# Hyperbases Exist

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**Abstract:** A hyperbasis is a combinatory basis for the lambda calculus which can represent all lambda terms in any infinite set of trees. The usual bases are not hyper. We show that a finite hyperbasis exists.

If **C** is a set of combinators let **C**+ be the set of all applicative combinations of members of **C**. To each member of **C**+ we assign a binary tree as follows; for Aε **C** the tree of A is the one point tree <>, and the tree of (MN) is <tree of M,tree of N> (it is true that this definition is in general ambiguous but the ambiguity is harmless). **C** is said to be a hyperbasis if, for every infinite set of trees **T**, for each combinator N there exists an Mε **C**+ so that M $=_\beta$ N and the tree of M ε **T**.

Example 1. The set {S,K} is not a hyperbasis. Let $\mathbf{T}_1$ = {<>, <<><>>, <<<><>><>>, <<<<><>><>><>>,.....}, then any applicative combination of S, K and x whose tree belongs to $\mathbf{T}_1$ has only head *w* -β reducts which can be written with at most 2 parenthesese. Thus, for example, λx. x(xx)(xx) is not definable this way. Compare this with [ 1 ] 7.4.7.

Example 2. Bohm's one point basis X = λx. xSKS is not a hyper basis. For let $\mathbf{T}_2$ = {<>,<<><>>, <<><<><>>>, <<><<><<><>>>>,.....} and let A be any one point basis. Then the sequence A, AA, A(AA), A(A(AA)),..... must either omit some combinator or repeat (modulo β conversion) for otherwise we can solve recursively the problem of conversion. Thus the sequence either omits some combinator or it is finite (modulo β conversion).

Lemma: Let **C** be a set of combinators containing I and let $\mathbf{T}_1$ and $\mathbf{T}_2$ be as in the previous examples. Suppose that for each i=1,2 and for each combinator N there exists Mε **C**+ such that N $=_\beta$ M and the tree of M belongs to $\mathbf{T}_i$, then **C** is a hyperbasis.

Proof: Let **T** be an infinite set of trees and let T be the union of all the trees in **T.** Since T is infinite ,by Konig's lemma ,T has an infinite path P. We distinguish two cases

Case 1; P has infinitely many steps to the left.

Then there is an infinite sequence of trees T(1),T(2),..., T(n),... so that T(n)ε **T** and T(n) has a path which steps left $l(n) \geq n$. Now each left subtree off any path of T(n) can be β reduced to nothing by substituting I for each of its leaves, and each right subtree off any path in T(n) can be β reduced to the one point tree by substituting I for all of its leaves except the rightmost leaf. Thus we can assume that **T** contains an infinite subset of $\mathbf{T}_1$. By a similar substitution for leaves we can assume that **T** actually contains $\mathbf{T}_1$ and thus by hypothesis for each combinator N there exists an Mε **C**+ such that $N =_\beta M$ and the tree of M belongs to **T.**

Case 2 ; P has only finitely many steps left.

For any tree T define $T^{(n)}$ by $T^{(0)} = T$ and $T^{(n+1)} = \langle T^{(n)}, \langle\rangle\rangle$.
By performing the substitutions of case 1 we can assume that there is an integer m and an infinite sequence T(0),T(1),...,T(n), ... of members of $\mathbf{T}_2$ such that each of the trees $T(n)^{(m)}$ belongs to **T**. Again by substitutions similar to those of case 1 we can assume that for each Tε $\mathbf{T}_2$ the tree $T^{(m)}$ belongs to **T**. Let N be given. By hypothesis there is an M ε **C**+ with tree Tε $\mathbf{T}_2$ so that $\underbrace{K(\ldots(K}_{m}N)\ldots) =_\beta M$. Then $N =_\beta M\underbrace{I\ldots I}_{m}$ and the tree of $M\underbrace{I\ldots I}_{m}$ is $T^{(m)}$ε **T.**

This completes the proof.

Theorem: The set {B,B',C,K,I,W,C*B,C*B',C*C,C*K,C*I,C*W} is a hyperbasis.

Proof: Let the designated set of combinators be **C**. The proof consists in first showing that for each N there exists an Mε {B,B',C,K,I,W}+ so that $N =_\beta M$ and the tree of M belongs to $\mathbf{T}_1$. Next we observe that $A_1 \ldots A_n =_\beta C^* A_n(\ldots(C^* A_2 A_1)\ldots)$ and the theorem follows from the lemma. Let N be given. By Church's theorem there exists an applicative combination P of B,C,K,I,W such that $N =_\beta P$. Now we prove by induction on P that $P =_\beta$

$A_1 \ldots A_n$ for $A_i \varepsilon \{B,B',C,K,I,W\}$. Indeed we have $A_1 \ldots A_n(A'_1 \ldots A'_m) =_\beta B(A_1 \ldots A_n)(A'_1 \ldots A'_{m-1})A'_m =_\beta BB'A_1A_2B'A_3 \ldots B'A_{n-1}BA_n(A'_1 \ldots A'_{m-1})A'_m$. This gives the induction step and completes the proof.

[1]Barendregt , The Lambda Calculus
North Holland 1984