat{\mathbf{f}}(x, y) = \hat{\mathbf{f}}(\alpha(x), \alpha(y)) \quad \text{when} \quad \mathbf{Q} := \nabla\alpha. \tag{4}$$

This equation should be valid for every Euclidean automorphisms α of $\mathcal{F}$ and all points $x, y \in \mathcal{F}$. Now choose a point $q \in \mathcal{F}$ arbitrarily and define $\hat{\mathbf{g}} : \mathcal{V} \longrightarrow \mathcal{V}$ by

$$\hat{\mathbf{g}}(\mathbf{u}) := \hat{\mathbf{f}}(q, q + \mathbf{u}) \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}. \tag{5}$$

Let $x, y \in \mathcal{F}$ be given. We apply (4) to the case when α is the translation $\mathbf{u} := q - x$ that carries x to q. Since the gradient of a translation is the identity and since $\mathbf{u}(x) = x + (q - x) = q$ and $\mathbf{u}(y) = y + (q - x) = q + (y - x)$, the equation (4) reduces to

$$\hat{\mathbf{f}}(x, y) = \hat{\mathbf{f}}(q + (y - x)) = \hat{\mathbf{g}}(y - x), \tag{6}$$

valid for all $x, y \in \mathcal{F}$. Recalling that the gradient $\mathbf{Q}$ of a given Euclidean automorphism α is characterized by the condition that $\alpha(x) - \alpha(y) = \mathbf{Q}(x - y)$ holds for all $x, y \in \mathcal{F}$, we conclude from (6) and (4) that

$$\mathbf{Q}\hat{\mathbf{g}}(\mathbf{u}) = \hat{\mathbf{g}}(\mathbf{Q}\mathbf{u}) \tag{7}$$

must be valid for all $\mathbf{u} \in \mathcal{V}$ and all orthogonal lineons $\mathbf{Q}$. Given $\mathbf{u} \in \mathcal{V}$, the equation (7) must be valid, in particular, for all orthogonal $\mathbf{Q}$ that leave $\mathbf{u}$ unchanged. Hence $\hat{\mathbf{g}}(\mathbf{u})$ must also remain unchanged by these $\mathbf{Q}$. Since the only

†) In the derivation that follows, we use the concepts, notations, and results of Sects.32, 33, and of Chapt.4 of [FDS].

vectors that have this property are scalar multiples of $\mathbf{u}$, we conclude that there is a function $g : \mathcal{V} \longrightarrow \mathbb{R}$ such that

$$\hat{\mathbf{g}}(\mathbf{u}) = g(\mathbf{u})\mathbf{u} \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}. \tag{8}$$

In view of (7), this function must have the property that

$$g(\mathbf{u}) = g(\mathbf{Qu}) \quad \text{for all orthogonal lineons} \quad \mathbf{Q}. \tag{9}$$

Now choose a unit vector $\mathbf{e}$ arbitrarily and define the function $h : \mathbb{P} \longrightarrow \mathbb{R}$ by

$$h(d) := g(d\mathbf{e}) \quad \text{for all} \quad d \in \mathbb{P}. \tag{10}$$

Given $\mathbf{u} \in \mathcal{V}$, it is easily seen that we can choose an orthogonal $\mathbf{Q}$ such that $\mathbf{u} = |\mathbf{u}|\mathbf{Qe} = \mathbf{Q}|\mathbf{u}|\mathbf{e}$. Using (9) with $\mathbf{u}$ replaced by $|\mathbf{u}|\mathbf{e}$, it follows that

$$g(\mathbf{u}) = h(|\mathbf{u}|)\mathbf{u} \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}. \tag{11}$$

Combining (11) with (8) and (6), we see that the equation (2) for the dependence of the force $\mathbf{f}$ on the locations x and y must reduce to the specific form

$$\mathbf{f} = \hat{\mathbf{f}}(x, y) = h(|x - y|)(x - y), \tag{12}$$

which justifies the assumptions (a) and (b) stated in Sect.1. Another consequence of (12) is $\hat{\mathbf{f}}(y, x) = -\hat{\mathbf{f}}(x, y)$, which expresses a case of what is usually called "the law of action and reaction".

For the derivation of (12) above, it was irrelevant that the Euclidean automorphisms α for which (4) holds may depend on time. However, such possible time-dependence plays a crucial role, for example, when applying the principle of frame-indifference to derive the specific form of the constitutive equation for linearly viscous fluids discussed in Sect.1.

It is possible to make the principle of material frame-indifference vacuously satisfied by using an intrinsic mathematical frame-work that does not use a frame-space at all when describing the internal interactions of a physical system. I did this for the continuum mechanics of simple †) materials in [N]. However, the mathematics that is needed to do this, although not necessarily complicated, is not familiar to many people and hence resisted by them as being "too abstract". Also, it seems that the action of the environment on a system cannot be described without using a frame of reference, and hence one must introduce such a frame in the end when dealing with specific problems.

5. Relativity.

Up to now, we have tacitly assumed the validity of the common-sense notions of time and distance. Specifically, we implicitly have taken for granted that the following statements are unambiguously valid:

†) "simple" must be understood in the technical sense defined in [N].

(1) Any two given events are either simultaneous or one of them precedes the other.
(2) To any given two events one can assign a time-lapse, which is zero if and only if they are simultaneous.
(3) To any two simultaneous events one can assign a distance between them.
A precise mathematical structure that describes a world in which these assumptions are valid is described in Sect.4.1 of [MN] under the name of *Pre-classical Spacetime*. In the present paper we have given an intuitive idea of what is meant by a frame of reference; in Sect.4.2 of [MN] one can find a more precise definition in the context of pre-classical spacetime.

In the Theory of Relativity (both Special and General) the common-sense notions of time and distance can no longer be used: Simultaneity becomes meaningless. There can be two events neither of which precedes the other. Even if one event does precede another, the timelapse between them may depend on a world-path connecting them. There is no simple notion of distance, and hence one cannot define unambiguously what is meant by a rigid system. Therefore, there is no relativistic counterpart of a frame of reference. The only correlation one can make is between inertial frames and world-directions (as defined in Sect.53 of [MN]).

In view of these remarks, it is not clear what a relativistic counterpart to the principle of material frame-indifference would be. The only proposal for such a counterpart that I know of is the *principle of non-sentient response* of Bragg [B]. The idea behind this principle is the following : The dependence of the state of a given material point X on the world-paths of the material points surrounding X can only involve information about these world-paths that can be obtained by signals originating from the worldpath of X and reflected back from these nearby world-paths.

6. Construction of frame-spaces.

In this section, I will describe the construction of a Euclidean space from a given metric set, as already announced in Sect.2.

Assume that a metric set $\mathcal{S}$, endowed with structure by the prescription of a function $d : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{P}$, is given. We also assume that $\mathcal{S}$ is isometric to some subset of some Euclidean space. This means that we can choose a Euclidean space $\mathcal{E}$ and a mapping $\kappa : \mathcal{S} \longrightarrow \mathcal{E}$ such that

$$d(x,y) = |\kappa(x) - \kappa(y)| \quad \text{for all} \quad x, y \in \mathcal{S}. \tag{13}$$

We may assume, without loss, that $\mathcal{E}$ is the flat span of the range $\mathrm{Rng}\kappa$ of κ, i.e. the smallest subspace of $\mathcal{E}$ that includes $\mathrm{Rng}\kappa$, because otherwise we could replace $\mathcal{E}$ by this flat span.

We now define the mapping $\Phi : \mathcal{E} \longrightarrow \mathrm{Map}(\mathcal{S}, \mathbb{P})$ †) by

$$\Phi(z)(x) := |z - \kappa(x)| \quad \text{for all} \ z \in \mathcal{E},\ x \in \mathcal{S}. \tag{14}$$

†) Given any two sets S and T, $\mathrm{Map}(S,T)$ denotes the set of all mappings from S to T.

Using Prop.7 of Sect.45 of [FDS] one can easily prove that the mapping Φ thus defined is injective and that $z \in \text{Rng}\kappa$ if and only if $0 \in \text{Rng}(\Phi(z))$. We now put

$$\mathcal{F} := \text{Rng}\Phi \subset \text{Map}(\mathcal{S}, \mathbf{P}) \tag{15}$$

and endow $\mathcal{F}$ with the structure of a Euclidean space by requiring that the invertible mapping $\Phi|^{\mathcal{F}} : \mathcal{E} \longrightarrow \mathcal{F}$ †) be a Euclidean isomorphism. Using Prop.5 of Sect.45 of [FDS] one can easily prove that the subset $\mathcal{F}$ of $\text{Map}(\mathcal{S}, \mathbf{P})$, its structure as a Euclidean space, and the injective mapping

$$\phi := \Phi|^{\mathcal{F}} \circ \kappa : \mathcal{S} \longrightarrow \mathcal{F} \tag{16}$$

are all independent of the initial choice of $\mathcal{E}$ and κ. We call $\mathcal{F}$ the **frame-space** of the given metric set $\mathcal{S}$.

We use the mapping (16) to imbed the metric set $\mathcal{S}$ into the frame-space $\mathcal{F}$ constructed from it. Then a given point $x \in \mathcal{S}$ becomes identified with the function $\phi(x) = d(x, \cdot)$, which gives the distances from x to all points in $\mathcal{S}$. Every point in the frame space $\mathcal{F}$ is identified with the function that gives its distances from all the points in $\mathcal{S}$.

The construction just given implies that locations in the frame-space constructed from a given rigid system $\mathcal{S}$ can be determined by distance measurements alone. In practice, however, it is often convenient to also use angle-measurements involving lines of sight.

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†) $\Phi|^{\mathcal{F}}$ is the mapping obtained from Φ by adjusting the codomain from the original $\text{Map}(\mathcal{S}, \mathbf{P})$ to the range $\mathcal{F}$.