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Abstract

A continuum model of coherent structures in two-dimensional magnetohydrodynamic turbulence is developed. These structures are macroscopic states which persist amongst the turbulent microscopic fluctuations, typically as magnetic islands with flow. They are modeled as statistical equilibrium states for the ideal (nondissipative) dynamics, which conserves energy and families of cross-helicity and flux integrals. The model predicts that an ideal magnetofluid will evolve into a turbulent relaxed state having steady mean magnetic and velocity fields, and Gaussian local fluctuations in these fields. Excellent qualitative and quantitative agreement is found with the results of direct numerical simulations. A rigorous justification of the theory is also provided, in the sense that the continuum model is derived from a lattice model in a fixed-volume, small-spacing limit. This construction uses the discrete Fourier transform to link the discretization of x-space with the truncation of k-space. The lattice model is defined by the most probable distribution on the discretized phase space that respects the approximated dynamical constraints. A concentration property shows that this distribution is equivalent to the microcanonical distribution in the continuum limit.

keywords: magnetohydrodynamics, coherent structure, turbulence, discrete Fourier transform, maximum entropy.

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1 Introduction

A conspicuous feature of high Reynolds number magnetohydrodynamics (MHD) is the formation of long-lived, large-scale organized states, or coherent structures, amid small-scale turbulent fluctuations. This turbulent relaxation phenomenon is clearly exhibited by two-dimensional systems, such as occur in the plane perperdicular to a strong magnetic field [4, 23]. In the 2D context, direct numerical simulations [5, 6, 7, 28] show that a freely evolving, slightly dissipative magnetofluid tends toward a relaxed state, in which the magnetic and velocity fields exhibit small-scale local fluctuations around a mean state that varies on the large spatial scale. The coherent structure defined by these mean fields typically assumes the form of one or more magnetic islands, generally with flow. A similar, but more subtle, phenomenon occurs in 2D hydrodynamics at high Reynolds number. In that setting, the coherent structure usually consists of an isolated vortex or a system of such vortices, which persist in the midst of small-scale vorticity fluctuations [24, 8]. This behavior is peculiar to 2D hydrodynamic turbulence, however, which is dominated by vorticity advection. By contrast, 2D and 3D magnetofluids share many basic features, and MHD turbulence in two dimensions is expected to be qualitatively similar to that in three dimensions [4].

These coherent structures in fluid or plasma turbulence can be modeled as statistical equilibrium states associated with the conservative (nondissipative) dynamics governing the system. Such an *ideal* model, in which the effects of fluid viscosity and electrical resistivity are ignored, is justified whenever the magnetic and kinetic Reynolds numbers are sufficiently large. Indeed, the organized states that emerge in the fully-developed turbulence vary on a spatial scale comparable to the domain size, and form on a temporal scale that is short compared to the corresponding dissipation time. Moreover, the small-scale local fluctuations attached to these coherent structures occur in some inertial range of scales, which is also controlled by the ideal dynamics. While the behavior of cascades and other transfer processes cannot be captured in such a statistical equilibrium model, the structure of the turbulent relaxed state itself can be characterized by the general principle that entropy be maximized subject to the constraints dictated by the ideal dynamics [1, 2].

Two fundamental difficulties are confronted, however, when the 2D MHD system is placed in this conceptual framework. First, the continuum system is infinite-dimensional, and hence requires some kind of discretization before it is amenable to a probabilistic treatment. Second, the ideal dynamics conserves not only energy, but also two infinite families of cross-helicity and flux integrals, which give the system its distinctive character. The key point in the construction of any statistical equilibrium model, therefore, becomes the compatible resolution of these difficulties. As is well explained in the review by Kraichnan and Montgomery [21], models of this kind consequently fall into two categories, which we shall call "k-space models" and "x-space models." Principally these categories refer to whether the system is discretized in wavenumber space or in physical space. But they also relate to the special way in which the conserved quantities are included in the model.

The k-space model for 2D MHD, indicated by Lee [22], is developed by Fyfe and Montgomery [13]. This statistical equilibrium theory is based upon a canonical ensemble for a truncated spectral representation of the ideal system. While this model captures some of the essential features of the turbulent state, it accounts only for the purely quadratic invariants of the ideal dynamics, ignoring the invariants that are not easily expressible in a spectral form. As a result of this simplification, it yields equilibrium distributions whose mean magnetic and velocity fields vanish identically, and hence it fails to predict a (nontrivial) coherent structure. Instead, it contains finite energy fluctuations in the lowest mode, which in the continuum limit contradicts the equivalence of ensembles postulate on which the theory rests. These defects of the k-space approach are removed in recent work by Gruzinov and Isichenko [17], who use a formal asymptotic analysis to build a steady mean state into the model. They obtain a meaningful continuum limit by appropriately rescaling the inverse temperature-like parameters in the canonical ensemble with the number of spectral modes. Their analysis, which ultimately relies on a separation of scales assumption, shows how the conserved integrals (energy, cross-helicity and flux) partition between the large-scale mean state and the small-scale fluctuations.

The x-space approach to modeling turbulent relaxed states in 2D MHD is taken by Montgomery *et al.* [27]. Their model uses a field-line discretization of the current density and vorticity, combined with an information-theoretic entropy [18]. It yields equilibrium equations for the most probable state by maximizing this entropy subject to constraints on the classical conserved quantities. The predicted macroscopic state is, however, not necessarily a steady solution of the MHD equations. This property, along with some arbitrariness in the construction of the maximum entropy principle itself, suggests that the underlying x-space discretization is too crude. A different x-space approach is proposed by the present authors [33]. Our model draws upon the ideas introduced by Robert [29] in the context of 2D hydrodynamic turbulence. The major innovation of this approach is the use of an x-parameterized probability measure (or Young measure) to provide the macroscopic description of the coherent structure. In the theory of 2D vortex structures, due independently to Robert *et al.* [29, 30] and Miller *et al.* [25, 26], the macrostate represents a local probability distribution on the values of the fluctuating vorticity field near each spatial point. In an analogous manner, our theory of turbulent relaxation in a 2D magnetofluid makes use of x-parameterized *joint* probability distributions on the values of the fluctuating magnetic and velocity fields. Such a description is valid under the hypothesis that the local fluctuations occur on an infinitesimal scale at each point, and that they are uncorrelated between distinct points. This separation of scales hypothesis leads directly to a continuum x-space model governed by a natural constrained maximum entropy principle. As is demonstrated in our paper [33], the most probable state is a steady solution of the 2D MHD equations, and the local fluctuations are Gaussian. These basic results of our x-space model agree with those obtained by Gruzinov and Isichenko [17] from their k-space model.

In the present paper we give a complete treatment of the ideal turbulent relaxation problem for 2D MHD, using a synthesis of the x-space and k-space methods. We examine the continuum model enunciated in our earlier paper [33] from two points of view. First, we analyze its predictions and compare them with the known results of direct numerical simulations. Second, we justify its formulation by deriving it as the limit of a lattice model. This derivation, along with the construction of the lattice model, is perhaps the main contribution of the paper. We construct the appropriate lattice model (for doubly-periodic boundary conditions) using the discrete Fourier transform, which allows us to exploit the relationship between truncation in k-space and discretization in x-space. With this natural correspondence between finite x-space and finite k-space, we are able to clarify the separation of scales hypothesis that is fundamental to the model. The lattice model is defined by what we term the "implicit canonical ensemble," which is not identical with the classical Gibbs ensemble for the discrete system. Nevertheless, we show that it is consistent with the Liouville property of the discrete dynamics, and that it satisfies an equivalence of ensembles property with the microcanonical ensemble for those dynamics.

The necessity for this novel treatment of the statistical equilibrium problem arises from the family of conserved quantities for 2D MHD, which include (general) flux and cross-helicity integrals as well as energy. On the one hand, the disparate weights that these different invariants put on the spectral modes makes a k-space representation essential to the analysis. On the other hand, an x-space formulation is demanded by these extra invariants, since they are (in general) nonlinear and nonquadratic. Moreover, the continuum model is achieved as a well-defined limit of the lattice model in x-space, while in k-space the model suffers the so-called ultraviolet catastrophe. These special features of the problem come together to motivate the separation of scales hypothesis, which lies at the heart of the model. In essence, our implicit canonical ensemble is the simplest measure on the discrete phase space that respects both the separation of scales condition and the complete family of dynamical invariants.

The paper is organized as follows. In Section 2, we pose the turbulent relaxation problem in a general spatial domain, and we formulate the x-space continuum model. The constrained maximum entropy principle that defines the model is developed in this section from intuitive, physical considerations. We then proceed in Section 2 to analyse the statistical equilibrium states governed by the continuum model. In general, these coherent structures solve a nonlinear, elliptic partial differential equation coupled with a set of integral constraints. We examine these solutions as thoroughly as possible, without resorting to numerical methods. This section extends the analysis published by Jordan [19] for the special case in which the family of cross-helicity integrals is reduced to the classical quadratic cross-helicity. Finally, in Section 4 we construct the lattice model and use it to give a rigorous derivation of the continuum model formulated heuristically in Section 1. In the course of this development, we introduce a new synthesis of x-space and k-space methods, which has not been previously explored in the context of statistical equilibrium theory.

2 Ideal magnetofluid turbulence

A. Microscopic dynamics. The equations governing ideal magnetohydrodynamics (MHD) are expressible in a normalized, dimensionless form as [4, 23]

$$\frac{\partial B}{\partial t} = \nabla \times (V \times B), \qquad \nabla \cdot B = 0, \tag{2.1}$$

$$\frac{\partial V}{\partial t} = V \times (\nabla \times V) + (\nabla \times B) \times B - \nabla P, \qquad \nabla \cdot V = 0, \tag{2.2}$$

where B is the magnetic induction field, V is the fluid velocity field, and $P = p + \frac{1}{2}V^2$ is the total pressure head. The fluid is incompressible with mass density normalized to unity. The conducting and flowing medium is ideal in the sense that its resistivity and viscosity are both ignored. Under these conditions the induced electric field is given by $E = -V \times B$, and therefore it does not enter into the governing equations. Similarly, the pressure head P is determined instantaneously in terms of B and V in response to the incompressibility constraint. For these reasons, the state of the magnetofluid is completely described by the vector field

$$Y(x,t) := (B, V), \tag{2.3}$$

which we shall call the *field-flow state.*

We are concerned exclusively with purely two-dimensional systems. We let the domain $D \subset R^2$ be the cross-section of the spatial region $D \times R$ occupied by the magnetofluid, and we assume that the field-flow state has the form $Y = (B_1e_1 + B_2e_2, V_1e_1 + V_2e_2)$ in which the components are functions of $x = (x_1, x_2) \in D$. For the sake of simplicity, we take the domain boundary ∂D to be finite and regular. On ∂D we impose the ideal boundary conditions $n \cdot B = 0, n \cdot V = 0$, with n normal to the boundary. Alternatively, in Section 4, we consider periodic boundary conditions on a rectangle D with fundamental periods L_1 and L_2 in x_1 and x_2, respectively. In the periodic case, we also require that Y averages to zero over D.

The induction equation (2.1) implies that the magnetic lines of force are frozen into the mass flow (the B-lines are advected by V) and that the magnetic flux of any tube composed of these lines is conserved by the flow (the flux of each B-tube is constant in time). In two dimensions these central properties of ideal MHD are expressed succinctly by the equation

$$\frac{\partial \psi}{\partial t} + V \cdot \nabla\, \psi = 0 \tag{2.4}$$

for the flux function ψ, which is defined by $B = \nabla \times \psi e_3$. The scalar advection equation (2.4) is equivalent to the primitive equations (2.1). In an analogous fashion, the primitive momentum equations (2.2) are reducible to the scalar equation

$$\frac{\partial \omega}{\partial t} + V \cdot \nabla \omega \;=\; B \cdot \nabla j\,, \tag{2.5}$$

in which $\omega = e_3 \cdot \nabla \times V$ is the vorticity, and $j = e_3 \cdot \nabla \times B$ is the current density. In contrast to the nonmagnetic situation, where the vorticity is rearranged by the flow it induces, the evolution equation (2.5) contains both advection and intensification, which arises from the $J \times B$ body force term in (2.2). These flux-vorticity equations for ideal MHD in two dimensions are often useful in deriving properties of solutions. Moreover, they are intimately connected to the (noncanonical) Hamiltonian structure of the conservative dynamical system (2.1) - (2.2) [15]. This structure is manifest once the stream function ϕ is introduced by $V = \nabla \times \phi e_3$. The flux-vorticity equations can then be written in the form

$$\frac{\partial \psi}{\partial t} + \partial(\psi, \phi) = 0\,, \qquad \frac{\partial \omega}{\partial t} + \partial(\omega, \phi) = \partial(j, \psi),$$

using the canonical bracket on R^2 defined by $\partial(\psi, \phi) = e_3 \cdot \nabla\psi \times \nabla\phi$.

For use throughout the paper, we introduce a notation for the linear operators that mediate between the primitive vector fields B and V and the derived scalar fields ψ, j and ϕ, ω. This is necessary because the statistical description of turbulent field-flow utilizes the primitive fields, while the conserved quantities and equilibrium equations involve the derived scalar fields. We let "curl" and "Curl" be defined by

$$\mathrm{curl} B \;:=\; e_3 \cdot \nabla \times B\,, \qquad \mathrm{Curl}\psi \;:=\; \nabla \times \psi e_3\,, \tag{2.6}$$

for any vector field B and any scalar field ψ. We note that curl acts on a vector and produces a scalar, while Curl acts on a scalar and produces a vector. These operators are adjoints in the sense that

$$\int_D \psi\, \mathrm{curl}\tilde{B}\, dx \;=\; \int_D \mathrm{Curl}\psi \cdot \tilde{B}\, dx \tag{2.7}$$

for any ψ and $\tilde{B}$ satisfying the ideal boundary conditions. We especially need the inverses of curl and Curl. Let G be the Green operator for the boundary-value problem

$$-\Delta\psi \;=\; j \;\text{ in }\; D, \quad \psi \;=\; 0 \;\text{ on }\; \partial D, \tag{2.8}$$

so that the solution is expressed as $\psi = Gj$. Using the identity $-\Delta = \mathrm{curl}\,\mathrm{Curl}$, we obtain the inverse operators

$$\mathrm{curl}^{-1} j := \mathrm{Curl}(Gj), \quad \mathrm{Curl}^{-1} B := G(\mathrm{curl} B), \tag{2.9}$$

for any scalar field j and any vector field B. We note that these formulas determine the inverse operators completely, without further integrability or gauge conditions. Indeed, $B = \mathrm{curl}^{-1} j$ is solenoidal ($\nabla \cdot B = 0$ in D and $n \cdot B = 0$ on ∂D) for an arbitrary density j, and $\psi = \mathrm{Curl}^{-1} B$ defines the flux function for the solenoidal part of an arbitrary field B. The identity (2.7) translates into the statement that curl^{-1} and Curl^{-1} are adjoints:

$$\int_D (\mathrm{Curl}^{-1} B)\,\tilde{j}\, dx = \int_D B \cdot \mathrm{curl}^{-1} \tilde{j}\, dx, \tag{2.10}$$

We use this reciprocal identity for several calculations in the sequel.

The complete family of conserved quantities for two-dimensional MHD is [35]:

$$E = \int_D \frac{1}{2}(|B|^2 + |V|^2) dx, \tag{2.11}$$

$$F_i = \int_D f_i(\psi) dx \qquad (\psi = \mathrm{Curl}^{-1} B), \tag{2.12}$$

$$K_i = \int_D \omega f_i(\psi) dx = \int_D B \cdot V f_i'(\psi) dx \qquad (\omega = \mathrm{curl} V). \tag{2.13}$$

The total energy integral, $E = E_{mag} + E_{kin}$, coincides with the Hamiltonian functional for the system. The families of conserved quantities F_i and K_i, indexed by i, are Casimir functionals that arise from the degeneracy of the (noncanonical) Hamiltonian structure [15]. In both of these families, $f_i(\psi)$ denotes an arbitrary (regular) real function defined on the range of the flux function $\psi(x,t)$. This range is invariant under evolution by (2.4). It suffices to let i be a continuous index, running over some interval I, that parameterizes the flux levels $\psi = \sigma_i$ realized by all the magnetic surfaces in D. The physical meaning of the Casimir functionals is revealed by choosing $f_i(\psi) = 1_{\{\psi > \sigma_i\}}$, the unit step function on the interior of the magnetic surface $\psi = \sigma_i$. Then F_i equals the mass and K_i equals the circulation inside the flux tube with index $i \in I$. That these quantities are constants of the motion can be verified directly from the flux and vorticity equations (2.4) and (2.5). We refer to the integrals F_i and K_i, in which $f_i(\psi)$ is a general (regular) real function, as the *generalized flux* and *generalized cross-helicity*, respectively. The second form of K_i noted in (2.13) follows upon an integration by parts. The classical cross-helicity integral results from the particular choice $f_i'(\psi) = 1$. (Here and throughout the sequel, prime denotes $d/d\psi$.) We expect that E, F_i, K_i exhaust the set of global conserved integrals for the ideal dynamics, apart from those that arise from special spatial symmetries depending on the domain geometry and boundary conditions.

Both for analytical and numerical reasons, it is convenient to approximate the *continuously infinite* families of generalized flux and cross-helicity integrals by the linear combination of a *finite* number of such integrals. Accordingly, we choose a finite basis of functions $f_i(\sigma)$, $i = 1, \ldots, h$, with argument $\sigma \in R$, having suitable regularity and growth properties, and we retain only the corresponding integrals F_i and K_i for $i = 1, \ldots, h$. We then retrieve an arbitrary $f(\psi)$ in the approximate sense that $f(\psi) \approx \sum_{i=1}^{h} c_i f_i(\psi)$ for an appropriate set of constants c_i. Even for a fairly small h, we expect this approximation to be quite accurate, owing to the natural regularity of ψ [12, 34]. We therefore adopt it throughout the sequel to ease the technical complications associated with continuously infinite families of constraints.

B. Macroscopic description. The evolution of an ideal magnetofluid is turbulent in the sense that the field-flow state $Y = Y(x,t)$ spontaneously develops finite-amplitude fluctuations on small scales, even when its initialization $Y^0 = Y(x,0)$ varies smoothly over the domain. This behavior of the deterministic governing equations (2.1) - (2.2) is strongly supported by direct numerical simulations, which necessarily pertain to a slightly dissipative perturbation of the ideal equations [4, 5, 6, 7, 28]. An extrapolation of these simulations to the ideal setting suggests that as time proceeds the fluctuations of Y reside on smaller and smaller scales around each point x, and that the correlations between these fluctuations at two separated points x and x' become weaker and weaker. In constructing a model of the fully-developed state of ideal turbulence, therefore, we are led to a description in which the fluctuations of $Y = (B, V)$ have finite variance, occur on an infinitesimal scale around each point x, and are uncorrelated between distinct points x and x'. This turbulent state is not homogeneous, however, as it exhibits a nonzero mean field-flow $\bar{Y} = (\bar{B}, \bar{V})$ that emerges on the macroscopic scale amongst the microscopic fluctuations. Indeed, high-resolution computational studies clearly demonstrate that the fully-developed state contains a coherent structure that varies on the scale of the domain size [5, 6, 7, 28]. In order to capture both the macroscopic mean and the microscopic fluctuations in our model of ideal turbulent relaxation, we adopt the hypothesis that there is a distinct separation of scales between the mean $\bar{Y}$ and the fluctuations Y' that comprise the microstate $Y = \bar{Y} + Y'$. In other words, we postulate that $\bar{Y}$ varies on the macroscopic scale, while Y' captures the local fluctuations on a microscopic scale.

We give these intuitive considerations a precise interpretation by introducing the probability distribution $p_x(dy)$ on the values $y \in R^4$ of the microstate $Y(x,t)$ at each point $x \in D$. This x-parameterized probability measure is defined by the property that

$$\int_{\mathcal{X}} dx \int_{\mathcal{Y}} p_x(dy) = \lim_{T\to\infty} \frac{1}{T} \int_0^T |\{x \in \mathcal{X} : Y(x,t) \in \mathcal{Y}\}|\, dt \tag{2.14}$$

for all measurable sets $\mathcal{X} \subset D$ and $\mathcal{Y} \subset R^4$; here, $|S|$ denotes the 2-volume of a subset S

of D. This definition means that for any spatial cell dx around x and any cell dy in the range of Y, the probability $p_x(dy)$ equals the average over dx of the statistical frequency (sampled over time) with which Y takes values in dy; or, in symbols,

$$p_x(dy) = \lim_{|dx|\to 0} \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{|x' \in dx : Y(x',t) \in dy|}{|dx|} \, dt. \tag{2.15}$$

Alternatively, this formula reveals an interpretation of $p_x(dy)$ as the statistical average of the local volume fraction around x over which Y lies in dy. As such, $p_x(dy)$ is identical with a so-called Young measure [29]. The local x-averaging is a crucial ingredient in the definition of $p_x(dy)$, as is the order of the limits in (2.15). Serious mathematical difficulties arise if the local spatial averaging is either omitted or interchanged with the temporal averaging.

With these definitions and interpretations in hand, we see that $p_x(dy)$ provides a macroscopic description of the field-flow state, which complements the microscopic description inherent in $Y(x,t)$. The macrostate $p_x(dy)$ constitutes a "coarse-grained" version of the "fine-grained" field-flow state Y. It ignores the arbitrarily complex local spatial arrangements realized by the microscopic fluctuations, but it contains all the information relevant to the turbulent relaxed state. Our separation of scales hypothesis allows us to use the single-point, local probability distribution $p_x(dy)$ rather than a joint probability distribution over many points $x^{(1)}, \ldots, x^{(N)}$, with $N \to \infty$ in the continuum limit.

Henceforth we shall use the probability density $\rho_x(y)$ of $p_x(dy)$ with respect to the 4-volume element dy as the macrostate for the system. The turbulent relaxed state is then defined by the identity

$$\int_D \int_{R^4} \eta(x,y)\rho_x(y)\, dxdy = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_D \eta(x, Y(x,t))\, dx \,, \tag{2.16}$$

which holds for all bounded, continuous test functions $\eta(x,y)$ on $D \times R^4$. This identity is clearly equivalent to the definition (2.14). It can also be recast into the statement that

$$\rho_x(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \delta(y - Y(x,t))dt, \tag{2.17}$$

where δ is the unit delta function on R^4, and the convergence is in the weak sense with respect to both x and y.

We obtain the local moments of Y from (2.16) as weak limits of the time-averaged microscopic quantities. In particular, the mean and variance of Y at $x \in D$, namely,

$$\bar{Y}(x) := \int_{R^4} y\rho_x(y)\, dy, \qquad \text{var}Y := \int_{R^4} |y - \bar{Y}(x)|^2 \rho_x(y)\, dy, \tag{2.18}$$

are constructed from (2.16) taking $\eta = \xi(x)y$ and $\eta = \xi(x)|y - \bar{Y}(x)|^2$, respectively, for all test functions $\xi(x)$ on D. (We assume some kind of uniform integrability [3] on the family of empirical measures defined by (2.17).) These local moments are not expected to be realized as strong (or pointwise) limits, however, since the construction of $\rho_x(y)$ requires local x-averaging. On the other hand, we expect that the potentials $\psi = \mathrm{Curl}^{-1}B$ and $\phi = \mathrm{Curl}^{-1}V$ converge strongly to their means, by virtue of the separation of scales hypothesis. Heuristically, both ψ and ϕ are insensitive to the local fluctuations, because these variations reside on an infinitesimal scale. Mathematically, the result follows from the compactness of the linear operator Curl^{-1} which maps weak convergence to strong convergence [29, 19]. On this basis, we assert that at each $x \in D$

$$\bar{\psi}(x) = \mathrm{Curl}^{-1}\bar{B} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \psi(x,t)\,dt, \qquad \mathrm{var}\,\psi(x) = 0. \tag{2.19}$$

The analogous expressions hold for $\phi(x)$. Thus, the flux function and stream function determine the mean field and mean flow. By an analogous reasoning, we see that the local fluctuations of j or ω are not quantified in this formulation.

C. Maximum entropy principle. While we defer a completely satisfactory derivation of the statistical equilibrium problem until Section 4, we are now able to formulate our continuum model of ideal turbulent relaxation. From the standard principles of statistical mechanics, we expect that the system relaxes into the most probable macrostate satisfying the prescribed values of the conserved quantities. Accordingly, we base the model on the macroscopic description of the system that follows from our separation of scales hypothesis. For this purpose, we need a suitable entropy functional, which we take to be

$$S(\rho) = -\int_D \int_{R^4} \rho_x(y) \log \rho_x(y)\,dxdy. \tag{2.20}$$

This classical Boltzmann-Gibbs-Shannon formula [1, 2] admits the usual interpretation as the logarithm of the number of microscopic realizations of the macrostate ρ. Alternatively, it is identical with the Kullback-Liebler entropy [11] of the measure $p_x(dy)dx$ on $D \times R^4$ relative to the spatially homogeneous measure $dxdy$. Its form is dictated by two properties of the ideal turbulence: 1) as an integral in x, it imposes independence on the fluctuations at distinct points in D; 2) as an integral in y, it uniformly weights the entire range R^4 of the local fluctuations. While these postulated properties are quite natural, their full justification rests on the Liouville property of the underlying dynamics, as is shown in Section 4.

The entropy serves as the objective functional in maximum entropy principle which determines the statistical equilibrium states. The constraints in this principle, which give the system its special character, are imposed on the following functionals of ρ:

$$E(\rho) = \int_D \int_{R^4} \frac{1}{2}(|b|^2 + |v|^2)\rho_x(y)\,dxdy\,, \tag{2.21}$$

$$F_i(\rho) = \int_D f_i(\bar{\psi}(x))\,dx\,, \tag{2.22}$$

$$K_i(\rho) = \int_D \int_{R^4} b \cdot v f_i'(\bar{\psi}(x)) \rho_x(y)\, dx dy\,, \tag{2.23}$$

in which b and v each run over R^2 with $y = (b, v)$, and $\bar{\psi}$ is defined by (2.19). These expressions are derived by applying the time-averaging identity (2.16) along with the property (2.19) to the conserved quantities (2.11), (2.12), and (2.13) for the microscopic dynamics. Again, the rigorous derivation is delayed until Section 4.

With the objective and constraint functionals in hand, we finally arrive at the maximum entropy principle for the statistical equilibrium state ρ:

$$S(\rho) \to \max \quad \text{over} \quad E(\rho) = E^0, \; F_i(\rho) = F_i^0, \; K_i(\rho) = K_i^0, \tag{2.24}$$

where the constraint values E^0, F_i^0, and K_i^0 are derived from a prescribed initial state Y^0. Of course, ρ must also satisfy the positivity condition, $\rho_x(y) \geq 0$, and the normalization constraint, $\int_{R^4} \rho_x(y)\, dy = 1$ for all $x \in D$. This constrained optimization problem completely determines the relaxed state ρ that emerges after a turbulent evolution from an initial state Y^0.

The development given in this section motivates the construction of our model, which is articulated in the maximum entropy principle (2.24). Of course, we cannot prove that the underlying dynamics drives the system toward a statistical equilibrium with the properties that we require. Nonetheless, we can proceed to examine the predictions of the model and compare them with the results of direct numerical simulations and physical observations.

3 Properties of statistical equilibrium states

A. Analysis of solutions The equilibrium equation satisfied by the solution to the maximum entropy problem is obtained by the Lagrange multiplier rule,

$$\delta S = \beta \delta E + \sum_i \alpha_i \delta F_i + \sum_i \gamma_i \delta G_i \tag{3.1}$$

on the subspace of variations $\delta\rho$ satisfying the local normalization $\int \delta\rho_x(y) dy = 0$. Here $\delta\Phi$ denotes the first variation of a functional $\Phi(\rho)$ with respect to ρ. The multipliers $\beta, \alpha_i, \gamma_i$ for the constraints are analogous to the "inverse temperature" in usual statistical mechanics [1, 2]. We calculate the functional derivatives appearing in (3.1) to be

$$\delta S = -\int\int [\log \rho_x(y)] \delta\rho_x(y)\, dx dy \tag{3.2}$$

$$\delta E = \int\int \frac{1}{2}(|b|^2 + |v|^2) \delta\rho_x(y)\, dx dy \tag{3.3}$$

$$\delta F_i = \int\int b \cdot \mathrm{curl}^{-1} f_i'(\bar{\psi}) \delta\rho_x(y)\, dx dy \tag{3.4}$$

$$\delta K_i = \int\int b \cdot \{f_i'(\bar\psi)v + \mathrm{curl}^{-1}[\overline{B\cdot V} f_i''(\bar\psi)]\}\delta\rho_x(y)\,dxdy, \tag{3.5}$$

where each double integral extends over $x \in D$ and $y = (b, v) \in R^4$. The calculations (3.2) and (3.3) are straightforward, while (3.4) and (3.5) deserve some explanation. We obtain (3.4) by the following steps:

$$\begin{aligned}\delta F_i &= \int f_i'(\bar\psi)\delta\bar\psi\,dx \\ &= \int f_i'(\bar\psi)\mathrm{Curl}^{-1}\delta\bar B\,dx \\ &= \int \mathrm{curl}^{-1} f_i'(\bar\psi)\cdot\delta\bar B\,dx \\ &= \int\int \mathrm{curl}^{-1} f_i'(\bar\psi)\cdot b\ \ \delta\rho_x(y)\,dxdy,\end{aligned}$$

in which we appeal to the adjoint formula (2.10). We get (3.5) by a similar sequence of steps, which now involve the mean cross-helicity density

$$\overline{B\cdot V}(x) := \int_{R^4} b\cdot v\rho_x(y)\,dy. \tag{3.6}$$

Upon substitution of these expressions into (3.1), we deduce that the statistical equilibrium macrostate ρ has the local canonical form

$$\rho_x(y) = Z_x^{-1}\exp(-\beta q(y;\lambda(\bar\psi),\mu(\bar\psi))) \tag{3.7}$$

with local partition function

$$Z_x = \int_{R^4}\exp(-\beta q(y;\lambda(\bar\psi),\mu(\bar\psi)))\,dy. \tag{3.8}$$

Here, we introduce the quadratic form

$$q(y;\lambda,\mu) := \frac{1}{2}(|b|^2+|v|^2) - b\cdot\mathrm{curl}^{-1}\lambda(\sigma) - b\cdot\{\mu(\sigma)v + \mathrm{curl}^{-1}[\overline{B\cdot V}\mu'(\sigma)]\}, \tag{3.9}$$

and the *profile functions*

$$\lambda(\sigma) := -\beta^{-1}\sum_i \alpha_i f_i'(\sigma), \qquad \mu(\sigma) = -\beta^{-1}\sum_i \gamma_i f_i'(\sigma). \tag{3.10}$$

We see immediately that the solution $\rho_x(y)$ is a Gaussian density in y for every $x \in D$. We also note that the covariance matrix for these local distributions depends implicitly on the mean field $\bar\psi(x)$. Moreover, this dependence is nonlinear and nonlocal, owing to the presence of the (nonlinear) profile functions λ and μ, and the (nonlocal) operator curl^{-1}.

We extract the mean field-flow $\bar Y = (\bar B, \bar V)$ and the fluctuations $Y' = Y - \bar Y$ from the basic formula (3.7) by introducing the variables $b' = b - \bar B(x), v' = v - \bar V(x)$, in terms of which we get the normal form

$$q = \frac{1}{2}(|b'|^2+|v'|^2) - b'\cdot v'\mu(\bar\psi) - \frac{1}{2}((1-\mu(\bar\psi)^2)|\bar B|^2), \tag{3.11}$$

and the mean field-flow equations

$$\bar{B} = \text{curl}^{-1}[\lambda(\bar{\psi}) + \overline{B \cdot V}\mu'(\bar{\psi})] + \mu(\bar{\psi})\bar{V} \tag{3.12}$$

$$\bar{V} = \mu(\bar{\psi})\bar{B}. \tag{3.13}$$

The equilibrium macrostate is then given by

$$\rho_x(y) = (\frac{\beta}{2\pi})^2[1-\mu(\bar{\psi})^2]\exp(-\frac{\beta}{2}\{|b'|^2 + |v'|^2 - 2\mu(\bar{\psi})b' \cdot v'\}). \tag{3.14}$$

The local partition function (3.8) is evaluated as $Z_x = (2\pi/\beta)^2[1-\mu(\bar{\psi})^2]^{-1}\exp(\frac{\beta}{2}(1-\mu(\bar{\psi})^2)|\bar{B}|^2)$. These equations for the mean field and mean flow together with the Gaussian distribution for the fluctuations of B and V completely determine the ideal turbulent relaxed state.

The multiplier β and the profile functions λ and μ are naturally determined by the energy, generalized flux and cross-helicity, respectively. The 4x4 covariance matrix for Y at $x \in D$ is readily deduced from (3.14), giving

$$\text{var}B_1 = \text{var}B_2 = \text{var}V_1 = \text{var}V_2 = \beta^{-1}[1-\mu(\bar{\psi})^2]^{-1} \tag{3.15}$$

$$\text{cov}(B_1, V_1) = \text{cov}(B_2, V_2) = \beta^{-1}\mu(\bar{\psi})[1-\mu(\bar{\psi})^2]^{-1}, \tag{3.16}$$

along with $\text{cov}(B_p, B_q) = \text{cov}(V_p, V_q) = \text{cov}(B_p, V_q) = 0$ for $p \neq q$. From these formulas we see that $\mu(\bar{\psi})$ is the magnetic-velocity correlation

$$\mu(\bar{\psi}) = \text{corr}(B_p, V_p) := \frac{\text{cov}(B_p, V_p)}{\sqrt{\text{var}B_p}\sqrt{\text{var}V_p}} \qquad (p = 1, 2) \tag{3.17}$$

Hence, apart from certain degenerate cases, $0 < \beta < \infty$ and $-1 < \mu(\bar{\psi}) < 1$.

The energy density and the cross-helicity density are given by

$$\frac{1}{2}(\overline{|B|^2} + \overline{|V|^2}) = \frac{1}{2}(|\bar{B}|^2 + |\bar{V}|^2) + 2\beta^{-1}[1-\mu(\bar{\psi})^2]^{-1}, \tag{3.18}$$

$$\overline{B \cdot V} = \bar{B} \cdot \bar{V} + 2\beta^{-1}\mu(\bar{\psi})[1-\mu(\bar{\psi})^2]^{-1}, \tag{3.19}$$

using (3.15) and (3.16), respectively. Consequently, the equilibrium expressions for the energy and generalized cross-helicity are:

$$E = \int \frac{1}{2}(|\bar{B}|^2 + |\bar{V}|^2)dx + \frac{2}{\beta}\int \frac{1}{1-\mu(\bar{\psi})^2}dx, \tag{3.20}$$

$$K_i = \int \bar{B} \cdot \bar{V} f_i'(\bar{\psi})dx + \frac{2}{\beta}\int \frac{\mu(\bar{\psi})}{1-\mu(\bar{\psi})^2}f_i'(\bar{\psi})dx. \tag{3.21}$$

These expressions exhibit the partitioning of E and K_i into mean and fluctuation parts. By contrast, the generalized flux F_i resides entirely in the mean.

The mean field-flow equations (3.12)-(3.13) reduce to a single equation for the mean magnetic field, upon eliminating $\bar{V}$ and substituting (3.19) for $\overline{B \cdot V}$:

$$[1-\mu(\bar{\psi})^2]\bar{B} = \mathrm{curl}^{-1}[\lambda(\bar{\psi}) + \mu(\bar{\psi})\mu'(\bar{\psi})\{|\bar{B}|^2 + 2\beta^{-1}[1-\mu(\bar{\psi})^2]^{-1}\}]. \tag{3.22}$$

In turn, this equation simplifies to a scalar equation for $\bar{\psi}$, by taking curl:

$$\bar{j} - \mu(\bar{\psi})\bar{\omega} = \Lambda(\bar{\psi}) \;:=\; \lambda(\bar{\psi}) + 2\beta^{-1}\mu(\bar{\psi})\mu'(\bar{\psi})[1-\mu(\bar{\psi})^2]^{-1}. \tag{3.23}$$

We conclude that the mean field-flow $(\bar{B}, \bar{V})$ is indeed a steady solution of the (deterministic) ideal MHD equations, with $\bar{V} = \mu\bar{B}$ and $\bar{B}\cdot\nabla\Lambda = 0$ in D.

We also obtain the following expression for the total pressure head $P = p + \frac{1}{2}V^2$:

$$P(\bar{\psi}) = \int_0^{\bar{\psi}} \lambda(\sigma)d\sigma + \beta^{-1}s(\bar{\psi}) + \text{const.}, \tag{3.24}$$

where $s(\bar{\psi})$ is the equilibrium entropy density

$$s(\bar{\psi}(x)) \;=\; -\int_{R^4} \rho_x(y)\log\rho_x(y)\,dy \;=\; 2 + \log\frac{4\pi^2}{\beta^2[1-\mu(\bar{\psi}(x))^2]}. \tag{3.25}$$

We verify this equilibrium relation by referring to the steady version of (2.2), from which we recognize that $\Lambda(\bar{\psi}) = dP/d\bar{\psi}$. The equivalence of (3.24) with the definition of Λ in (3.23) then follows immediately. Accordingly, we see that the profile function $\lambda(\bar{\psi})$ represents the contribution to the pressure gradient from the mean field, while $\beta^{-1}ds/d\bar{\psi}$ represents the contribution from the fluctuations.

B. Dual variational principles. In general, there is a nontrivial coupling between the mean field-flow and the fluctuations in the statistical equilibrium problem. Indeed, the equilibrium equation for the mean flux function $\bar{\psi}$ contains the inverse temperature β and and the correlation μ, while the energy and generalized cross-helicity constraints, which determine those parameters, also involve $\bar{\psi}$. For this reason, we cannot expect to solve the mean field-flow equations independently of the global constraints on E, F_i and K_i, even though they are identical with the steady ideal MHD equations. Rather the profile functions occurring in those equations, which are arbitrary from a deterministic standpoint, must be required to satisfy the family of constraints in the maximum entropy principle.

In order to cast the nonlinear and nonlocal equations governing statistical equilibrium in a more tractable form, we now formulate a dual pair of variational problems associated with the maximum entropy principle. The primal problem determines the mean field $\bar{\psi}$ given the thermal parameters β and γ_i, while the dual problem determines the parameters β and γ_i given the field $\bar{\psi}$. Of course, we cannot actually solve the full problem with prescribed energy, cross-helicity and flux constraints in the general case by solving these dual subproblems separately. Nevertheless, we can suggest a numerical method that solves the fully-coupled problem by iterating between these subproblems. However, we shall not pursue this line of development in the present paper. Rather, we shall show how these dual variational principles can be used to infer some of the qualitative properties of solutions in certain special cases.

Our dual variational principles are built from the Lagrangian functional associated with the maximum entropy principle (2.24); namely,

$$L := S - \beta(E - E^0) - \sum_i \gamma_i(K_i - K_i^0). \tag{3.26}$$

Rather than consider L as a functional of ρ, however, we use the expressions (3.20), (3.21) and (3.25) to write L in terms of the mean field $\bar{B}$ and the parameters β and γ_i that quantify the fluctuations. The required calculation is

$$\begin{aligned} L &= \int s(\bar{\psi})dx - \beta \int \{\frac{1}{2}(|\bar{B}|^2 + |\bar{V}|^2) - \mu(\bar{\psi})\bar{B}\cdot\bar{V}\}\,dx - 2|D| \\ &\quad + \beta E^0 + \sum_i \gamma_i K_i^0 \\ &= \int \log\frac{4\pi^2}{\beta^2[1-\mu(\bar{\psi})^2]}\,dx - \beta\int\frac{1}{2}[1-\mu(\bar{\psi})^2]|\mathrm{Curl}\,\bar{\psi}|^2\,dx \\ &\quad + \beta E^0 + \sum_i \gamma_i K_i^0 . \end{aligned} \tag{3.27}$$

From the definition of μ given in (3.10), we see that $L = L(\bar{\psi}; \beta, \gamma_1, \dots, \gamma_h)$. We obtain an alternative functional dependence $L = L(\bar{\psi}; \beta, \mu)$ by noticing that the expression

$$E^0 + \sum_i \frac{\gamma_i}{\beta}K_i^0 = \int_D \{\frac{1}{2}|B^0|^2 + \frac{1}{2}|V^0|^2 - \mu(\bar{\psi})B^0\cdot V^0\}dx,$$

can be substituted into (3.27).

The variational principle for the mean field can be stated as follows: given β and $\gamma_1 \dots, \gamma_h$, the mean flux function solves

$$L(\bar{\psi}; \beta, \gamma_1 \dots, \gamma_h) \to \max \quad \text{over} \quad F_i(\bar{\psi}) = F_i^0 \tag{3.28}$$

Here the flux constraints are retained from the maximum entropy principle, because they involve only the mean field. The dual variational principle for the thermal parameters can be stated as follows: given $\bar{\psi}$, the inverse temperatures β and $\gamma_1 \dots, \gamma_h$ satisfy

$$L(\bar{\psi}; \beta, \gamma_1, \dots, \gamma_h) \to \min\,. \tag{3.29}$$

The given $\bar{\psi}$ is assumed to satisfy the flux constraints, and the minimization of L in $\beta, \gamma_1, \cdots, \gamma_h$ yields the constraints $E = E^0, K_1 = K_1^0, \dots, K_h = K_h^0$. We shall now sketch proofs of these variational characterizations.

To verify (3.28) we make a variation in $\bar{\psi}$ with $\delta\beta = \delta\gamma_i = 0$, and thereby arrive at the functional derivatives:

$$\delta L = \int \delta\bar{B}\cdot[\mathrm{curl}^{-1}(\mu(\bar{\psi})\mu'(\bar{\psi})\{\beta|\bar{B}|^2 - 2[1-\mu(\bar{\psi})^2]^{-1}\}) \;-\; \beta[1-\mu(\bar{\psi})^2]\bar{B}]\,dx\,,$$

$$\delta F_i = \int \delta\bar{B}\cdot\mathrm{curl}^{-1} f_i'(\bar{\psi})\,dx\,.$$

Substituting these formulas into the variational equation

$$\delta L - \sum_i \alpha_i \delta F_i = 0\,, \tag{3.30}$$

and recalling the definition of the profile function λ in (3.10), we retrieve precisely the mean field equation (3.22).

To verify (3.29) we make variations in $\beta, \gamma_1, \ldots, \gamma_h$ with $\delta\bar{\psi} = 0$. Noting that $\frac{\partial}{\partial\beta}\mu(\bar{\psi}) = -\mu(\bar{\psi})/\beta$ and $\frac{\partial}{\partial\gamma_i}\mu(\bar{\psi}) = -f_i'(\bar{\psi})/\beta$, we obtain the derivatives

$$\frac{\partial L}{\partial\beta} \;=\; E^0 - \frac{2}{\beta}\int\frac{1}{1-\mu(\bar{\psi})^2}\,dx \;-\; \int\frac{1}{2}[1+\mu(\bar{\psi})^2]|\bar{B}|^2 dx$$

$$\frac{\partial L}{\partial\gamma_i} \;=\; K_i^0 - \frac{2}{\beta}\int\frac{\mu(\bar{\psi})}{1-\mu(\bar{\psi})^2}\,dx - \int \mu(\bar{\psi})|\bar{B}|^2 f_i'(\bar{\psi})\,dx\,.$$

Referring to (3.20) and (3.21), we retrieve the constraints $E = E^0, K_i = K_i^0$ in the variational form

$$\frac{\partial L}{\partial\beta} = 0, \;\; \frac{\partial L}{\partial\gamma_i} = 0 \qquad (i = 1, \ldots, h). \tag{3.31}$$

That L is maximized in (3.28) and L is minimized in (3.29) is a standard property of dual variational problems [16]. An examination of the form of L in simple cases shows that this mathematical property is physically natural.

C. Free solutions. The statistical equilibrium problem is greatly simplified when we free the constraints on E, K_i, and instead fix the values of the inverse temperatures β, γ_i. Given $\beta > 0$ and $\gamma \in R^h$ such that the magnetic-velocity correlation profile satisfies $|\mu(\bar{\psi})| < 1$ everywhere, we can restate the variational principle (3.28) in terms of the (extended) free energy functional

$$\Phi(\bar{\psi}) \;=\; \int\frac{1}{2}[1-\mu(\bar{\psi})^2]|\mathrm{Curl}\bar{\psi}|^2 dx - \beta^{-1}\int\log\frac{4\pi^2}{\beta^2[1-\mu(\bar{\psi})^2]}dx. \tag{3.32}$$

(This free energy is extended in the sense that it also contains the cross-helicities.) As is evident from (3.27), the mean field is determined by solving

$$\Phi(\bar{\psi};\beta,\mu) \to \min \;\text{ over }\; F_i(\bar{\psi}) = F_i^0 \;\; (i = 1, \ldots, h), \tag{3.33}$$

The local fluctuations of B and V around the mean field $\bar{B} = \text{curl}\bar{\psi}$ and flow $\bar{V} = \mu\bar{B}$ are described by the Gaussian distribution (3.14) with inverse temperature β and correlation μ. The free solution is thus completely specified.

The variational equation for (3.33) is a second-order, nonlinear elliptic partial differential equation in $\bar{\psi}$ identical with the equilibrium equation (3.23). The profile function $\lambda(\bar{\psi}) = \sum_i \lambda_i f_i'(\bar{\psi})$ is determined by the multipliers λ_i for the flux constraints in (3.33). The mean field problem can therefore be viewed as a nonlinear elliptic eigenvalue problem with many eigenparameters λ_i. Numerical algorithms are available to solve this kind of problem [12, 34].

The simplest case of this general problem is the magnetostatic case, which occurs when $\mu = 0$. This choice results in a mean field that solves the Grad-Shafranov equation [4, 23]

$$-\Delta\bar{\psi} = \lambda(\bar{\psi}) \qquad (\text{where } \lambda = dp/d\bar{\psi}\,). \tag{3.34}$$

The energy $E(\bar{\psi}) = \int \frac{1}{2}|\text{Curl}\bar{\psi}|^2 dx$ is therefore the minimum energy compatible with the given flux constraints $F_i(\bar{\psi}) = F_i^0$. The inverse temperature then determines the equilibrium energy according to

$$E = E(\bar{\psi}) + 2\beta^{-1}|D|. \tag{3.35}$$

Moreover, $E_{mag} = E(\bar{\psi}) + \beta^{-1}|D|$ and $E_{kin} = \beta^{-1}|D|$. Thus it is evident that the difference between the total energy E and the energy of the coherent structure $E(\bar{\psi})$ resides in the local fluctuations, where it is equipartitioned into magnetic and kinetic parts equal to $\beta^{-1}|D|$. This result is clearly in good agreement with the picture of ideal turbulence as a sea of high-wavenumber random Alfven waves [4]. A further consequence of this result is the prediction that the relaxed value of the ratio E_{mag}/E_{kin} is greater than one.

The general case with nonvanishing B-V correlations admits a free formulation in which the profile function $\mu(\sigma)$ satisfying $-1 < \mu(\sigma) < 1$ is prescribed. The objective functional $\Phi(\bar{\psi})$ simplifies in this case, at least in the limit as the number of basis functions $f_i(\bar{\psi})$ tends to infinity. Indeed, since the flux constraints in (3.33) discretize the continuous family of constraints on all functionals of the form $\int f(\bar{\psi})dx$, we see that the entropy term in Φ is irrelevant in the minimization, being fixed by the constraints. Accordingly, the mean flux function is determined by the variational principle (3.33) with Φ given by the first member in (3.32). With $\bar{B}$ and $\bar{V}$ determined, the energy and cross-helicity values are calculated from (3.20) and (3.21). Thus, the free solutions with given β and $\mu(\sigma)$ are determined.

Of course, this reduction of the mean field problem presupposes that the number h of flux constraints is not too small. If, say for numerical purposes, only a few of these constraints are imposed, then it is presumably better to retain the entropy term in Φ. Since we are mainly concerned with conceptual issues here, we will assume that the finite family of flux constraints approximates the complete family of constraints to sufficient accuracy. Experience with the corresponding deterministic variational problem supports this view, unless h is quite small [12, 34].

In this context the mean field problem can be brought to a form identical with the magnetostatic case by appropriately relabeling the magnetic surfaces [17]. A glance at the functional Φ suggests the substitution

$$\bar{\Psi} = \int_0^{\bar{\psi}} \sqrt{1-\mu(\sigma)^2}\, d\sigma . \tag{3.36}$$

We can then express the variational problem (3.33) in terms of $\bar{\Psi}$:

$$\Phi(\bar{\Psi}) \;=\; \frac{1}{2}\int |\mathrm{Curl}\bar{\Psi}|^2\, dx \to \min \quad \text{over} \quad \int \tilde{f}_i(\bar{\Psi})\, dx = \tilde{F}_i^0, \tag{3.37}$$

allowing for a change in the basis functions $\tilde{f}_i(\sigma)$. Our earlier analysis of the case $\mu = 0$ now applies to this problem in the transformed variable $\bar{\Psi}$. The reparameterization of the flux constraints using the given profile function $\mu(\sigma)$ is the only complication.

D. Constrained solutions. From the standpoint of comparison with physical experiment or numerical simulation, however, these free solutions are deficient, because they contain the arbitrarily chosen constant β and function $\mu(\sigma)$, which are actually determined by the turbulent relaxed state. We are therefore compelled to consider the constrained solutions of our basic maximum entropy problem. This complete problem, however, appears to require a numerical method of solution – say, an iterative procedure that alternates between solves in $\bar{\psi}, \alpha_1, \ldots, \alpha_h$ and solves in $\beta, \gamma_1, \ldots, \gamma_h$. We shall not attempt to develop such a method in the present paper. Instead, we shall be content to describe some of the general qualitative features of the constrained solutions.

In the magnetostatic case, for which $\mu = 0$, the constrained solutions are easy to construct. This case corresponds to cross-helicity constraint values $K_i^0 = 0$. The mean field $\bar{\psi}$ is the minimizer of energy $E(\bar{\psi})$ subject to the given flux constraints. The temperature β^{-1} is then determined by (3.35), in which $E = E^0$. Thus, the constrained solution with vanishing cross-helicity is completely specified for any energy value with $E^0 \geq E(\bar{\psi})$.

For these magnetostatic solutions, we observe a qualitative difference between the high energy regime ($E^0 \gg E(\bar{\psi})$) and the low energy regime ($E^0 \approx E(\bar{\psi})$). On the one hand, as $E^0 \to +\infty$, we find that $\beta \to 0$, and hence that the variance of B and V diverges. The relaxed state therefore resembles homogeneous turbulence: the variance of its local fluctuations is large and constant over the domain D, while its mean field is bounded. The coherent structure is thus obliterated by the fluctuations. On the other hand, as $E^0 \to E(\bar{\psi})$, we have that $\beta \to +\infty$, and hence that the variance of B and V tends to zero. This means that the field-flow state relaxes into a nonturbulent magnetostatic equilibrium, in which the fluctuations about the coherent mean field are negligible. These features of the relaxed state are in excellent agreement with the results of direct numerical simulations with small μ [5, 6, 7].

The effects of B-V correlations are readily examined by considering a simplified version of the maximum entropy problem in which the classical *quadratic* cross-helicity constraint, $K^0 = \int \overline{B \cdot V}\, dx$ is imposed, rather than the full family of generalized cross-helicities. As this simplified version of the statistical equilibrium problem is analyzed in detail in a companion paper [19], we content ourselves here with a brief discussion of its main predictions. Most importantly, the magnetic-velocity correlation defined in (3.17) becomes the constant, $\mu = -\gamma/\beta$, where γ is the multiplier for the quadratic cross-helicity constraint. As a result, the equilibrium density $\rho_x(y)$ has a *constant* covariance matrix. Furthermore, the mean field-flow equations (3.12,3.13) simplify when μ is constant. In fact, as in the magnetostatic case, the mean magnetic field is a critical point of the magnetic energy functional, $E(\bar{\psi})$, subject to the given flux constraints. Finally, the energy and cross-helicity constraints take the simple form:

$$E^0 = (1+\mu^2)E(\bar{\psi}) + \frac{2|D|}{\beta(1-\mu^2)} \tag{3.38}$$

$$K^0 = 2\mu E(\bar{\psi}) + \frac{2\mu|D|}{\beta(1-\mu^2)}, \tag{3.39}$$

We note that μ has the same sign as K^0, and that μ tends to zero as $K^0 \to 0$.

The entropy of the statistical equilibrium density ρ is found to be

$$S(\rho) \;=\; |D| \log\{[E^0 - (1+\mu^2)E(\bar{\psi})]^2 - [K^0 - 2\mu E(\bar{\psi})]^2\} \;+\; \text{const.}, \tag{3.40}$$

where the constant depends only on $|D|$. This expression is obtained by substituting (3.38) and (3.39) into the general formula (3.25). From it we draw the interesting conclusion that the relaxed state ρ balances two competing tendences: the one, to maximize the fluctuation part of the energy; the other, to minimize the fluctuation part of the cross-helicity. Since the mean field $\bar{B}$ is fixed by the flux constraints, this balance is achieved by the correlation μ, which is the only free parameter in the entropy expression (3.40). Hence, we find that μ is the solution of the associated critical point equation

$$E(\bar{\psi})\mu^3 - (E(\bar{\psi}) + E^0)\mu + K^0 = 0. \tag{3.41}$$

Under some restrictions on E^0 and K^0, this equation can be shown to admit a unique solution $\mu \in (-1,1)$. Once μ is known, β is found from either (3.38) or (3.39), and the statistical equilibrium state ρ is thus completely determined. The details are presented in [19].

Of particular interest is the regime in which B and V are strongly correlated. Provided that $E^0 > 2E(\bar{\psi})$, the above analysis shows that the correlation μ approaches 1, and the variance $\beta^{-1}(1-\mu^2)^{-1}$ approaches $E^0 - 2E(\bar{\psi})$, as the prescibed ratio K^0/E^0 goes to 1, its largest possible value. Consequently, the field and the flow become completely correlated in this limit, and the marginal distributions of $B(x)$ and $V(x)$ converge to the same Gaussian distribution with mean $\bar{B}$ and componentwise variance $E^0 - 2E(\bar{\psi})$. In short, the field and the flow become statistically indistinguishable in this degenerate limit. This vivid effect is confirmed by direct numerical simulations [32], which show that the field and flow tend to align dynamically when the initial ratio of quadratic cross-helicity to energy is taken above a certain threshold value. Similarly, as K^0/E^0 goes to -1, its smallest possible value, the correlation μ tends to -1, and the field and the flow become anti-aligned.

In the general case in which the profile function $\mu(\sigma)$ is not constant, the properties of the constrained solutions are difficult to extract from the statistical equilibrium equations. Nevertheless, we can infer the following qualitative features. First, we note that in all regimes the equilibrium ratio E_{mag}/E_{kin} is greater than one, regardless of its initial value. The model, therefore, captures the amplification and saturation of the magnetic field observed in turbulence simulations when the initial ratio is small [6]. Second, we see that the functional relationship $\bar{V} = \mu\bar{B}$ between the mean magnetic field and the mean velocity field is always predicted. Moreover, the model identifies the scalar factor between the mean field and flow as the correlation $\mu(\bar{\psi})$ for their fluctuations. Third, we deduce from (3.15) and (3.16) that large fluctuations of both B and V are expected wherever this alignment/correlation is strong. This prediction is especially interesting when $\mu(\sigma)$ is nonconstant, since then the variance of the fluctuations is enlarged near those flux surfaces where $\mu(\bar{\psi})^2 \approx 1$. Furthermore, the current density is predicted to peak at these flux surfaces, owing to the presence of the second term in the profile function Λ exhibited in (3.23). This behavior is in accord with the observed behavior of turbulent relaxed states, which show the concentration of fluctuations around definite magnetic surfaces, usually connected with X-points [5, 28]. The analysis of this noteworthy effect, however, requires that we determine variability of the equilibrium profile function $\mu(\sigma)$ in the constrained solutions. We defer such an analysis to subsequent work.

4 Justification for the continuum model

A. Discrete model. The formulation of the continuum model in Section 2 relies on a fundamental hypothesis - the separation of scales between the microscopic fluctuations and the macroscopic mean field-flow. Under this hypothesis we are able to describe the turbulent relaxed state in terms of the local probability density $\rho_x(y)$. Our derivation of the maximum entropy principle (2.24) that determines the most probable macrostate $\rho_x(y)$, however, is heuristic rather than rigorous. In three particular respects the formulation requires further justification: 1) the neglect of correlations between points, which allows us to use the single-point distributions; 2) the specific form of the entropy functional, which is based on an a priori distribution that is uniform on $D \times R^4$; 3) the vanishing of the variance of ψ, which permits us to express magnetic surface quantities in terms of the mean flux function $\bar{\psi}$. In this section, we therefore develop a lattice model of ideal magnetofluid turbulence, which tends in the limit to the continuum model. In the course of this development, we justify the continuum model rigorously, and we connect it to some standard concepts in statistical mechanics [1, 2, 11] and fluid turbulence [10, 21].

Throughout this section we restrict our attention to a doubly-periodic spatial geometry. These familiar boundary conditions allow us to develop the relationship between the discretization of x-space and the truncation of k-space in a simple and concrete way. We let $D = \{x = (x_1, x_2) \in R^2 : |x_1| < L_1/2, |x_2| < L_2/2\}$ be the fundamental period domain, and we normalize its area $|D| = L_1 L_2 = 1$. For large, even integers N_1 and N_2, we introduce the discrete domain D_N consisting of equally spaced nodes $x_n = (n_1 \Delta x_1, n_2 \Delta x_2)$ with $\Delta x_1 = L_1/N_1$ and $\Delta x_2 = L_2/N_2$, where the multi-index $n = (n_1, n_2)$ runs over the integral lattice with $-N_1/2 < n_1 \le N_1/2$, $-N_2/2 < n_2 \le N_2/2$. The discrete grid D_N then contains $N := N_1 N_2$ nodes.

Any microstate $Y(x)$, defined for $x \in D$, can be sampled at the nodes $x_n \in D_N$ to produce the discretized state vector $Y = (Y_n)$, whose components represent the nodal values $Y(x_n)$. The discretized phase space is therefore R^{4N}. The Euclidean inner product on this space will be denoted by

$$(Y, \tilde{Y})_N := \frac{1}{N} \sum_n Y_n \cdot \tilde{Y}_n,$$

where the factor $1/N$ is included to maintain contact with the continuum limit. The Euclidean norm will be written as $||Y||_N^2 = (Y, Y)_N$. Here and throughout the sequel, sums over n are understood to extend over the N nodes in D_N.

The discrete Fourier transform of a microstate Y on D_N involves the N wavevectors: $k_m = (m_1 \Delta k_1, m_2 \Delta k_2)$ with $\Delta k_1 = 2\pi/L_1$ and $\Delta k_2 = 2\pi/L_2$, where $m = (m_1, m_2)$ is a multi-index running over $-N_1/2 < m_1 \le N_1/2$, $-N_2 < m_2 \le N_2/2$. The inversion formula for the transform reads

$$Y_n = \sum_k \hat{Y}_k e^{ik \cdot x_n} \quad \text{with} \quad \hat{Y}_k := \frac{1}{N} \sum_n Y_n e^{-ik \cdot x_n}. \tag{4.1}$$

Here, and henceforth, sums over k extend over the N wavevectors $k = k_m$. This formula establishes a linear, one-to-one correspondence between the real state vectors Y and those complex amplitude vectors $\hat{Y}$ which satisfy the compatibility conditions $\hat{Y}_{-k} = \hat{Y}_k^*$. The Parseval identity for (4.1) is

$$\frac{1}{N} \sum_n |Y_n|^2 = \sum_k |\hat{Y}_k|^2, \tag{4.2}$$

in which $|\hat{Y}_k|^2 = \hat{Y}_k \cdot \hat{Y}_k^*$ denotes the Hermitian dot product. By virtue of this identity, the discrete Fourier transform constitutes an isometry between Euclidean space R^{4N} and the subspace of the Hermitian space C^{4N} defined by the compatibility conditions. We refer the reader to standard texts [31, 9] for detailed treatments.

While the doubly-periodic boundary conditions are natural and convenient in formulating a discrete model, they suffer from the minor drawback that they permit arbitrary x-translations. Rather than including additional conserved quantities as constraints in the statistical equilibrium model, we shall eliminate these x-translations by imposing some finite symmetry conditions on the microstates. These conditions amount to posing the problem on the quarter domain $D^+ = D \bigcap \{x_1, x_2 > 0\}$ with ideal boundary conditions on ∂D^+. The continuous microstate $Y(x) = (\mathrm{Curl}\psi, \mathrm{Curl}\phi)$ then possesses the symmetries in x_1 and x_2 inherited from those of ψ and ϕ, on which we impose $\psi = 0 = \phi$ on ∂D^+, and which we extend to D by odd reflection in x_1 and x_2. In other words, we imagine the physical domain to be D^+ and assume that ∂D^+ is a perfectly conducting and slipping boundary. The Fourier representations of ψ and ϕ then become sine expansions. The components of Y are similarly extended to D by reflection (even or odd depending on the component and the variable), and so are represented by reduced expansions (cosine or sine expansions). In what follows we shall enforce these symmetry conditions on the continuous and discrete versions of the problem. We note specifically that all the $k = 0$ Fourier amplitudes vanish.

We can take the discretized microscopic dynamics to be governed by the truncated spectral form of the primitive equations. This form, which is often described in the literature, defines the evolution of the complex amplitudes $\hat{Y}_k(t)$. The detailed equations for $\hat{Y}_k$ are not needed for our purposes. We use only the Liouville property satisfied by the truncated k-space equations: the phase flow conserves the (Hermitian) phase volume. This crucial property is demonstrated by several authors [21, 22]. By invoking the inverse discrete Fourier transform we can map this dynamics from the truncated k-space to the discrete x-space. In this way we can define the microscopic dynamics in the discretized primitive variables Y_n over D_N. Since this mapping is a linear isometry between $\hat{Y}$ and Y, we infer immediately that the induced phase flow on the x-space R^{4N} is volume preserving. In other words, our discrete x-space dynamics satisfy the Liouville property.

We now approximate the conserved quantities for the continuum equations by their discrete analogues, namely,

$$E = \frac{1}{2N} \sum_n |B_n|^2 + |V_n|^2 = \frac{1}{2} \sum_k |\hat{B}_k|^2 + |\hat{V}_k|^2, \tag{4.3}$$

$$F_i = \frac{1}{N} \sum_n f_i(\psi_n), \tag{4.4}$$

$$K_i = \frac{1}{N} \sum_n B_n \cdot V_n f_i'(\psi_n). \tag{4.5}$$

The energy E is conserved exactly by the discretized dynamics, as an analysis of the detailed equations shows. The generalized fluxes F_i and cross-helicities K_i, however, are conserved only in the limit as $N \to \infty$, unless very particular basis functions are used ($f(\psi) = \psi^2/2$ for F, and $f(\psi) = \psi$ for K). Even though the nonlinear and nonlocal dependence of these quantities on ψ destroys their exact conservation, nevertheless, we shall adopt them as the approximate dynamical invariants for our discrete model.

We note that the flux function used in (4.4) and (4.5) is defined by the k-space relation

$$\hat{\psi}_k = i|k|^{-2} k \times \hat{B}_k. \tag{4.6}$$

Together with the discrete Fourier transform, this relation provides the discretized form of the operator Curl^{-1}. We shall denote this x-space operator by $\mathcal{A}$, and its adjoint with respect to $(\cdot,\cdot)_N$ by $\mathcal{A}^\dagger$.

B. Separation of scales. With a discrete microscopic model in hand, we now consider a macroscopic description in terms of a probability distribution $p^N(dY)$ on the phase space R^{4N}. For the sake of definiteness, we write $p^N(dY) = \rho^N(Y)dY$, and take the macrostate to be the probability density $\rho^N(Y)$ with respect to $4N$-volume dY. Any functional $A(Y)$ on the discrete phase space then defines a random variable whose expectation (mean) is denoted by

$$\langle A \rangle = \int_{R^{4N}} A(Y)\rho^N(Y)dY.$$

We seek a probability density that describes the turbulent relaxed state for the discrete dynamics. In accord with standard practice, we define a statistical equilibrium state to be the density ρ^N that maximizes the entropy

$$S(\rho^N) = \frac{-1}{N} \int_{R^{4N}} \rho^N(Y) \log \rho^N(Y) dY \tag{4.7}$$

subject to the constraints associated with the conserved quantities E, F_i, K_i [1, 2, 18]. The form of this entropy functional is dictated by the Liouville property for the discrete dynamics, which demands that the density $\rho^N(Y)$ be relative to the invariant measure dY. Thus, the objective functional in our maximum entropy principle is uniquely determined up to a multiplicative factor, which we take to be $1/N$ in order to have a finite entropy in the continuum limit.

Unlike the usual canonical ensemble, however, the constraint functionals are *not* simply taken to be the expectations $\langle E \rangle, \langle F_i \rangle, \langle K_i \rangle$ of the functionals in (4.3) (4.4) (4.5). This unorthodox aspect of our model is dictated by the form of the constraints themselves, and hence by the special nature of ideal turbulence. In addition to the constraint on the energy E, our statistical equilibrium problem also depends upon two families of constraints on F_i and K_i, and these functionals involve the flux function ψ, which has distinctly different statistical behavior from that of B and V. Consequently, the usual canonical ensemble formed from a linear combination of E, F_i and K_i can fail to provide a meaningful continuum limit as $N \to \infty$.

Before giving our formulation, we briefly review the theory of absolute equilibrium distributions [13, 21], which is derived from the usual canonical ensemble based on energy E, *quadratic* flux F (with $f(\psi) = \psi^2/2$) and *classical* cross-helicity K (with $f(\psi) = \psi$). First, these statistical equilibrium distributions have zero mean: $\langle \hat{Y}_k \rangle = 0$ for all k. Second, they predict that the given energy E^0 splits into a part θE^0 that equipartitions among all the modes $k_{\min} < |k| \le k_{\max}$, and a part $(1-\theta)E^0$ that resides in fluctuations at $|k| = k_{\min}$. From these properties, we see that the theory does not predict a nontrivial coherent structure in the mean. Instead, it puts some energy into a fluctuating eigenmode at the lowest wavenumber. While the equipartition of energy among the high wavenumbers is consistent with the ideal model, the presence of finite fluctuations at the lowest wavenumber violates the principle on which the canonical ensemble rests, since the fluctuations of the energy about its prescribed mean value do not tend to zero as $k_{\max} \to \infty$. In other words, the equivalence of ensembles breaks down. Moreover, the prediction of a zero mean state contradicts the microcanonical ensemble, when E^0 is close to the minimum energy subject to given values of F^0 and K^0. In such cases the microscopic constraint surface becomes disconnected into two components distinguished by the sign of the eigenmode at $k = k_{\min}$. The microcanonical ensemble for one fixed component then produces a nonzero mean state with small fluctuations in every mode. The usual canonical ensemble, however, predicts finite fluctuations around the zero state in the lowest mode.

These considerations motivate our particular formulation of the constraints on flux and cross-helicity in the maximum entropy principle. We base our model on the *separation of scales hypothesis*, which we now articulate precisely. In our discrete model, we separate the modal energy density $\frac{1}{2}\langle |\hat{Y}_k|^2 \rangle$ into mean and fluctuation parts, which we require to satisfy

$$\frac{1}{2}|\langle \hat{Y}_k \rangle|^2 \le \epsilon_k \quad \text{where} \quad \sum_k \epsilon_k \le E^0, \tag{4.8}$$

$$\frac{1}{2}\langle |\hat{Y}_k - \langle \hat{Y}_k \rangle|^2 \rangle \le CE^0/N, \tag{4.9}$$

for a sequence of constants ϵ_k and a constant C, both independent of N as $N \to \infty$. These conditions imply that the given energy $E^0 = \langle E(Y) \rangle$ resides partly in a uniformly square-integrable mean state $\langle Y \rangle$ and partly in fluctuations, where it is partitioned uniformly (up to a constant factor) among all the N modes. For large N, therefore, we see that the separation of scales imposed in (4.8) - (4.9) causes the mean to concentrate in the low wavenumbers, and the fluctuations to spread out to the high wavenumbers. We emphasize that these conditions constitute a hypothesis used to derive the model, not a conclusion deduced from it. Nevertheless, this single hypothesis is the only special ingredient needed to arrive at our model.

A crucial consequence of the separation of scales is the asymptotic smallness of the fluctuations of ψ. Recalling (4.6), we obtain the estimate

$$\begin{aligned} \langle \| \psi' \|_N^2 \rangle &= \sum_k \langle |\hat{\psi}_k'|^2 \rangle \\ &\leq \sum_k |k|^{-2} \langle |\hat{Y}_k'|^2 \rangle \\ &\leq CE^0 N^{-1} \sum_k |k|^{-2} \\ &= O(N^{-1} \log N), \end{aligned} \tag{4.10}$$

where the last sum is evaluated over the two-dimensional lattice of wavevectors $k = k_m$. Here and in what follows we denote fluctuations by a prime, $Y' = Y - \langle Y \rangle$ and so forth. We conclude that in a strong sense the fluctuations of ψ are negligible for large N. Of course, an analogous statement holds for ϕ.

We can now infer the correct form of the constraints on F_i and K_i, by making a formal expansion in ψ around $\langle \psi \rangle$. For the flux constraints we get

$$\begin{aligned} \langle F_i(Y) \rangle &= \frac{1}{N} \sum_n f_i(\langle \psi_n \rangle) + \frac{1}{2N} \sum_n f_i''(\langle \psi_n \rangle) \langle (\psi_n')^2 \rangle + \ldots \\ &= F_i(\langle Y \rangle) + O(N^{-1} \log N), \end{aligned} \tag{4.11}$$

assuming that f_i'' is uniformly bounded over the range of the mean flux function. For the cross-helicity constraints, using a similar but more involved analysis, we obtain

$$\langle K_i(Y) \rangle = \frac{1}{N} \sum_n \langle B_n \cdot V_n \rangle f_i'(\langle \psi_n \rangle) + O(N^{-\frac{1}{2}} (\log N)^{\frac{1}{2}}). \tag{4.12}$$

In view of these expansions, we formulate our statistical equilibrium model by retaining the leading terms for $\langle F_i \rangle$ and $\langle K_i \rangle$, rather than the full expressions for each of them. This procedure, which we prove in Section 4D to be consistent with the desired properties of the ensemble, effectively enforces the separation of scales conditions on the solution $\rho(Y)$ to the maximum entropy problem.

C. Implicit canonical ensemble. We define a statistical equilibrium macrostate $\rho^N(Y)$ to be a solution to the multiconstrained optimization problem

$$S(\rho^N) \rightarrow \max \qquad \text{subject to}$$

$$\frac{1}{2N} \sum_n \langle |B_n|^2 + |V_n|^2 \rangle = E^0, \tag{4.13}$$

$$\frac{1}{N} \sum_n f_i(\langle \psi_n \rangle) = F_i^0, \tag{4.14}$$

$$\frac{1}{N} \sum_n \langle B_n \cdot V_n \rangle f_i'(\langle \psi_n \rangle) = K_i^0. \tag{4.15}$$

Unlike the usual maximum entropy principle, the density ρ^N enters into the constraints (4.14) and (4.15) *nonlinearly*. Consequently, we refer to the solution as the "implicit canonical measure," emphasizing that it is determined by solving the statistical equilibrium equations implicitly in ρ^N. This implicit dependence acts through the mean flux function $\langle\psi\rangle = \mathcal{A}\langle B\rangle$, with the result that the nonlinearity occurs in the mean field equations. The purely quadratic dependence of the constraints (4.13) and (4.15) on the primitive variables B and V results in Gaussian fluctuations. These properties, which reflect the separation of scales, make the problem tractable.

The variational equations for this discrete model parallel those for the continuum model (Section 3A), with the difference that $\rho^N(Y)$ is now a joint probability density on the phase space R^{4N}. In terms of the discrete inner product $(\cdot,\cdot)_N$ and norm $\|\cdot\|_N$, and the linear operators $\mathcal{A}$ and $\mathcal{A}^\dagger$, which are the discretizations of Curl^{-1} and curl^{-1}, we find that

$$\delta S = \frac{-1}{N}\int \log\rho^N \; \delta\rho^N \, dY \tag{4.16}$$

$$\delta E = \int \frac{1}{2}(\| B \|_N^2 + \| V \|_N^2)\, \delta\rho^N \, dY \tag{4.17}$$

$$\delta F_i = \int (B, \mathcal{A}^\dagger f_i'(\langle\psi\rangle))_N \, \delta\rho^N \, dY \tag{4.18}$$

$$\delta K_i = \int (B, f_i'(\langle\psi\rangle)V + \mathcal{A}^\dagger[\langle B\cdot V\rangle f_i''(\langle\psi\rangle)])_N \, \delta\rho^N \, dY. \tag{4.19}$$

As in the continuum calculations, we let $\beta, \alpha_i, \gamma_i$ be the multipliers corresponding to E, F_i, K_i, respectively, and we introduce the profile functions $\lambda(\sigma)$ and $\mu(\sigma)$ defined as in (3.10). We then obtain the (implicit) partition function

$$Z = \int_{R^{4N}} \exp(-\frac{\beta N}{2} Q(Y))dY, \tag{4.20}$$

where Q is the quadratic form in B_n, V_n given by

$$Q = \frac{1}{2}\| B \|_N^2 + \frac{1}{2}\| V \|_N^2 - (B, \mathcal{A}^\dagger\lambda)_N - (B, \mu V + \mathcal{A}^\dagger[\langle B\cdot V\rangle\mu'])_N. \tag{4.21}$$

Here, $\lambda_n = \lambda(\langle\psi_n\rangle)$, and similarly for μ. A straightforward calculation yields the implicit canonical density

$$\begin{aligned} \rho^N(Y) &= \prod_n \rho_n(Y_n) \\ &= \prod_n (\frac{\beta}{2\pi})^2(1-\mu_n^2)\exp(-\frac{\beta}{2}\{|B_n'|^2 + |V_n'|^2 - 2\mu_n B_n'\cdot V_n'\}), \end{aligned} \tag{4.22}$$

with mean field $\langle B\rangle$ and mean flow $\langle V\rangle$ satisfying

$$\langle B\rangle = \mathcal{A}^\dagger(\lambda + \langle B\cdot V\rangle\mu') + \mu\langle V\rangle, \tag{4.23}$$

$$\langle V \rangle = \mu \langle B \rangle. \tag{4.24}$$

(A prime denotes $d/d\sigma$, when applied to a basis function f_i or a profile function λ or μ.)

The decisive result of these calculations is the statistical independence of the Gaussian fluctuations Y_n' at each node $x_n \in D_N$. This property of the maximum entropy solution ρ^N is a direct consequence of the particular form of the constraints (4.13) - (4.15). We see therefore that the desired separation of scales follows from these implicit dynamical constraints. The nodal densities $\rho_n(Y_n)$ are clearly a discretization of the local probability densities $\rho_x(y)$ in the continuum model.

The analysis now follows exactly as in the continuous model. Equations (4.23) and (4.24), being the analogue of (3.12) and (3.13), are reducible to a mean field equation that is the discrete version of (3.22). This equation determines $\langle B \rangle$ (or equivalently $\langle \psi \rangle$) implicitly in terms of the profile functions $\lambda(\sigma)$ and $\mu(\sigma)$, which in turn are determined by the constraints. In equilibrium, the constraints for the discrete model are

$$E(\langle B \rangle, \langle V \rangle) + \frac{2}{\beta N} \sum_n \frac{1}{1-\mu_n^2} = E^0, \tag{4.25}$$

$$F_i(\langle B \rangle) = F_i^0, \tag{4.26}$$

$$K_i(\langle B \rangle, \langle V \rangle) + \frac{2}{\beta N} \sum_n \frac{\mu_n}{1-\mu_n^2} f_i'(\langle \psi_n \rangle) = K_i^0, \tag{4.27}$$

using an analysis as in Section 3A. The comments made in Section 3 about how to solve the statistical equilibrium problem in either its free or constrained form now carry over to this discrete case without any substantive changes.

D. Concentration property. We now complete the justification of our continuum model by showing that as $N \to \infty$ the implicit canonical density ρ^N concentrates around the manifold defined by the microscopic constraints: $E(Y) = E^0, F_i(Y) = F_i^0, K_i(Y) = K_i^0$. In other words, we establish an equivalence between the microcanonical ensemble and our implicit canonical ensemble. This rigorous demonstration complements the formal procedure used in Section 4B to motivate the construction of the implicit canonical measure.

Our proof of the concentration result assumes that the (B_n, V_n) distributions are non-degenerate, in the sense that the correlation μ_n is bounded away from ± 1 uniformly over $x_n \in D_N$ as $N \to \infty$. In the discussion that follows, therefore, we impose the condition that $\max_n |\mu_n| \le \mu_* < 1$ for all N. This limit condition for the discrete model amounts to a regularity condition on the solution to the continuum model, as is evident from the discussion in Section 3. We therefore adopt the view that it can be verified *a posteriori*. Under this condition on the B-V correlation, the mean field-flow $\langle Y_n \rangle$ and the variance $\langle |Y_n'|^2 \rangle$ remain uniformly bounded over D_N as $N \to \infty$. Moreover, since the distributions of the Y_n are Gaussian, all higher moments are similarly bounded in terms of the mean and variance.

We begin our analysis by showing that the variance of ψ_n at each node x_n tends to zero as $N \to \infty$. This property is of independent interest, since it constitutes the discrete version of (2.19). We first express the flux function as a convolution over D_N,

$$\psi_n = \frac{1}{N} \sum_\nu G^N(x_n - x_\nu) \times B_\nu$$

using the Green function

$$G^N(x_n) = \sum_k i|k|^{-2} k e^{ik\cdot x_n}.$$

This formula follows easily from (4.6). Next, we calculate the variance of each component of B_n as $\beta^{-1}(1-\mu_n^2)^{-1}$, using the discrete analogue of (3.15). Since the B_n are statistically independent, we then conclude that

$$\begin{aligned} \operatorname{var}\psi_n &\leq \frac{1}{\beta(1-\mu_*^2)N^2} \sum_\nu |G^N(x_n - x_\nu)|^2 \\ &= \frac{1}{\beta(1-\mu_*^2)N} \sum_k |\hat{G_k^N}|^2 \\ &= O(N^{-1}\log N), \end{aligned} \tag{4.28}$$

using the fact that $|\hat{G_k^N}| = O(|k|^{-1})$.

We claim that the statistical equilibrium density ρ^N satisfies

$$\langle [E(Y) - E^0]^2 \rangle, \ \ \langle [F_i(Y) - F_i^0]^2 \rangle, \ \ \langle [K_i(Y) - K_i^0]^2 \rangle \to 0 \tag{4.29}$$

as $N \to \infty$. This property clearly means that the implicit canonical measure is a consistent approximation to the statistical behavior of the underlying microscopic dynamics, assuming the ergodicity of these dynamics over the microcanonical manifold. We proceed to give the proof of each of these three limits.

The energy functional is treated in a routine fashion, since $Y = (Y_n)$ consists of independent random vectors. Namely,

$$\begin{aligned} \langle [E(Y) - E^0]^2 \rangle &= \langle E(Y)^2 \rangle - \langle E(Y) \rangle^2 \\ &= \frac{1}{4N^2} \sum_n \sum_\nu \langle |Y_n|^2 |Y_\nu|^2 \rangle - \langle |Y_n|^2 \rangle \langle |Y_\nu|^2 \rangle \\ &= \frac{1}{4N^2} \sum_n \langle |Y_n|^4 \rangle - \langle |Y_n|^2 \rangle^2 \\ &= O(N^{-1}). \end{aligned} \tag{4.30}$$

The generalized flux functional is expanded in ψ_n using the estimate (4.28). We assume that the basis functions $f_i'(\sigma)$ are bounded for all $\sigma \in R$, and we obtain

$$\langle [F_i(Y) - F_i^0]^2 \rangle = \langle [\frac{1}{N} \sum_n f_i(\psi_n) - f_i(\langle \psi_n \rangle)]^2 \rangle \tag{4.31}$$

$$
\begin{aligned}
&= \langle [\frac{1}{N}\sum_n f_i'(\tilde{\psi}_n)\psi_n']^2\rangle \\
&\leq \langle [\frac{1}{N}\sum_n f_i'(\tilde{\psi}_n)^2] \max_n(\psi_n')^2\rangle \\
&\leq C \max_n \operatorname{var}\psi_n \\
&= O(N^{-1}\log N),
\end{aligned}
$$

where $\tilde{\psi}_n$ denotes some intermediate value between ψ_n and $\langle\psi_n\rangle$.

The generalized cross-helicity functional is analyzed by combining the techniques used in (4.30) and (4.31). First, we write $K_i(Y) = G_i(Y) + H_i(Y)$, where

$$
G_i(Y) := \frac{1}{N}\sum_n B_n \cdot V_n f_i'(\langle\psi_n\rangle)
$$

$$
H_i(Y) := \frac{1}{N}\sum_n B_n \cdot V_n[f_i'(\psi_n) - f_i'(\langle\psi_n\rangle)].
$$

Then we see that

$$
\begin{aligned}
\langle [K_i(Y) - K_i^0]^2\rangle &= \langle G_i^2\rangle - \langle G_i\rangle^2 + 2\{\langle G_i H_i\rangle - \langle G_i\rangle\langle H_i\rangle\} + \langle H_i^2\rangle \\
&\leq \langle G_i^2\rangle - \langle G_i\rangle^2 + 4\langle G_i^2\rangle^{\frac{1}{2}}\langle H_i^2\rangle^{\frac{1}{2}} + \langle H_i^2\rangle,
\end{aligned}
$$

noting that the constraint is equivalent to $\langle G_i\rangle = K_i^0$.

It suffices therefore to estimate $\langle G_i^2\rangle - \langle G_i\rangle^2$ and $\langle H_i^2\rangle$, the treatments of which follow (4.30) and (4.31), respectively. First, we have

$$
\begin{aligned}
\langle G_i^2\rangle - \langle G_i\rangle^2 &\leq \frac{1}{N^2}\sum_n\{\langle (B_n\cdot V_n)^2\rangle - \langle B_n\cdot V_n\rangle^2\} f_i'(\langle\psi_n\rangle)^2 \\
&\leq \frac{C}{N^2}\sum_n \langle |Y_n|^4\rangle + \langle |Y_n|^2\rangle^2 \\
&= O(N^{-1}).
\end{aligned}
$$

Second, we assume that the basis functions $f_i''(\sigma)$ are bounded for all $\sigma \in R$, and we get

$$
\begin{aligned}
\langle H_i^2\rangle &= \langle [\frac{1}{N}\sum_n B_n\cdot V_n f_i''(\tilde{\psi}_n)\psi_n']^2\rangle \\
&\leq \langle [\frac{1}{N}\sum_n (B_n\cdot V_n)^2 f_i''(\tilde{\psi}_n)^2]\max_n(\psi_n')^2\rangle \\
&\leq C[\frac{1}{N}\sum_n \langle |Y_n|^8\rangle]^{\frac{1}{2}} \max_n \langle (\psi_n')^4\rangle^{\frac{1}{2}} \\
&\leq C\max_n \operatorname{var}\psi_n \\
&= O(N^{-1}\log N).
\end{aligned}
$$

Putting these estimates together, we obtain

$$\langle [K_i(Y) - K_i^0]^2 \rangle = O(N^{-\frac{1}{2}}(\log N)^{\frac{1}{2}}). \tag{4.32}$$

The conditions we place on the basis functions in this analysis are mild. Indeed, these functions define the flux constraints, which actually apply over the range of the mean flux function $\bar{\psi}$. But this range is fixed *a priori* by the initial state Y^0 that determines the constraint values. Consequently, we can merely extend the basis functions $f_i(\sigma)$ to be defined for all $\sigma \in R$ with the desired growth conditions.

E. Continuum limit. We end our analysis of the lattice model by indicating how the statistical equilibrium measures $\rho^N = \prod \rho_n$ converge to the solution $\rho_x(y)$ to the continuous maximum entropy problem (2.24). For each N, we define an x-parameterized probability density $\sigma_x^N(y)$ on $y \in R^4$ by interpolating the discrete measure ρ^N for the lattice model. It suffices to set $\sigma_x^N(y) = \rho_n(y)$ whenever x lies in the $\Delta x_1 \times \Delta x_2$ cell at the node x_n. This piecewise constant macrostate $\sigma_x^N(y)$ then has the same entropy as ρ^N:

$$\begin{aligned} S(\sigma^N) &= -\int_D \int_{R^4} \sigma_x^N(y) \log \sigma_x^N(y)\, dx dy \\ &= -\frac{1}{N} \sum_n \int_{R^4} \rho_n(y) \log \rho_n(y)\, dy \\ &= -\frac{1}{N} \int_{R^{4N}} \rho^N(Y) \log \rho^N(Y)\, dY \\ &= S(\rho^N), \end{aligned}$$

By standard probability theory [3], we can ensure that the densities $\sigma_x^N(y)$ converge in the usual weak sense to a density $\rho_x(y)$ along a sequence $N \to \infty$. Then, by virtue of the continuity of the constraint functionals and the upper semicontinuity of the entropy functional, we can infer that $\rho_x(y)$ is the solution of the continuous maximum entropy problem (2.24). Thus, we can demonstrate that ρ^N converges in an appropriate weak sense to a solution of the continuous problem (2.24). We omit the details of this proof.

5 Conclusions

We have formulated, analyzed and justified a statistical equilibrium model of relaxed states in two-dimensional magnetofluid turbulence that accounts for the complete family of conserved integrals for the ideal dynamics. By adopting the hypothesis of a separation of scales between the microscopic fluctuations and the macroscopic mean inherent in the relaxed state, we are able to describe it in terms of the x-local probability densities $\rho_x(y)$ on the fluctuating field-flow state $Y(x) = (B, V)$. The constrained maximum entropy principle defining our continuum model, which is derived intuitively in section 3 and substantiated rigorously in section 4, determines the most probable macrostate $\rho_x(y)$. We have found that this statistical equilibruim state is a Gaussian probability density, and that its mean is a steady solution of the ideal MHD equations.

The separation of scales condition imposed on our model means that the fluctuations of the magnetic flux function ψ and of the velocity stream function ϕ are vanishingly small compared with those of the magnetic field B and velocity field V. This crucial property implies that the mean field alone contributes to the flux integrals. We therefore conclude that the mean field for the relaxed state satisfies the same flux constraints as the initial state. On the other hand, we find that the energy and cross-helicity integrals for the mean field-flow are, in general, different from those of the initial state. The remaining energy and cross-helecity are transferred to the turbulent Gaussian fluctuations, where they can be considered as lost to dissipation. This phenomenon has an analogue in 2D vorticity dynamics, in which a portion of the (generalized) enstrophy integrals is stored in the (generally non-Gaussian) fluctuations of the vorticity, while the energy resides entirely in the mean vorticity field [29, 30, 25, 26].

To the extent that comparisons between our model and numerical studies of magnetofluid turbulence are meaningful, we have found a rather striking agreement. The detailed studies of 2D magnetofluid turbulence by Biskamp *et al.* [5, 6, 7], in particular, support the conclusions of our model in several respects. First, these simulations confirm the Gaussianity of the local fluctuations in the magnetic and velocity fields. Second, they verify a direct cascade of energy to small scales, and an inverse cascade of (quadratic) flux to large scales. While these nonequilibrium cascades have no place in our equilibrium model, they are strongly indicative of the formation of macroscopic coherent structures. They also support the separation of scales hypothesis on which our continuun model is built. Third, the simulations clearly display the relaxation of the ratio of magnetic to kinetic energy to an equilibrium value greater than 1, even for initial ratios on the order of 0.01. We have deduced this striking effect as a quantitative prediction of our model.

The model also captures the qualitative behavior of the relaxed states over the range of admissible energy values. For small energy values, the simulations of Biskamp *et al.* reveal a dominant, large-scale structure in the mean magnetic field, which forms through the process of quasi-static coalescence of flux tubes. For large values of initial energy, on the other hand, a relaxed state resembles homogeneous turbulence in the sense fluctuations predominate. In the low energy regime the turbulent fluctuations only perturb the coherent structure, while in the high energy regime they effectively eradicated the spatial structure. Our analysis of the statistical equilibrium states has exhibited similar qualitative behavior in these regimes. Indeed, when E^0 is close to its minimum possible value, determined by the deterministic steady state, the relaxed state consists of a macroscopic mean state (a magnetic island with flow) together with small-variance fluctuations. On the other hand, when E^0 is much greater than the minimum possible energy, the variance of the fluctuations is correspondingly large, and so the relaxed state is dominated by turbulence.

In a similar manner, the model predicts the dependence of the turbulent relaxed state on the value of cross-helicity relative to energy. We have shown that the proportionality between the mean fields $\bar{B}$ and $\bar{V}$ is identical with the B-V correlation. For small cross-helicity values, therefore, the equilibria are nearly magnetostatic, and the model becomes a perturbation of flux-conserving equilibrium theory [23]. For large cross-helicity values, however, the mean field is nearly equipartitioned into magnetic and kinetic parts, and the fluctuations in B and V are highly correlated. This result of the model explains the B-V alignment effect observed in numerical simulations [32].

When the generalized flux and cross-helicity integrals (or at least some finite basis of them) and the energy integral are imposed as constraints in the continuum model, the resulting equilibrium equations become analytically untractable. The essential difficulty arises from the coupling between the mean field equation for $\bar{\psi}$ and the inverse temperature-like parameters β and γ_i, which are determined by the energy and generalized cross-helicity constraints. This coupling is strongest when the relaxed state contains relative velocity-magnetic shear in its mean field-flow. The covariance of turbulent fluctuations is then nonconstant across the mean flux surfaces. In this general case, the statistical equilibrium problem itself requires a numerical method of solution. Rather than pursue this topic in the present paper, which is concerned with theoretical issues, we have stated a dual pair of variational principles for the solutions. One principle provides the mean field-flow associated with free solutions, which have prescribed inverse temperature-like parameters. The other principle characterizes those parameters for constrained solutions with a given mean field-flow. With these principles, we have inferred the properties of solutions in the special cases of greatest interest. But they also suggest the form of an iterative method for solving the problem with general constraints, the development of which we defer to future investigations.

Our statistical equilibrium theory has the virtue that it can be formulated directly as a continuum model in x-space. Nonetheless, its full justification requires both a discretization of x-space and a dual representation in k-space. In this way, it can be connected to the traditional concepts of statistical mechanics and turbulence theory. For these reasons, we have constructed a lattice model with N grid nodes, and we have proved that it tends to the continuum model as $N \to \infty$. With the aid of the discrete Fourier transform, we are able to represent the separation of scales hypothesis for the model in a k-space form. This hypothesis then allows us to deduce the constraints for the lattice model, using an approximation consistent with the continuum model. We then define our "implicit canonical ensemble" to be the most probable distribution on the discrete phase space subject to these constraints. It follows that our lattice model itself satisfies the separation of scales condition in the strong sense that the discretized field-flow state is statistically independent from node to node. The equilibrium state is therefore easy to calculate. We find that it shares all of the properties of the relaxed state $\rho_x(y)$ for the continuum model, but on the discrete level.

Even though the lattice model is not determined from the classical Gibbs ensemble, we have proved that it has a concentration property with respect to the microcanonical constraint manifold. In particular, we have estimated the variances of the energy, generalized flux and generalized cross-helicity for our implicit canonical ensemble, and we have established that they go to zero as $N \to \infty$. This property, combined with the demonstrated Liouville property for the discretized dynamics, constitutes a rigorous justification of our model.

There are two aspects of the model and its discretization that we have not addressed. First, it would be theoretically attractive to construct a discrete dynamics that conserves M quantities *exactly* for finite N, with $M \to \infty$ as $N \to \infty$. Instead, our model imposes *approximate* dynamical constraints whose truncation errors go to zero as $N \to \infty$. In this respect, our approach to discretization is similar to the "pseudo-spectral method" [9], which is often used to compute the microscopic evolution. Second, it would be interesting to build lattice models which approach the continuum model asymptotically as $N \to \infty$, but which have nonvanishing spatial correlations for finite N. The drawbacks noted in the case of purely quadratic constraints indicate, however, that some restriction like our separation of scales condition would be required. A model of this kind might give a more realistic picture of the turbulent relaxed state for a slightly dissipative magnetofluid. Among such models, our lattice model would be merely the simplest discretization of the continuous statistical equilibrium problem.

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