Characterizations of the Finite Cover Property and Stability

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Abstract

Let T be a first order theory, and let $\lambda \geq |T|^+$. T is called almost categorical in λ iff there exists an expansion T_1 of T (of cardinality $\leq |T| + \aleph_0$) such that for every $M \models T_1$ of cardinality λ if M is $|T|^+$-saturated then $M \restriction L(T)$ is saturated.

Theorem A. *Let T be a complete theory. The following are equivalent:*
(1) T does not have the finite cover property (f.c.p.).
(2) $\forall \lambda \geq |T|^+$ T is almost categorical in λ.
(3) $\exists \lambda > 2^{|T|}$ such that $\lambda^{|T|} = \lambda$ and T is almost categorical in λ. (e.g. $\lambda = (2^{|T|})^+$)

Let $\mu \geq |T|$ we say that the condition $(*)_{\mu,T}$ holds iff there exists an expansion T_1 of T (of cardinality $\leq |T| + \aleph_0$) such that for every $M \models T_1$ if M is μ^+-saturated then $M \restriction L(T)$ is 2^μ -saturated.

Theorem B. *Suppose that there exists a cardinality $\mu \geq |T|$ such that $2^\mu > \mu^+$. For a complete theory T, the following are equivalent:*
(1) T is stable.
(2) $\forall \mu \geq |T|$ such that $\mu^+ < 2^\mu$, $(*)_{\mu,T}$ holds.
(3) $\exists \mu \geq |T|$ such that $\mu^+ < 2^\mu$ and $(*)_{\mu,T}$ holds.

We also present a local version and discuss some related questions.

1 Introduction

This paper considers the characterization of properties of first order theories T in a language L in terms of certain saturation properties of expansions of models of T to a language L_1. The finite cover property and stability are each characterized in this way.

The finite cover property was introduced by Keisler in [Ke], to produce unsaturated ultrapowers. In some sense our result could be viewed as a continuation of his study. Despite the fact that ultraproducts are not mentioned in the statement of Theorem A, they do play a central role in its proof. The finite cover property was studied extensively by Shelah in [Sh 10] and [Sh c] (mainly in §4 of Chapter II). The construction of the theories T_1 here uses technology developed by Shelah, mainly in Chapter VI and §4 of Chapter II. The other direction of the proof relies on the following theorem of Keisler:

Theorem 6.1.8 (of [CK]) *If D is an $\aleph_1$-incomplete good ultrafilter on I then M^I/D is $|I|^+$-saturated.*

Our notation generally follows [Sh c] with a few minor exceptions: $|T|$ is the number of symbols in $|L(T)|$ plus $\aleph_0$. We do not distinguish between finite sequences and elements, i.e. we write $a \in A$ to represent that the elements of the finite sequence a are from the set A. References of the form IV x.y are to [Sh c].

There are several equivalent formulations of the finite cover property, we prefer to state the one that looks like a strengthening of the compactness theorem.

Definition 1.1 *T does not have the finite cover property iff for every $\phi(x;y) \in L(T)$ there exists $k := k(\phi) < \omega$ such that for every set of parameters A and every $p \subseteq \{\phi(x;a), \neg\phi(x;a) | a \in A\}$ we have that the following implication holds*
if $\forall q \subseteq p[|q| \leq k \Rightarrow q$ is consistent] then p is consistent.

Fact 1.2 *Let T be a complete first order theory.*

1. (Theorem II 4.2(1)) *If T is stable then T does not have the f.c.p.*
2. (Theorem II 4.4) *Suppose T is stable, the following are equivalent:*

(a) T does not have the f.c.p.

(b) There exists a formula $\phi(x; y; z) \in L(T)$ and there exists $\{a_n : n < \omega\}$ such that for every a, $\phi(x; y; a)$ is an equivalence relation and for every $n < \omega$ we have that

$$n < |\mathfrak{C}/\phi(x; y; a_n)| < \aleph_0.$$

Definition 1.3 *Let M be an L-structure, $\Delta \subseteq_{finite} L$ and $\lambda \geq |L|$.*

1. *M is (Δ, λ)-saturated iff for every $A \subseteq M$ of cardinality less than λ, every $p \in \mathbf{S}_\Delta(A, M)$ is realized in M.*

2. *M is λ-locally saturated iff for each $\Delta \subseteq_{finite} L$, M is (Δ, λ)-saturated.*

The following easy consequence of the definition of the finite cover property will be used:

Fact 1.4 *Let T be a complete first order theory without the f.c.p. If $M \models T$ is a locally saturated model then for every $\Delta \subseteq_{finite} L(T)$ there exists an integer k_Δ such that if $I \subseteq M$ is a set of Δ-indiscernibles of cardinality at least k_Δ then there exists $J \subseteq M$ a set of Δ-indiscernibles extending I of cardinality $\|M\|$.*

The major new concept of this paper is 'almost categoricity'.

Definition 1.5 *Let T be a first order theory, and let $\lambda \geq |T|^+$. T is called almost categorical in λ iff there exists an expansion T_1 of T (of cardinality $\leq |T| + \aleph_0$) such that for every $M \models T_1$ of cardinality λ, if M is $|T|^+$-saturated then $M \restriction L(T)$ is saturated.*

In the second section we prove Theorem A; in the third section we discuss some related issues and generalizations. Theorem B is proved in the fourth section; we present a local version and some other formulations in the fifth section.

2 Proof of Theorem A

Theorem A. *Let T be a complete theory. The following are equivalent:*
(1) T does not have the finite cover property (f.c.p.).
(2) $\forall\lambda \geq |T|^+$ T is almost categorical in λ.
(3) $\exists\lambda > 2^{|T|}$ such that $\lambda^{|T|} = \lambda$ and T is almost categorical in λ. (e.g. $\lambda = (2^{|T|})^+$)

Since it is obvious that (2) $\Rightarrow$ (3), we need only to prove that (1) $\Rightarrow$ (2), and that (3) $\Rightarrow$ (1). This is carried out in the following two subsections:

2.1 Proof of (1) $\Rightarrow$ (2):

Let T be a theory without the f.c.p.. By Fact 1.2(1) we know that T is stable. Let $\lambda \geq |T|$ be given, in order to show that T is almost categorical in λ it suffices to find an expansion T_1 of T as in the definition of almost categoricity.

Let $L_1 := L(T)\bigcup\{F\}$ when F is a binary function symbol. The theory T_1 consists of T and the following axioms:

1. For each x, the function $F(x, \cdot)$ is injective.
2. For every finite $\Delta \subseteq L(T)$, let k_Δ be the integer from Fact 1.4, if I is a finite set of Δ-indiscernibles of cardinality at least k_Δ then there exists an x_I such that
   (a) the range of $F(x_I, \cdot)$ contains I, and
   (b) the range of $F(x_I, \cdot)$ is a set of Δ-indiscernibles.

It should be clear that the above axioms can be formulated in first order logic in the similarity type L_1.

Claim 2.1 *The theory T_1 is consistent.*

Proof: By stability of T, T has a saturated model M of cardinality $2^{|T|}$. By the compactness theorem it is enough to show that every $T^* \subseteq_{finite} T_1$

has a model. Let $\Delta \subseteq_{finite} L(T)$ be large enough to contain all the $L(T)$-formulas appearing in T^*. We will show that M can be expanded to a model of T^*.

Fix a bijection $G : S_{<\aleph_0}(Fml(L(T)) \times S_{<\aleph_0}(|M|) \to |M|$.
Let $I \subseteq_{finite} M$ be an arbitrary finite set of Δ-indiscernibles. By Fact 1.4 if I is a finite set of Δ-indiscernibles of cardinality at least k_Δ then there exists $J \subseteq M$ of cardinality $\|M\|$ which is a set of Δ-indiscernibles extending I. Let $a_I := G(\Delta, I)$, and pick a bijection $F^M(a_I, \cdot) : |M| \to J$. Verify that $\langle M, F^M \rangle \models T^*$. ■

Now suppose that $N^* \models T_1$ is a $|T|^+$-saturated model of cardinality λ, let N be the reduct of N^* to $L(T)$. It is enough to show that for every $A \subseteq N$ of cardinality less than λ and every 1-type p over the set A, p is realized in N. Using the stability of T there exists $p^* \in S(|N|)$ which is a non forking extension of p (over the set A). There exists $B \subseteq N$ such that p^* is the unique nonforking extension (over B) of the type $p^* \restriction B$, and $|B| < \kappa(T) \leq |T|$.

Using the $|T|^+$-saturation (in fact we are using are using only $\kappa(T)$-saturation) of N choose by induction on $n < \omega$) a sequence $\{b_n : n < \omega\} \subseteq M$ such that for every $n < \omega$ the type:
$tp(b_n, B \bigcup \{b_k : k < n\})$ is the nonforking extension of $p^* \restriction B$. Note that by the uniqueness of nonforking extensions (of the type $p^* \restriction B$) we have that $I := \{b_n : n < \omega\}$ is a set of indiscernibles.

Claim 2.2 *In order to find a realization of the type p in the model N it is enough to find J with $I \subseteq J \subseteq N$ of cardinality λ which is a set of indiscernibles over the empty set.*

Proof: Since N is $|T|^+$-saturated, in particular it is $(\kappa(T) + \aleph_1)$-saturated. Since our hypothesis is that $dim(I, N) = dim(J, N) = \lambda$. We are done by applying the argument in the proof of Lemma III 3.10(1). ■

In order to finish this stage we need to verify that the hypothesis of the previous Claim holds (this is where the expansion T_1 is used), namely it suffices to prove:

Claim 2.3 *For every infinite set $I := \{b_n : n < \omega\} \subseteq N$ of indiscernibles, there exists a set of indiscernibles $J \subseteq N$ of cardinality λ extending I.*

Proof: Let us consider the following set of formulas:
$q(x) := \{\exists y[b_n = F(x, y)] : n < \omega\} \bigcup$
$\{[$" the range of $F(x, \cdot)$ is a set of Δ-indiscernibles"$] : \Delta \subseteq_{finite} L(T)\}$.

Note that since in the second half of the definition of $q(x)$ we limit attention to finite Δ, $q(x)$ is a set of first order formulas in $L(T_1)$. Since the model N has cardinality λ, from the axioms of T_1 it is clear that if $a \in N$ realizes the type $q(x)$ then $J := \{F(a, b) : b \in N\}$ is as required.

Since N^* is $|T|^+$-saturated, in particular it is $\aleph_1$-saturated. To show that $q(x)$ is realized in N^* it is sufficient to show that $q(x)$ is consistent. This is done by verifying that every finite $q^* \subseteq q(x)$ is realized in N^*. Let $q^* \subseteq_{finite} q(x)$ be given. Let Δ be a finite subset of $L(T)$ such that all the $L(T)$-formulas from q^* appear in Δ. Let $m < \omega$ be sufficiently large so that all the elements of I appearing in q^* are among $\{b_0, \ldots, b_{m-1}\}$, and $m \geq k_\Delta$.

It is enough to show that the following is true:

$$N^* \models \exists x[\bigwedge_{n<m} \exists y[b_n = F(x, y)] \wedge \exists y_0 \cdots \exists y_{m-1}[\bigwedge_{i<j} y_i \neq y_j \wedge$$

$$\text{"}\{F(x, y_0), \ldots, F(x, y_{m-1})\}\text{is a set of } \Delta\text{-indiscernibles"}]].$$

Since the last sentence is an instance of an axiom of T_1, by the assumption that $m \geq k_\Delta$ there exists an element $a \in N^*$ satisfying the above. ■

2.2 Proof of (3) ⇒ (1):

Let $\lambda > 2^{|T|}$ be such that $\lambda = \lambda^{|T|}$ and T is almost categorical in λ; suppose that $T_1 \supseteq T$ witnesses almost categoricity of T.

Claim 2.4 *The theory T is stable.*

Proof: Suppose T is unstable. Since $\lambda = \lambda^{|T|} > 2^{|T|}$, by Theorem VIII 3.2 there are $\{M_i^1 \models T_1 : i < 2^\lambda\}$ such that for every $i < 2^\lambda$ $\|M_i^1\| = \lambda$, and

M_i^1 is $|T|^+$-saturated, but $i \neq j \Rightarrow M_i^1 \restriction L(T) \not\cong M_j^1 \restriction L(T)$. Namely, at most one of the models $M_i^1 \restriction L(T)$ can be saturated, this is a contradiction to the hypothesis of almost categoricity.

For the sake of contradiction suppose that T does have the f.c.p. By the previous claim, we have established that T is stable. We can apply Fact 1.2(2). Fix $\phi(x;y;z) \in L(T)$ and $\{a_n : n < \omega\}$ such that for every a, $\phi(x;y;a)$ is an equivalence relation, and for every $n < \omega$ we have
$n < |\mathfrak{C}/\phi(x;y;a_n)| < \aleph_0$.

Let $M \models T_1$ be a model of cardinality λ containing the set
$\{a_n : n < \omega\}$. We use M to construct another model M^* of T_1 also of cardinality λ such that M^* is $|T|^+$-saturated but its reduct to $L(T)$ is not even $(2^{|T|})^+$-saturated (recall that $\|M\| = \lambda > 2^{|T|}$). Let $I := |T|$, by Kunen's theorem (see [Ku], or Theorem 6.1.4 in [CK]) there exists an $\aleph_1$-incomplete good ultrafilter D on I. Define $M^* := M^I/D$. By Theorem 6.1.8 (of [CK]) the model M^* is $|T|^+$-saturated. The cardinality of M^* is λ, since $\lambda = \|M\| \leq \|M^*\| \leq \|M\|^{|T|} = \lambda^{|T|} = \lambda$.

Since T is stable, applying Fact 1.2 ii), there is a formula $\phi(x;y;z)$ such that for each $a \in N_0$, $\phi(x;y;a)$ defines an equivalence relation and for arbitrarily large n, there exists a_n such that $\phi(x;y;a_n)$ has n equivalence classes. We define below a sequence $\overline{a} = \langle \overline{a}[i] : i < \tau \rangle$ of elements of M such that each $\overline{a}[i]$ is one of the a_n and, using $\aleph_1$-incompleteness, there is no $n < \omega$ such that $\{i : \overline{a}[i] = a_n\}$ is in D. Let $\overline{a}$ also denote the element it determines in M^*. Then, as we show in detail below, $\phi(x;y;\overline{a})$ has infinitely many equivalence classes. Clearly, $\phi(x;y;\overline{a})$ has at most 2^τ equivalence classes and so (since $|M^*| > 2^\tau$), $M^* \restriction L$ is not $(2^\tau)^+$-saturated. Here are the details.

For every $N \models T$, and every $a \in N$ denote by $ecn(a, N)$ the cardinality of $|N/\phi(x;y;a)|$, i.e. it is the number of equivalence classes determined by the equivalence relation $\phi(x;y;a)$ in N. In order to show that $M^* \restriction L(T)$ is not $(2^{|T|})^+$-saturated it is enough to show that there exists $\overline{a} \in M^*$ such that $\aleph_0 \leq ecn(\overline{a}, M^*) \leq 2^{|T|}$. [[Why? Suppose that $\{b_i : i < \alpha \leq 2^{|T|}\} \subseteq M^*$ is a complete set of representatives of the equivalence relation $\phi(x;y;\overline{a})$. Namely we have $i \neq j \Rightarrow M^* \models \neg\phi[b_i, b_j, \overline{a}]$ and for every $b \in M^*$ there exists $i < \alpha$ such that $M^* \models \phi[b, b_i, \overline{a}]$. We just established that the type $p(x) := \{\neg\phi(x, b_i, \overline{a}) : i < \alpha\}$ is omitted by M^*. (Note that α needs to be infinite for $p(x)$ to be consistent.)]]

Since D is $\aleph_1$-incomplete, fix $\{I_n : n < \omega\} \subseteq D$ such that for every $n < \omega$ we have that $I_{n+1} \subseteq I_n$, and $\bigcap_{n<\omega} I_n = \emptyset$.

Define $\overline{a} : I \to |M|$. Given $i \in I$ let

$$\overline{a}[i] := \begin{cases} a_n & \text{if } n = \max\{m \in \omega \ : \ i \in I_m\} \\ a_0 & \text{if } i \in I - I_0. \end{cases}$$

Note that by the assumption on $\{I_n : n < \omega\}$, $\overline{a}$ is well defined. Let $n_\zeta := ecn(\overline{a}[\zeta], M)$. Since $\phi(x; y, \overline{a})$ is an equivalence relation, by the definition of the ultraproduct M^* we have:

$$ecn(\overline{a}, M^*) \leq \prod_{\zeta<|T|} ecn(\overline{a}[\zeta], M) = \prod_{\zeta<|T|} n_\zeta \leq \prod_{\zeta<|T|} \aleph_0 = \aleph_0^{|T|} = 2^{|T|}.$$

To finish that, it is enough to show that $ecn(\overline{a}, M^*)$ is infinite. Suppose to the contrary $k < \omega$ satisfies $k = ecn(\overline{a}, M^*)$. Namely

$$M^* \models \exists x_0 \cdots \exists x_{k-1}[\bigwedge_{i \neq j} \neg\phi(x_i; x_j; \overline{a}) \ \wedge \ \forall y \bigvee_{i<k} \phi(x_i; y; \overline{a})].$$

By the definition of M^* we have that the set

$$\{\zeta \in I : M \models \exists x_0 \cdots \exists x_{k-1}[\bigwedge_{i \neq j} \neg\phi(x_i; x_j; \overline{a}[\zeta]) \ \wedge \ \forall y \bigvee_{i<k} \phi(x_i; y; \overline{a}[\zeta])]\}$$

is in D. Denote by J_k the intersection of the last set with I_k. Clearly $J_k \in D$. By the definition of $\overline{a}[\zeta]$, it is a_l for some $l \geq k$ (when $\zeta \in I_k$). Pick any $\zeta \in J_k$. By the above we have shown that $k = ecn(\overline{a}[\zeta], M) = ecn(a_l, M) > l \geq k$. This is a contradiction . ■

3 Extensions and Limitations to Theorem A

We first note a peculiarity caused by the implication in the definition of almost categorical. In essence, this remark means that almost categoricity in λ is interesting only for λ which satisfy $\lambda^{|T|} = \lambda$.

Lemma 3.1 *Let T be a complete first order theory. If $|T| \leq \lambda < \lambda^{|T|}$, then T is almost categorical in λ.*

Proof: There is an expansion T_1 of T with $|T_1| = |T|$ such that every $|T|^+$-saturated model of T_1 with cardinality at least λ has cardinality at least $\lambda^{|T|}$. Thus T is vacuously almost categorical in power λ. To find T_1, add to T a binary function and let T_1 assert that for every pair of finite sets I, J of the same cardinality there is an $x_{I,J}$ such that $F(x_{I,J}, \cdot)$ maps I one-one onto J. Now if $N_1 \models T_1$, for each pair b, c of injections from $|T|$ into N_1 there is a type $q_{b,c}(x)$ asserting that $F(x, \cdot)$ maps the range of b one-one onto the range of c. If N_1 is $|T|^+$-saturated, for each b, c we can choose $a_{b,c}$ in N_1 to realize $q_{b,c}$. Fix a particular b_0. Then the collection of $a_{b_0,c}$ as c ranges through injections of $|T|$ into N_1 is a subset of N_1 with cardinality $\lambda^{|T|}$. $\blacksquare_{3.1}$

The proof of Theorem A yields somewhat more than is asserted by the definition of almost categoricity in λ. The theory T_1 which is found in the implication i) implies ii) does not depend on λ.

The notion of almost categoricity in λ concerns three cardinal numbers: λ, $|T_1|$, and κ - the amount of saturation required. We examine the effect of varying each of these.

It is natural to ask whether in the definition of almost categoricity, the requirement of $|T|^+$-saturation could be weakened. Careful examination of the proof shows that in the choice of the sequence of the b_i, $\kappa(T)$ saturation is needed while the choice of the parameter a requires $\aleph_1$-saturation. So we could replace $|T|^+$-saturation by $\kappa(T) + \aleph_1$-saturation. This observation yields the following result.

Theorem 3.2 *Suppose that T is superstable. The following are equivalent.*

1. *T does not have the f.c.p.*
2. *There exists an expansion T_1 of T by a single function symbol such that for every $\aleph_1$-saturated model $M \models T_1$, the model $M \restriction L(T)$ is saturated.*

Since we used only one additional function symbol in Theorem A, the key point in Theorem 3.2 is the restriction to $\aleph_1$. But further improvement is not possible in general. Indeed, Theorem VIII.3.5 shows that for a countable stable but not superstable theory T and any extension T_1 of T, there are many $\aleph_0$-saturated models in power $\lambda > \aleph_1$ whose reducts are not saturated.

The requirement that $|T_1| = |T|$ is made solely to simplify notation. Clearly the direction (1) implies (2) is just made easier if the cardinality of T_1 is allowed to increase. (2) implies (3) is completely unaffected and we showed in Theorem 2.2 that (3) implies (1) holds with no bound on $|T_1|$.

The methods used here are similar to those of E. Casanovas [Ca]. He defines a model to be expandable if every consistent expansion of $\mathrm{Th}(M)$ with at most $|M|$ additional symbols can be realized as an expansion of M. His results are orthogonal to those here. He shows for countable stable T that T has an expandable model which is not saturated of cardinality greater than the continuum if and only if T is not superstable or T has the finite cover property.

4 Proof of Theorem B

We introduce the following technical notation.

Definition 4.1 *Let $\mu \geq |T|$; we say that the condition $(*)_{\mu,T}$ holds iff there exists an expansion T_1 of T (of cardinality $\leq |T| + \aleph_0$) such that for every $M \models T_1$ <u>if</u> M is μ^+-saturated <u>then</u> $M \restriction L(T)$ is 2^μ -saturated.*

Theorem B. *Suppose that there exists a cardinal $\mu \geq |T|$ such that $2^\mu > \mu^+$. For a complete theory T, the following are equivalent:*
(1) T is stable.
(2) $\forall \mu \geq |T|$ such that $\mu^+ < 2^\mu$, $(*)_{\mu,T}$ holds.
(3) $\exists \mu \geq |T|$ such that $\mu^+ < 2^\mu$ and $(*)_{\mu,T}$ holds.

Since it is obvious that (2) $\Rightarrow$ (3), we need only to prove that (1) $\Rightarrow$ (2), and that (3) $\Rightarrow$ (1). This is carried out in the following two subsections:

4.1 Proof of (1) $\Rightarrow$ (2):

Proof: Suppose T is a stable theory, and let $\mu \geq |T|$ be such that $\mu^+ < 2^\mu$. It suffices to find an expansion T_1 of T as in the definition of $(*)_{\mu,T}$.

Add one additional binary predicate $E(x, y)$ and axioms asserting that E codes all finite sets. (I.e. for every set of k elements x_i there is a unique y such that $E(z, y)$ if and only if z is one of the x_i.) For any model M_1 of T_1 and any element b of M_1, let $[b] := \{a \in M_1 : M_1 \models E[a, b]\}$.

Now let M_1 be a μ^+-saturated model of T_1 and M the reduct of M_1 to L. Suppose $A \subseteq M$ has cardinality less than 2^μ and $p \in S^1(A)$. We must show p is realized in M. By the definition of $\kappa(T)$ there exists $B \subseteq A$ of cardinality less than $\kappa(T)$ such that p does not fork over B. Let $\hat{p} \in S(M)$ be an extension of p that does not fork over B. Since $\mu^+ > |T| \geq \kappa(T)$ by μ^+-saturation of M there exists $I := \{a_n : n < \omega\} \subseteq M$ such that $a_n \models \hat{p} \restriction (B \cup \{a_k : k < n\})$. We now define a T_1-type q over I so that if b realizes q, $[b]$ is a set of indiscernibles equivalent to I:

$q(x) := \{\exists y_0 \cdots \exists y_{n-1}[\bigwedge_{i<j<n} y_i \neq y_j \wedge \bigwedge_{i<n} E(y_i, x) | n < \omega\} \cup$

$\{\forall y_0 \cdots \forall y_{n-1}[[\bigwedge_{i<j<n} y_i \neq y_j \wedge \bigwedge_{i<n} E(y_i, x) \rightarrow$

$\{y_0, \ldots, y_{n-1}, a_0, \ldots, a_{n-1}\}$ is a set of ϕ-indiscernibles $| \; n < \omega, \phi \in L(T)\}$.

Since the relation E codes finite sets, and I is a set of indiscernibles q is a T_1-type. By the $\aleph_1$-saturation of M_1 there exists $b \in M$ realizing the type q. If $[b]$ has 2^μ elements we are finished since for each formula $\phi(x, \overline{y})$ and each $\overline{a} \in A$ with $\phi(x, \overline{a}) \in p$, only finitely many elements of $[b]$ satisfy $\neg\phi(x, \overline{a})$. To show $[b]$ is big enough define types q_σ for $\sigma \in 2^{<\mu}$. For $\sigma, \tau \in 2^{<\mu}$ we denote 'τ is an initial segment of σ' by $\sigma \lessdot \tau$. The requirements are: If c_σ (c_τ) realizes q_σ (q_τ):

1. $[c_\sigma] \subseteq [b]$
2. $[c_\sigma]$ is a nonempty set of indiscernibles realizing the same types over the empty set as I.
3. $\sigma \lessdot \tau$ implies $[c_\sigma] \supseteq [c_\tau]$.
4. If σ and τ are incomparable $[c_\sigma] \cap [c_\tau] = \emptyset$.

Now for $s \in 2^\mu$ let $X_s = \bigcap_{\sigma \lessdot s}[c_\sigma]$. Each X_s is nonempty by μ^+-saturation; they are disjoint by construction. But any member of $\bigcup_{s \in 2^\mu} X_s$ is in $[b]$ so we finish. $\blacksquare_{4.2}$.

4.2 Proof of (3) $\Rightarrow$ (1):

For the amusement of the reader we will present two different arguments. The first is short, but depends on "heavy artillery"-namely Theorem VIII 3.2; the second argument is much more elementary. Note that the first argument could be adapted to deduce (3) implies (1) in Theorem A.

First Proof: For the sake of contradiction suppose T is an unstable theory and that there exists $\mu \geq |T|$ such that $\mu^+ < 2^\mu$ and there is a $T_1 \supseteq T$ such that for every model M of T_1 which is μ^+-saturated its reduct to $L(T)$ must be 2^μ-saturated. By the instability of T, since $(2^\mu)^{<\mu^+} = 2^\mu > \mu^+ > |T|$. We can apply Theorem VIII 3.2 to get $\{M_i^1 \models T_1 : i < 2^{(2^\mu)}\}$ all μ^+-saturated of cardinality 2^μ, such that $i \neq j \Rightarrow M_i^1 \restriction L(T) \not\cong M_j^1 \restriction L(T)$. However by $(*)_{\mu,T}$, for every i, the model $M_i^1 \restriction L(T)$ is 2^μ-saturated.

Second Proof: For the sake of contradiction suppose T is an unstable theory and that there exists $\mu \geq |T|$ such that $\mu^+ < 2^\mu$ and there is a $T_1 \supseteq T$ such that if M is a μ^+-saturated model of T_1 its reduct to $L(T)$ is 2^μ-saturated.

Fix M_0 which is a μ^+-saturated model of T_1. To ease notation, we will use a, b to denote k-tuples in this argument and write $<$ for the $2k$-ary formula which witnesses the order property in T. Thus, without loss of generality we may assume that $<$ defines a dense linear order on a subset of M_0. Let κ denote μ^+. Now define by induction a sequence $< M_i, a_i, b_i >$ for $i < \kappa$ such that:

1. The M_i are a continuous increasing chain of κ-saturated models.
2. $a_i, b_i \in M_{i+1} - M_i$.
3. $a_i < a_{i+1} < b_{i+1} < b_i$.
4. $[a_{i+1}, b_{i+1}] \cap M_i = \emptyset$.

The only difficult part of the construction is to guarantee clause 4) but this is done by choosing a_i, b_i to realize the same $<$-cut in M_i. Now, let M be the union of the M_i. Since κ is regular, M is κ-saturated. But, M is not even κ^+-saturated since the type

$$p = \{a_i < x < b_i : i < \kappa\},$$

is consistent but not realized. (If it were realized it would be realized in some M_i but this would contradict clause 4.)

5 A local characterization

Since Fact 1.4 depends only on local saturation rather than saturation, we can get a local version of Theorem A. We also characterize stability in terms of the spectrum of locally saturated models.

Theorem 5.1 *Let T be a complete first order theory. If T is stable then for every $\mu \geq |T|$ there exists a locally saturated model of cardinality μ.*

Proof: Let $|M| = \lambda \geq |T|$. For a given set X of stationary types over a subset of M define a new model $M'(X)$ as follows. For each finite Δ and each definition of a Δ-type over M fix a complete type p in $S(M)$ with that definition for $p \restriction \Delta$. Let $M'(X)$ realize the nonforking extension to M of each type in X and each of the types over M just specified. Using Theorem I.2.2 (if T is stable then for every finite Δ $|S_\Delta(M)| \leq \|M\| = \lambda$) there exists $M'(X)$ as above of cardinality λ.

Now define M_i for $i < \lambda$ by induction. $M_0 = M$; each M_{i+1} is $M'_i(X)$ where X is the set of types fixed in constructing M_i. Take unions at limits. Note that M_λ has cardinality λ. Now, we claim that M_λ is locally saturated. For, let A be a subset of M with $|A| < \lambda$, Δ an arbitrary finite set of formulas and $p \in S_\Delta(A)$. Let $\hat{p} \in S(M_\lambda)$ be a nonforking extension of p. The $\hat{p} \restriction \Delta$ is defined over some M_i for $i < \lambda$ and so some type $q \in S(M)$ with $q \restriction \Delta = \hat{p} \restriction \Delta \restriction M$ is chosen at stage i. Thus a sequence of λ indiscernibles realizing q has been chosen in M_λ. Since T is stable each formula in p is satisfied by all but finitely many of members of this sequence, so one of these points must realize p. $\blacksquare_{5.1}$

Corollary 5.2 *(*Assume that $\exists\kappa\ \ 2^\kappa > (2^{<\kappa})^+ \geq |T|$.*) For a complete theory T the following are equivalent:*

1. *T is stable.*

2. *for every $\mu \geq |T|$ and every A such that $|A| \leq \mu$, T has a locally saturated model $M \supseteq A$ of cardinality μ.*

3. *For some μ satisfying $2^\kappa > \mu > 2^{<\kappa}$, for every A of cardinality less or equal to μ there exists a locally saturated model $M \supseteq A$ of cardinality $|A|$.*

Remark 5.3 The following particular case may clarify the last statement:

Assuming that the continuum hypothesis is false then we have shown that for T countable: T has a locally saturated model of cardinality $\aleph_1$ iff T is stable (just substitute $\kappa = \aleph_0$ in part 3.).

Proof: (1)$\Rightarrow$(2): In the previous proof start with $M_0 \supseteq A$.
(2)$\Rightarrow$(3): Trivial.
$\neg$(1)$\Rightarrow\neg$(3): Since T is unstable, by Theorem II 2.13(3) there exists an unstable formula $\phi(x;y)$. By Theorem II 2.2(5) there exists $A := \{a_\eta : \eta \in^{\kappa>} 2\}$ s.t. for every $\eta \in^\kappa 2$, the type $p_\eta := \{\phi(x; a_{\eta\restriction\beta})^{\eta[\beta]} | \beta < \kappa\}$ is consistent. Since $|A| = 2^{<\kappa} < \mu$, if there exists a locally saturated model $M \supseteq A$ of cardinality μ all the types in $\mathbf{S}_\phi(A)$ should be realized in M. But since

$$|\mathbf{S}_\phi(A)| \geq |\{p_\eta | \eta \in^\kappa 2\}| = 2^\kappa > \mu,$$

we have a contradiction. ■

Definition 5.4 *Let T be a first order theory, and let $\lambda \geq |T|$. We say that T is almost locally categorical in λ iff there exists an expansion T_1 of T (of cardinality $\leq |T| + \aleph_0$) such that for every $M \models T_1$ of cardinality λ if M is $|T|^+$-local saturated then $M \restriction L(T)$ is λ-locally saturated.*

Now slightly varying our earlier arguments we have the following.

Theorem 5.5 *Let T be a complete theory. The following conditions are equivalent:*

1. *T does not have the finite cover property (f.c.p.).*

2. *$\forall\lambda \geq |T|^+$ T is almost categorical in λ.*

3. *$\exists\lambda > 2^{|T|}$ such that $\lambda^{|T|} = \lambda$ and T is almost categorical in λ.*

4. *$\forall\lambda \geq |T|^+$ T is almost locally categorical in λ.*

5. *$\exists\lambda > 2^{|T|}$ such that $\lambda^{|T|} = \lambda$ and T is almost locally categorical in λ. (e.g. $\lambda = (2^{|T|})^+$).*

The following notation enables us to show the connection of the two main results.

Definition 5.6 *The property $P_T(\mu, \kappa)$ holds if there is an expansion T_1 of the theory T such that if M is a μ-saturated model of T_1, $M \restriction L(T)$ is κ-saturated.*

Now we can restate Theorems A and B as

Theorem 5.7 *1. T does not have the f.c.p. if and only if $(\forall\lambda \geq |T|)P_T(|T|^+, \lambda)$ iff $\exists\lambda > 2^{|T|}\ \lambda^{|T|} = \lambda\ \&\ P_T(|T|^+, \lambda)$.*

2. Assume: $\exists\kappa\ \ 2^\kappa > (2^{<\kappa})^+ \geq |T|$. T is stable if and only if $(\exists\mu)(\mu^+ < 2^\mu)\ \&\ P_T(\mu, 2^\mu)$.

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