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Quasiconvexification in $W^{1,1}$ and Optimal Jump Microstructure in BV Relaxation

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Abstract

An integral representation for the relaxation in $BV(\Omega; \mathbb{R}^p)$ of the functional

$$u \mapsto \int_\Omega W(\nabla u(x))dx + \mathcal{H}^{N-1}(S(u))$$

with respect to BV weak $*$ convergence is obtained. The bulk term in the integral representation reduces to QW, the quasiconvexification of W, and it is shown exactly how optimal approximating sequences behave along $S(u)$, for scalar valued u.

Keywords: quasiconvex, lower semicontinuous, microstructure, bounded variation, relaxation

AMS Classifications: 49K10, 49N60, 49Q15, 73V25

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1 Introduction

Given a functional $E:\mathcal{F}\to\mathbb{R}$, where $\mathcal{F}$ is a function space, a central problem is to understand what the convergence of $u_n \to u$ in $\mathcal{F}$ implies about $\{E(u_n)\}$ and $E(u)$. In particular, the direct method of the Calculus of Variations can only be applied if E is lower semicontinuous with respect to the appropriate notion of convergence in $\mathcal{F}$. When there is failure of this lower semicontinuity, a natural object to look at is the relaxed functional $I:\mathcal{F}\to\mathbb{R}$ defined by

$$I(u) := \inf\left\{\liminf_{n\to\infty} E(u_n) : u_n \to u\right\}.$$

Often, a priori known properties of E and $\mathcal{F}$ ensure lower semicontinuity of I.

When E stands for physical energy and $\mathcal{F}$ represents states of a physical system, there is an interesting consequence of this failure of lower semicontinuity: the apparent, macroscopic state may not accurately reflect the actual, microscopic state or properties of the system. That is, if the macroscopic view corresponds to $u \in \mathcal{F}$, all that is known is that the system is in a small neighborhood of u in $\mathcal{F}$. If E were lower semicontinuous, then for small enough neighborhoods, the state might as well be u, as there is no need to develope microstructure to lower the energy. Lack of lower semicontinuity suggests that it might be energetically necessary to develope this infinitely fine detail.

In the case of failure of lower semicontinuity, relaxing the energy has two benefits. First, it tells us the actual energy of a macroscopic state, since given a macroscopic configuration, the system will choose a state that is close, but with as low energy as possible. The second benefit comes from finding integral representations of I. Often, if we let the underlying domain A for functions in $\mathcal{F}$ vary, $A \mapsto I(u|_A)$ is the trace of a measure, in particular, a Radon measure absolutely continuous with respect to $\mathcal{L}^N + |Du|$, where $\mathcal{L}^N$ stands for the N dimensional Lebesgue measure in $\mathbb{R}^N$. In finding the density of this measure with respect to $\mathcal{L}^N$ and $|Du|$, one is led to understand the local behavior of optimal approximating sequences.

We take $\mathcal{F}$ to be the space of functions of *bounded variation* $BV(\Omega;\mathbb{R}^p)$, where $\Omega \subset \mathbb{R}^N$ is open and bounded. This space includes Sobolev spaces, yet allows jumps and other kinds of variation. For $u \in BV(\Omega;\mathbb{R}^p)$, $|Du|$ denotes the total variation measure of Du. We also set

$$E(u) := \int_\Omega W(\nabla u)dx + \mathcal{H}^{N-1}(S(u)),$$

where $S(u)$ is the complement of the set of Lebesgue points of u, and $\mathcal{H}^{N-1}$ is the $N-1$ dimensional Hausdorff measure. For $u \in BV(\Omega;\mathbb{R}^p)$, we use the representation $Du = \nabla u\mathcal{L}^N + [u]\otimes\nu\mathcal{H}^{N-1}\lfloor S(u) + C(u)$, where $[u]$ is the jump in u, i.e., $u^+ - u^-$, where u^+ and u^- are the traces of u on either side of $S(u)$ (see, e.g., [10] and [17]), ν is the normal to $S(u)$, and $C(u)$ is the so-called *Cantor part*. If $C(u) = 0$, we say $u \in SBV(\Omega;\mathbb{R}^p)$, the space of *special functions of bounded variation* introduced in [9]. Since functions with $Du = C(u)$ are dense in L^1, if $\min W = W(0)$ then the relaxation of $E(\cdot,\Omega)$ would reduce to $\mathcal{L}^N(\Omega)W(0)$. We avoid this pathology by only considering sequences in SBV, which is equivalent to relaxing $E(\cdot,\Omega)+\infty|C(\cdot)|(\Omega)$. This corresponds to allowing macroscopic states with Cantor part, but not microscopic states.

In [3], Ambrosio analyzed the energy functional on SBV given by

$$E(u) := \int_\Omega W(x,u,\nabla u)dx + \int_{S(u)} \phi(u^+,u^-,\nu)d\mathcal{H}^{N-1}(x),$$

under the hypotheses that W is Carathéodory and has superlinear growth in ∇u, and under conditions on ϕ that, in particular, allow ϕ to be any positive constant. A result is that if W is quasiconvex, and if certain assumptions on ϕ are met, then E is L^1_{loc} lower semicontinuous in SBV. This model is particularly relevant in image segmentation, where the issue is the optimal placement of edges around objects in a photograph, as well as to smooth out each object and denoise the initial picture. This problem can be formulated as a search for minimizers of an energy. Specifically, one looks for minimizers of

$$E(u) = \int_\Omega \left(|\nabla u|^2 + |u-g|^2\right)dx + \mathcal{H}^{N-1}(S(u)),$$

where $g \in L^\infty(\Omega)$ is a given function representing the initial photographic image and $u \in SBV(\Omega)$.

In this paper, we assume that W has linear growth and $\phi \equiv 1$. Physically, this last assumption corresponds to weighing jumps, or cracks, only by the size of the jump set $S(u)$, with no dependence on the orientation ν or size of the jump $[u]$. The analysis of the model where $\phi \equiv 1$ and W has superlinear growth was carried out in [11]. The linear growth of W allows for interaction along the jump set; approximating sequences of u need not jump at $S(u)$, but might have much more complicated behavior consisting of combinations of smooth growth and jumps. The study of the case where W and ϕ both have linear growth was undertaken in [6] (see also [7]).

The main new contributions in this paper are the expressions for QW in Section 3, a new method for showing the upper bound inequality for the jump density in Section 4, Lemma 5.1 in Section 5 which allows us to blow up in such a way that the rescaled variation measures do not lose mass as they converge weakly $*$, and finally a method for finding the optimal jump microstructure for scalar valued functions in Section 6. This last result allows us to exhibit the optimal behavior of approximating sequences along $S(u)$. The method is applicable not only when the jump energy density is a constant, but also when it depends in a positive homogenous degree one way on the jump (see Theorem 6.6).

This paper is organized as follows: in Section 2 we discuss preliminaries and state the relaxation theorem, the essence of which is the integral representation

$$I(u,A) = I^*(u,A) := \int_A QW(\nabla u)dx + \int_{S(u)} h([u],\nu)d\mathcal{H}^{N-1} + \int_A (QW)^\infty(dC(u)),$$

where QW is the quasiconvexification of W (see Section 2), and h and $(QW)^\infty$ are defined in Section 2.

In Section 3 we show that, although QW is defined in terms of sequences in Sobolev spaces, there are equivalent definitions in terms of certain sequences in BV. In particular,

$$QW(F) = \inf\left\{\liminf_{n\to\infty}\int_Q W(\nabla u_n)dx : u_n \in BV(Q;\mathbb{R}^p), u_n \to Fx \text{ in } L^1, \text{ and } |D_s u_n|(Q) \to 0\right\}$$

and

$$QW(F) = G(F) := \inf\left\{\liminf_{n\to\infty}\int_Q W(\nabla u_n)dx : u_n \in SBV(Q;\mathbb{R}^p), u_n \to Fx \text{ in } L^1, \text{ and } \mathcal{H}^{N-1}(S(u_n)) \to 0\right\},$$

where $D_s u$ is the singular part of Du with respect to $\mathcal{L}^N$. An analogous lower semicontinuity result for superlinear W was obtained by Ambrosio in [3], Theorem 3.3.

In Section 4, and in order to show the upper bound inequality $I \le I^*$, we first prove that $I(u,\cdot)$ is a finite Borel regular measure, absolutely continuous with respect to $\mathcal{L}^N + |Du|$. This follows largely from [12]. The remaining issue in this section is the upper bound for $I(u,\cdot)\lfloor S(u)$, for which we introduce a new argument. There is some difficulty with this step because $\mathcal{H}^{N-1}\lfloor S(u)$ is, in general, not a Radon measure, and so taking derivatives with respect to $\mathcal{H}^{N-1}\lfloor S(u)$ is not possible. The usual method for showing upper bound inequalities for jump densities is based on [4] and [5], and involves approximating jump sets with boundaries of sets with finite perimeter. The technique here is based on looking at the intersection of the jump set with certain sets of finite perimeter. We consider level sets E_t of the components of u, such that E_t has finite perimeter and $|D_j u| := |D_s u|\lfloor S(u)$ concentrates on $S(u) \cap \partial_* E_t$ as we blow-up. We then see that the analysis on $S(u) \cap \partial_* E_t$ is much easier than on $S(u)$. The rest follows from constructing functions in a reasonable way, and by using a suitable covering argument.

Section 5 deals with the proof of the lower bound inequality $I \geq I^*$, which is a modified version of the corresponding argument in (a draft of) [6]. The changes include choosing the rescaling factors so that the weak $*$ limit measure μ does not see the boundary of the rescaled unit cube, and so that as the rescaled variation measures converge weakly $*$ on a cube, they do not lose any mass (see Lemma 5.1).

In Section 6 we find the optimal microstructure along the jump set of u, for scalar valued u. The proof relies on a coarea formula and a covering argument. It turns out that the proof may be easily extended to the case where the jump density is a positive homogeneous degree one function of $[u]\nu$, and also when the jump density is just a function of the normal.

2 Preliminaries and the Relaxation Theorem

We consider a bounded, open set $\Omega \subset \mathbb{R}^N$, and we define the Sobolev spaces $W^{1,1}(\Omega)$ and $W^{1,\infty}(\Omega)$, and the space of functions of bounded variation $BV(\Omega)$ in the usual way (see, e.g., [10] and [17]). We denote by ρ_m, or alternatively ρ_ϵ, the standard mollifier, and for $E \subset \Omega$, χ_E stands for the characteristic function of E. Given two sets A and B, we define the *symmetric difference* $A \triangle B := (A \backslash B) \cup (B \backslash A)$.

We say that a set $E \subset \Omega$ has *finite perimeter* in Ω if $\chi_E \in BV(\Omega)$. For such an E, the *measure theoretic boundary* in Ω, $\partial_* E$, is defined as

$$\left\{ x \in \Omega : \limsup_{\delta \to 0^+} \frac{\mathcal{L}^N(B(x,\delta) \cap E)}{\mathcal{L}^N(B(x,\delta))} > 0 \text{ and } \limsup_{\delta \to 0^+} \frac{\mathcal{L}^N(B(x,\delta) \backslash E)}{\mathcal{L}^N(B(x,\delta))} > 0 \right\}, \tag{2.1}$$

where $B(x,\delta)$ is the closed ball in $\mathbb{R}^N$ centered at x with radius δ. We denote by $\nu_E(x)$ the *measure theoretic normal* to E at $x \in \partial_* E$ (for properties of this normal, see [10] or [17]). The *reduced boundary* $\partial^* E$ is the set of $x \in \partial_* E$ such that x is a Lebesgue point for ν_E, with respect to the Radon measure $\mathcal{H}^{N-1}\lfloor \partial_* E$. Given a set E of finite perimeter, we define on $\partial_* E$ the following:

$$H(x) := \{ y \in \mathbb{R}^N : \nu_E(x) \cdot (y - x) = 0 \},$$

$$H^+(x) := \{ y \in \mathbb{R}^N : \nu_E(x) \cdot (y - x) \geq 0 \},$$

and

$$H^-(x) := \{ y \in \mathbb{R}^N : \nu_E(x) \cdot (y - x) \leq 0 \}.$$

For $u \in BV(\Omega; \mathbb{R}^p)$, we write $Du = D_{ac}u + D_s u$, where $D_{ac}u$ and $D_s u$ stand for, respectively, the absolutely continuous and singular part of Du with respect to $\mathcal{L}^N$. We also consider the set $S(u)$ of points which are not Lebesgue points for u. We set $D_j u := D_s u \lfloor S(u)$ and use the representations $D_{ac}u = \nabla u \mathcal{L}^N$ and $D_j u = [u] \otimes \nu \mathcal{H}^{N-1}\lfloor S(u)$, where $[u]$ is the jump in u across $S(u)$, i.e., $[u] = u^+ - u^-$, where u^+, u^-, and ν are such that

$$\lim_{\delta \to 0^+} \frac{1}{\delta^N} \int_{\{y \in B(x,\delta) : (y-x)\cdot\nu(x) > 0\}} |u(y) - u^+(x)|^{\frac{N}{N-1}} dy = 0$$

and

$$\lim_{\delta \to 0^+} \frac{1}{\delta^N} \int_{\{y \in B(x,\delta) : (y-x)\cdot\nu(x) < 0\}} |u(y) - u^-(x)|^{\frac{N}{N-1}} dy = 0$$

for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$. It is convenient to define $C(u) := D_s u - D_j u$, and if $C(u) = 0$, then we say u is a *special function of bounded variation*, and we write $u \in SBV(\Omega)$. This space was introduced in [9].

We denote by $\tilde{u}$ the *precise representative* of u defined $\mathcal{H}^{N-1}$-a.e. by

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \text{ is a Lebesgue point of } u \\ \frac{u^+(x)+u^-(x)}{2} & \text{if } x \in S(u). \end{cases}$$

We set $\mathbb{R}^+ := [0, \infty)$ and $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. We denote the space of $p \times N$ matrices by $\mathbb{M}^{p\times N}$, and, for $W : \mathbb{M}^{p\times N} \to \mathbb{R}$, we define the *recession function* $W^\infty : \mathbb{M}^{p\times N} \to \bar{\mathbb{R}}$ by

$$W^\infty(F) := \limsup_{t\to\infty} \frac{W(tF)}{t}.$$

We say a function $f : \mathbb{M}^{p\times N} \to \mathbb{R}$ is *quasiconvex* if

$$f(F) \leq \int_A f(\nabla\phi)dx$$

for all $\phi \in Fx + C_0^\infty(A)$ and all $F \in \mathbb{M}^{p\times N}$, where A is open and $\mathcal{L}^N(A) = 1$ (see [15]). We denote by QW the quasiconvex envelope of W and by CW the convex envelope, i.e.,

$$QW(F) := \sup\{f(F) : f \leq W \text{ and } f \text{ is quasiconvex}\},$$

$$CW(F) := \sup\{f(F) : f \leq W \text{ and } f \text{ is convex}\}.$$

We know (see [13]) that for W satisfying (2.2) below, we have

$$QW(F) = \inf\left\{\liminf_{n\to\infty} \int_A W(\nabla u_n)dx : \{u_n\} \subset W^{1,1}(A;\mathbb{R}^p), u_n \to Fx \text{ in } L^1(A;\mathbb{R}^p)\right\}$$

for any open set A with $\mathcal{L}^N(A) = 1$.

For a unit vector $\nu \in \mathbb{R}^N$, we denote by Q_ν any open unit cube centered at 0 with two faces normal to ν, and S_ν is the set

$$\{x \in \mathbb{R}^N : |x \cdot \nu| < 1/2\}.$$

If $f : \mathbb{M}^{p\times N} \to \mathbb{R}$ is positive homogeneous of degree one and μ is a $\mathbb{M}^{p\times N}$-valued measure, we use the notation

$$\int f(d\mu) := \int f\left(\frac{d\mu}{d|\mu|}\right) d|\mu|,$$

where $|\mu|$ is the total variation measure of μ.

Theorem 2.1 *Assume that $W : \mathbb{M}^{p\times N} \to \mathbb{R}^+$ is continuous and such that for some $C_0, C_1 > 0$ and all $F \in \mathbb{M}^{p\times N}$*

$$C_0|F| - \frac{1}{C_0} \leq W(F) \leq C_1(1 + |F|). \tag{2.2}$$

Assume further that there exist $m \in (0,1), L > 0$, and $C > 0$ such that

$$\left|W^\infty(F) - \frac{W(tF)}{t}\right| \leq \frac{C}{t^m} \tag{2.3}$$

for all $F \in \mathbb{M}^{p\times N}$ with $|F| = 1$, and all $t > L$. For $A \subset \Omega$ open and $u \in BV(A;\mathbb{R}^p)$, set

$$E(u, A) := \int_A W(\nabla u)dx + \mathcal{H}^{N-1}(S(u) \cap A)$$

and

$$I(u,A) := \inf\Big\{ \liminf_{n\to\infty} E(u_n,A) : \{u_n\} \subset SBV(A;\mathbb{R}^p), u_n \to u \text{ in } L^1(A;\mathbb{R}^p)\Big\}.$$

Then

$$I(u,A) = \int_A QW(\nabla u)dx + \int_{S(u)} h([u],\nu)d\mathcal{H}^{N-1} + \int_A (QW)^\infty(dC(u)), \tag{2.4}$$

where

$$h(\xi,\nu) := \inf\Big\{ \int_{Q_\nu} W^\infty(\nabla v)dx + \mathcal{H}^{N-1}(S(v)) : v \in SBV(Q_\nu;\mathbb{R}^p),$$

$$v = \xi \text{ if } x \in \partial Q_\nu \text{ and } x\cdot\nu \geq 0, \text{ and } v = 0 \text{ if } x \in \partial Q_\nu \text{ and } x\cdot\nu < 0\Big\}. \tag{2.5}$$

The proof of this theorem will be carried out in Sections 3, 4, and 5, and will use the following alternative expression for the density h:

$$h(\xi,\nu) = h_1(\xi,\nu) := \inf\Big\{ \int_{Q_\nu} W^\infty(\nabla v)dx + \mathcal{H}^{N-1}(S(v)\cap Q_\nu) : v \in SBV_{loc}(S_\nu;\mathbb{R}^p),$$

$$v(y) = 0 \text{ if } y\cdot\nu = -\frac{1}{2}, v(y) = \xi \text{ if } y\cdot\nu = \frac{1}{2},$$

$$v \text{ is 1-periodic in the directions } \nu_1,\ldots,\nu_{N-1}\Big\}. \tag{2.6}$$

Clearly, $h \geq h_1$ since admissible functions for h have the necessary periodicity. The other inequality follows from the fact that, after scaling, an admissible function for h_1 will have the correct trace for h after altering it to jump on a set of small $\mathcal{H}^{N-1}$ measure. Specifically, suppose v is an admissible function for h_1. Define v_n, admissible for h_1, by

$$v_n(x) := \begin{cases} v(nx) & \text{if } x \in \frac{1}{n}S_\nu \\ \xi & \text{if } x\cdot\nu > \frac{1}{2n} \\ 0 & \text{if } x\cdot\nu < -\frac{1}{2n}. \end{cases}$$

Then $\int_{Q_\nu} W^\infty(\nabla v_n(x))dx + \mathcal{H}^{N-1}(S(v_n)\cap Q_\nu) =$

$$= n^{N-1}\int_{\frac{1}{n}Q_\nu} W^\infty(\nabla v_n(x))dx + n^{N-1}\mathcal{H}^{N-1}\Big(S(v_n)\cap \frac{1}{n}Q_\nu\Big)$$

$$= n^{N-1}\int_{\frac{1}{n}Q_\nu} W^\infty(n\nabla v(nx))dx + n^{N-1}\mathcal{H}^{N-1}\Big(\frac{1}{n}[S(v)\cap Q_\nu]\Big)$$

$$= n^{N-1}\int_{Q_\nu} nW^\infty(\nabla v(x))\frac{1}{n^N}dx + n^{N-1}\frac{1}{n^{N-1}}\mathcal{H}^{N-1}(S(v)\cap Q_\nu)$$

$$= \int_{Q_\nu} W^\infty(\nabla v(x))dx + \mathcal{H}^{N-1}(S(v)\cap Q_\nu).$$

It is easy to prove that v_n has the correct trace for h after modifying it to jump across a set with $\mathcal{H}^{N-1}$ measure of order $\frac{1}{n}$ near $\partial Q_\nu \cap \frac{1}{n}S_\nu$.

It will also be useful to consider the function $G : \mathbb{M}^{p\times N} \to \mathbb{R}^+$ defined by

$$G(F) := \inf\Big\{ \liminf_{n\to\infty}\int_Q W(\nabla u_n)dx : \{u_n\} \subset SBV(Q;\mathbb{R}^p), u_n \to Fx$$

$$\text{in } L^1(Q;\mathbb{R}^p), \text{ and } \mathcal{H}^{N-1}(S(u_n)) \to 0\Big\}.$$

As we will see in the next section, it turns out that $G = QW$.

3 Characterizations of QW for Sequences in BV

The goal of this section is to prove that $G = QW$. Note the connection to Theorem 4.5 in [3], where Ambrosio deals with a W with superlinear growth together with a more general jump density.

We begin with

Lemma 3.1 *Suppose that $W : \mathbb{M}^{p\times N} \to \mathbb{R}^+$ is a Borel measurable function such that $C_0|F| - \frac{1}{C_0} \le W(F) \le C_1(1+|F|)$ for every $F \in \mathbb{M}^{p\times N}$, and some $C_0, C_1 > 0$. Then*

$$QW(F) = Q^*W(F) := \inf\Big\{ \liminf_{n\to\infty} \int_Q W(\nabla u_n(x))dx : \{u_n\} \subset BV(Q;\mathbb{R}^p),$$

$$u_n \to Fx \text{ in } L^1(Q;\mathbb{R}^p) \text{ and } |D_s u_n|(Q) \to 0\Big\}$$

for all $F \in \mathbb{M}^{p\times N}$.

Proof. Although this follows from a straightforward application of the lower semicontinuity theorem of [13] in BV, we prefer here to provide a direct proof.

We need only show $QW \le Q^*W$, since the other inequality follows from the growth condition on W and standard relaxation theory (see Dacorogna [8] and Acerbi and Fusco [1]). Let $\{u_n\}$ be an admissible infimizing sequence for $Q^*W(F)$. The idea is to find a sequence $\{v_n\} \subset W^{1,\infty}(Q;\mathbb{R}^p)$ such that $v_n \to Fx$ in $L^1(Q;\mathbb{R}^p)$ and

$$\liminf_{n\to\infty} \int_Q W(\nabla v_n(x))dx \le \liminf_{n\to\infty} \int_Q W(\nabla u_n(x))dx, \tag{3.1}$$

since then the coercivity hypothesis implies that $\{v_n\}$ is bounded in $W^{1,1}$, and using the lower semicontinuity theorem of Fonseca and Müller [13] in $W^{1,1}$ for the L^1 topology, we have

$$QW(F) \le \liminf_{n\to\infty} \int_Q W(\nabla v_n)dx.$$

Let $\varepsilon > 0$ be given. Choose $m > \frac{1}{\varepsilon}$ such that

$$|D_s u_m|(Q) < \varepsilon \tag{3.2}$$

and

$$\int_Q W(\nabla u_m(x))dx < \liminf_{n\to\infty} \int_Q W(\nabla u_n(x))dx + \varepsilon.$$

Let $i \in \{1,\ldots,p\}$. We now use maximal functions to find a good Lipschitz approximation for $(u_m)_i$. The following is based on an alteration of the proof of claim # 1 in Theorem 2 of Section 6.6.2 of [10].

For $\lambda > 0$, set

$$R^\lambda := \left\{ x \in Q : \frac{|D(u_m)_i|(B(x,r))}{r^N} \le \lambda \text{ for all } r > 0 \right\}.$$

By Vitali's Covering Theorem, we can choose disjoint balls $\{B(x_j,r_j)\}_{j=1}^\infty$ such that

$$Q \backslash R^\lambda \subset \bigcup_{j=1}^\infty B(x_j, 5r_j)$$

and

$$\frac{|D(u_m)_i|(B(x_j,r_j))}{r_j^N} > \lambda.$$

Then

$$\begin{aligned}\mathcal{L}^N(Q\backslash R^\lambda) &\leq \mathcal{L}^N(\cup_{j=1}^\infty B(x_j, 5r_j)) \qquad (3.3)\\ &\leq 5^N\alpha(N)\sum_{j=1}^\infty r_j^N \\ &\leq \frac{5^N\alpha(N)}{\lambda}|D(u_m)_i|(\cup_{j=1}^\infty B(x_j,r_j)) \\ &\leq \frac{5^N\alpha(N)}{\lambda}|D(u_m)_i|(Q),\end{aligned}$$

where $\alpha(N)$ is the volume of an N dimensional unit ball. Note that we have

$$\mathcal{L}^N(\cup_{j=1}^\infty B(x_j,r_j)) \leq \frac{\alpha(N)}{\lambda}|D(u_m)_i|(Q). \qquad (3.4)$$

For $\delta > 0$, put

$$\lambda(\delta) := \max\left\{\frac{5^N\alpha(N)}{\delta}, \frac{\alpha(N)}{\delta}|D(u_m)_i|(Q)\right\}$$

and

$$B(\delta) := \bigcup_{j=1}^\infty B(x_j,r_j),$$

where the $B(x_j,r_j)$ are chosen for $\lambda = \lambda(\delta)$. By claim # 2 of Theorem 2 in Section 6.6.2 of [10], there exists a Lipschitz function $w_{i,\delta}$ such that $w_{i,\delta} = (u_m)_i$ on $R^{\lambda(\delta)}$ and $\text{Lip}(w_{i,\delta}) \leq c\lambda(\delta) \leq \frac{C}{\delta}(1 + |D(u_m)_i|(Q))$ for some constants c, C depending only on N. Note, however, that by (3.2), $|D(u_m)_i|(Q) < |D_{ac}(u_m)_i|(Q) + \varepsilon$, and

$$|D_{ac}u_m|(Q) = \int_Q |\nabla u_m|dx \leq \frac{1}{C_0}\int_Q \left(W(\nabla u_m) + \frac{1}{C_0}\right)dx$$

which is bounded since $\{u_m\}$ is infimizing. Hence, $\text{Lip}(w_{i,\delta}) \leq \frac{C}{\delta}$ for a constant C depending only on N and the sequence $\{u_m\}$. It follows from (3.4) that $\mathcal{L}^N(B(\delta)) \to 0$ as $\delta \to 0^+$, and so

$$\limsup_{\delta\to 0}|D(u_m)_i|(B(\delta)) \leq |D_s(u_m)_i|(Q).$$

Note that $R^{\lambda(\delta_1)} \subset R^{\lambda(\delta_2)}$ for $\delta_1 \geq \delta_2$. Hence, we may choose $y \in Q$ such that $w_{i,\delta}(y) = (u_m)_i(y)$ for all δ sufficiently small. Fix $\delta > 0$ such that

$$|D(u_m)_i|(B(\delta)) < |D_s(u_m)_i|(Q) + \varepsilon$$

for all $i \in \{1,\ldots,p\}$, and so that

$$\int_E |u_m(x) - (w_{1,\delta}(y),\ldots,w_{p,\delta}(y))|dx < \varepsilon$$

for some $y \in Q$ and all $E \subset Q$ with $\mathcal{L}^N(E) < 2p\delta 5^N\varepsilon$. By (3.3) and the choice of δ, we have

$$\begin{aligned}\mathcal{L}^N(Q\backslash R^{\lambda(\delta)}) &< \delta 5^N(|D_s(u_m)_i|(Q) + \varepsilon) \qquad (3.5)\\ &< 2\delta 5^N\varepsilon \text{ (by (3.2))}\end{aligned}$$

for all $i \in \{1,\ldots,p\}$. Setting $v_m(x) := (w_{1,\delta}(x),\ldots,w_{p,\delta}(x))$, it follows that v_m is Lipschitz with $\text{Lip}(v_m) \leq p^{\frac{1}{2}}\frac{C}{\delta}$. Setting $T := \{x \in Q : v_m(x) \neq u_m(x)\}$, (3.5) yields $\mathcal{L}^N(T) < 2p\delta 5^N\varepsilon$, so

$$\begin{aligned}\int_T |\nabla v_m(x)|dx &\leq \mathcal{L}^N(T)\text{Lip}(v_m) \\ &< 2p^{\frac{3}{2}}C5^N\varepsilon\end{aligned}$$

and

$$\begin{aligned}\int_T |v_m(x) - u_m(x)|dx &= \int_T |(v_m(x) - v_m(y)) - (u_m(x) - v_m(y))|dx \\ &\le \int_T |v_m(x) - v_m(y)|dx + \int_T |u_m(x) - v_m(y)|dx \\ &< \mathcal{L}^N(T) \max |v_m(x) - v_m(y)| + \varepsilon \\ &\le \mathcal{L}^N(T)\sqrt{N}\mathrm{Lip}(v_m) + \varepsilon \\ &< 2p^{\frac{3}{2}}C5^N\sqrt{N}\varepsilon + \varepsilon.\end{aligned}$$

From Theorem 3 and Remark (i) of Theorem 4 in Section 6.1.3 of [10], we know that for all $u \in BV$,

$$\nabla u = 0 \ \mathcal{L}^N\text{-a.e. on } \{u = 0\}.$$

Hence,

$$\nabla v_m - \nabla u_m = 0 \ \mathcal{L}^N\text{-a.e. outside } T,$$

and we have

$$\begin{aligned}\int_{\{\nabla v_m \neq \nabla u_m\}} W(\nabla v_m(x))dx &\le \int_T C_1(1 + |\nabla v_m(x)|)dx \\ &< C_1(\mathcal{L}^N(T) + 2p^{\frac{3}{2}}C5^N\varepsilon) \\ &< C_1(2p\delta 5^N\varepsilon + 2p^{\frac{3}{2}}C5^N\varepsilon).\end{aligned}$$

Since we can do this for a sequence $\varepsilon \to 0$, we conclude (3.1). □

We also need the following lemma in order to show $QW = G$.

Lemma 3.2 *Let $W : \mathbb{M}^{p\times N} \to \mathbb{R}^+$ be a Borel measurable function with $W(F) \le C_1(1 + |F|)$ for all $F \in \mathbb{M}^{p\times N}$ and some $C_1 > 0$, and let $f \in L^\infty(Q;\mathbb{R}^p)$ and $\varepsilon > 0$ be given. Then for all $\{u_n\} \subset SBV(Q;\mathbb{R}^p)$ with $||u_n||_{L^1(Q;\mathbb{R}^p)} + |D_{ac}u_n|(Q) \le R$ for all $n \in \mathbb{N}$ and some $R > 0$, there exists a sequence $\{v_n\} \subset SBV(Q;\mathbb{R}^p)$ uniformly bounded in $L^\infty(Q;\mathbb{R}^p)$ such that*

$$S(v_n) \subset S(u_n),$$

$$||v_n - f||_{L^1(Q;\mathbb{R}^p)} \le ||u_n - f||_{L^1(Q;\mathbb{R}^p)},$$

and

$$\liminf_{n\to\infty} \int_Q W(\nabla v_n(x))dx \le \liminf_{n\to\infty} \int_Q W(\nabla u_n(x))dx + \varepsilon.$$

Proof. The proof is a simpler version of the proof of Lemma 3.7 in [6], which relies on a truncation argument proposed by De Giorgi. Set $\lambda := [\ln(||f||_\infty + 1)] + 1$, where $[\cdot]$ is integer part, and fix $k \in \mathbb{N}$ with $k \ge \lambda$. Let $i \in \{\lambda, \ldots, k\}$ be given. Define $\phi_i \in W^{1,\infty}(\mathbb{R}^p;\mathbb{R}^p)$ by

$$\phi_i(x) := \begin{cases} x & \text{if } |x| \le e^i \\ \frac{x}{e-1}\left(\frac{e^{i+1}}{|x|} - 1\right) & \text{if } e^i < |x| < e^{i+1} \\ 0 & \text{if } |x| \ge e^{i+1}. \end{cases}$$

Set

$$u_n^i := \phi_i \circ u_n.$$

Then $||u_n^i||_\infty \le e^i$. Since $||\nabla\phi_i||_\infty = 1$ (see Ambrosio [3] and Vol'pert [16]), we have

$$u_n^i \in SBV(Q;\mathbb{R}^p),$$

$$|D_{ac}u_n^i|(Q) \leq |D_{ac}u_n|(Q),$$

and

$$S(u_n^i) \subset S(u_n).$$

Furthermore, by the choice of λ we have

$$\begin{aligned} ||u_n^i - f||_{L^1(Q;\mathbb{R}^p)} &= \int_{\{|\tilde{u}_n|<e^i\}} |u_n(x) - f(x)|dx + \int_{\{|\tilde{u}_n|\geq e^i\}} |\phi_i(u_n(x)) - \phi_i(f(x))|dx \\ &\leq ||u_n - f||_{L^1(Q;\mathbb{R}^p)}, \end{aligned}$$

where we used the fact that $\mathrm{Lip}(\phi_i) = 1$ and $\phi_i \circ f = f$. Now, fix $n \in \mathbb{N}$ and set

$$Q_i := \{x \in Q : |\tilde{u}_n(x)| < e^i\}.$$

Note that we have

$$\int_{Q\backslash Q_i} W(\nabla u_n^i(x))dx \leq C_1(\mathcal{L}^N(Q\backslash Q_i) + |D_{ac}u_n^i|(Q\backslash Q_i))$$

where $\mathcal{L}^N(Q\backslash Q_i) \leq \frac{R}{e^i}$ and

$$\begin{aligned} \sum_{i=\lambda}^{k} |D_{ac}u_n^i|(Q\backslash Q_i) &\leq \sum_{i=\lambda}^{k} |D_{ac}u_n^i|(\{e^i \leq |\tilde{u}_n(x)| < e^{i+1}\}) \\ &\leq |D_{ac}u_n|(Q) \\ &\leq R. \end{aligned}$$

We now have that

$$\begin{aligned} \sum_{i=\lambda}^{k} \int_{Q\backslash Q_i} W(\nabla u_n^i(x))dx &\leq \sum_{i=\lambda}^{k} C_1 \frac{R}{e^i} + C_1 R \\ &\leq C_1 R \left(\frac{1}{e^{\lambda-1}(e-1)} + 1 \right) \end{aligned}$$

so that

$$\sum_{i=\lambda}^{k} \int_Q W(\nabla u_n^i(x))dx \leq (k - \lambda + 1) \int_Q W(\nabla u_n(x))dx + C_1 R \left(\frac{1}{e^{\lambda-1}(e-1)} + 1 \right),$$

and by the choice of λ,

$$\begin{aligned} \frac{1}{k-\lambda+1} \sum_{i=\lambda}^{k} \int_Q W(\nabla u_n^i(x))dx \leq & \int_Q W(\nabla u_n(x))dx \\ & + \frac{C_1 R}{k - [\ln(||f||_\infty + 1)]} \left(1 + \frac{1}{(||f||_\infty + 1)(e-1)} \right). \end{aligned}$$

Choosing k large enough so that

$$\frac{C_1 R}{k - [\ln(||f||_\infty + 1)]} \left(1 + \frac{1}{(||f||_\infty + 1)(e-1)} \right) < \frac{\varepsilon}{2},$$

we see that there must be an $i \in \{\lambda, \ldots, k\}$ so that

$$\int_Q W(\nabla u_n^i(x))dx \leq \int_Q W(\nabla u_n(x))dx + \varepsilon$$

with $||u_n^i||_\infty \le e^k$, where the above choice of k does not depend on n. Hence, this can be done for all $n \in \mathbb{N}$, giving the same L^∞ bound of e^k. □

We now recall the definition of G, given in Section 2:

$$G(F) := \inf\left\{ \liminf_{n\to\infty} \int_Q W(\nabla u_n)dx : \{u_n\} \subset SBV(Q;\mathbb{R}^p), u_n \to Fx \right.$$

$$\left. \text{in } L^1(Q;\mathbb{R}^p), \text{ and } \mathcal{H}^{N-1}(S(u_n)) \to 0 \right\}.$$

Proposition 3.3 *Suppose that $W : \mathbb{M}^{p\times N} \to \mathbb{R}^+$ is a Borel measurable function such that $C_0|F| - \frac{1}{C_0} \le W(F) \le C_1(1+|F|)$ for all $F \in \mathbb{M}^{p\times N}$ and some $C_0, C_1 > 0$. Then*

$$QW = G.$$

Proof. We need only show that

$$QW(F) \le G(F), \tag{3.6}$$

since the admissible class of $\{u_n\}$ for $G(F)$ includes that for $QW(F)$, and so $QW(F) \ge G(F)$. Choose $\{u_n\} \subset SBV(Q;\mathbb{R}^p)$ such that $u_n \to Fx$ in $L^1(Q;\mathbb{R}^p)$, $\mathcal{H}^{N-1}(S(u_n)) \to 0$, and

$$\lim_{n\to\infty} \int_Q W(\nabla u_n(x))dx = G(F).$$

Because $||u_n||_{L^1(Q;\mathbb{R}^p)} \to ||Fx||_{L^1(Q;\mathbb{R}^p)}$, we obtain

$$\sup_{n\in\mathbb{N}} ||u_n||_{L^1(Q;\mathbb{R}^p)} < \infty.$$

Furthermore, since $W(\nabla u_n(x)) \ge C_0|\nabla u_n(x)| - \frac{1}{C_0}$, we deduce that

$$\begin{aligned} |D_{ac}u_n|(Q) &= \int_Q |\nabla u_n(x)|dx \\ &\le \tfrac{1}{C_0}\left(\int_Q W(\nabla u_n(x))dx + \tfrac{1}{C_0}\right) \to \tfrac{1}{C_0}\left(G(F) + \tfrac{1}{C_0}\right) < \infty, \end{aligned}$$

so

$$\sup_{n\in\mathbb{N}} |D_{ac}u_n|(Q) < \infty.$$

Let $\varepsilon > 0$ and apply Lemma 3.2 to $f := Fx$, $R := \sup_{n\in\mathbb{N}}(||u_n||_{L^1(Q;\mathbb{R}^p)} + |D_{ac}u_n|(Q))$, and the above ε and $\{u_n\}$. We now have

$$\lim_{n\to\infty} \int_Q W(\nabla v_n(x))dx \le G(F) + \varepsilon$$

for some $\{v_n\}$ with the same properties as $\{u_n\}$ and, in addition, $||v_n||_\infty \le M < \infty$ for all $n \in \mathbb{N}$. Hence, $|D_s v_n|(Q) \le 2M\mathcal{H}^{N-1}(S(v_n)) \to 0$. By Lemma 3.1, we conclude that

$$QW(F) \le \lim_{n\to\infty} \int_Q W(\nabla v_n(x))dx \le G(F) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have (3.6). □

4 Upper Bound

In this section we prove an inequality leading to (2.4). Precisely,

$$I(u,A) \leq \int_A QW(\nabla u)dx + \int_{S(u)} h([u],\nu)d\mathcal{H}^{N-1} + \int_A QW^\infty(dC(u)).$$

We first need the following result.

Proposition 4.1 *Suppose that $u \in BV(A;\mathbb{R}^p)$, where A is a bounded, open subset of Ω. Then $I(u,\cdot)$ extends to a nonnegative, finite, Borel regular measure on A which is absolutely continuous with respect to $\mathcal{L}^N + |Du|$.*

Proof. By using an argument similar to that for Theorem 3.2 in [12], we know that $I(u,\cdot)$ is a Radon measure on A. It remains to prove that $I(u,\cdot)$ is finite and absolutely continuous with respect to $\mathcal{L}^N + |Du|$. Let $B \subset A$ be open. By Theorem 5.3.3 of [17] and Theorem 2 in Section 5.2.2 of [10], we choose $u_n \in C^\infty(B;\mathbb{R}^p)$ such that $u_n \to u$ in $L^1(B;\mathbb{R}^p)$ and $|Du_n|(B) \to |Du|(B)$. Since the u_n are smooth, we have

$$\begin{aligned}
I(u,B) &\leq \liminf_{n\to\infty} E(u_n,B) \\
&= \liminf_{n\to\infty} \int_B W(\nabla u_n)dx \\
&\leq \liminf_{n\to\infty} \int_B C_1\Big[1 + |\nabla u_n|\Big]dx \\
&= \liminf_{n\to\infty} C_1\Big[\mathcal{L}^N(B) + |Du_n|(B)\Big] \\
&= C_1\Big[\mathcal{L}^N(B) + |Du|(B)\Big],
\end{aligned}$$

which, in particular, implies that $I(u,A) < \infty$ for all $u \in BV(A;\mathbb{R}^p)$. □

Fix $A \subset \Omega$ open and $u \in BV(A;\mathbb{R}^p)$. Note that we have

$$I(u,A) \leq \inf\left\{\liminf_{n\to\infty} \int_A W(\nabla u_n)dx : \{u_n\} \subset W^{1,1}(A;\mathbb{R}^p), u_n \to u \text{ in } L^1(A;\mathbb{R}^p)\right\},$$

so from [14] we know that

$$I(u,A) \leq \int_A QW(\nabla u)dx + \int_{S(u)\cap A} (QW)^\infty(dD_ju) + \int_A (QW)^\infty(dC(u));$$

hence, it only remains to prove that

$$I(u,S(u)) \leq \int_{S(u)} h([u](x),\nu(x))d\mathcal{H}^{N-1}(x). \tag{4.1}$$

The jump set $S(u)$ is, in general, not so easy to deal with. Indeed, there exist functions $u \in BV((0,1)^2)$ with jump set $\{(x,y) \in (0,1)^2 : x \in \mathbb{Q}\}$. Furthermore, although for such u one has $E(u) = \infty$, we know that $I(u) \leq C_1[1 + |Du|((0,1)^2)] < \infty$. However, measure theoretic boundaries of sets of finite perimeter are much easier to handle and, for our purposes, there are connections between $S(u)$ and certain sets of finite perimeter that we can exploit.

Remark 4.2 Let $u \in BV(\Omega)$ and let $D \subset \mathbb{R}$ be dense. Then

$$S(u) = \bigcup_{t \in D} S(u) \cap \partial_* E_t = \bigcup_{\substack{t_1, t_2 \in D \\ t_1 \neq t_2}} \partial_* E_{t_1} \cap \partial_* E_{t_2},$$

where $E_t := \{x \in \Omega : u(x) > t\}$.

If $u \in BV(\Omega; \mathbb{R}^p)$, we denote the t level set of u_i by E^i_t. Also, if $u \in BV(\Omega)$, then E_t has finite perimeter for $\mathcal{L}^1$-a.e. t, and $\{x \in S(u) : u^-(x) < t < u^+(x)\} \subset \partial_* E_t$ (see, e.g., the proof of Theorem 1 in Section 5.9 of [10]). We also point out that for $u \in BV(\Omega; \mathbb{R}^p)$, we have $S(u) = \bigcup_{i=1}^p S(u_i)$.

If $T \subset \Omega$ has finite perimeter, then $\mathcal{H}^{N-1} \lfloor \partial_* T$ is a Radon measure. Since $S(u)$ is $\mathcal{H}^{N-1}$ measurable, we conclude that $\chi_{S(u)} \in L^1(\Omega, \mathcal{H}^{N-1} \lfloor \partial_* T)$. So, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u) \cap \partial_* T$ we have

$$\begin{aligned}
\lim_{\delta \to 0^+} \frac{\mathcal{H}^{N-1}(B(x,\delta) \cap S(u) \cap \partial_* T)}{\alpha(N-1)\delta^{N-1}} &= \lim_{\delta \to 0^+} \frac{\mathcal{H}^{N-1}(B(x,\delta) \cap S(u) \cap \partial_* T)}{\mathcal{H}^{N-1}(B(x,\delta) \cap \partial_* T)} \\
&= \lim_{\delta \to 0^+} \fint_{B(x,\delta)} \chi_{S(u)} d\mathcal{H}^{N-1} \lfloor \partial_* T \\
&= 1,
\end{aligned}$$

where the first equality follows from Corollary 1 (ii) in Section 5.7.2 of [10]. Hence, if $D \subset \mathbb{R}$ is countable and dense and such that E^i_t has finite perimeter for all $t \in D$ and all $i \in \{1, \ldots, p\}$, then for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$, for all $t \in D \cap (u_i^-(x), u_i^+(x))$ we have

$$\lim_{\delta \to 0^+} \frac{\mathcal{H}^{N-1}(B(x,\delta) \cap S(u) \cap \partial_* E_t)}{\alpha(N-1)\delta^{N-1}} = \lim_{\delta \to 0^+} \frac{\mathcal{H}^{N-1}(B(x,\delta) \cap S(u) \cap \partial_* E_t)}{\mathcal{H}^{N-1}(B(x,\delta) \cap \partial_* E_t)} = 1. \tag{4.2}$$

Furthermore, since $[u] \in L^1(\Omega, \mathcal{H}^{N-1} \lfloor (S(u) \cap \partial_* E_t))$, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$, for all $t \in D \cap (u_i^-(x), u_i^+(x))$ we have

$$\fint_{B(x,\delta) \cap S(u) \cap \partial_* E_t} |[u](y) - [u](x)| d\mathcal{H}^{N-1}(y) = 0.$$

Note that the same is true if $B(x,\delta)$ is replaced by $Q(x,\delta) := x + \delta Q_{\nu(x)}$. Hence, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$, for all $t \in D \cap (u_i^-(x), u_i^+(x))$ we have

$$\lim_{\delta \to 0^+} \frac{1}{\delta^{N-1}} \int_{Q(x,\delta) \cap S(u) \cap \partial_* E_t} |[u](y)| d\mathcal{H}^{N-1}(y) = |[u](x)|. \tag{4.3}$$

On the other hand, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$,

$$\lim_{\delta \to 0^+} \frac{1}{\delta^{N-1}} \int_{Q(x,\delta) \cap S(u)} |[u](y)| d\mathcal{H}^{N-1}(y) = |[u](x)|,$$

which, together with (4.3), shows that for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$, for all $t \in D \cap (u_i^-(x), u_i^+(x))$ we have

$$\lim_{\delta \to 0^+} \frac{|D_j u|(Q(x,\delta) \backslash \partial_* E_t)}{\delta^{N-1}} = 0. \tag{4.4}$$

We are now ready to prove (4.1). First, we note that $W^\infty(\xi \otimes \nu)$ is continuous since the limit W^∞ is obtained uniformly (see (2.3)), hence $h(\xi, \nu)$ is continuous. Note further that, for E_t as above, $[u]$ and ν are $\mathcal{H}^{N-1} \lfloor (S(u) \cap \partial_* E_t)$-measurable, and $h \leq 1$, so

$$h([u](\cdot), \nu(\cdot)) \in L^1(\Omega, \mathcal{H}^{N-1} \lfloor (S(u) \cap \partial_* E_t)).$$

Let $x_0 \in S(u) \cap \Omega$ and $t \in \mathbb{R}$ be given such that $u_i^-(x_0) < t < u_i^+(x_0)$, E_t has finite perimeter,

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{N-1}} \int_{Q(x_0,\delta)} |\nabla u| dx = 0,$$

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{N-1}} |C(u)|(Q(x_0,\delta)) = 0,$$

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{N-1}} |D_j u|(Q(x_0,\delta)\backslash\partial_* E_t) = 0,$$

$$\lim_{\delta\to 0^+} \frac{\mathcal{H}^{N-1}(Q(x_0,\delta) \cap S(u) \cap \partial_* E_t)}{\delta^{N-1}} = 1, \tag{4.5}$$

and

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{N-1}} \int_{Q(x_0,\delta)\cap S(u)\cap\partial_* E_t} |h([u](x),\nu(x)) - h([u](x_0),\nu(x_0))| d\mathcal{H}^{N-1}(x) = 0. \tag{4.6}$$

Note that the above can be done for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$ (the last three follow from (4.4) and (4.2)).

Let $\varepsilon > 0$ be given and choose $\delta_{x_0} \in (0,\varepsilon)$ such that if $\delta \in (0,\delta_{x_0})$, then the above equalities hold to within ε. If, given $\delta \in (0,\delta_{x_0})$ and $n \in \mathbb{N}$, we can find $v \in SBV(Q(x_0,\delta);\mathbb{R}^p)$ such that

$$||v - u||_{L^1(Q(x_0,\delta);\mathbb{R}^p)} \leq \frac{1}{n}$$

and

$$\frac{E(v,Q(x_0,\delta))}{\delta^{N-1}} \leq h([u](x_0),\nu(x_0)) + O(\varepsilon),$$

then it follows that

$$\lim_{\delta\to 0^+} \frac{I(u,Q(x_0,\delta))}{|D_j u|(Q(x_0,\delta))} = \lim_{\delta\to 0^+} \frac{I(u,Q(x_0,\delta))}{|[u](x_0)|\delta^{N-1}} \leq \frac{h([u](x_0),\nu(x_0))}{|[u](x_0)|},$$

and so

$$I(u,S(u)) \leq \int_{S(u)} \frac{h([u](x),\nu(x))}{|[u](x)|} |[u](x)| d\mathcal{H}^{N-1}(x)$$

and we have (4.1). Let $\delta \in (0,\delta_{x_0})$ and $n \in \mathbb{N}$ be given. Choose $A \subset Q(x_0,\delta)$ open such that

$$S(u) \cap \partial_* E_t \cap Q(x_0,\delta) \subset A \text{ and } \mathcal{L}^N(A) < \min\{\varepsilon, 1\}.$$

For $\mathcal{H}^{N-1}$-a.e. $x \in S(u) \cap \partial_* E_t \cap Q(x_0,\delta)$, choose $r_x > 0$ such that $B(x,r_x) \subset A$ and $r \in (0,r_x)$ implies, using (4.2) and (4.6),

$$\int_{B(x,r)\cap H^-(x)} |u(y) - u^-(x)| dy < \frac{\alpha(N)r^N}{6n},$$

$$\int_{B(x,r)\cap H^+(x)} |u(y) - u^+(x)| dy < \frac{\alpha(N)r^N}{6n},$$

$$\left| \frac{\alpha(N-1)r^{N-1}}{\mathcal{H}^{N-1}(B(x,r) \cap S(u) \cap \partial_* E_t)} - 1 \right| < \varepsilon, \tag{4.7}$$

and

$$\int_{B(x,r)\cap S(u)\cap\partial_* E_t} |h([u](y),\nu(y)) - h([u](x),\nu(x))| d\mathcal{H}^{N-1}(y) < \varepsilon\alpha(N-1)r^{N-1}. \tag{4.8}$$

In addition, for $\mathcal{H}^{N-1}$-a.e. $x \in Q(x_0,\delta) \cap S(u) \cap \partial_* E_t$,

$$\frac{1}{r^N}\int_0^r \int_{\partial B(x,r)\cap H^-(x)} |\tilde{u}(y) - u^-(x)| d\mathcal{H}^{N-1}(y) dr = \frac{1}{r^N}\int_{B(x,r)\cap H^-(x)} |u(y) - u^-(x)| dy \to 0 \tag{4.9}$$

and similarly for $u^+(x)$. So, we consider the set T of those points in $S(u) \cap \partial_* E_t \cap Q(x_0,\delta)$ such that we can choose r_x as above, and (4.9) holds for $u^-(x)$ and $u^+(x)$. For $x \in T$, set

$$R_x := \Big\{ r \in (0,r_x) : \int_{\partial B(x,r)\cap H^-(x)} |\tilde{u}(y) - u^-(x)| d\mathcal{H}^{N-1}(y) < \varepsilon\delta^{N-1} r^{N-1},$$

$$r \text{ is a point of approximate continuity for the function}$$

$$r \mapsto \int_{\partial B(x,r)\cap H^-(x)} |\tilde{u}(y) - u^-(x)| d\mathcal{H}^{N-1}(y),$$

$$\text{and similarly for } u^+ \Big\}.$$

Since the family $\{B(x,r) : x \in T, r \in R_x\}$ is a fine cover of T, by Besicovitch's Covering Theorem, there is a finite, disjoint subfamily $\mathcal{H}$ such that

$$\mathcal{H}^{N-1}(T \setminus \cup_{B\in\mathcal{H}} B) < \varepsilon \tag{4.10}$$

and

$$|D_j u|(T \setminus \cup_{B\in\mathcal{H}} B) < \varepsilon\delta^{N-1}. \tag{4.11}$$

Put $k := \#\mathcal{H}$. We now extend the diameter of each ball $B \in \mathcal{H}$ by at most $\frac{\varepsilon}{k|[u](x_B)|}$, obtaining a ball $B^e \subset A$, so that the following hold:

i) the B^e are mutually disjoint;

ii) $\mathcal{H}^{N-1}(S(u) \cap \partial B^e) = 0$;

iii) $\displaystyle\int_{\partial B^e \cap H^-(x_B)} |\tilde{u}(x) - u^-(x_B)| d\mathcal{H}^{N-1}(x) < \varepsilon\delta^{N-1} r_B^{N-1}$ and similarly for $u^+(x_B)$;

iv) $\displaystyle\int_{(B^e\setminus B)\cap H^-(x_B)} |\tilde{u}(x) - u^-(x_B)| dx < \frac{\alpha(N) r_{B^e}^N}{12n}$, and similarly for $u^+(x_B)$.

Define u_0 by $u_0 := u$ on $Q(x_0,\delta) \setminus \cup_{B\in\mathcal{H}} B^e$, and

$$u_0(x) = \begin{cases} u^+(x_B) & \text{if } x \in B^e \cap H^+(x_B) \\ u^-(x_B) & \text{if } x \in B^e \cap H^-(x_B). \end{cases}$$

Again, by Theorem 5.3.3 of [17] and Theorem 2 in Section 5.2.2 of [10], we can choose

$$u_m = u_0 * \rho_m \in C^\infty(Q(x_0,\delta) \setminus \cup_{B\in\mathcal{H}} B; \mathbb{R}^p)$$

such that

$$u_m \to u_0 \text{ in } L^1(Q(x_0,\delta) \setminus \cup_{B\in\mathcal{H}} B; \mathbb{R}^p)$$

and

$$|Du_m|(Q(x_0,\delta) \setminus \cup_{B\in\mathcal{H}} B) \to |Du_0|(Q(x_0,\delta) \setminus \cup_{B\in\mathcal{H}} B).$$

Let m be sufficiently large so that, for all $B \in \mathcal{H}$,

$$r_{B^e} - r_B > \frac{1}{m},$$

$$\mathcal{H}^{N-1}\left(\left\{ x \in \partial B\left(x_B, r_{B^e} - \frac{1}{m}\right) : \operatorname{dist}\big(x, H(x_B)\big) < \frac{1}{m} \right\}\right) < \frac{\varepsilon}{k}, \tag{4.12}$$

$$\Big| |Du_m|(Q(x_0,\delta) \setminus \cup_{B\in\mathcal{H}} B) - |Du_0|(Q(x_0,\delta) \setminus \cup_{B\in\mathcal{H}} B) \Big| < \varepsilon\delta^{N-1}, \tag{4.13}$$

$$\int_{Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B^\epsilon} |u_m - u|dx < \frac{1}{3n}, \tag{4.14}$$

and

$$\int_{\cup_{B\in\mathcal{H}}(B^\epsilon\setminus B)} |u_m - u_0|dx < \frac{1}{6n}. \tag{4.15}$$

We now slightly alter u_m as follows: on $B\Big(x_B, r_{B^\epsilon} - \frac{1}{m}\Big)\setminus B$, we set $u_m := u_0$. Note that this introduces a jump on $H(x_B)\cap B\Big(x_B, r_{B^\epsilon} - \frac{1}{m}\Big)\setminus B$, and on $\{x \in \partial B(x_B, r_{B^\epsilon} - \frac{1}{m}) : \text{dist}(x, H(x_B)) < \frac{1}{m}\}$. By (4.12), we know this last set is of small $\mathcal{H}^{N-1}$ measure.

We extend u_m to $\cup_{B\in\mathcal{H}}B$ as follows: for each $B\in\mathcal{H}$, choose v_B admissible for $h([u](x_B), \nu(x_B))$ such that

$$\int_{Q_{\nu(x_B)}} W^\infty(\nabla v_B)dx + \mathcal{H}^{N-1}(S(v_B)) < h([u](x_B), \nu(x_B)) + \frac{\varepsilon}{k}. \tag{4.16}$$

Put

$$v_{B,k}(x) := v_B(kx) \in SBV\Big(\frac{1}{k}Q_{\nu(x_B)}; \mathbb{R}^p\Big).$$

We can select $a_j \in H(x_B)\cap B$ such that $a_j + \frac{1}{k}Q_{\nu(x_B)} \subset B$ and are mutually disjoint, and $\mathcal{H}^{N-1}\Big(H(x_B)\cap B\setminus \cup_j (a_j + \frac{1}{k}Q_{\nu(x_B)})\Big) < O(\frac{1}{k})$. We define

$$u_m^k(x) := \begin{cases} u^-(x_B) + v_{B,k}(x - a_j) & \text{if } x \in a_j + \frac{1}{k}Q_{\nu(x_B)} \\ u^+(x_B) & \text{if } x \in B\cap H^+(x_B)\setminus \cup (a_j + \frac{1}{k}Q_{\nu(x_B)}) \\ u^-(x_B) & \text{if } x \in B\cap H^-(x_B)\setminus \cup (a_j + \frac{1}{k}Q_{\nu(x_B)}). \end{cases}$$

Note that as $k\to\infty$, $u_m^k \to u_0$ in $L^1(\cup_{B\in\mathcal{H}}B; \mathbb{R}^p)$. Furthermore,

$$E(u_m^k, B) = \sum_j E\Big(u_m^k, a_j + \frac{1}{k}Q_{\nu(x_B)}\Big) + \mathcal{L}^N\Big(B\setminus \cup_j \Big(a_j + \frac{1}{k}Q_{\nu(x_B)}\Big)\Big)W(0) \tag{4.17}$$

and

$$\begin{aligned} E\Big(u_m^k, a_j + \frac{1}{k}Q_{\nu(x_B)}\Big) &= \int_{\frac{1}{k}Q_{\nu(x_B)}} W(k\nabla v_B(kx))dx + \mathcal{H}^{N-1}(S(v_{B,k})) \\ &= \int_{Q_{\nu(x_B)}} W(k\nabla v_B(x))\frac{1}{k^N}dx + \frac{1}{k^{N-1}}\mathcal{H}^{N-1}(S(v_B)). \end{aligned}$$

Since, by (2.3),

$$\begin{aligned} \int_{Q_{\nu(x_B)}} \Big|\frac{1}{k}W(k\nabla v_B(x)) - W^\infty(\nabla v_B(x))\Big|dx &\le \int_{Q_{\nu(x_B)}\cap\{k|\nabla v_B|>L\}} \frac{C}{L^{m-1}k} \\ &\quad + \int_{Q_{\nu(x_B)}\cap\{k|\nabla v_B|\le L\}} \frac{1}{k}C_1[1+L] \\ &\quad \to 0 \end{aligned}$$

as $k\to\infty$, we can choose k so that, setting $u_m := u_m^k$, we have

$$\int_{\cup_{B\in\mathcal{H}}(B\cap H^-(x_B))} |u_m(x) - u^-(x_B)|dx < \frac{1}{12n}, \text{ and similarly for } u^+(x_B), \tag{4.18}$$

and, using (4.16) and (4.17),

$$\left|E(u_m,B)-\left[\alpha(N-1)r_B^{N-1}h([u](x_B),\nu(x_B))+\alpha(N)r_B^N W(0)\right]\right|<\frac{\varepsilon}{k}. \tag{4.19}$$

We now have

$$\begin{aligned}||u_m-u||_{L^1(Q(x_0,\delta);\mathbb{R}^p)} &= \int_{\cup_{B\in\mathcal{H}}B}|u_m-u|dx+\int_{Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B^e}|u_m-u|dx\\
&+\int_{\cup_{B\in\mathcal{H}}(B^e\setminus B)}|u_m-u|dx\\
&<\int_{\cup_{B\in\mathcal{H}}B}|u_m(x)-u^{-,+}(x_B)|dx+\int_{\cup_{B\in\mathcal{H}}B}|u(x)-u^{-,+}(x_B)|dx\\
&+\frac{1}{3n}+\int_{\cup_{B\in\mathcal{H}}(B^e\setminus B)}|u_m-u_0|dx+\int_{\cup_{B\in\mathcal{H}}(B^e\setminus B)}|u_0-u|dx\\
&\qquad\text{(by (4.14))}\\
&<\frac{1}{6n}+\frac{1}{6n}+\frac{1}{3n}+\frac{1}{6n}+\int_{\cup_{B\in\mathcal{H}}(B^e\setminus B)}|u(x)-u^{-,+}(x_B)|dx\\
&\qquad\text{(by (4.18) and (4.15))}\\
&<\tfrac{1}{n}.\ \text{(by property iv) of } B^e)\end{aligned}$$

Finally, we have

$$\left|h([u](x_0),\nu(x_0))-\frac{E(u_m,Q(x_0,\delta))}{\delta^{N-1}}\right|\leq$$

$$\leq\left|h([u](x_0),\nu(x_0))-\frac{1}{\delta^{N-1}}E(u_m,\cup_{B\in\mathcal{H}}B)\right|+\frac{1}{\delta^{N-1}}E(u_m,Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B)=:I_1+I_2,$$

and

$$\begin{aligned}I_1 &\leq\left|h([u](x_0),\nu(x_0))-\frac{1}{\delta^{N-1}}\sum_{B\in\mathcal{H}}\alpha(N-1)r_B^{N-1}h([u](x_B),\nu(x_B))\right|+O(\varepsilon)\ \text{(by (4.19))}\\
&\leq\left|h([u](x_0),\nu(x_0))-\frac{1}{\delta^{N-1}}\sum_{B\in\mathcal{H}}\mathcal{H}^{N-1}(B\cap S(u)\cap\partial_*E_t)h([u](x_B),\nu(x_B))\right|+O(\varepsilon)\\
&\qquad\text{(by (4.7))}\\
&\leq\frac{1}{\delta^{N-1}}\left|\delta^{N-1}h([u](x_0),\nu(x_0))-\sum_{B\in\mathcal{H}}\int_{B\cap S(u)\cap\partial_*E_t}h([u](x),\nu(x))d\mathcal{H}^{N-1}(x)\right|+O(\varepsilon)\\
&\qquad\text{(by (4.8))}\\
&\leq\frac{1}{\delta^{N-1}}\sum_{B\in\mathcal{H}}\int_{B\cap S(u)\cap\partial_*E_t}|h([u](x_0),\nu(x_0))-h([u](x),\nu(x))|d\mathcal{H}^{N-1}(x)+O(\varepsilon)\\
&\qquad\text{(by (4.5) and (4.10))}\\
&\leq\frac{1}{\delta^{N-1}}\int_{Q(x_0,\delta)\cap S(u)\cap\partial_*E_t}|h([u](x_0),\nu(x_0))-h([u](x),\nu(x))|d\mathcal{H}^{N-1}(x)+O(\varepsilon)\\
&<O(\varepsilon);\ \text{(by (4.6))}\end{aligned}$$

$$
\begin{aligned}
I_2 &= \frac{1}{\delta^{N-1}}\left[\int_{Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B} W(\nabla u_m)dx + \mathcal{H}^{N-1}(S(u_m)\cap[Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B])\right]\\
&\leq \frac{1}{\delta^{N-1}}\int_{Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B} C_1(1+|\nabla u_m|)dx + O(\varepsilon) \text{ (by the choice of } B^e \text{ and (4.12))}\\
&= \frac{1}{\delta^{N-1}}\left[C_1\mathcal{L}^N(Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B) + C_1|D_{ac}u_m|(Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B)\right] + O(\varepsilon)\\
&\leq \frac{1}{\delta^{N-1}}C_1|Du_0|(Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B) + O(\varepsilon) \text{ (since } \delta<\varepsilon \text{ and by (4.13))}\\
&= \frac{1}{\delta^{N-1}}C_1\left[|Du|(Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B^e) + |Du_0|(\cup_{B\in\mathcal{H}}(B^e\setminus B))\right] + O(\varepsilon)\\
&\qquad\text{(by the definition of } u_0)\\
&\leq \frac{1}{\delta^{N-1}}C_1\Big[|D_{ac}u|(Q(x_0,\delta)) + |C(u)|(Q(x_0,\delta)) + |D_ju|(Q(x_0,\delta)\setminus\cup_{B\in\mathcal{H}}B)\\
&\quad + |D_{ac}u_0|(\cup_{B\in\mathcal{H}}(B^e\setminus B)) + |D_ju_0|(\cup_{B\in\mathcal{H}}(B^e\setminus B))\Big] + O(\varepsilon)\\
&\qquad\text{(by the choice of } \delta_{x_0} \text{ and } u_0)\\
&\leq \frac{1}{\delta^{N-1}}C_1\int_{\cup_{B\in\mathcal{H}}\partial B^e} |[u_0](x)|d\mathcal{H}^{N-1}(x) + O(\varepsilon)\\
&\qquad\text{(by (4.11) and the size of the extension } B^e)\\
&= \frac{1}{\delta^{N-1}}C_1\sum_{B\in\mathcal{H}}\left[\int_{\partial B^e\cap H^-(x_B)} |\tilde{u}(x)-u^-(x_B)|d\mathcal{H}^{N-1}(x)\right.\\
&\qquad\left.+\int_{\partial B^e\cap H^+(x_B)} |\tilde{u}(x)-u^+(x_B)|d\mathcal{H}^{N-1}(x)\right] + O(\varepsilon)\\
&< O(\varepsilon) \text{ (by property iii) of } B^e)
\end{aligned}
$$

and we have (4.1). □

5 Lower Bound

In this section we prove that

$$I(u,\Omega) \geq \int_\Omega QW(\nabla u)dx + \int_{S(u)} h([u],\nu)d\mathcal{H}^{N-1} + \int_\Omega QW^\infty(dC(u)).$$

As mentioned in the introduction, we rely heavily on [6], and we use the blow-up method introduced by Fonseca and Müller in [14].

Let $u_n \in SBV(\Omega;\mathbb{R}^p)$ be given such that $u_n \to u$ in $L^1(\Omega,\mathbb{R}^p)$ and

$$\liminf_{n\to\infty}\left[\int_\Omega W(\nabla u_n)dx + \mathcal{H}^{N-1}(S(u_n)\cap\Omega)\right] =$$

$$\lim_{n\to\infty}\left[\int_\Omega W(\nabla u_n)dx + \mathcal{H}^{N-1}(S(u_n)\cap\Omega)\right] < \infty.$$

Define a sequence of Radon measures by

$$< \mu_n, \psi >:= \int_\Omega \psi(x) W(\nabla u_n(x))dx + \int_{S(u_n)} \psi(x) d\mathcal{H}^{N-1}(x), \text{ for all } \psi \in C_0(\Omega).$$

Since $\sup_n |\mu_n|(\Omega) < \infty$, there exists a subsequence (not relabeled) and a finite Radon measure μ such that $\mu_n \stackrel{*}{\rightharpoonup} \mu$, i.e., for all $\psi \in C_0(\Omega)$,

$$< \mu, \psi >= \lim_{n\to\infty} \left[\int_\Omega \psi(x) W(\nabla u_n)dx + \int_{S(u_n)} \psi(x) d\mathcal{H}^{N-1} \right].$$

The Radon-Nikodym Theorem allows us to write

$$\mu = \mu_a \mathcal{L}^N + \mu_J |[u]| \mathcal{H}^{N-1} \lfloor S(u) + \mu_c |C(u)| + \mu_s,$$

where $\mu_s \geq 0$. In view of Proposition 3.3, we need only show that

a) $\mu_a(x_0) \geq G(\nabla u(x_0))$ for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$,

b) $\mu_J(x_0) \geq \frac{h([u](x_0), \nu(x_0))}{|[u](x_0)|}$ for $\mathcal{H}^{N-1}$-a.e. $x_0 \in S(u) \cap \Omega$,

and

c) $\mu_c(x_0) \geq G^\infty(\frac{dC(u)}{d|C(u)|}(x_0))$ for $|C(u)|$-a.e. $x_0 \in \Omega$.

Proof of a):

Let $x_0 \in \Omega$ be given such that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \fint_{B(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)| dx = 0,$$

$$\mu_a(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(Q(x_0,\varepsilon))}{\varepsilon^N} \text{ exists and is finite,}$$

and note that the above hold for $\mathcal{L}^N$-a.e. $x \in \Omega$. Also, since $\mu(\Omega) < \infty$ we have $\mu(\Omega \cap \partial Q(x_0, \delta)) = 0$ except for countably many $\delta > 0$, and we choose $\delta_k < \frac{1}{k}$ such that $\bar{Q}(x_0, \delta_k) \subset \Omega$ and $\mu(\partial Q(x_0, \delta_k)) = 0$. Then

$$\begin{aligned}
\mu_a(x_0) &= \lim_{k\to\infty} \frac{\mu(Q(x_0,\delta_k))}{\delta_k^N} \\
&= \lim_{k\to\infty} \frac{1}{\delta_k^N} \lim_{n\to\infty} \left[\int_{Q(x_0,\delta_k)} W(\nabla u_n(x))dx + \mathcal{H}^{N-1}(Q(x_0,\delta_k) \cap S(u_n)) \right] \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \left[\int_Q W(\nabla u_n(x_0 + \delta_k y))dy + \frac{1}{\delta_k} \mathcal{H}^{N-1} \left(Q \cap \frac{S(u_n) - x_0}{\delta_k} \right) \right],
\end{aligned}$$

where the second equality follows from Section 1.9 Theorem 1(iii) of [10] modified for Radon measures on Ω (the modification being the requirement that $\bar{B} \subset \Omega$).

Define $u_{n,k}(y) := \frac{u_n(x_0+\delta_k y) - u(x_0)}{\delta_k}$ for $y \in Q$, and note that

$$\begin{aligned}
&\int_Q W(\nabla u_n(x_0 + \delta_k y))dy + \frac{1}{\delta_k} \mathcal{H}^{N-1} \left(\frac{S(u_n) - x_0}{\delta_k} \cap Q \right) \\
&= \int_Q W(\nabla u_{n,k}(y))dy + \frac{1}{\delta_k} \mathcal{H}^{N-1}(S(u_{n,k}) \cap Q).
\end{aligned}$$

Then
$\lim_{k\to\infty}\lim_{n\to\infty}||u_{n,k}-\nabla u(x_0)y||_{L^1(Q;\mathbb{R}^p)}=$

$$= \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\delta_k}\int_Q |u_n(x_0+\delta_k y)-u(x_0)-\delta_k\nabla u(x_0)y|dy$$

$$\leq \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\delta_k}\int_Q |u_n(x_0+\delta_k y)-u(x_0+\delta_k y)|dy$$

$$+ \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\delta_k}\int_Q |u(x_0+\delta_k y)-u(x_0)-\delta_k\nabla u(x_0)y|dy$$

$$= \lim_{k\to\infty}\frac{1}{\delta_k}\int_Q |u(x_0+\delta_k y)-u(x_0)-\delta_k\nabla u(x_0)y|dy$$

$$= 0,$$

and

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\frac{1}{\delta_k}\mathcal{H}^{N-1}(S(u_{n,k})\cap Q) =: M < \infty.$$

Choose a subsequence of δ_k, not relabeled, such that

$$\lim_{n\to\infty}\left[\int_Q W(\nabla u_n(x_0+\delta_k y))dy + \frac{1}{\delta_k}\mathcal{H}^{N-1}(S(u_{n,k})\cap Q)\right] < \mu_a(x_0)+\frac{1}{k},$$

$$\lim_{n\to\infty}||u_{n,k}-\nabla u(x_0)y||_{L^1(Q;\mathbb{R}^p)} < \frac{1}{k}, \text{ and}$$

$$\limsup_{n\to\infty}\frac{1}{\delta_k}\mathcal{H}^{N-1}(S(u_{n,k})\cap Q) < M+1.$$

Then select $n_k > k$ such that each of the above inequalities remains true. It follows that

$$v_k := u_{n_k,k} \to \nabla u(x_0)y \text{ in } L^1(Q;\mathbb{R}^p),$$

$$\mathcal{H}^{N-1}(S(v_k)\cap Q)\to 0,$$

and we have

$$\mu_a(x_0) \geq \limsup_{k\to\infty}\int_Q W(\nabla v_k)$$

$$\geq G(\nabla u(x_0)).$$

□

Proof of b):

Let $x_0 \in S(u)\cap\Omega$ be given such that

$$\lim_{\delta\to 0^+}\frac{1}{\delta^{N-1}}\int_{S(u)\cap(x_0+\delta Q_{\nu(x_0)})}|u^-(x)-u^+(x)|d\mathcal{H}^{N-1}(x) = |u^-(x_0)-u^+(x_0)|,$$

$$\lim_{\delta\to 0^+}\frac{1}{\delta^N}\int_{\{x\in B(x_0,\delta):(x-x_0)\cdot\nu(x_0)>0\}}|u(x)-u^+(x_0)|^{\frac{N}{N-1}}dx = 0,$$

$$\lim_{\delta\to 0^+}\frac{1}{\delta^N}\int_{\{x\in B(x_0,\delta):(x-x_0)\cdot\nu(x_0)<0\}}|u(x)-u^-(x_0)|^{\frac{N}{N-1}}dx = 0, \text{ and}$$

$$\mu_J(x_0) = \lim_{\delta\to 0^+}\frac{\mu(x_0+\delta Q_{\nu(x_0)})}{|u^+-u^-|\mathcal{H}^{N-1}\lfloor S(u)(x_0+\delta Q_{\nu(x_0)})} \text{ exists and is finite,}$$

and note that the above hold for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$. For simplicity of notation, we write $Q := Q_{\nu(x_0)}$ and $Q(x_0, \delta) := x_0 + \delta Q$. As in a), choose $\delta_k < \frac{1}{k}$ such that $\bar{Q}(x_0, \delta_k) \subset \Omega$ and $\mu(\partial Q(x_0, \delta_k)) = 0$. Then

$$
\begin{aligned}
\mu_J(x_0) &= \lim_{k\to\infty} \frac{\mu(Q(x_0,\delta))}{|u^+ - u^-|\mathcal{H}^{N-1}\lfloor S(u)(Q(x_0,\delta))} \\
&= \frac{1}{|[u](x_0)|} \lim_{k\to\infty} \frac{1}{\delta_k^{N-1}} \int_{Q(x_0,\delta_k)} d\mu(x) \\
&= \frac{1}{|[u](x_0)|} \lim_{k\to\infty} \frac{1}{\delta_k^{N-1}} \lim_{n\to\infty} \left[\int_{Q(x_0,\delta_k)} W(\nabla u_n(x))dx + \mathcal{H}^{N-1}(S(u_n) \cap Q(x_0,\delta_k)) \right] \\
&= \frac{1}{|[u](x_0)|} \lim_{k\to\infty} \lim_{n\to\infty} \left[\delta_k \int_Q W(\nabla u_n(x_0 + \delta_k y))dy + \mathcal{H}^{N-1}\left(\frac{S(u_n) - x_0}{\delta_k} \cap Q \right) \right].
\end{aligned} \tag{5.1}
$$

For $y \in Q$ define $u_{n,k}(y) := u_n(x_0 + \delta_k y)$ and

$$
u_0(y) := \begin{cases} u^+(x_0) & \text{if } y \cdot \nu(x_0) > 0 \\ u^-(x_0) & \text{if } y \cdot \nu(x_0) \le 0. \end{cases}
$$

Since $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$ we obtain

$$
\begin{aligned}
\lim_{k\to\infty}\lim_{n\to\infty} \int_Q |u_{n,k}(y) - u_0(y)|dy &= \lim_{k\to\infty} \frac{1}{\delta_k^N} \int_{\{x\in Q(x_0,\delta_k):(x-x_0)\cdot\nu(x_0)>0\}} |u(x) - u^+(x_0)|dx \\
&\quad + \lim_{k\to\infty} \frac{1}{\delta_k^N} \int_{\{x\in Q(x_0,\delta_k):(x-x_0)\cdot\nu(x_0)<0\}} |u(x) - u^-(x_0)|dx \\
&= 0.
\end{aligned}
$$

Note that $S(u_{n,k}) = \frac{S(u_n) - x_0}{\delta_k}$. Then, by (5.1) we have

$$
\begin{aligned}
\mu_J(x_0) = \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to\infty}\lim_{n\to\infty} \Bigg[\int_Q W^\infty(\nabla u_{n,k}(y))dy \\
+\mathcal{H}^{N-1}(Q \cap S(u_{n,k})) + \int_Q \left(\delta_k W\left(\frac{\nabla u_{n,k}(y)}{\delta_k} \right) - W^\infty(\nabla u_{n,k}(y)) \right) dy \Bigg].
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty} \int_{Q\cap\{||\nabla u_{n,k}||\le\delta_k L\}} \left| \delta_k W\left(\frac{\nabla u_{n,k}}{\delta_k}\right) - W^\infty(\nabla u_{n,k}) \right| dx \\
&+ \limsup_{k\to\infty}\limsup_{n\to\infty} \int_{Q\cap\{||\nabla u_{n,k}||>\delta_k L\}} \left| \delta_k W\left(\frac{\nabla u_{n,k}}{\delta_k}\right) - W^\infty(\nabla u_{n,k}) \right| dx \\
&\qquad \le \limsup_{k\to\infty}\limsup_{n\to\infty} \int_{Q\cap\{||\nabla u_{n,k}||\le\delta_k L\}} \delta_k C[1+L] + \delta_k L \\
&\qquad + \limsup_{k\to\infty}\limsup_{n\to\infty} \int_{Q\cap\{||\nabla u_{n,k}||>\delta_k L\}} C||\nabla u_{n,k}||^{1-m}\delta_k^m dy \\
&\qquad \le \limsup_{k\to\infty}\limsup_{n\to\infty} C\delta_k^m \left(\int_Q ||\nabla u_{n,k}(y)||dy \right)^{1-m} = 0,
\end{aligned}
$$

where the last equality follows from the bound on $\{\int_Q ||\nabla u_{n,k}(y)||dy\}$ due to (5.1). We now have

$$
\mu_J(x_0) = \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to\infty}\lim_{n\to\infty} \left[\int_Q W^\infty(\nabla u_{n,k}(y))dy + \mathcal{H}^{N-1}(Q \cap S(u_{n,k})) \right].
$$

As in a), choose a subsequence of δ_k, not relabeled, and $n_k > k$ such that $v_k := u_{n_k,k} \to u_0$ in $L^1(Q;\mathbb{R}^p)$ and

$$\mu_J(x_0) = \frac{1}{|u^+(x_0) - u^-(x_0)|} \lim_{k\to\infty} \left[\int_Q W^\infty(\nabla v_k(x))dx + \mathcal{H}^{N-1}(Q \cap S(v_k)) \right].$$

Since we can assume the trace of v_k = the trace of u_0 (see [13]), it follows that

$$\mu_J(x_0) \geq \frac{h([u](x_0), \nu(x_0))}{|u^+(x_0) - u^-(x_0)|}.$$

□

Proof of c):

We will use the following:

Lemma 5.1 *Let λ be a Radon measure on $\Omega \subset \mathbb{R}^N$. Then, for λ-a.e. $x \in \Omega$, given any open, bounded convex set C containing the origin, there is a sequence $\delta_i \to 0^+$ and a Radon measure γ on C such that*

$$\lambda_{\delta_i}(\cdot) := \frac{\lambda(x + \delta_i \cdot)}{\lambda(x + \delta_i C)} \overset{*}{\rightharpoonup} \gamma \text{ on } C, \text{ and } \gamma(C) = 1.$$

Proof. We first show that for λ-a.e. $x \in \Omega$ we have

$$\liminf_{\delta\to 0^+} \frac{\lambda(x + \delta C)}{\delta^N} > 0 \tag{5.2}$$

for all C as in the statement of the lemma. It is sufficient to consider (5.2) for an open ball B containing the origin, since $\delta B \subset C$ for small enough δ. Put

$$A := \left\{ x \in \Omega : \liminf_{\delta\to 0^+} \frac{\lambda(x + \delta B)}{\delta^N} = 0 \right\}.$$

Let $\varepsilon > 0$ be given, and using Besicovitch's Covering Theorem, choose a countable family of disjoint balls $x_i + \delta_i B \subset \Omega$ such that $\lambda(A \setminus \cup (x_i + \delta_i B)) = 0$ and $\lambda(x_i + \delta_i B) < \varepsilon \delta_i^N$. It follows that $\lambda(A) < \varepsilon \frac{\mathcal{L}^N(\Omega)}{\mathcal{L}^N(B)}$, which implies $\lambda(A) = 0$.

Fix $x \in \Omega$ for which (5.2) is satisfied. Without loss of generality, we can assume $x = 0$. Let $\eta \in (0, 1)$ be given and set $\delta_i := \eta^i$. Suppose that

$$\limsup_{\delta_i \to 0^+} \lambda_{\delta_i}(\eta C) < \eta^N.$$

Then we can choose $j \in \mathrm{N}$ and $\alpha \in (0, \eta^N)$ such that if $i \geq j$, then

$$\frac{\lambda(\delta_i \eta C)}{\lambda(\delta_i C)} < \alpha.$$

We now have

$$\frac{\lambda(\delta_i C)}{\delta_i^N} \leq \frac{\lambda(\delta_j C)\alpha^{i-j}}{[\eta^{i-j}\delta_j]^N} \to 0$$

as $i \to \infty$, which contradicts (5.2). Hence, we may extract a subsequence, not relabeled, and choose a Radon measure γ so that $\lambda_{\delta_i} \overset{*}{\rightharpoonup} \gamma$ on C and $\lambda_{\delta_i}(\eta C) > \alpha$, where $\alpha < \eta^N$. Choose $\beta \in (\eta, 1)$ such that $\gamma(\partial \beta C) = 0$. Then, for a subsequence and a Radon measure $\bar{\gamma}$, $\lambda_{\beta\delta_i} \overset{*}{\rightharpoonup} \bar{\gamma}$ on C, and for any Borel set $A \subset C$ we have

$$\begin{aligned} \lambda_{\beta\delta_i}(A) &= \frac{\lambda(\beta\delta_i A)}{\lambda(\beta\delta_i C)} \\ &< \frac{1}{\alpha} \frac{\lambda(\beta\delta_i A)}{\lambda(\delta_i C)} \\ &= \frac{1}{\alpha} \lambda_{\delta_i}(\beta A). \end{aligned}$$

Let $\varepsilon > 0$ be given and let $D \subset C$ be an open neighborhood of $\partial\beta C$ such that $\gamma(\bar{D}) < \varepsilon$. Then

$$\begin{aligned}\limsup_{\delta_i \to 0+} \lambda_{\beta\delta_i}\left(\frac{\bar{D}}{\beta} \cap C\right) &\leq \frac{1}{\alpha} \limsup_{\delta_i \to 0+} \lambda_{\delta_i}(\bar{D}) \\ &\leq \frac{1}{\alpha}\gamma(\bar{D}) \\ &< \frac{\varepsilon}{\alpha}.\end{aligned}$$

Hence,

$$\begin{aligned}\bar{\gamma}(C) &\geq \bar{\gamma}\left(C \backslash \frac{D}{\beta}\right) \\ &\geq \limsup_{\delta_i \to 0+} \lambda_{\beta\delta_i}\left(C \backslash \frac{D}{\beta}\right) \\ &\geq \liminf_{\delta_i \to 0+} \lambda_{\beta\delta_i}\left(C \backslash \frac{\bar{D}}{\beta}\right) \\ &> 1 - \frac{\varepsilon}{\alpha}.\end{aligned}$$

From the arbitrariness of ε, it follows that $\bar{\gamma}(C) = 1$. □

Now let $x_0 \in \Omega$ be given such that

$$\lim_{\delta \to 0+} \frac{|Du|(Q(x_0,\delta))}{|C(u)|(Q(x_0,\delta))} = 1,$$

$$\lim_{\delta \to 0+} \frac{|Du|(Q(x_0,\delta))}{\delta^{N-1}} = 0, \quad \lim_{\delta \to 0+} \frac{|Du|(Q(x_0,\delta))}{\delta^{N}} = \infty,$$

$$A_0 := \lim_{\delta \to 0+} \frac{Du(Q(x_0,\delta))}{|Du|(Q(x_0,\delta))} \text{ exists and } ||A_0|| = 1, A_0 = a \otimes \nu, \text{ and}$$

$$\mu_c(x_0) = \lim_{\delta \to 0+} \frac{\mu(Q(x_0,\delta))}{|C(u)|(Q(x_0,\delta))} = \lim_{\delta \to 0+} \frac{\mu(Q(x_0,\delta))}{|Du|(Q(x_0,\delta))}.$$

Note that the above hold for $|C(u)|$ a.e. $x \in \Omega$, where the statements regarding A_0 are due to Alberti [2]. Without loss of generality, assume that $\nu = e_N$ and $|a| = 1$. Choose $\delta_k < \frac{1}{k}$ such that, setting

$$z_k(x) := \frac{\delta_k^{N-1}}{|Du|(Q(x_0,\delta_k))}\left[u(x_0 + \delta_k x) - \frac{1}{\delta_k^N}\int_{Q(x_0,\delta_k)} u(y)dy\right],$$

the sequence $\{\delta_k\}$ is selected according to Lemma 5.1 so that, with $\lambda := |Du|$ and γ equal to the weak $*$ limit of $|Dz_k|$, we have $\gamma(Q) = \lim_{k\to\infty} |Dz_k|(Q) = 1$. Note that, using notation from the proof of Lemma 5.1, the δ_k chosen here are of the form $\beta\delta_i$. So, by choosing an appropriate β in the proof of Lemma 5.1, the δ_k can also be chosen such that $\mu(\partial Q(x_0,\delta_k)) = 0$ for all k. Then

$$\begin{aligned}\mu_c(x_0) &= \lim_{k\to\infty} \frac{\mu(Q(x_0,\delta_k))}{|Du|(Q(x_0,\delta_k))} \\ &= \lim_{k\to\infty} \frac{1}{|Du|(Q(x_0,\delta_k))} \lim_{n\to\infty} \int_{Q(x_0,\delta_k)} d\mu_n \\ &= \lim_{k\to\infty} \frac{1}{|Du|(Q(x_0,\delta_k))} \lim_{n\to\infty}\left[\int_{Q(x_0,\delta_k)} W(\nabla u_n(x))dx + \mathcal{H}^{N-1}(Q(x_0,\delta_k) \cap S(u_n))\right].\end{aligned} \tag{5.3}$$

Also,

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{\delta_k^{N-1}}{|Du|(Q(x_0,\delta_k))}\int_Q\left|u_n(x_0+\delta_k x)-\frac{1}{\delta_k^N}\int_{Q(x_0,\delta_k)}u_n(y)dy\right.$$

$$\left.-\left[u(x_0+\delta_k x)-\frac{1}{\delta_k^N}\int_{Q(x_0,\delta_k)}u(y)dy\right]\right|dx=0. \tag{5.4}$$

By (5.3) and (5.4), using a standard diagonalization argument, choose a subsequence $\{u_k\}$ such that

$$\mu_c(x_0)=\lim_{k\to\infty}\frac{1}{|Du|(Q(x_0,\delta_k))}\left[\int_{Q(x_0,\delta_k)}W(\nabla u_k(x))dx+\mathcal{H}^{N-1}(S(u_k)\cap Q(x_0,\delta_k))\right]$$

and

$$||\bar{u}_k-z_k||_{L^1(Q;\mathbb{R}^p)}\to 0, \tag{5.5}$$

where

$$\bar{u}_k(x):=\frac{\delta_k^{N-1}}{|Du|(Q(x_0,\delta_k))}\left[u_k(x_0+\delta_k x)-\frac{1}{\delta_k^N}\int_{Q(x_0,\delta_k)}u_k(y)dy\right],$$

$$z_k(x):=\frac{\delta_k^{N-1}}{|Du|(Q(x_0,\delta_k))}\left[u(x_0+\delta_k x)-\frac{1}{\delta_k^N}\int_{Q(x_0,\delta_k)}u(y)dy\right].$$

Setting

$$t_k:=\frac{|Du|(Q(x_0,\delta_k))}{\delta_k^N}\to\infty,$$

$$\theta_k:=\frac{|Du|(Q(x_0,\delta_k))}{\delta_k^{N-1}}\to 0$$

we conclude that

$$\mu_c(x_0)=\lim_{k\to\infty}\left[\frac{1}{t_k}\int_Q W(t_k\nabla\bar{u}_k(x))dx+\frac{1}{\theta_k}\mathcal{H}^{N-1}(S(\bar{u}_k)\cap Q)\right].$$

Then

$$\mathcal{H}^{N-1}(S(\bar{u}_k)\cap Q)\to 0 \text{ (since } \theta_k\to 0^+) \tag{5.6}$$

and

$$\begin{aligned}\mu_c(x_0)&\geq\limsup_{k\to\infty}\frac{1}{t_k}\int_Q W(t_k\nabla\bar{u}_k(x))dx\\&=\limsup_{k\to\infty}\int_Q W^\infty(\nabla\bar{u}_k(x))dx\end{aligned} \tag{5.7}$$

just as in b). Since

$$\int_Q z_k(x)dx=\int_Q\bar{u}_k(x)dx=0$$

and

$$|D\bar{u}_k|(Q)=|Dz_k|(Q)=1,$$

by (5.5) and Poincaré's inequality, there exist subsequences (not relabeled) $\{z_k\},\{\bar{u}_k\}$, and there exists $u_0\in BV(Q;\mathbb{R}^p)$ such that

$$z_k,\bar{u}_k\to u_0 \text{ in } L^1(Q;\mathbb{R}^p).$$

Now,

$$Dz_k(Q)=\frac{Du(Q(x_0,\delta_k))}{|Du|(Q(x_0,\delta_k))}\to A_0=a\otimes e_N$$

and $|Dz_k|(Q) = 1$ so, by Proposition A.1 of [14], it follows that

$$|Dz_k - (Dz_k \cdot A_0)A_0|(Q) \to 0,$$

from which we conclude that

$$|Dz_k \cdot e_i|(Q) \to 0 \text{ for } i = 1, \dots, N-1.$$

Since

$$|Du_0 \cdot e_i|(Q) \le \liminf_{k\to\infty} |Dz_k \cdot e_i|(Q) = 0,$$

we obtain

$$u_0(x) = \hat{u}_0(x_N) \in BV\left(\left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^p\right).$$

Note that, in general, if $\mu_k \overset{*}{\rightharpoonup} \eta$, $|\mu_k| \overset{*}{\rightharpoonup} \gamma$, and $\gamma(Q) = \lim_{k\to\infty} |\mu_k|(Q)$, then $\eta(Q) = \lim_{k\to\infty} \mu_k(Q)$. Here we have $Dz_k \overset{*}{\rightharpoonup} \eta$ and $\gamma(Q) = \lim_{k\to\infty} |Dz_k|(Q)$, so that $\eta(Q) = A_0$. On the other hand, $z_k \to u_0$ in $L^1(Q;\mathbb{R}^p)$, which implies that $Du_0 = \eta$ in Q, and so $Du_0(Q) = A_0$. Thus, $u_0(x) - A_0(x) = p(x_N) + c$, where $p(-1/2) = p(1/2) = 0$, and $u_0(x) - A_0x$ can be extended periodically to $\mathbb{R}^N$. Without loss of generality, we can assume that the trace of $\bar{u}_k$ equals the trace of u_0, so that $\bar{u}_k - A_0x$ can be extended periodically, and we call this extension w_k. Set

$$v_k^j(x) := A_0x + \frac{1}{j} w_k(jx)$$

and note that, for $x \in Q_j := (-\frac{1}{2j}, \frac{1}{2j})^N$, we have $\nabla v_k^j(x) = \nabla \bar{u}_k(jx)$. By (5.6) we may choose $k(j) > j$ such that $\mathcal{H}^{N-1}(S(v_{k(j)}^j) \cap Q) < \frac{1}{j}$, and we have

$$v_j := v_{k(j)}^j \to A_0x \text{ in } L^1(Q;\mathbb{R}^p)$$

and

$$\mathcal{H}^{N-1}(S(v_j) \cap Q) \to 0.$$

Furthermore,

$$\int_Q W^\infty(\nabla v_j(x))dx = j^N \int_{Q_j} W^\infty(\nabla \bar{u}_{k(j)}(jx))dx = \int_Q W^\infty(\nabla \bar{u}_{k(j)}(x))dx$$

and so, by (5.7), we need only show that

$$G^\infty(A_0) \le \limsup_{j\to\infty} \int_Q W^\infty(\nabla v_j(x))dx.$$

By (2.3) we have

$$\begin{aligned}
G^\infty(A_0) &\le \limsup_{t\to\infty} \frac{1}{t} \limsup_{j\to\infty} \int_Q W(t\nabla v_j(x))dx \\
&= \limsup_{t\to\infty} \limsup_{j\to\infty} \int_Q \frac{W(t\nabla v_j(x))}{t} dx \\
&\le \limsup_{t\to\infty} \limsup_{j\to\infty} \left\{ \int_{\{t|\nabla v_j|>L\}} \left[W^\infty(\nabla v_j(x)) + \frac{C}{L^{m-1}t} \right] + \int_{\{t|\nabla v_j|\le L\}} \frac{1}{t} C_1[1+L] \right\} \\
&\le \limsup_{j\to\infty} \int_Q W^\infty(\nabla v_j(x))dx.
\end{aligned}$$

□

6 Optimal Jump Microstructure for Scalar Valued Functions

In Section 3 we proved that, for sequences whose energy approaches the infimum G, it is not necessary to allow a singular part of the variation measures for $\{u_n\}$, provided we know that $|D_s u_n| \to 0$, or if we know that $\mathcal{H}^{N-1}(S(u_n)) \to 0$ in the case where $u_n \in SBV$. In other words, there is no gap when considering the infimum over smooth sequences and over sequences "almost" smooth (see Lemma 3.1 and Proposition 3.3).

The question now is, what behavior is it necessary to allow for admissible functions for h? That is, how do infimizing sequences behave? Below, we answer this question completely for scalar valued functions.

Looking at the definition of $h(\rho, \nu)$ (see (2.5)), we see that admissible functions may have both jumps and nonzero gradient. Is this necessary? Is it possible that there is an admissible function v that jumps and has nonzero gradient, and the energy of v is below the infimum over functions that just jump, and below the infimum over functions in $W^{1,1}$? The answer to this question is "yes", and we will see that the natural example illustrates the behavior of infimizing sequences. The square in Figure 2 on page 30 represents Q_ν for $N = 2$. Suppose that $CW^\infty(\rho\nu) \gg 1$ and $W^\infty(\rho\mu) \ll 1$ for some $\rho \in \mathbb{R}^+$ and unit vectors $\nu, \mu \in \mathbb{R}^2$, where $\nu \cdot \mu > 0$. We then see that a function that is 0 below $\Gamma := \Gamma_1 \cup \Gamma_2$ and ρ above, with a jump across Γ_1 and affine growth across a narrow extension of Γ_2, has lower energy than the infimum over functions that just jump (this infimum is 1), and the infimum over functions in $W^{1,1}$ (this infimum is $CW^\infty(\rho\nu)$). Note that this example fails if CW^∞ is isotropic. We show that this behavior is optimal. The idea is this: first, we give a coarea formula which allows us to consider, for any admissible function for $h(\rho, \nu)$, the bulk energy as an integral over measure-theoretic boundaries of level sets. We may then choose a "good" level set. Next, we prove that it is energetically better for the jump part of the boundary, i.e., $S(u)$ intersected with the boundary, to be connected and flat. As we will show in Lemma 6.1 below, we can assume that W^∞ is convex without changing the infimum of the energy, in which case we will prove that the remainder of the boundary might as well be flat, and we conclude that Figure 1 on page 30 captures the geometry of minimizing sequences.

We begin with

Lemma 6.1

$$h_W = h_{CW}.$$

Proof. Since $CW \leq W$, it follows that $h_{CW} \leq h_W$. Conversely, let u be an admissible function for h_{CW}. By the relaxation theorem (Theorem 2.1), we have

$$I(u, Q) \leq \int_Q CW(\nabla u)dx + \mathcal{H}^{N-1}(S(u) \cap Q),$$

where we use the fact that $h \leq 1$. It also follows from Theorem 2.1 that

$$I_{CW}(u, Q) = \int_Q CW(\nabla u)dx + \int_{S(u)\cap Q} h_{CW}([u], \nu)d\mathcal{H}^{N-1}.$$

By the lower semicontinuity of I, we have

$$\int_Q CW(\nabla u)dx + \int_{S(u)\cap Q} h_W([u], \nu)d\mathcal{H}^{N-1} \leq \int_Q CW(\nabla u)dx + \int_{S(u)\cap Q} h_{CW}([u], \nu)d\mathcal{H}^{N-1},$$

which implies $h_W \leq h_{CW}$. □

Lemma 6.2 *Let λ be a finite Borel regular measure on Q and let $f : Q \to \mathbb{R}^N$ be λ measurable with $||f||_\infty < \infty$. Then there is a sequence $\{f_n\} \subset C_0^\infty(Q; \mathbb{R}^N)$ such that $f_n \to f$ λ-a.e. and $||f_n||_\infty \le ||f||_\infty$ for all n.*

Proof. By a corollary to Lusin's Theorem (see Corollary 1 to Theorem 2 in Section 1.2 of [10]), for each $\varepsilon > 0$ there exists a continuous function $f^\varepsilon : Q \to \mathbb{R}^N$ such that

$$\lambda(\{x \in Q : f^\varepsilon(x) \neq f(x)\}) < \varepsilon,$$

and, by truncation, we can assume that $||f^\varepsilon||_\infty \le ||f||_\infty$. By Theorem 1 (ii) in Section 4.2.1 of [10], there exists $\{f_n^\varepsilon\} \subset C^\infty(Q; \mathbb{R}^N)$ such that $||f_n^\varepsilon||_\infty \le ||f||_\infty$ and

$$f_n^\varepsilon \to f^\varepsilon \text{ uniformly on compact subsets of } Q.$$

Choose an increasing sequence of compact sets $C_n \subset Q$ such that $\lambda(Q \backslash C_n) < \frac{1}{n}$. By cutting off f_n^ε outside C_n so that $f_n^\varepsilon \in C_0^\infty(Q; \mathbb{R}^N)$, we have

$$f_n^\varepsilon \to f^\varepsilon \ \lambda\text{-a.e.}$$

It follows that we can extract a diagonal subsequence $\{f_n\}$ such that

$$f_n \to f \ \lambda\text{-a.e.}$$

□

We now recall some notation: for $u \in BV(Q)$, set

$$E_t := \{x \in Q : u(x) > t\}.$$

For $x \in \partial_* E_t \subset Q$ (see (2.1)), we denote by $\nu_{E_t}(x)$ the measure theoretic unit *inner* normal (see Theorem 1, Section 5.8 of [10]), i.e.

$$\int_{E_t} \text{div}\phi(x)dx = -\int_{\partial_* E_t} \phi(x) \cdot \nu_{E_t}(x) d\mathcal{H}^{N-1}(x)$$

for all $\phi \in C_0^1(Q; \mathbb{R}^N)$.

Lemma 6.3 (Coarea Formula) *Let $u \in BV(Q)$ be given, and let $f : Q \times \mathbb{M}^{1\times N} \to \mathbb{R}$ be a Carathéodory function, where measurability is with respect to $|Du|$, and positive homogeneous of degree one in the last variable. Assume further that $f\left(x, \frac{dDu(x)}{d|Du|(x)}\right) \in L^\infty(Q, |Du|)$. Then*

$$\int_Q f(x, dDu(x)) = \int_{\mathbb{R}} \int_{\partial_* E_t} f(x, \nu_{E_t}(x)) d\mathcal{H}^{N-1}(x) dt. \tag{6.1}$$

Remark 6.4 Note that for fixed $u \in SBV(Q)$, if we have a Carathéodory function $g : Q \times \mathbb{R} \times \mathbb{M}^{1\times N} \to \mathbb{R}$, where measurability is as above, and which is positive homogeneous of degree one in the last variable, then we can take f in the coarea formula to be

$$f(x, \cdot) := g(x, \hat{u}(x), \cdot),$$

assuming $\hat{u}$ is $|Du|$ measurable and $g\left(x, \hat{u}(x), \frac{dDu(x)}{d|Du|(x)}\right) \in L^\infty(Q, |Du|)$.

If we want to consider a representative $\hat{u}$ of u, it must be defined $|Du|$-a.e. and be $|Du|$ measurable. For example, we could take $\hat{u}$ to be the precise representative of u, or just the precise representative of u on $Q \backslash S(u)$ and u^+ or u^-, or even $[u]$ on $S(u)$. In the former case, and if $u \in SBV$, we have

$$g(x, \tilde{u}(x), dDu(x)) = \chi_{Q\backslash S(u)} g(x, \tilde{u}(x), dDu(x)) + \chi_{S(u)} g(x, \tilde{u}(x), dDu(x))$$

so that (6.1) reduces to

$$\int_Q g(x,\tilde{u}(x),\nabla u(x))dx + \int_{S(u)} g(x,\tilde{u}(x),[u](x)\nu(x))d\mathcal{H}^{N-1}(x) =$$

$$\int_{\mathbf{R}} \left[\int_{\partial_* E_t \setminus S(u)} g(x,t,\nu_{E_t}(x))d\mathcal{H}^{N-1}(x) + \int_{\partial_* E_t \cap S(u)} g(x,\tilde{u}(x),\nu_{E_t}(x))d\mathcal{H}^{N-1}(x) \right] dt.$$

Proof of the Coarea Formula. First we show that for any $|Du|$ measurable set $A \subset \Omega$,

$$|Du|(A) = 0 \text{ implies } \mathcal{H}^{N-1}(A \cap \partial_* E_t) = 0 \text{ for } \mathcal{L}^1\text{-a.e. t.} \tag{6.2}$$

Note that for all $B \subset Q$ open (see Theorem 1 (ii) of Section 5.5 of [10])

$$\begin{aligned} |Du|(B) &= \int_{\mathbf{R}} |\partial E_t|(B)dt \\ &= \int_{\mathbf{R}} \mathcal{H}^{N-1}(B \cap \partial_* E_t)dt. \end{aligned}$$

So,

$$\begin{aligned} |Du|(A) &= \inf_{\substack{B \supset A \\ B \text{ open}}} |Du|(B) \\ &= \inf_{\substack{B \supset A \\ B \text{ open}}} \int_{\mathbf{R}} \mathcal{H}^{N-1}(B \cap \partial_* E_t)dt \\ &= 0 \end{aligned}$$

which implies that

$$\inf_{\substack{B \supset A \\ B \text{ open}}} \mathcal{H}^{N-1}(B \cap \partial_* E_t) = 0 \text{ for } \mathcal{L}^1\text{-a.e. } t,$$

and so

$$\mathcal{H}^{N-1}(A \cap \partial_* E_t) = 0 \text{ for } \mathcal{L}^1\text{-a.e. } t.$$

We know (see claim 1 in the proof of Theorem 1, Section 5.5 in [10]) that

$$\int_Q u(x)\text{div}\phi(x)dx = \int_{\mathbf{R}} \int_{E_t} \text{div}\phi(x)dxdt$$

for all $\phi \in C_0^1(Q;\mathbb{R}^N)$. Hence,

$$\int_Q \phi(x) \cdot \sigma(x)d|Du|(x) = \int_{\mathbf{R}} \int_{\partial_* E_t} \phi(x) \cdot \nu_{E_t}(x)d\mathcal{H}^{N-1}(x)dt \tag{6.3}$$

for all $\phi \in C_0^1(Q;\mathbb{R}^N)$, where

$$\sigma(x) := \frac{dDu(x)}{d|Du|(x)}.$$

We now show that for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, we have

$$\sigma(x) = \nu_{E_t}(x) \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial_* E_t. \tag{6.4}$$

Using Lemma 6.2, choose $\sigma_n \in C_0^1(Q;\mathbb{R}^2)$ such that $\sigma_n(x) \to \sigma(x)$ $|Du|$-a.e. (and so, by (6.2), $\mathcal{H}^{N-1}\lfloor\partial_* E_t$-a.e. for $\mathcal{L}^1$-a.e. t) and $|\sigma_n| \leq 1$. Note that $\sigma_n \cdot \nu_{E_t}$ is $\mathcal{H}^{N-1}\lfloor\partial_* E_t$ measurable since ν_{E_t} is, and

$$t \mapsto \int_{\partial_* E_t} \sigma_n(x) \cdot \nu_{E_t}(x)d\mathcal{H}^{N-1}(x) = \int_{E_t} \text{div}\sigma_n(x)dx$$

is $\mathcal{L}^1$ measurable (see, e.g., the proof of Lemma 1 in Section 5.5 of [10]). Then, by (6.3) and the dominated convergence theorem,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\partial_* E_t} \sigma(x)\cdot\nu_{E_t}(x)d\mathcal{H}^{N-1}(x)dt &= \lim_{n\to\infty}\int_{\mathbb{R}}\int_{\partial_* E_t} \sigma_n(x)\cdot\nu_{E_t}(x)d\mathcal{H}^{N-1}(x)dt \\ &= \lim_{n\to\infty}\int_Q \sigma_n(x)\cdot\sigma(x)d|Du|(x) \\ &= \int_Q d|Du| \\ &= \int_{\mathbb{R}}\int_{\partial_* E_t} d\mathcal{H}^{N-1}(x)dt.\end{aligned}$$

Since $\sigma\cdot\nu_{E_t}\le 1$, we have (6.4).

Using Lemma 6.2 once more, choose $\phi_n \in C_0^1(Q;\mathbb{R}^N)$ such that $\phi_n(x)\to f(x,\sigma(x))\sigma(x)$ $|Du|$-a.e. and $||\phi_n||_\infty \le ||f(\cdot,\sigma(\cdot))||_\infty$. Then, as above,

$$\begin{aligned}\int_Q f(x,dDu(x)) &= \int_Q f(x,\sigma(x))d|Du|(x) \\ &= \int_Q f(x,\sigma(x))\sigma(x)\cdot\sigma(x)d|Du|(x) \\ &= \lim_{n\to\infty}\int_Q \phi_n(x)\cdot\sigma(x)d|Du|(x) \\ &= \lim_{n\to\infty}\int_{\mathbb{R}}\int_{\partial_* E_t}\phi_n(x)\cdot\nu_{E_t}(x)d\mathcal{H}^{N-1}(x)dt \text{ (by (6.3))} \\ &= \int_{\mathbb{R}}\int_{\partial_* E_t} f(x,\nu_{E_t}(x))d\mathcal{H}^{N-1}(x)dt. \text{ (by (6.4))}\end{aligned}$$

□

Now we introduce another infimum, similar to h, but which includes only very simple functions in its admissible class. Given $\nu\in S^{N-1}$, we consider the family Q_ν^2 of squares with unit edge length, centered at zero, with two edges normal to ν (with ν in the plane of the square). Without loss of generality, we will assume $\nu = e_N$. Now consider the square $S\in Q_\nu^2$ with the remaining two edges having normal e_1 (in the plane of the square), and we consider all curves $\Gamma\subset S$ of the form $\Gamma=\Gamma_1\cup\Gamma_2$, where Γ_1 is the line segment from $(-1/2,0,\ldots,0)$ to some point p in the square, and Γ_2 is the line segment connecting p to $(1/2,0,\ldots,0)$ (see Figure 1). For other $S\in Q_\nu^2$, with two edges not having normal e_1, we consider analogous $\Gamma\subset S$. Set

$$H(\rho,\nu) := \inf\{\mathcal{H}^1(\Gamma_1)+\rho\mathcal{H}^1(\Gamma_2)CW^\infty(\mu) :$$

$$S\in Q_\nu^2, \Gamma\subset S \text{ is as above, and } \mu \text{ is the unit normal to } \Gamma_2 \text{ such that } \mu\cdot\nu\ge 0\}.$$

Remark 6.5 Note the following:

i) $H(\rho,\nu)\le 1$ since we can take $\Gamma=\Gamma_1$.

ii) The infimum H is attained since CW^∞ is continuous.

iii) If CW^∞ is isotropic, then the minimizing Γ equals Γ_1 or Γ equals Γ_2.

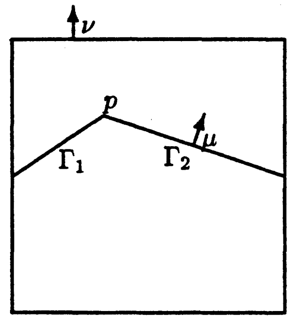

Figure 1: admissible "function" for H

Theorem 6.6 *If $p = 1$ then*

$$h = H.$$

Proof. By Lemma 6.1, we may assume that $W = (CW)^\infty$. Furthermore, $(CW)^\infty$ is convex, since if $A, B \in \mathbb{M}^{1\times N}$ and $\lambda \in (0,1)$, then

$$\begin{aligned}(CW)^\infty(\lambda A + (1-\lambda)B) &= \limsup_{t\to\infty} \frac{CW(t[\lambda A + (1-\lambda)B])}{t} \\ &\le \limsup_{t\to\infty} \frac{\lambda CW(tA)}{t} + \limsup_{t\to\infty} \frac{(1-\lambda)CW(tB)}{t} \\ &= \lambda (CW)^\infty(A) + (1-\lambda)(CW)^\infty(B).\end{aligned}$$

Hence, in the sequel we will take W to be convex and positive homogeneous of degree one.

Step 1. We show that $h \le H$.

Case a) $N = 2$.
Fix $S \in Q^2_\nu$ and $\Gamma \subset S$ as in the definition of H, and consider functions in $SBV(S;\mathbb{R})$ that are zero below Γ, ρ above, jump at Γ_1 and are affine across a narrow extension of Γ_2, with another jump connection near the intersection of the boundary with Γ_2, and we see that these functions are admissible for $h(\rho,\nu)$ and their energy E approaches $\mathcal{H}^1(\Gamma_1) + \rho\mathcal{H}^1(\Gamma_2)W(\mu)$ (see Figure 2).

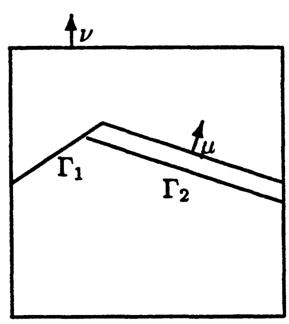

Figure 2: $h \le H$

Case b) $N > 2$:
Again, let any S and Γ as in the definition of H be given. Choose Q_ν such that the remaining normal to S (besides ν) is normal to Q_ν. Extend Γ to Q_ν by $\bar{\Gamma} := \{x \in Q_\nu :$

proj(x) onto $S \in \Gamma$}. We then construct functions as in the case $N = 2$, where here there is a jump connection near the intersection of $\bar{\Gamma}_2$ with the face of Q_ν having the same normal as S. By (2.6), these functions have the necessary periodicity to be admissible for $h(\rho, \nu)$. Again, we see that the energy E approaches $\mathcal{H}^1(\Gamma_1) + \rho\mathcal{H}^1(\Gamma_2)W(\mu)$.

Step 2. We now show that $h \geq H$.

Let u be an admissible function for $h(\rho, \nu)$, i.e.,

$$u \in SBV(Q_\nu; \mathbb{R}), u = \rho \text{ if } x \in \partial Q_\nu \text{ and } x \cdot \nu \geq 0, \text{ and } u = 0 \text{ if } x \in \partial Q_\nu \text{ and } x \cdot \nu < 0.$$

For simplicity, we refer to Q_ν as Q. Applying Lemma 6.3 to

$$f(x, dDu(x)) := \chi_{Q \setminus S(u)}(x) W(dDu(x)),$$

so that $\left| f\left(x, \frac{dDu(x)}{d|Du|(x)}\right) \right| \leq C_1$, we get

$$\int_Q W(\nabla u(x))dx = \int_{\mathbb{R}} \int_{\partial_* E_t \setminus S(u)} W(\nu_{E_t}(x)) d\mathcal{H}^{N-1}(x) dt.$$

Choose $t_0 \in (0, \rho)$ such that

$$\int_{\partial_* E_{t_0} \setminus S(u)} W(\nu_{E_{t_0}}(x)) d\mathcal{H}^{N-1}(x) \leq \frac{1}{\rho} \int_Q W(\nabla u(x)) dx. \tag{6.5}$$

Note that the coercivity of W guarantees that $\partial_* E_{t_0}$ has finite perimeter. Set

$$\beta := \mathcal{H}^{N-1}(\partial_* E_{t_0} \setminus S(u)) \text{ and } \bar{\nu} := \frac{1}{\beta} \int_{\partial_* E_{t_0} \setminus S(u)} \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x)$$

so that, by Jensen's inequality,

$$\beta W(\bar{\nu}) \leq \int_{\partial_* E_{t_0} \setminus S(u)} W(\nu_{E_{t_0}}(x)) d\mathcal{H}^{N-1}(x). \tag{6.6}$$

We can assume $\nu = e_N$ and $\bar{\nu} \cdot e_i = 0$ for $i \in \{2, \ldots, N-1\}$. Note that we can also assume that Q has e_1 normal to two of its faces, for the following reason: let Q_1 be a cube with normals e_N and e_1. We can rescale Q and u, and almost cover $\{x \in Q_1 : x_N = 0\}$ with the cubes $a_i + \delta Q$, where $a_i \in \{x \in Q_1 : x_N = 0\}$. Define $v \in SBV(Q_1)$ by

$$v(x) := \begin{cases} u\left(\frac{x - a_i}{\delta}\right) & \text{if } x \in a_i + \delta Q \\ 0 & \text{if } x_N < 0 \text{ and } x \notin \cup(a_i + \delta Q) \\ \rho & \text{if } x_N \geq 0 \text{ and } x \notin \cup(a_i + \delta Q). \end{cases}$$

Using the homogeneity of W, we now have $E(v, Q_1) - E(u, Q) \leq \mathcal{H}^{N-1}(\{x \in Q_1 : x_N = 0\} \setminus \cup_i (a_i + \delta Q))$, yet $\bar{\nu}$, defined as for u, remains unchanged.

We first wish to show that

$$\int_{\partial_* E_{t_0}} \nu_{E_{t_0}}(x) \cdot e_1 d\mathcal{H}^{N-1}(x) = 0 \tag{6.7}$$

and

$$\int_{\partial_* E_{t_0}} \nu_{E_{t_0}}(x) \cdot e_N d\mathcal{H}^{N-1}(x) = 1. \tag{6.8}$$

From Theorem 1 (ii) of Section 5.8 in [10], we know that if $E \subset \mathbb{R}^N$ has locally finite perimeter in $\mathbb{R}^N$, then

$$\int_E \operatorname{div}\phi(x)dx = -\int_{\partial_* E} \phi(x) \cdot \nu_E(x) d\mathcal{H}^{N-1}(x) \tag{6.9}$$

for all $\phi \in C_0^1(\mathbb{R}^N; \mathbb{R}^N)$, where, as before, ν_E is the *inner* unit normal. Define $E \subset \mathbb{R}^N$ by

$$E := E_{t_0} \cup \{x \in \mathbb{R}^N \backslash Q : x \cdot e_N > 0\}.$$

Then E is locally of finite perimeter in $\mathbb{R}^N$, and we claim that

$$\mathcal{H}^{N-1}([\partial_* E] \Delta C) = 0, \tag{6.10}$$

where $C := \partial_* E_{t_0} \cup \{x \in \mathbb{R}^N \backslash Q : x \cdot e_N = 0\}$. It is clear that $C \subset \partial_* E$ and that

$$(\partial_* E) \backslash C \subset \partial Q,$$

so the idea is to show that $\mathcal{H}^{N-1}(\partial Q \cap \partial_* E) = 0$. Let $x \in \partial Q$ be given such that $x \cdot e_N > 0$ and

$$\lim_{r \to 0} ⨍_{B(x,r) \cap Q} |u(y) - \rho| dy = 0. \tag{6.11}$$

We need only show that

$$\limsup_{r \to 0} \frac{\mathcal{L}^N(B(x,r) \backslash E)}{r^N} = 0,$$

since then $x \notin \partial_* E$. We have

$$\begin{aligned} \limsup_{r \to 0} \frac{\mathcal{L}^N(B(x,r) \backslash E)}{r^N} &= \limsup_{r \to 0} \frac{1}{r^N} \mathcal{L}^N(B(x,r) \cap \{y \in Q : u(y) \leq t_0\}) \\ &\leq \limsup_{r \to 0} \frac{1}{r^N} \frac{1}{|\rho - t_0|} \int_{B(x,r) \cap Q} |u(y) - \rho| dy \\ &= 0. \end{aligned}$$

Since, by Theorem 2 of Section 5.3 of [10], (6.11) holds $\mathcal{H}^{N-1}$-a.e. on the upper half of ∂Q, and, dealing with the case $x \cdot e_N < 0$ similarly, we find $\mathcal{H}^{N-1}(\partial Q \cap \partial_* E) = 0$.

Choose $\phi \in C_0^1(\mathbb{R}^N; \mathbb{R}^N)$ such that $\phi_i = 0$ for all $i \in \{2, \ldots, N\}$, $\phi_1 = 1$ on Q. Clearly, $\operatorname{div}\phi = \frac{\partial \phi_1}{\partial x_1}$, $\operatorname{div}\phi = 0$ on $\bar{Q}$, and $\nu_E = e_N$ on $\mathbb{R}^N \backslash \bar{Q}$. So, $\int_E \operatorname{div}\phi(x)dx = 0$. For example, we can take $\phi := \rho_{\frac{1}{2}} * \chi_{2Q} e_1$. By (6.9), (6.10), and the fact that E is locally of finite perimeter, we have

$$\begin{aligned} 0 &= \int_{\partial_* E} \phi(x) \cdot \nu_E(x) d\mathcal{H}^{N-1}(x) \\ &= \int_{\partial_* E_{t_0}} e_1 \cdot \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) + \int_{\{x \cdot e_N = 0\} \backslash Q} \phi(x) \cdot e_N d\mathcal{H}^{N-1}(x) \\ &= \int_{\partial_* E_{t_0}} e_1 \cdot \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x), \end{aligned}$$

and we conclude (6.7).

Equation (6.8) follows by considering, for $\varepsilon > 0$, $\phi \in C_0^1((-1/2 - \varepsilon, 1/2 + \varepsilon)^N; \mathbb{R}^N)$ such that $\phi_i = 0$ for all $i < N$, $\phi = -e_N$ on Q and $-1 \leq \phi \cdot e_N \leq 0$. For example, take $\phi := -\rho_\varepsilon * \chi_{(-1/2-\varepsilon, 1/2+\varepsilon)^N} e_N$. By (6.9) and (6.10) we have

$$\int_E \operatorname{div}\phi(x)dx = \int_{\partial_* E_{t_0}} e_N \cdot \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) - \int_{\{x \cdot e_N = 0\} \backslash Q} \phi(x) \cdot e_N d\mathcal{H}^{N-1}(x).$$

By the choice of ϕ, we know that, writing $x = (x', x_N)$, if $(x', 0) \in \bar{Q}$, then

$$\int_{\frac{1}{2}}^{\infty} \frac{\partial \phi_N}{\partial x_N}(x', y) dy = 1$$

and, for $(x', 0) \in (-1/2 - \varepsilon, 1/2 + \varepsilon)^N \setminus \bar{Q}$, we have

$$0 \leq \int_0^{\infty} \frac{\partial \phi_N}{\partial x_N}(x', y) dy \leq 1.$$

We then see that

$$1 < \int_E \operatorname{div} \phi(x) dx < (1 + 2\varepsilon)^{N-1}$$

and

$$\left| \int_{\{x \cdot e_N = 0\} \setminus Q} \phi(x) \cdot e_N d\mathcal{H}^{N-1}(x) \right| < (1 + 2\varepsilon)^{N-1} - 1.$$

The arbitrariness of ε yields

$$\int_{\partial_* E_{t_0}} e_N \cdot \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) = 1.$$

We now recall some facts about sets of finite perimeter (see [10]), which we apply to E_{t_0}:

$$\partial^* E_{t_0} \subset \partial_* E_{t_0}, \ \ \mathcal{H}^{N-1}(\partial_* E_{t_0} \setminus \partial^* E_{t_0}) = 0,$$

and if $x \in \partial^* E_{t_0}$, then we have

$$\lim_{r \to 0} \frac{\alpha(N-1) r^{N-1}}{\mathcal{H}^{N-1}(B(x,r) \cap \partial^* E_{t_0})} = 1$$

and

$$\lim_{r \to 0} \fint_{\partial^* E_{t_0} \cap B(x,r)} \nu_{E_{t_0}}(y) d\mathcal{H}^{N-1}(y) = \nu_{E_{t_0}}(x),$$

where $\alpha(N-1)$ is the volume of an $N-1$ dimensional ball with radius 1. Let $\varepsilon > 0$ be given. Since $\mathcal{H}^{N-1} \lfloor \partial^* E_{t_0}$ is a Radon measure, we can choose an open set $A \subset Q$ such that

$$A \supset (S(u) \cap \partial^* E_{t_0}) \text{ and } \mathcal{H}^{N-1}((A \setminus S(u)) \cap \partial^* E_{t_0}) < \varepsilon.$$

For each $x \in \partial^* E_{t_0} \cap S(u)$, let $r_x > 0$ be such that $0 < r < r_x$ implies

$$\left| \frac{\alpha(N-1) r^{N-1}}{\mathcal{H}^{N-1}(B(x,r) \cap \partial^* E_{t_0})} - 1 \right| < \varepsilon \tag{6.12}$$

and

$$\left| \fint_{\partial^* E_{t_0} \cap B(x,r)} \nu_{E_{t_0}}(y) d\mathcal{H}^{N-1}(y) - \nu_{E_{t_0}}(x) \right| < \varepsilon. \tag{6.13}$$

The family $\{B(x,r) \subset A : x \in \partial^* E_{t_0} \cap S(u), r \in (0, r_x)\}$ is a fine cover of $\partial^* E_{t_0} \cap S(u)$. By Besicovitch's Covering Theorem, choose a countable, disjoint subcollection $\mathcal{G}$ such that

$$\mathcal{H}^{N-1}(\partial^* E_{t_0} \cap S(u) \setminus \cup_{B \in \mathcal{G}} B) = 0.$$

We now do some calculations to show that

$$\mathcal{H}^{N-1}(S(u)) \geq [(\beta \bar{\nu} \cdot e_1)^2 + (1 - \beta \bar{\nu} \cdot e_N)^2]^{1/2}.$$

For $B \in \mathcal{G}$, denote the center of B by x_B and the radius by r_B. From (6.7) we have

$$|\beta\bar{\nu} \cdot e_1| = \left| e_1 \cdot \int_{\partial^* E_{t_0} \setminus S(u)} \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) \right| = \left| e_1 \cdot \int_{\partial^* E_{t_0} \cap S(u)} \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) \right|,$$

and so

$$\begin{aligned} |\beta\bar{\nu} \cdot e_1| &\le \left| \sum_{B\in\mathcal{G}} e_1 \cdot \int_{B\cap\partial^* E_{t_0}} \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) \right| \\ &< \left| \sum_{B\in\mathcal{G}} e_1 \cdot \nu_{E_{t_0}}(x_B) \mathcal{H}^{N-1}(B \cap \partial^* E_{t_0}) \right| + \varepsilon \mathcal{H}^{N-1}(\partial^* E_{t_0}) \quad \text{(by (6.13))} \\ &< \left| \sum_{B\in\mathcal{G}} e_1 \cdot \nu_{E_{t_0}}(x_B) \alpha(N-1) r_B^{N-1} \right| + 2\varepsilon \mathcal{H}^{N-1}(\partial^* E_{t_0}). \quad \text{(by (6.12))} \end{aligned} \tag{6.14}$$

Similarly, from (6.8) we have

$$|1-\beta\bar{\nu} \cdot e_N| = \left| 1 - e_N \cdot \int_{\partial^* E_{t_0} \setminus S(u)} \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) \right| = \left| e_N \cdot \int_{\partial^* E_{t_0} \cap S(u)} \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) \right|$$

and so

$$\begin{aligned} |1-\beta\bar{\nu} \cdot e_N| &\le \left| \sum_{B\in\mathcal{G}} e_N \cdot \int_{B\cap\partial^* E_{t_0}} \nu_{E_{t_0}}(x) d\mathcal{H}^{N-1}(x) \right| \\ &< \left| \sum_{B\in\mathcal{G}} e_N \cdot \nu_{E_{t_0}}(x_B) \mathcal{H}^{N-1}(B \cap \partial^* E_{t_0}) \right| + \varepsilon \mathcal{H}^{N-1}(\partial^* E_{t_0}) \quad \text{(by (6.13))} \\ &< \left| \sum_{B\in\mathcal{G}} e_N \cdot \nu_{E_{t_0}}(x_B) \alpha(N-1) r_B^{N-1} \right| + 2\varepsilon \mathcal{H}^{N-1}(\partial^* E_{t_0}) \quad \text{(by (6.12))}. \end{aligned} \tag{6.15}$$

Finally,

$$\begin{aligned} \mathcal{H}^{N-1}(S(u)) &\ge \mathcal{H}^{N-1}(\partial^* E_{t_0} \cap S(u)) \\ &> \sum_{B\in\mathcal{G}} \mathcal{H}^{N-1}(B \cap \partial^* E_{t_0}) - \varepsilon \text{ (by the choice of } A) \\ &> \sum_{B\in\mathcal{G}} \alpha(N-1) r_B^{N-1} - \varepsilon - \varepsilon \mathcal{H}^{N-1}(\partial^* E_{t_0}) \text{ (by (6.12))} \\ &\ge \sum_{B\in\mathcal{G}} [(e_1 \cdot \nu_{E_{t_0}}(x_B) \alpha(N-1) r_B^{N-1})^2 \\ &\quad + (e_N \cdot \nu_{E_{t_0}}(x_B) \alpha(N-1) r_B^{N-1})^2]^{1/2} - \varepsilon - \varepsilon \mathcal{H}^{N-1}(\partial^* E_{t_0}) \\ &\ge \left[\left(\sum_{B\in\mathcal{G}} e_1 \cdot \nu_{E_{t_0}}(x_B) \alpha(N-1) r_B^{N-1} \right)^2 \right. \\ &\quad \left. + \left(\sum_{B\in\mathcal{G}} e_N \cdot \nu_{E_{t_0}}(x_B) \alpha(N-1) r_B^{N-1} \right)^2 \right]^{1/2} - \varepsilon - \varepsilon \mathcal{H}^{N-1}(\partial^* E_{t_0}) \\ &\ge [(\beta\bar{\nu} \cdot e_1)^2 + (1 - \beta\bar{\nu} \cdot e_N)^2]^{1/2} - \varepsilon - 5\varepsilon \mathcal{H}^{N-1}(\partial^* E_{t_0}), \end{aligned}$$

(by (6.14) and (6.15))

hence, letting $\varepsilon \to 0$, we conclude that

$$\mathcal{H}^{N-1}(S(u)) \geq [(\beta\bar{\nu}\cdot e_1)^2 + (1-\beta\bar{\nu}\cdot e_N)^2]^{1/2}. \tag{6.16}$$

Suppose that $\bar{\nu}\cdot e_N \leq 0$. Then (6.16) implies that $\mathcal{H}^{N-1}(S(u)) \geq 1$. Therefore, $E(u,Q) \geq 1 \geq H(\rho,\nu)$. Assume now that $\bar{\nu}\cdot e_N > 0$. Consider the square in the e_1-e_N plane with normals e_1 and e_N and, suppressing e_i for $i \in \{2,\ldots,N-1\}$, take $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_2 is the line segment with right endpoint $(1/2,0)$, unit normal $\frac{\bar{\nu}}{|\bar{\nu}|}$, and length $|\bar{\nu}|\beta$ (if $\beta\bar{\nu}\cdot e_N \geq 1$, redefine $\beta := \frac{1-\varepsilon}{\bar{\nu}\cdot e_N}$). Γ_1 is then the line segment from the left endpoint of Γ_2 to $(-1/2,0)$. Note that the length of Γ_1 is $[(\beta\bar{\nu}\cdot e_1)^2 + (1-\beta\bar{\nu}\cdot e_N]^2)^{1/2}$, and so, by (6.16), we have $\mathcal{H}^1(\Gamma_1) \leq \mathcal{H}^{N-1}(S(u))$. Finally, by (6.5) and (6.6), we conclude that

$$\begin{aligned} H(\rho,\nu) &\leq \rho|\bar{\nu}|\beta W(\tfrac{\bar{\nu}}{|\bar{\nu}|}) + \mathcal{H}^1(\Gamma_1) \\ &\leq \rho\beta W(\bar{\nu}) + \mathcal{H}^{N-1}(S(u)) \\ &\leq \int_Q W(\nabla u(x))dx + \mathcal{H}^{N-1}(S(u)). \end{aligned}$$

Due to the arbitrariness of u, we have $h \geq H$. □

Remark 6.7 Suppose that the energy of the admissible functions for h is given by

$$E(u,Q) = \int_Q W(\nabla u)dx + \int_{S(u)} \phi([u]\nu)d\mathcal{H}^{N-1},$$

where W and ϕ are convex and positive homogeneous of degree one. Then the conclusion of Theorem 6.6 holds. Taking

$$f(x,dDu) := \chi_{Q\setminus S(u)}W(dDu) + \chi_{S(u)}\phi(dDu)$$

we have

$$\int_Q f(x,dDu) = \int_Q W(\nabla u)dx + \int_{S(u)} \phi([u]\nu)d\mathcal{H}^{N-1} = E(u,Q),$$

and, by Lemma 6.3,

$$E(u,Q) = \int_{\mathbf{R}} \left[\int_{\partial_* E_t\setminus S(u)} W(\nu_{E_t}(x))d\mathcal{H}^{N-1} + \int_{\partial_* E_t\cap S(u)} \phi(\nu_{E_t}(x))d\mathcal{H}^{N-1} \right] dt.$$

The rest of the proof of Theorem 6.6 follows with the obvious alterations.

Remark 6.8 (Behavior of Infimizing Sequences) Let $\{u_n\}$ be an infimizing sequence for $h(\rho,\nu)$. Then there is a subsequence and a minimizer Γ for $H(\rho,\nu)$ such that

$$\mathcal{H}^{N-1}(S(u_n)) \to \mathcal{H}^1(\Gamma_1) \tag{6.17}$$

and

$$\int_Q W^\infty(\nabla u_n)dx \to \rho\mathcal{H}^1(\Gamma_2)CW^\infty(\mu). \tag{6.18}$$

Indeed, for each u_n, using Lemma 6.3 as in the proof of Theorem 6.6, choose a "good" level set E_{t_n}, and construct Γ_n as above. We refer to the points separating $\Gamma_{n,1}$ and $\Gamma_{n,2}$ as p_n and the squares containing them as S_n. Then the p_n's have a limit point, p, and the S_n's have a corresponding limit square, S. We do not relabel the corresponding subsequence of

$\{u_n\}$. By the continuity of CW^∞, p determines a minimizer $\Gamma \subset S$ of $H(\rho, \nu)$. By the constructions of Γ_n, we know that

$$\mathcal{H}^{N-1}(S(u_n)) \geq \mathcal{H}^1(\Gamma_{n,1}) \to \mathcal{H}^1(\Gamma_1)$$

and

$$\int_Q W^\infty(\nabla u_n)dx \geq \rho\mathcal{H}^1(\Gamma_{n,2})CW^\infty(\mu_n) \to \rho\mathcal{H}^1(\Gamma_2)CW^\infty(\mu).$$

Since $\{u_n\}$ is infimizing, and using Theorem 6.6, we have (6.17) and (6.18).

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