Behavior at a Corner for Solutions of the One Dimensional Heat Equation

by

R. N. Pederson

Carnegie Mellon University
Department of Mathematics
Pittsburgh, PA, USA

Research Report No. 95-181 2
August, 1995

Behavior at a Corner for Solutions of the One Dimensional Heat Equation

R. N. Pederson

August 28, 1995

Let us consider the behavior as $(x,t) \longrightarrow (0,0)$ of solution of

$$u_{+} = u_{xx}, \quad 0 < x < a, \quad t > 0, \tag{1}$$

under the initial and boundary conditions

$$u\left(x,0\right) = f\left(x\right), \quad u\left(0,t\right) = \varphi\left(t\right), u\left(a,0\right) = \Psi\left(t\right), \ a > 0. \tag{2}$$

Our solution uses the error function in the form

$$v\left(x,t\right) = \frac{2}{\sqrt{\pi}} \int_0^{x/2t^{1/2}} e^{-\partial^2} d\partial. \tag{3}$$

We shall show that, if the data is continuous, but not necessarily matching at the corners, then as $x \searrow 0, \ t \searrow 0$,

$$u\left(x,t\right) = v\left(x,t\right) f\left(0\right) + \left(1 - v\left(x,t\right)\right) \varphi\left(0\right) + o\left(1\right). \tag{4}$$

In particular, if $f\left(0\right) = \varphi\left(0\right)$, then u is continuous at $(x,t) = (0,0)$. Moreover, these results are true whether a is finite or infinite.

Let us prove this for the case $a = +\infty$. Then, as $(x,t) \longrightarrow (0,0)$, for any $A > 0$,

$$\begin{aligned} u\left(x,t\right) &= \frac{f\left(0\right)}{\sqrt{4\pi t}} \int_0^A \left[e^{-(x-y)^L/4t} - e^{-(x+y)^L/4t}\right] dy \\ &\quad + \frac{\varphi\left(0\right)}{2\pi^{1/L}} \int_0^t \frac{x e^{-x^2/4(t-s)}}{(t-s)^{3/2}} ds + o\left(1\right). \end{aligned} \tag{5}$$

The trick now is to change variables so that each of the three integrands is the same, say $e^{-\partial^2}$. Then we have

$$\begin{aligned} u(x,t) &= \frac{f(0)}{\sqrt{\pi}} \int_{\frac{-x}{2t^{1/L}}}^{(A-x)/2t^{1/L}} e^{-\partial^2} d\partial - \int_{x/2t^{1/L}}^{(A+x)/2t^{1/L}} \\ &\quad + \frac{2\varphi(0)}{\sqrt{\pi}} \int_{x/t^{1/L}}^{\infty} e^{-\partial^2} d\partial + o(1). \end{aligned} \tag{6}$$

After noting that the contributions of the first two integrands from the interval $\left(x/2t^{1/L}, (A-x)\,2^{1/2}\right)$ cancel, we see that the result (4) follows.

In this case $a < \infty$, we use the representation in terms of the Jacobi's Theta Function, θ_3, see Fulks [2], Hartman and Wintner [3] and Goursat [5], and Friedman [9].

Note that the contribution from the infinite series for $k \geq 1$ and from the right boundary integral tend to zero uniformly. This leaves the expression (5).

Moreover, since $\nu(x,t)$ satisfies the Heat Equations, we may solve the problem numerically for continuous data and then add on the easily compatible function ν. Dr. Myron Sussman has informed me that he has found the solvability of continuous boundary value problems more stable.

References

[1] Doetch, Theory und Andwendung der Laplace-Transformation, New York, 1943.

[2] Fulks, W., On the Boundary Values of Solutions of the Heat Equation, *Pacific Journal of Math.* (**2**), 141-145, 1952.

[3] Hartman and Wintner, On the solution of the Equation of Heat Conduction, *American Journal of Math.,* 367, 395 (1950).

[4] Gevrey, M., Sur les équations aux de derivier du type parabolique, *Journal of Mathematique*, ser. 6, **9**, 1913, 305-471.

[5] Goursat, *Cours d'Analyse Mathematique,* Vol. 1-4, Gauther-Villors, Paris, 1945.

[6] Weyl, H., Uebec gevohnliche Differentialglichungen mit Singularitätem und die Zugenhöugen Entvicklungen wilkürliche Analen, *Mathematische Annalen,* **68**, (190) 222-269.

[7] Sussman, Myron, personal communication.

[8] Pederson, R. N., Class Notes, Partial Differential Equations, CMU, Spring, 1983.

[9] Friedman, A., Partial Differential Equations of Parabolic Type, Prentice Hall, 1964.