On universal graphs without cliques or without large bipartite graphs

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ABSTRACT. For every uncountable cardinal λ, suitable negations of the Generalized Continuum Hypothesis imply:

(a) There is no universal $K_{\alpha,\beta}$-free graphs in λ for all infinite α and β.

(b) There is no universal K_α-free graph in λ for all $\alpha \geq 3$.

The instance K_{ω,ω_1} for $\lambda = \aleph_1$ was settled in [KP] from a strengthening of CH.

§0 Introduction

The Generalized Continuum Hypothesis, GCH, is an extremely useful assumption in infinite graph theory in general, and in the theory of universal graphs in particular. One consequence of the GCH is the existence of universal graphs in all infinite powers.

In this paper *negations* of the GCH are used to settle a few problem in the theory of universal graphs. Some of these problems were treated in the past with the GCH, and some were not.

The theory of universal graphs began with Rado's construction [R] of a strongly universal countable graph. The research in this area has advanced considerably since Rado's paper, mainly in studying universality in monotone classes of graphs, or, equivalently, in classes of the form $\mathrm{Forb}(\Gamma)$, all graphs omitting a set Γ of "forbidden" configurations. A good source for the development in this area is the survey paper [KP1] in which the authors suggest a generalization of universality they name "complexity": the least number of elements in the class needed to embed all other members in the class as induced subgraphs. The complexity of a class is 1 exactly when a universal member exists in the class.

In this paper omissions of infinite cliques and infinite complete bipartite graphs are studied. The omissions of K_ω and of $K_{\omega,\omega}$ were studied in [DHV] and in [HK]. Omission of K_α for uncountable α was treated in [KS] using the GCH. Omissions of $K_{\alpha,\beta}$ for α finite, $\alpha \le \beta$, were settled in [KP] for all infinite powers λ from the GCH, and the omission of K_{ω,ω_1} was settled negatively for $\lambda = \aleph_1$ from the principle $\diamondsuit(\omega_1)$ in the same paper.

The omission K_α for all $\alpha \ge 3$ and the omission of $K_{\alpha,\beta}$ for all infinite $\alpha \le \beta$ is settled here in all uncountable powers from suitable negations of the GCH. This complements and extends the results of Diestel-Halin-Vogler, Komjath-Pach and Komjath-Shelah.

NOTATION

Write $G_1 \le G_2$ if the graph G_1 is isomorphic to an induced subgraph of the graph G_2 and $G_1 \le_w G_2$ if G_1 is isomorphic to a subgraph of G_2. A class $\mathcal{G}$ of graphs is *monotone* if $G_1 \le_w G_2 \in \mathcal{G} \Rightarrow G_1 \in \mathcal{G}$. For a set of graphs Γ, let $\mathrm{Forb}(\Gamma)$ be the class of all graphs G satisfying $H \not\le_w G$ for all $H \in \Gamma$. Every monotone class is of the form $\mathrm{Forb}(\Gamma)$ for some class Γ of graphs.

Write $\mathcal{G}_\lambda$ and $\mathrm{Forb}_\lambda(\Gamma)$ for the set of all isomorphism types of cardinality λ in $\mathcal{G}$

and in Forb(Γ) respectively. Let $\mathrm{cp}\,\mathcal{G}_\lambda$, the *complexity* of $\mathcal{G}_\lambda$, be the least cardinality of a collection $D \subseteq \mathcal{G}_\lambda$ satisfying that for all $G \in \mathcal{G}_\lambda$ there exists $G' \in D$ such that $G \leq G'$. Replacing $\leq$ by $\leq_w$ in the last definition we obtain $\mathrm{wcp}\,\mathcal{G}_\lambda$, the *weak complexity* of $\mathcal{G}_\lambda$. For every class $\mathcal{G}$ and cardinal λ it holds that $\mathrm{wcp}\,\mathcal{G}_\lambda \leq \mathrm{cp}\,\mathcal{G}_\lambda$. The complexity $\mathrm{cp}\,\mathcal{G}_\lambda$ is 1 iff there is a strongly universal graph in $\mathcal{G}_\lambda$ and similarly for $\mathrm{wcp}\,\mathcal{G}_\lambda$.

Let κ, λ be cardinals. By $\mathrm{cf}\,\kappa$ we denote the *cofinality* of κ. The power set $\mathcal{P}(\kappa)$ is the set of all subsets of κ. By $[\kappa]^\lambda$ we denote the set of all subsets of κ whose cardinality is λ. Let $\mathrm{cf}\,\langle [\kappa]^\lambda, \subseteq\rangle$, the *cofinality* of the partially ordered set $\langle [\kappa]^\lambda, \subseteq\rangle$ (the partial ordering is set inclusion), be the least cardinality of a collection $D \subseteq [\kappa]^\lambda$ satisfying that for all $X \in [\kappa]^\lambda$ there exists $Y \in D$ such that $X \subseteq Y$.

§1 The results

1.1 Definition: Let θ be infinite. For $A \subseteq \mathcal{P}(\theta)$ let the *incidence graph* of A, denoted Γ_A, be the bipartite graph with left side θ, right side A and edge relation given by $\in$, the membership relation (a set is connected to its members by edges).

1.2 Theorem: *Suppose $\theta < \lambda$ are infinite cardinals and $\mathcal{G}$ is a class of graphs that contains all incidence graphs Γ_A for $A \in [\mathcal{P}(\theta)]^\lambda$. If $\mathrm{cp}\,\mathcal{G}_\lambda \leq 2^\theta$ then $\mathrm{cf}\,\langle [2^\theta]^\lambda, \subseteq\rangle \leq 2^\theta$.*

Proof. Suppose $\mathcal{F}$ is a family of graphs, each of cardinality λ, such that $|\mathcal{F}| \leq 2^\theta$ and every $G \in \mathcal{G}_\lambda$ is embeddable as an induced subgraph in some member of $\mathcal{F}$.

For every $A \in [\mathcal{P}(\theta)]^\lambda$ fix an embedding $f_A : \Gamma_A \to G_A$ for some $G_A \in \mathcal{F}$. Given a graph $G \in \mathcal{F}$ the number of functions $g : \theta \to G$ is at most $\lambda^\theta \leq 2^{\theta^\theta} = 2^\theta$.

For every $G \in \mathcal{F}$ and every function $g : \theta \to G$, define

$$S(G, g) \stackrel{\text{def}}{=} \bigcup \{A \in [\mathcal{P}(\theta)]^\lambda : G = G_A \ \&\ f_A | \theta = g\}$$

The family $\mathcal{F}^* = \{S(G, g) : G \in \mathcal{F}, g \in G^\theta\}$ has cardinality $\leq 2^\theta$ and covers $[\mathcal{P}(\theta)]^\lambda$ because $A \subseteq S(G_A, f_A|\theta)$. Since $|\mathcal{P}(\theta)| = 2^\theta$, the proof will be done once we prove that every member of $\mathcal{F}^*$ has cardinality $\leq \lambda$. Suppose that $x, y \in S(G, g)$ are distinct, and let A, B be such that $x \in A$, $y \in B$ and $f_A|\theta = f_B|\theta = g$. Since x and y are distinct, there is a point $z \in \theta$ such that $z \in x \Leftrightarrow z \notin y$. As $f_A(z) = f_B(z)$ and both functions preserve edges and non-edges, it follows that $\{g(z), f_A(x)\} \in E^G \Leftrightarrow \{g(z), f_B(y)\} \notin E^G$. Hence

$f_A(x) \neq f_B(y)$. We have shown, then, that $f = \bigcup\{f_A^{-1} : f_A|\theta = g\}$ is a surjection from G onto $S(G, g)$, and therefore $|S(G, g)| \leq \lambda$. $\triangle$

1.3 Remark: The condition $\operatorname{cp} \mathcal{G}_\lambda \leq 2^\theta$ in 1.2 can be weakened to "there exist $\leq 2^\theta$ many graphs, each of cardinality λ, not all of which necessarily belonging to $\mathcal{G}_\lambda$, such that every member of $\mathcal{G}_\lambda$ is isomorphic to an induced subgraph of at least one of them".

1.4 Corollaries: *Suppose θ is infinite and $\mathcal{G}$ is a class of graphs that contains all incidence graphs of subsets of $\mathcal{P}(\theta)$. Then:*

(0) If $\operatorname{cf} 2^\theta \leq \lambda < 2^\theta$ then $\operatorname{cp} \mathcal{G}_\lambda > 2^\theta$.

(1) If $\operatorname{cf} 2^\theta = \theta^+$ then $\mathcal{G}$ possesses no universal elements in any cardinal λ satisfying $\theta < \lambda < 2^\theta$; in fact $\operatorname{cp} \mathcal{G}_\lambda \geq 2^{\theta^+}$.

(2) It is impossible to compute $\operatorname{cp} \mathcal{G}_\lambda$ in ZFC or to prove the existence of a universal element in $\mathcal{G}_\lambda$ for all cardinals $\lambda > \theta$.

Proof: To prove (1) it is enough, by Theorem 1.2, to show that if $\operatorname{cf} 2^\theta \leq \lambda$ then $\operatorname{cf} \langle [2^\theta]^\lambda, \subseteq \rangle > 2^\theta$. This is a standard diagonalization argument: for every list of 2^θ many members of $[R]^\lambda$ construct in λ many steps a subset of 2^θ of size $\operatorname{cf} 2^\theta$ which is not contained in any of the members in the list.

(2) follows from (1).

To prove (3) we recall that, by Easton's results [E], for every cardinal μ with $\operatorname{cf} \mu > \theta$ it is consistent with the axioms of set theory that GCH holds below θ and $2^\theta = \mu$. Given any cardinal $\lambda > \theta$, there are infinitely many cardinals $\mu > \lambda$ whose cofinality is, say, θ^+. Thus by (1) the complexity $\operatorname{cp} \mathcal{G}_\lambda$ may assume infinitely many different values, all larger than λ. $\triangle$

Omitting complete subgraphs. We apply 1.2 to omissions of complete graphs:

1.5 Theorem: *If $\alpha \geq 3$ is a cardinal then corollaries (0)–(2) above hold for* $\operatorname{Forb}(K_\alpha)$. *In particular, for no uncountable λ and $\alpha \leq \lambda$ can one prove from the usual axioms of set theory the existence of a universal K_α-free graph in power λ.*

Proof: For every $A \subseteq \mathcal{P}(\theta)$ the incidence graph of A is K_α-free for all $\alpha \geq 3$ and $\theta \geq \aleph_0$. $\triangle$

Hajnal and Komjath showed in [HK] that the complexity of $\operatorname{Forb}_{\aleph_0}(K_\omega)$ equals exactly $\aleph_1$ (see [KS]§2 for a generalization of this). This shows that $\theta < \lambda$ cannot be relaxed to

$\theta \leq \lambda$ in Theorem 1.2 and in 1.4(1),(2). Komjath and Shelah showed that from the GCH it follows that Forb (K_α) has a universal graph in $\lambda \geq \alpha$ iff $\mathrm{cf}\,\lambda < \mathrm{cf}\,\alpha$. Theorem 1.5 above settles the problem negatively from suitable negations of GCH, namely for all $\lambda \in [\mathrm{cf}\, 2^\theta, 2^\theta)$ for some θ.

One may ask whether the condition $\mathrm{cf}\, 2^\theta \leq \lambda$ in Corollary 1.4(0) is necessary, or can be replaced by $\theta < \lambda$. Shelah constructs a model of set theory in [S2] in which $2^{\aleph_0} > \lambda$ for a prescribed regular uncountable λ and a universal graph (in the class of all graphs) exists in power λ. This was generalized by Mekler [M] to classes of structures including Forb (K_n) for all n. Komjath and Shelah [KS] construct a model in which GCH holds up to κ, 2^κ is large and $\mathrm{cp}\,\mathrm{Forb}_\kappa(K_{\omega_1}) = \kappa^+$. Since Corollaries (0)-(2) hold for the class of all graphs, Forb (K_n) and Forb (K_{ω_1}), the singularity assumption is needed for each of these classes.

Omitting complete bipartite subgraphs

We turn now to omissions of complete bipartite graphs. Theorem 1.2 does not apply to Forb $(K_{\alpha,\beta})$ for infinite α and β, because incidence graphs may contain copies of $K_{\alpha,\beta}$. But an easy variation on the proof handles this.

Let θ be an infinite cardinal.

1.6 Definition: A family $\mathcal{A} \subseteq \mathcal{P}(\theta)$ is *θ-almost disjoint* if $|\bigcap A| < \theta$ for every $A \in [\mathcal{A}]^\theta$.

The cardinal arithmetic assumption $\theta = 2^{<\theta}$ implies the existence of a θ-almost disjoint $\mathcal{A} \subseteq \mathcal{P}(\theta)$ of cardinality $|\mathcal{A}| = 2^\theta$.

1.7 Problem: Is it true that a θ-almost disjoint family of size 2^θ exists over every infinite cardinal θ?

1.8 Fact: If $\mathcal{A} \subseteq \mathcal{P}(\theta)$ is θ-almost disjoint and $A \subseteq \mathcal{A}$ then the incidence graph Γ_A is $K_{\theta,\theta}$-free.

1.9 Theorem: *If $\theta \leq \alpha \leq \beta$ are infinite cardinals and $2^{<\theta} = \theta$ then Corollaries (0)–(2) hold for $\mathcal{G} =$ Forb $(K_{\alpha,\beta})$. In particular, for all uncountable λ one cannot prove in ZFC the existence of a universal $K_{\alpha,\beta}$-free in power λ for all $\beta \geq \alpha \geq \omega$.*

Proof. It is enough to prove that Theorem 1.2 holds for all classes $\mathcal{G}$ that contain all $K_{\theta,\theta}$-free incidence graphs of $A \in [\mathcal{P}(\theta)]^\lambda$. Using $2^{<\theta}$ fix $\mathcal{A} \subseteq \mathcal{P}(\theta)$, θ-almost disjoint of cardinality 2^θ. In the proof of 1.2 consider only $A \in [\mathcal{A}]^\lambda$. For such A, the incidence graph Γ_A is $K_{\theta,\theta}$-free, and therefore belongs to Forb $(K_{\alpha,\beta})$. The proof shows that cf $\langle [\mathcal{A}]^\lambda, \subseteq \rangle \leq 2^\theta$. Since $|\mathcal{A}| \leq 2^\theta$, also cf $\langle [2^\theta]^\lambda, \subseteq \rangle \leq 2^\theta$. $\triangle$

By a theorem of Diestel, Halin and Vogler [DHV], for every non-empty set Γ so that every $G \in \Gamma$ contains an infinite path, $\mathrm{wcp}_{\aleph_0} \mathrm{Forb}(\Gamma) > \aleph_0$. The proof generalizes readily to give $\mathrm{wcp}\,\mathrm{Forb}_\lambda(\Gamma) \geq \lambda^+$. Since $K_{\omega,\omega}$ contains an infinite path, putting $\Gamma = \{K_{\omega,\omega}\}$ we obtain from Diestel-Halin-Vogler that there is no universal $K_{\omega,\omega}$-free graph in λ for all infinite cardinals λ. Komjath and Pach use the principle $\diamondsuit(\omega_1)$ to prove that $\mathrm{wcp}\,\mathrm{Forb}_{\omega_1}(K_{\omega,\omega_1}) > \omega_1$. The omission of $K_{\alpha,\beta}$ is settled from negations of GCH for all infinite $\alpha \leq \beta$ by 1.9 above.

Discussion The structure of embeddability in a pretty broad spectrum of monotone classes is seen to be sensitive to the exponent function $\theta \mapsto 2^\theta$: there are no universal graphs in those classes in a cardinal λ belonging to an interval $[\mathrm{cf}\, 2^\theta, 2^\theta)$. Shelah's consistency results show that a tighter connection to the exponent function, one which does not necessitate the singularity of 2^θ, is not possible for the same spectrum of classes. It is reasonable to ask if there are monotone classes of graphs in which the complexity in power λ is greater than or equal to 2^θ for some smaller θ, not assuming anything about the cofinality of 2^θ. The answer to this is yes. In [K] a class of graphs is defined by forbidding a set of countable configurations, and the complexity at an uncountable regular $\lambda > \aleph_1$ is shown to be at least $2^{\aleph_0}$ by means of a *representation Theorem*, asserting the existence of a surjective homomorphism from the relation of embeddability over the class onto the relation of set inclusion over all subsets of reals of cardinality λ.

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