New Results in the Simplex Method in Linear Programming

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"Notation is important. It can even solve problems. But, at some point, you must do some work yourself." K. O. Friedrichs.

1. Introduction and Statement of the Problem.

Without using any symbols at all, we can give a precise statement of the problem by saying that it is to find the maximum, if it exists, of a linear function of a finite number of real variables on a convex plane polyhedron of the same variables. The simplex method of solving the problem is then to find a vertex of the polyhedron and then to proceed along edges from one vertex to the next, in a manner that the linear function increases, until the maximum is reached. All the data needed to state and solve the problem can be stored in an $(m+1) \times (n+1)$ matrix $\mathcal{A}$.

The analytical statement of the problem then is to find the maximum of the objective function

$$\sum_{j=1}^{n} \mathcal{A}_{m+1,j} X_j + \mathcal{A}_{m+1,n+1} \tag{1.1}$$

subject to the constraints

$$\sum_{j=1}^{n} \mathcal{A}_{ij} X_j + \mathcal{A}_{i,n+1} \geq 0, \quad i = 1, ..., m. \tag{1.2}$$

By defining A to be the matrix comprising the first m rows and first n columns of $\mathcal{A}$ and b to be the transpose of $\mathcal{A}_{1,n+1,..,}\mathcal{A}_{m,n+1}$, the constraint (1.2) takes the simpler form

$$Ax + b \geq 0, \tag{1.3}$$

meaning, of course, that each component of the column vector is non-negative. The vector x is superfluous for the purpose of applying the simplex algorithm. But, working only with the matrix $\mathcal{A}$, can lead to misconceptions as we shall see in the next section. But first, let us find another notation for the constraint set by using $\mathcal{A}_i$ to denote the rows of A. Then (1.3) can be replaced by

$$L_i(x) = (\mathcal{A}_{i,}x) + b_i \geq 0, \quad i = 1, ..., m \tag{1.4}$$

where b_i is the i^{th} coordinate of b and $(,)$ represents the canonical inner product.

2. But, Those Slack Variables are Unnecessary.

Let us re-write (1.3) as

$$(A\,C)\left(C^{-1}X\right) + b \geq 0 \tag{2.1}$$

where C is any non-singular $n \times n$ matrix, noting that this does not require an equality. Now, assuming A has rank n, we may apply elementary column operations to reduced echelon form. If C is the product of the corresponding elementary column matrices and $y = C^{-1}X$, the first n coordinate of (2.1) are

$$y_i + b_i \geq 0. \tag{2.2}$$

Then, by making the translation $z_i = y_i + b_i$, we may assume the constraint set to be in, what is commonly called, canonical form. Furthermore, if for one $j, 1 \leq j \leq n$, we put $x_j = z_j - \beta_j$ in (1.1),(1.2) we see that this corresponds to multiplying the j^{th} column of the full matrix $\mathcal{A}$ by β_i and subtracting it from the $(n+1)^{\text{th}}$ column; that is, it is an elementary column operation. I prefer doing elementary row operation on the transpose. Thus the simplex method reduces to transposing the matrix $\mathcal{A}$ and applying elementary row operation until the first n column are in reduced echelon form, with the restriction that the pivots are to be picked from the first n rows of $\mathcal{A}^T$. The only question that remains is when to start using the simplex pivoting strategy. After the system is in canonical form, we must use the simplex strategy; before that we may use instead the standard Gaussian Elimination Strategy. Note that the simplex strategy requires picking the maximum positive element of the current column and hence is a partial pivoting strategy. We shall have more to say about this in Section 5.

3. Empty Sets, Redundant Constraints and Lower Dimensional Sets.

Let us now suppose that the normals of the first n constraints form a linearly independent set. Then, for any $k > n$,

$$\mathcal{A}_k = \sum_{i=1}^{n} \alpha_{ki} \mathcal{A}_i \tag{3.1}$$

and hence

$$L_k(x) = \sum_{i=1}^{n} \alpha_{ki} L_i(x) + \Delta_k \tag{3.2}$$

with

$$\Delta_k = \mathcal{A}_{k,n+1} - \sum_{i=1}^{n} \alpha_{ki} \mathcal{A}_{i,n+1}. \tag{3.3}$$

It follows from (1.4) and (3.2) that if $(\alpha_{k1}, ., ..., \alpha_{k,n}, \Delta_k)$ are all non-negative, the k^{th} constraint is redundant and that if they are all negative the set is empty. If for some $i \leq n$, $\alpha_{ki} > 0$, $\alpha_{kj} \leq 0$ for $j \neq i$ and $\Delta_k < 0$, then the i^{th} constraint is redundant.

In all other cases where none of the numbers $(\alpha_{k1}, ., ..., \alpha_{k,n}, \Delta_k)$ is zero it is easily shown that the set formed for the first n and the k^{th}, is non-empty. The other important special case occurs when $\Delta_k = 0$ and $\alpha_{ki} \leq 0$ for $i = 1, ..., n$. Then the entire constraint set is contained in the set where $L_k(x) = 0$. Hence, we may use this constraint to eliminate a variable and obtain a lower dimensional set. This means that, by reducing the number of dimensions, we may assume that this case does not occur.

We note from (3.2) and (3.3) that, when the constraint set is in canonical form, $\mathcal{A}_{i,n+1} = 0$, $i = 1, ..., n$, so the α_{ki}'s and Δ_k are just the coefficients of the constraint equation. From this point on we shall assume that the set is in canonical form. The origin will be called the basic vertex, the first n constraints the basic constraints and the rest of the constraints the non-basic constraints.

4. The Simplex Algorithm with a Non-Degenerate Basic Vertex.

A vertex which is the intersection of more than n-planes is called a degenerate vertex. This means that, when the basic vertex is non-degenerate, all of the non-basic constraints have non-zero constants. The simplex strategy then is to increase by one the number of positive constants among these until they are all positive and then to increase the constant in the objective function.

Let us assume that the constraints are ordered so that

$$\begin{aligned} &\mathcal{A}_{i,n+1} > 0, \quad i < p \qquad\qquad \text{and if } p < m \\ &\mathcal{A}_{i,n+1} < 0, \quad p \le i \le m. \end{aligned} \tag{4.1}$$

Our first objective is to increase p by one when it is less than m. The first step is to choose k to maximize

$$\{\mathcal{A}_{p,j} : \mathcal{A}_{p,j} > 0,\ \ 1 \le j \le n\}. \tag{4.2}$$

When $p \le m$, the results of Section 3 insure that we may assume the above set to be non-empty; when $p = m + 1$, it is only empty when we have found the maximum.

Next, we choose ℓ to maximize the negative numbers

$$\left\{ \frac{\mathcal{A}_{\nu,n+1}}{\mathcal{A}_{\nu,k}} : \nu \le p - 1,\ \mathcal{A}_{\nu,k} < 0 \right\}. \tag{4.3}$$

Suppose that the above set is empty. If $p = m + 1$ and $\mathcal{A}_{m+1,k} > 0$ there is no maximum while if $\mathcal{A}_{m+1,k} < 0$ we may set $x_k = 0$ and continue in one less dimension. If $p \le m$ we simply set $\ell = p$, observing that the simplex method requires only one step.

Next, we interchange the ℓ^{th} non-basic constraint with the k^{th} basic constraint and put the constraint set back into canonical form. This requires applying Gaussian elimination to the k^{th} column of $\mathcal{A}^T$. The new elements of the matrix then are

$$\mathcal{A}'_{\ell,k} = \frac{1}{\mathcal{A}_{\ell,k}} \tag{4.4}$$

$$\mathcal{A}'_{\ell,j} = -\frac{\mathcal{A}_{\ell j}}{\mathcal{A}_{\ell,k}}, \ \ j \ne k \tag{4.5}$$

and when $i \ne \ell$,

$$\mathcal{A}'_{i,k} = \frac{\mathcal{A}_{i,k}}{\mathcal{A}_{\ell,k}}, \tag{4.6}$$

$$\mathcal{A}'_{ij} = \mathcal{A}_{ij} - \frac{\mathcal{A}_{ik}\mathcal{A}_{\ell j}}{\mathcal{A}_{\ell,k}}, \ \ j \ne k. \tag{4.7}$$

In particular,

$$\mathcal{A}'_{\ell,n+1} = -\frac{\mathcal{A}_{\ell,n+1}}{\mathcal{A}_{\ell,k}} > 0 \tag{4.8}$$

since, whether $\ell = p$ or $\ell < p, \mathcal{A}_{\ell,n+1}$ and $\mathcal{A}_{\ell k}$ have opposite signs. If $i \neq \ell$ and $\ell < p-1$, we see from (4.7) that $\mathcal{A}'_{i,n+1}$ is the sum of two positive numbers when $\mathcal{A}_{ik} > 0$ and that when $\mathcal{A}_{ik} < 0$ it is positive as a consequence of the choice (4.3) of ℓ. Hence, in any case, the first $p-1$ constants remain positive and if $\ell = p, \mathcal{A}_{p,n+1}$ is also positive and we have increased p by one. But we also see from (4.7) that if $\ell < p$,

$$\mathcal{A}'_{p,n+1} > \mathcal{A}_{p,n+1} \tag{4.9}$$

Since the constraint set has only a finite number of vertices, we shall, in a finite number of steps either find the set to be empty, prove that $\mathcal{A}'_{p,n+1} > 0$ or arrive at a degenerate vertex.

5. The Case of a Degenerate Vertex.

The case of a degenerate vertex occurs when there are zero constants $\mathcal{A}_{i,n+1} = 0$. Suppose that we apply the previous strategy to the basic constraints and the non-basic constraints with non-zero constants. Then we see from (4.7) that when $\mathcal{A}_{i,n+1} = 0$,

$$\mathcal{A}'_{i,n+1} = -\left(\frac{\mathcal{A}_{\ell j}}{\mathcal{A}_{\ell k}}\right)\mathcal{A}_{ik} \tag{5.1}$$

and since $\mathcal{A}_{\ell j} < 0,\ \mathcal{A}_{\ell k} > 0$, we have $\mathcal{A}'_{i,n+1} > 0$ whenever $\mathcal{A}_{i,k} > 0$. There is no reason that this should be the case, but, by applying the simplex strategy to the first n column of $\mathcal{A}$, with the k^{th} playing the roll of the constants, we can use the simplex strategy to achieve this. Because the algorithm is slightly more complicated when the degeneracy is of higher order, it is convenient to introduce constants α_k, β_k satisfying, after reordering the constraints and variables

$$\begin{aligned} \mathcal{A}_{i,k} &= 0, \quad n+1 \leq i < \alpha_k \\ &> 0, \quad \alpha_k \leq i < \beta_k \\ &< 0, \quad \beta_k \leq i < \alpha_{k+1} \end{aligned} \tag{5.2}$$

with $\alpha_{n+2} = m$. The cases $\alpha_k = n+1,\ \beta_k = \alpha_k$ and $\beta_k = \alpha_{k+1}$ are used to indicate that the corresponding set is empty.

Now we apply the following algorithm to the constraint set in canonical form.

[1] $k = n+1$

[2] Reorder the constraints so that (5.2) is satisfied.

Now, we are ready to pick the current constraint indexed by p. The choice agrees with (4.1) when $k = n+1$.

[3] If $k = n+1$ or $\beta_k < \alpha_{k+1}$, set $p = \beta_k$ and proceed to [5].

Now, when we arrive at line [4], we have $k < n+1$ and $\beta_k = \alpha_{k+1}$. This means that the elements of the pivot column below the zeros in the k^{th} row of $\mathcal{A}^T$ are all zero so we can take advantage of the remark preceding (5.1) noting that, because $k < n$ the current pivot row has already been chosen in the line [5].

[4] Replace k by $k+1$ and proceed to [7].

Now we are ready to pick the current pivot row of $\mathcal{A}^7$.

[5] Reorder the variables so that $\mathcal{A}_{p,k-1}$ maximizes the positive coefficients $\mathcal{A}_{p,i},\ i \leq i \leq k-1$ when it is non-empty. If it is empty proceed to [10].

If $\alpha_k = n+1$, we are ready to begin the updating subroutine. Otherwise, we decrease k by 1 and return to [2].

[6] If $\alpha_k > n+1$, decrease k by 1 and return to [2].

When we arrive at line [7] we know that the $k - 1^{\text{th}}$ row of $\mathcal{A}^T$ is the current pivot row and, before updating, we must find the current pivot column.

[7] If the set $\{i < p : \mathcal{A}_{i,k-1} < 0\}$ is non-empty, choose ℓ to maximize the ratios $\mathcal{A}_{i,k}|\mathcal{A}_{i,k-1}$. Otherwise set $\ell = p$.

Now, we are ready to interchange the constraints indexed by $k-1$ and ℓ and then put the matrix back into canonical form.

[8] Return the matrix to canonical form by applying Gaussian elimination to reduced echelon form to $\mathcal{A}^T$ using the element indexed by $\ell, k-1$ as pivot.

We note that, since the elements $\mathcal{A}_{\ell,j},\ j > k-1$, are all zero the elementary row operation correspond to adding zero to the rows of A^T indexed by $j > k$. Hence the α_j's and β_j's, $j > k$ so they are unchanged. We now redefine the α_j's and β_j's for $j \leq k$ returning to [2].

[9] Return to [2].

The program will terminate at [10].

[10] The maximum is $\mathcal{A}_{m+1,n+1}$.

We have tacitly assumed the maximum to exist, leaving to the reader the task of adding the lines, explained in Section 3, regarding empty sets, redundant constraints, lower dimensional problems and problems with no maximum.

6. Small Pivots and Degenerate Vertices.

In running the above algorithm, it is crucial that one distinguish between non-zero numbers and zeros represented by round-off errors. The author has studied this problem extensively on the Radio Shack Color Computer and on the Tandy 1000. Computing, respectively, to 9 and 16 places, base 10. The Random Number generator was used to supply the data and, computing to places base 10, the test for determining whether or not a number is zero was by comparison with $10^{p-\gamma}$, $2 \leq \gamma \leq p/2$. In order to increase the probability that the set is not empty, the probability that the origin satisfying a constraint is set at π, $0 \leq \pi \leq 1$. With no other restriction, a degenerate vertex has never been found. By building in the condition of degeneracy, e.g. by applying a similarity transformation to a known degenerate situation and adding more constraints, the program seems to work as well as in the non-degenerate case. The problem, in each case, is checked by re-running the program on the constraints forming the final basic vertex and by evaluating the objective function at the intersection of their planes.

We have also never found an ill-conditioned matrix with the random number generator. By putting in the Hilbert matrix [2], prob. 169, p. 337, we find the obvious difficulty. However, by computing to a sufficient number of places, we have always been able to overcome the difficulty.

7. Further Methods of Speeding Up the Program.

The Simplest Method of Speeding Up the Program is to remove the redundant constraint using the test of Section 3, noting that the test requires only sign-tests of quantities that are computed anyway. Its disadvantage is that a constraint that shows up as redundant in one coordinate system does not necessarily in another. The number of degenerate constraints can be increased by adding the condition that the objective function be greater than its value at the current basic vertex.

Another method of possibly speeding up the program is to use the fact that once a vertex has been found we know that the constraint set is non-empty. Then we can eliminate a variable using any of the constraints. If the constraint used was redundant, the new set will be empty. Otherwise, we obtain the maximum on an $(n-1)$-dimensional face. The weakness of this method is that we lose time when we use a redundant constraint to eliminate a variable.

8. The Statement of the Condition that the Set be Empty or Contain a Redundant Constraint.

In this section we iterate the formulas (4.4) - (4.8) for the constraint set

$$\sum_{j=1}^{n} \mathcal{A}_{ij}x_j + \mathcal{A}_{i,n+1} \geq 0,\ i = 1,...,m \tag{8.1}$$

in canonical form. That is,

$$\mathcal{A}_{ij} = \delta_{ij},\ i = 1,...,n+1,\ \ j = 1,...,n. \tag{8.2}$$

Specifically, we generalize the condition that the set is empty when $\mathcal{A}_{kj} < 0$ for all $j = 1,...,n+1$ and contains a redundant constraint when the set $\{\mathcal{A}_{k1},...,\mathcal{A}_{kn},\mathcal{A}_{k,n+1}\}$ consists only or non-negative elements or $\mathcal{A}_{k,n+1} < 0$ and $\mathcal{A}_{kj} > 0$ for exactly one $j \leq n$.

In this section we shall use the above stated condition to obtain a result for appropriate union by obtaining explicit formulas for the coefficients in the constraints when the constraints

$$k_1,...,k_r,\ \ k_i \geq n+1,\ r \leq n \tag{8.3}$$

have been interchanged with the constraints

$$\ell_1,...,\ell_r,\ \ell_i \leq n, \tag{8.4}$$

in the order $k_i, \ell_i,\ i = 1,...,r$ and the constraint set is returned to canonical form at each step. In order to state the formulas, we denote by

$$f_\sigma\left(i_1,...,i_\sigma : j_1,...,j_\sigma\right) \tag{8.5}$$

the minor determinant of $\mathcal{A}_{ij},\ i = m+1,...,m,\ j = 1,...,n$ indexed by the rows $i_1,...,i_n$ and the columns $j_1,...,j_r$. Then, with $\mathcal{A}^0_{ij}$ representing the original matrix and $\mathcal{A}^r_{ij}$ the matrix after the constraints indexed by $k_1,...,k_r$ have replaced those indexed by $\ell,...,\ell_r$,

$$\begin{aligned} \mathcal{K}_r &= \{k_1,...,k_r\},\mathcal{K}'_r = [1,m]-\mathcal{K}_r,\ \mathcal{L}_r = \{\ell_1,...,\ell_r\},\ \mathcal{L}'_r = [1,n+1]-\mathcal{L}_r, \\ D_r &= f_r\left(k_1,...,k_i : \ell_1,...,\ell_r\right) \end{aligned} \tag{8.6}$$

we have the formulas for $i \in \mathcal{K}'_r$,

$$\mathcal{A}^r_{ij} = f_{r+1}\left(k_1,...,k_r,\ i : \ell_1,...,\ell_r,j\right)/D_r,\ \ j \in \mathcal{L}'_r, \tag{8.7}$$

$$\mathcal{A}^r_{i,\ell_j} = (-1)^{r-j} f_r\left(k_1,...,k_{j-1},k_{j+1},...,k_r,i : \ell_1,...,\ell_r\right)/D_r, \quad \ell_j \in \mathcal{L}_r, \tag{8.8}$$

and for $k_i \in \mathcal{K}_r$,

$$\mathcal{A}^r_{ki,j} = (-1)^{r+1-i} f_r\left(k_1,...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...,\ell_r,j\right)/D_r, \quad j \in \mathcal{L}'_r, \tag{8.9}$$

$$\mathcal{A}^r_{k_i,\ell_j} = (-1)^{i+j} f_{r-1}\left(k_1,...,k_{j-1},k_{j+1},...,k_r : \ell_1,...\ell_{i-1},\ell_{i+1},...,\ell_r\right)/D_r, \quad \ell_j \in \mathcal{L}_r. \tag{8.10}$$

Before stating the condition for redundant constraints or empty sets, we shall prove the following theorem.

Theorem 8.1. *The formulas (8.7) - (8.10) are invariant under permutation of $k_1,...,k_r$ or $\ell_1,...,\ell_r$ in the sense the sign of either (8.7), (8.8) or (8.9), (8.10) for fixed i and $j = 1,...,n+1$ are invariant.*
This makes it possible to state the condition for empty sets or redundant constraints using only the pair (8.7), (8.8) in the order $r = 1,2,...,n$.

Proof. First let us note that we may assume that the k's and l's are in increasing order. This follows from the fact that when $k_1,...,k_n$ are permutations of the same set, then $k_1,...,k_{j-1},k_{j+1},...k_r, j = 1,...,n$ are merely written down in a different order.

To prove this by induction, let $\sigma == (k_1,...,k_r)$ and $\sigma_j = (k_1,...,k_{j-1},k_{j+1},...,k_r)$ and suppose that the largest element y of σ is indexed by ℓ. Then after interchanging the y with the last elements of σ and $\sigma_j, j \neq \ell$, the sign of the ratio σ_j/σ is retained when $j < \ell$, changes when $j > \ell$ and is multiplied by $(-1)^{r-\ell}$ when $j = \ell$. Hence by moving the ℓ^{th} ratio to the end of the list and decreasing the order of those indexed by $k, \ell+1 \leq k \leq r$, we obtain a valid induction proof. Similarly for the l's. ■

Theorem 8.2. *In applying the empty set or redundant constraint test, it is sufficient to scan (8.7), (8.8) for all permutation $(k_1,...,k_r)$ and $(\ell_1,...,\ell_r)$ in increasing order of r.*

Proof. In proceeding from r to $r+1$ we interchange the constraints indexed by k_{r+1} and ℓ_{r+1}. A simple computation shows that in an $(n+1)$ constraint set in canonical form, an interchange of the $(n+1)^{\text{st}}$ constraint with a basic constraint can't change the sign test indicating an empty set or redundant constraint[1]. But, by Theorem 8.1, we may assume that any k_i and ℓ_i were interchanged.

[1]See Section 13, #2.

9. The Recursion Formula.

Assuming that we have computed the matrix $\mathcal{A}_{ij}^r$, the matrix $\mathcal{A}_{ij}^{r+1}$ is obtained by interchanging the constraints indexed by k_{r+1}, ℓ_{r+1} and updating the matrix as in [1]. The result is

$$\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r+1} = 1/\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r} \tag{9.1}$$

$$\mathcal{A}_{k_{r+1},j}^{r+1} = -\mathcal{A}_{k_{r+1},j}^{r}/\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r}, \quad j \neq \ell_{r+1} \tag{9.2}$$

and for $i \neq k_{r+1}$,

$$\mathcal{A}_{i,\ell_{r+1}}^{r+1} = \mathcal{A}_{i,\ell_{r+1}}^{r} \,/\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r}, \tag{9.3}$$

$$\mathcal{A}_{ij}^{r+1} = \mathcal{A}_{ij}^{r} - \frac{\mathcal{A}_{i,\ell_{r+1}}^{r}\mathcal{A}_{k_{r+1},j}^{r}}{\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r}}, \quad j \neq \ell_{r+1}. \tag{9.4}$$

Note, in particular, that (9.4) is the ratio of a 2×2 minor and a 1×1 minor and when $r = 0$, it agrees with (8.10). Also, when $r = 0$, (9.2), (9.3) agree with (8.8) (8.9). In order to make (9.1) agree with (8.6) we make the convention $f_0 = 1$. Before proving the general result, we shall develop some lemmas on determinants.

10. Some Lemmas on Determinants.

Let us use the usual convention that $\mathcal{B}_{ij}$ is the co-factor of b_{ij}. Then our first and main lemma is:

Lemma 10.1. *Let $\mathcal{B} = (b_{ij})$ be a $k \times k$ matrix and let $\mathcal{C}$ be the $(k-1) \times (k-1)$ matrix*

$$\mathcal{C} = \left(b_{ij} - \frac{b_{i,k}b_{kj}}{b_{kk}}\right), \quad 1 \leq i, \;\; j \leq k-1. \tag{10.1}$$

Then

$$\det \mathcal{C} = \det \mathcal{B}/b_{kk}. \tag{10.2}$$

Proof. Define:

$$\varphi(\epsilon) = \det\left(b_{ij} - \epsilon\, b_{i,k}b_{kj}\right), \quad 1 \leq i, j \leq k-1. \tag{10.3}$$

Now we use the fact that the derivative of a determinant is the sum of the determinants obtained by differentiating one row of the matrix. When we differentiate the i^{th} row of $\mathcal{C}$, the new i^{th} row is

$$-b_{ik}\left(b_{k1}, b_{k2}, ..., b_{k,k-1}\right). \tag{10.4}$$

■

If we interchange this row with each of those indexed by $i+1, ..., k-1$, we have the matrix obtained by deleting the i^{th} row from the first k columns of $\mathcal{B}$. Hence, when we take the determinant, we obtain

$$b_{ik}\mathcal{B}_{ik}. \tag{10.5}$$

It follows that

$$\varphi'(0) = \sum_{i=1}^{k-1} b_{ik}\mathcal{B}_{ik}. \tag{10.6}$$

When we differentiate twice we obtain a sum of determinants of matrices having two rows equal. Hence $\varphi''(\epsilon) \equiv 0$ so $\varphi^{(j)}(\epsilon) = 0$ for $j \geq 2$.

Since $\varphi(0) = \mathcal{B}_{kk}$ we then have

$$\varphi(\epsilon) = \mathcal{B}_{kk} + \epsilon \sum_{i=1}^{k-1} b_{ik}\mathcal{B}_{ik}. \tag{10.7}$$

Putting $\epsilon = 1/b_{kk}$ gives (10.2).

The recursion formula (9.4) with $i > k_{r+1},\ j > \ell_{r+1}$ can be rewritten

$$\mathcal{A}_{ij}^{r+1} = \frac{\det\left(\mathcal{A}_{\mu\nu}^{r}\right)_2}{\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r}} \tag{10.8}$$

where the numerator is the determinant of the 2×2 matrix indexed by $\mu = k_{r+1}, i$ and $\nu = \ell_{r+1,j}$ and is, in fact, just the Lemma 10.1 with $k = 2$ after a change of indices. More generally, we can use Lemma 10.1 to prove inductively that, for $1 \leq p \leq r+1$,

$$\mathcal{A}_{ij}^{r+1} = \frac{\det\left(\mathcal{A}_{\mu\nu}^{r+1-\rho}\right)_{\rho+1}}{\mathcal{A}_{k_{r+2-\rho},\ell_{r+2-\rho}}^{r+1-\rho} \cdots \mathcal{A}_{k_{r+1},\ell_{r+1}}^{r}} \tag{10.9}$$

where the numerator is the determinant of the $(\rho+1) \times (\rho+1)$matrix indexed by $\mu = k_{r+2-\rho}, ..., k_{r+1}, i$ and $\nu = \ell_{r+1-\rho}, ..., \ell_{r+1}, j$. In particular, when $\rho = r+1$, (10.9) reduces, in view of (8.5), to

$$\mathcal{A}_{ij}^{r+1} = \frac{f_{r+2}\left(k_1, ..., k_{r+1},\ i : \ell_1, ..., \ell_{r+1}, j\right)}{\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r} \mathcal{A}_{k_r,\ell_r}^{r-1} \cdots \mathcal{A}_{k_1,\ell_1}^{0}} \tag{10.10}$$

for any $i \in \mathcal{K}_{r+1},\ j \in \mathcal{L}'_{r+1}$. In particular,

$$\mathcal{A}^{r+1}_{k_{r+2},\ell_{r+2}} = \frac{D_{r+2}}{\mathcal{A}^{r}_{k_{r+1},\ell_{r+1}} \cdots \mathcal{A}^{0}_{k_1,\ell_1}} \tag{10.11}$$

or

$$D_{r+2} = \mathcal{A}^{0}_{k_1,\ell_1}\mathcal{A}^{1}_{k_2,\ell_2}\cdots\mathcal{A}^{r+1}_{k_{r+2},\ell_{r+2}}. \tag{10.12}$$

Since this is true for each r, we have proved (8.7) with $i\ \mathcal{K}'_{r+1},\ j \in \mathcal{L}'_{r+1}$ as a consequence of (10.10) and (10.11) with r replaced by $r-1$.

By eliminating $\mathcal{A}^{r+1}_{ij}$ between (10.9), (10.10), setting $i = k_{r+2},\ j = \ell_{r+2}$, and using 8.6 for $\mathcal{A}^{r+1-\rho}_{\mu\nu}$, we obtain the interesting identity

$$\begin{aligned} & f_{r+2}\left(k_1, ..., k_{r+2} : \ell_1, ..., \ell_{r+2}\right)\left(D_{r+1-\rho}\right)^{\rho} \\ = \ & \det\left(f_{r+2-\rho}\left(k_1, ..., k_{r+1-\rho}, \mu : \ell_1, ..., \ell_{r+1-\rho}, \nu\right)\right)_{\rho+1} \end{aligned} \tag{10.13}$$

with μ and ν ranging over the indices $k_{r+2-\rho}, ..., k_{r+2}$ and $\ell_{r+2-\rho}, ..., \ell_{r+2}$. The main use we make of this identity is:

Theorem 10.2. *Consider the identity (10.13) with $\rho = 1$. If three of the four minors comprising the determinant on the right have sign opposite the fourth then $D_r \neq 0$ and the sign of f_{r+2} is determined by the identity.*

We shall also need the following identity

$$\begin{aligned} & f_{r+1}\left(k_1, ..., k_{r+1} : \ell_1, ..., \ell_{i-1}, \ell_{i+1}, ..., \ell_{r+1,j}\right) f_r\left(k_1, ..., k_r : \ell_i, ...\ell_r\right) \\ - \ & f_{r+1}\left(k_1, ..., k_{r+1} : \ell_1, ..., \ell_r, j\right) f_r\left(k_1, ..., k_r : \ell_1, ..., \ell_{i-1}, \ell_{i+1}, ..., \ell_{r+1}\right) \\ + \ & f_{r+1}\left(k_1, ..., k_{r+1} : \ell_1, ..., \dot{\ell}_{r+1}\right) f_r\left(k_1, ..., k_r : \ell_1, ..., \ell_{i-1}, \ell_{i+1}, ..., \ell_r, j\right) = 0. \end{aligned} \tag{10.14}$$

If we suppress the dependence on $k_1, ..., k_{r-1}$ and $\ell_1, ..., \ell_{r-1}$, the left side of (10.14) is the 3×3 determinant of the matrix with rows indexed by (k_1, k_1, k_2) and column indexed by ℓ_r, ℓ_{r+1}, j. Since the first two rows are equal the determinant is zero.

11. Completion of the Proofs of the Identities.

We now have the main tools sufficient for the proofs of (8.7),...,(8.10) by induction. Note that we have proved (8.7) for all r and $i \notin \mathcal{K}_r,\ j \notin \mathcal{L}_r$, the proofs of the cases (8.8), (8.9), (8.10). Hence, we may use $i = k_{r+1},\ j = \ell_{r+1}$ in (8.7) to express (9.1) as

$$\mathcal{A}^{r+1}_{k_{r+1},\ell_{r+1}} = \frac{f_r\left(k_1,...,k_r : \ell_1,...,\ell_r\right)}{D_{r+1}}. \tag{11.1}$$

This is the promoted version of (8.10) with $i = r+1,\ j = r+1$. By putting $i = k_{r+1},\ j \notin \mathcal{L}_{r+1}$ into (8.7) and substituting (11.1) into (8.10), we obtain

$$\mathcal{A}^{r+1}_{k_{r+1},j} = -\frac{f_{r+1}\left(k_1,...,k_{r+1} : \ell_1,...,\ell_r,j\right)}{D_r} \tag{11.2}$$

which is the formula (8.9) corresponding to the pair $k_{r+1,j}$ with $j \notin \mathcal{L}_r$. It follows from (9.4) that for $k_i \in \mathcal{K}_r,\ j \notin \mathcal{L}_{r+1}$.

$$\mathcal{A}^{r+1}_{k_i,j} = \mathcal{A}^{r}_{k_i,j} - \frac{\mathcal{A}^{r}_{k_i,\ell_{r+1}}\mathcal{A}^{r}_{k_{r+1},j}}{\mathcal{A}^{r}_{k_{r+1},\ell_{r+1}}}. \tag{11.3}$$

After substituting (8.7) and (8.9), and setting the result equal to (8.9) with r replaced by $r+1$, we obtain the identity (10.14). This completes the proof of the remaining cases in (8.9). The proof of the promoted version of (8.8) is isomorphic.

There remains the case indexed by $k_i \in \mathcal{K}_r$ and $\ell_j \in \mathcal{L}_r$. We obtain from (9.4), (8.8), (8.9), (8.10) and (10.11) with r replaced by $r-1$ into (11.4), we obtain

$$\begin{aligned}\mathcal{A}^{r+1}_{k_i,k_j} &= \tfrac{(-1)^{i+j}}{D_r}\{f_{r-1}\left(k_1,...,k_{j-1},k_{i+1},...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...,\ell_r\right)\\ &+ \frac{f_r\left(k_1,...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...,\ell_{r-1}\right) f_r\left(k_1,k_{i-1},k_{i+1},...,k_{i-1} : \ell_1,...,\ell_{r+1}\right)}{f_{r+1}\left(k_1,...,k_{r+1} : \ell_1,...,\ell_{r+1}\right)}\}.\end{aligned} \tag{11.4}$$

We now apply (10.13) in the form

$$\begin{aligned}&f_{r+1}\left(k_1,...,k_{r+1} : \ell_1,...,\ell_{r+1}\right) f_{r-1}\left(k_1,...,k_{j-1},k_{j+1},...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...\ell_r\right)\\ =\ &f_r\left(k_1,...,k_{j-1},k_{j+1},...,k_{r+1} : \ell_1,...,\ell_{i-1},\ell_{i+1},\ell_{r+1}\right) f_r\left(k_1,...,k_r : \ell_1,...,\ell_r\right)\\ &-f_r\left(k_1,...,k_{j-1},k_{j+1},...,k_{r+1} : \ell_1,...,\ell_r\right) f_r\left(k_1,...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...,\ell_r\right).\end{aligned} \tag{11.5}$$

After substituting (11.5) into (11.4), we have the promoted version of (8.10) for $k_i \in \mathcal{K}_i,\ \ell_j \in \mathcal{L}_j$. Since the case of $k_{r+1} \subset \mathcal{K}_{r+1}, \ell_{r+1} \in \mathcal{L}_{r+1}$ has already been disposed of, the proof is complete.

12. Duplications of Constraints.

The formulas (8.7) - (8.10) are derived under the assumption that the sets $(k_1, ..., k_r)$ and $(\ell_1, ..., \ell_r)$ are distinct. In particular the ℓ's are a subset of $(1, ..., n)$ so we must have $r \leq n$. On the other hand, it follows from the recursion formulas (9.1) - (9.4) that we may, at any time, start over with a new matrix and continue until there is a duplication in either the k's or the ℓ's. In this section we resolve the question of such a duplication in the second step.

The new matrix coefficients, after interchanging the i^{th} nonbasic constraint with the ℓ^{th} basic constraint and then returning to reduced echelon form by the use of elementary column operations, are

$$\mathcal{A}'_{i\ell} = 1/\mathcal{A}_{i\ell} \tag{12.1}$$

$$\mathcal{A}'_{ij} = -\mathcal{A}_{ij}/\mathcal{A}_{i\ell}, \;\; j \neq \ell \tag{12.2}$$

and for $k \neq i$,

$$\mathcal{A}'_{k\ell} = \mathcal{A}_{k\ell}/\mathcal{A}_{i\ell}, \tag{12.3}$$

$$\mathcal{A}'_{kj} = \mathcal{A}_{kj} - \mathcal{A}_{k\ell}\,\mathcal{A}_{ij}/\mathcal{A}_{i\ell}, \;\; j \neq \ell. \tag{12.4}$$

Now let us interchange the new k^{th} constraint with the ℓ^{th} basic constraint. By analogy with (12.1), (12.2) the coefficients for the new k^{th} constraint are

$$\mathcal{A}''_{k\ell} = 1/\mathcal{A}'_{k\ell} \tag{12.5}$$

and

$$\mathcal{A}''_{kj} = \mathcal{A}'_{kj}/\mathcal{A}'_{k\ell}, \;\; j \neq \ell. \tag{12.6}$$

After substituting from (12.1) - (12.4) there becomes

$$\mathcal{A}''_{k\ell} = \mathcal{A}_{i\ell}/\mathcal{A}_{k\ell} \tag{12.7}$$

and

$$\mathcal{A}''_{kj} = \mathcal{A}_{ij} - \mathcal{A}_{i\ell}\;\;\mathcal{A}_{kj}/\mathcal{A}_{k\ell}, \;\; j \neq \ell. \tag{12.8}$$

These are just the parameters obtained after interchanging the i^{th} constraint with the ℓ^{th} and returning to reduced echelon form. But they are in the position of the k^{th}. The new coefficients for the i^{th} constraint are

$$\mathcal{A}''_{i\ell} = \mathcal{A}'_{i\ell}/\mathcal{A}'_{k\ell}, \tag{12.9}$$

$$\mathcal{A}''_{ij} = \mathcal{A}'_{ij} - \mathcal{A}'_{i\ell}\,\mathcal{A}'_{kj}/\mathcal{A}'_{k\ell}, \;\; j \neq \ell. \tag{12.10}$$

Again, after substituting from (12.1) - (12.4) and taking into account cancellations, these become

$$\mathcal{A}''_{i\ell} = 1/\mathcal{A}_{k\ell}, \tag{12.11}$$

$$\mathcal{A}''_{ij} = -\mathcal{A}_{kj} \,/\, \mathcal{A}'_{k\ell}, \;\; j \neq \ell. \tag{12.12}$$

They are the coefficients for the k^{th} constraint after interchanging the k^{th} constraint with the ℓ^{th}, and they are in the position of the i^{th}.

For $r \neq k$ or i, $r > n$, the new coefficients for the r^{th} constraint are

$$\mathcal{A}''_{r\ell} = \mathcal{A}'_{k\ell} \,/\, \mathcal{A}'_{k\ell}, \tag{12.13}$$

$$\mathcal{A}''_{rj} = \mathcal{A}'_{rj} - \mathcal{A}'_{r\ell}\, \mathcal{A}'_{kj} \,/\, \mathcal{A}'_{k\ell}, \;\; j \neq \ell \tag{12.14}$$

After substituting from (12.1) - (12.4), these become

$$\mathcal{A}''_{r\ell} = \mathcal{A}_{r\ell} \,/\, \mathcal{A}_{k\ell}, \tag{12.15}$$

$$\mathcal{A}''_{rj} = \mathcal{A}_{rj} - \mathcal{A}_{r\ell}\, \mathcal{A}_{rj} \,/\, \mathcal{A}_{k\ell}, \;\; j \neq \ell \tag{12.16}$$

which are just the coefficient obtained after interchanging the r^{th} constraint with the ℓ^{th} in the original matrix. This together with the remarks following (12.8) and (12.12) yields a proof of the following theorem.

Theorem 12.1. *Interchanging the i^{th} non-basic constraint with the ℓ^{th}, updating and then interchanging the k^{th} with the ℓ^{th} and updating is equivalent to merely interchanging the k^{th} with the ℓ^{th} in the original matrix, updating and then interchanging the i^{th} and k^{th}.*

Now let us determine the effect of interchanging one non-basic constraint with two different basic constraints. If after obtaining the formulas (12.1) - (12.4), we interchange the i^{th} constraint with the q^{th} basic constraint, $q \neq i$, the new parameter for the i^{th} constraint are

$$\mathcal{A}''_{iq} = 1/\mathcal{A}'_{iq}, \tag{12.17}$$

$$\mathcal{A}''_{i\ell} = -\mathcal{A}'_{i\ell} \,/\, \mathcal{A}'_{iq}, \tag{12.18}$$

and

$$\mathcal{A}'_{qj} = -\mathcal{A}'_{ij} \,/\, \mathcal{A}'_{iq}, \; j \neq q, \ell. \tag{12.19}$$

The formulas (12.17) - (12.19), after substituting from (12.1) - (12.4) are just the formulas obtained after interchanging the i^{th} with the q^{th} in the original matrix. For $k \neq q$,

$$\mathcal{A}''_{kq} = \mathcal{A}'_{kq} \,/\, \mathcal{A}'_{iq} \tag{12.20}$$

$$\mathcal{A}'_{k\ell} = \mathcal{A}'_{k\ell} - \mathcal{A}'_{kq}\,\mathcal{A}'_{i\ell} \,/\, \mathcal{A}'_{iq} \tag{12.21}$$

and

$$\mathcal{A}'_{kj} = \mathcal{A}'_{kj} - \mathcal{A}'_{kq}\,\mathcal{A}'_{ij} \,/\, \mathcal{A}'_{iq}, \;\; j \neq q, \ell. \tag{12.22}$$

Again, after substituting from (12.1) - (12.4), these are just the formula for the k^{th} constraint after interchanging the k^{th} with the q^{th} in the original matrix except that the q^{th} and ℓ^{th} variables have been interchanged.

Theorem 12.2. *If we interchange the i^{th} non-basic constraint with the ℓ^{th} basic constraint, update and then interchange the new i^{th} constraint with the q^{th}, $q \neq i$, and update, this is equivalent to merely interchanging the i^{th} with the q^{th} updating and permutatin the q^{th} and ℓ^{th} variable.*

13. The Case of $(n+2)$ Constraints.

Let us assume the constraint set to be in canonical form. If α is any subset of the non-basic indices, we shall denote by S_α the corresponding set of non-basic constraints and by $\bar{S}_\alpha$ the set S_α together with the basic constraints. For a single index i we define

$$S_i^+ = \{j \leq n : \mathcal{A}_{ij} > 0\} \tag{13.1}$$

and

$$S_i^- = \{j \leq n : \mathcal{A}_{ij} < 0\}\,. \tag{13.2}$$

The cardinality of set S shall be denoted by $|S|$. We shall assume that our constraint set contains no degenerate vertices and that minior determinants used in counting are always non-zero.

Our $(n+1)$ constraint set S_i is empty when $a_{i,n+1} < 0$ and $\left|S_i^-\right| = n$, hence $\left|S_i^+\right| = 0$. It contains a redundant constraint when $a_{i,n+1} > 0$ and $\left|S_i^+\right| = n$ or $a_{i,n+1} < 0$ and $\left|S_i^+\right| = 1$. Now suppose that $\left|S_i^+\right| = \sigma, 0 \leq \sigma \leq n$. If $\sigma < n$ there exists an $\ell \leq n$ such that $\mathcal{A}_{i\ell} < 0$. If we interchange the i^{th} and ℓ^{th} constraints and put the set back into canonical form we obtain the constraint set $\hat{\mathcal{S}}_i$ with $\left|\hat{\mathcal{S}}_i\right| = \sigma$. If $\sigma > 0$ there exists an index $\ell \leq n$ with $a_{i,\ell} > 0$. After interchanging the i^{th} and ℓ^{th} constraint and putting the set back into canonical form the set $\hat{\mathcal{S}}_i$ has $\left|\hat{\mathcal{S}}_i\right| = n+1-\sigma$. It follows that interchanging two constraints in an $(n+1)$ - constraint set cannot change its status relative to being empty, or having a redundant constraint. Hence, if neither $\mathcal{S}_i$ nor $\mathcal{S}_j$ has this property, we can find an empty set or redundant constraint in an $(n+1)$ constraint set only by interchanging S_i with a basic constraint and examining $\bar{\mathcal{S}}_j, j \neq i$ or conversely. In particular, after making this interchange, the new constant term is

$$\begin{aligned} \mathcal{A}'_{j,n+1} &= \mathcal{A}_{j,n+1} - \frac{\mathcal{A}_{j\ell}}{\mathcal{A}_{i\ell}}\mathcal{A}_{i,n+1} \\ &= -\frac{\mathcal{A}_{j\ell}}{\mathcal{A}_{i\ell}}\left(\mathcal{A}_{i,n+1} - \frac{\mathcal{A}_{i\ell}}{\mathcal{A}_{j\ell}}\mathcal{A}_{j,n+1}\right) \end{aligned} \quad . \tag{13.3}$$

Hence, that constant term in the i^{th} constraint, after interchanging the j^{th} and the ℓ^{th}, has the same or opposite sign as the j^{th} constant, after interchanging the i^{th} and the ℓ^{th}, according to whether $\mathcal{A}_{j\ell}$ and $\mathcal{A}_{i\ell}$ have the opposite or the same signs.

Let us now study the j^{th} constraint after interchanging the i^{th} and the ℓ^{th} with $\mathcal{A}_{i\ell} < 0$ and $\mathcal{A}_{j,\ell} > 0$. This requires analyzing the signs of

$$\mathcal{A}'_{j,\ell} = \frac{\mathcal{A}_{j\ell}}{\mathcal{A}_{i\ell}} \tag{13.4}$$

and

$$\mathcal{A}'_{j,k} = \mathcal{A}_{j,k} - \frac{\mathcal{A}_{j\ell}}{\mathcal{A}_{i\ell}}\mathcal{A}_{ik}. \tag{13.5}$$

Since $\mathcal{A}_{j\ell}$ and $\mathcal{A}_{i\ell}$ have opposite signs, it follows from (13.4) that

$$\mathcal{A}'_{j\ell} < 0 \tag{13.6}$$

and from (13.5) that

$$\mathcal{A}'_{jk} > 0, \quad k \in S_i^+ \cap S_j^+ \tag{13.7}$$

and

$$\mathcal{A}'_{jk} < 0, \quad k \in S_i^- \cap S_j^-. \tag{13.8}$$

For $k \in S_i^- \cap S_j^+ - \{\ell\}$, we may make the signs of

$$\mathcal{A}'_{jk} = \left(\frac{\mathcal{A}_{jk}}{\mathcal{A}_{ik}} - \frac{\mathcal{A}_{j\ell}}{\mathcal{A}_{i\ell}}\right) \cdot \mathcal{A}_{ik} \tag{13.9}$$

all negative or all positive, without violating (13.6), by choosing ℓ to minimize or maximize the ratios

$$\frac{\mathcal{A}_{jk}}{\mathcal{A}_{ik}}, \; k \in S_i^- \cap S_j^+. \tag{13.10}$$

Since, by (13.6), $\mathcal{A}'_{j\ell} < 0$, we can't achieve an empty set or redundant constraint unless $\mathcal{A}'_{j,n+1} < 0$. This rules out the possibility $\mathcal{A}_{i,n+1} > 0$ and $\mathcal{A}_{j,n+1} > 0$. When $\mathcal{A}_{i,n+1} < 0$, $\mathcal{A}_{j,n+1} > 0$, this cannot be the case unless it is true for either $\mathcal{S}_i$ or $\mathcal{S}_j$.

There remains the cases where $\mathcal{A}_{i,n+1}$ and $\mathcal{A}_{j,n+1}$ have opposite signs. By applying the results (13.6) - (13.10), we see that $\mathcal{S}_{ij}$ is empty or contains a redundant constraint if ℓ minimizes the ratios (13.10),

$$\left|S_i^+ \cap S_k^+\right| + \left|\left\{k \in S_i^+ \cap S_j^- : \mathcal{A}_{jk} > 0\right\}\right| \leq 1 \tag{13.11}$$

and either

$$\mathcal{A}_{j,n+1} < 0, \;\; \mathcal{A}_{i,n+1} > 0, \frac{\mathcal{A}_{j,n+1}}{\mathcal{A}_{i,n+1}} < \frac{\mathcal{A}_{j\ell}}{\mathcal{A}_{i\ell}} \tag{13.12}$$

or

$$\mathcal{A}_{j,n+1} > 0, \;\; \mathcal{A}_{i,n+1} < 0, \;\; \frac{\mathcal{A}_{j,n+1}}{\mathcal{A}_{i,n+1}} > \frac{\mathcal{A}_{j\ell}}{\mathcal{A}_{i\ell}}. \tag{13.13}$$

14. The Case of Three Variables.

We shall focus our attention on the case of proving that the constraint set is non-empty by finding a vertex that satisfies all constraints. Thus, suppose that

$$\mathcal{A}_{i,n+1} > 0, \;\; n+1 \leq i \leq p-1, \;\; \mathcal{A}_{p,n+1} < 0. \tag{14.1}$$

Definition 14.1. *For each $k \in S_p^+$, we define*

$$\mathcal{T}_k^+ = \left\{i : \mathcal{A}_{i,k} < 0, \;\; \mathcal{A}_{p,n+1} - \frac{\mathcal{A}_{pk}}{\mathcal{A}_{ik}}\mathcal{A}_{i,n+1} > 0\right\} \tag{14.2}$$

and

$$\mathcal{T}_k^- = \left\{i : \mathcal{A}_{ik} < 0, \;\; \mathcal{A}_{p,n+1} - \frac{\mathcal{A}_{pk}}{\mathcal{A}_{ik}}\mathcal{A}_{i,n+1} < 0\right\}. \tag{14.3}$$

The simplex strategy makes $\mathcal{A}_{p,n+1}$ increase until it is either positive or the set has been demonstrated to be empty. That this strategy requires more than one step requires that $\mathcal{T}_k^-$ be non-empty for each $k \in S_p^+$. Otherwise, if $S_k^- = \emptyset$, we may achieve our objective by interchanging the k^{th} basic constraint with the p^{th} and putting the matrix back into reduced echelon form.

From now on we shall assume the number of dimensions to be three. By making one Simplex step and permutating the variables, we assume that

$$\mathcal{A}_{p1} > 0, \ \ \mathcal{A}_{p2} > 0, \ \ \mathcal{A}_{p3} < 0, \ \ \mathcal{A}_{p4} < 0. \tag{14.4}$$

If there is a constraint indexed by $i < p$ for which $1 \in S_p^-$ and $2 \in S_p^-$, it follows from (13.11) - (13.13) that there is a redundant constraint. Let us then assume that there exist two non-basic constraints indexed by i and j for which

$$i \in \mathcal{T}_1^- \cap \mathcal{T}_2^+, \ \ j \in \mathcal{T}_1^+ \cap \mathcal{T}_2^-. \tag{14.5}$$

The constraints indexed by i, j, p have the following sign configuration

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ i & \ominus & - & + & + \\ j & - & \ominus & + & + \\ k & + & + & - & - \end{array}. \tag{14.6}$$

The circled and uncircled minuses referring to $\mathcal{T}_k^-$ and $\mathcal{T}_k^+$ respectively. We then have

$$\mathcal{A}_{p4} - \frac{\mathcal{A}_{p1}}{\mathcal{A}_{i1}}\mathcal{A}_{i4} < 0, \tag{14.7}$$

$$\mathcal{A}_{p4} - \frac{\mathcal{A}_{p2}}{\mathcal{A}_{i2}}\mathcal{A}_{i4} > 0, \tag{14.8}$$

$$\mathcal{A}_{p4} - \frac{\mathcal{A}_{p2}}{\mathcal{A}_{j2}}\mathcal{A}_{j4} < 0 \tag{14.9}$$

and

$$\mathcal{A}_{p4} - \frac{\mathcal{A}_{p1}}{\mathcal{A}_{j1}}\mathcal{A}_{j4} > 0. \tag{14.10}$$

These are equivalent to

$$\begin{aligned} f_2(i,p:1,4) &> 0, \ \ f_2(i,p:2,4) < 0, \ \ f_2(j,p:2,4) > 0, \\ f_2(j,p:1,4) &< 0. \end{aligned} \tag{14.11}$$

It follows from (14.7), (14.8); (14.9), (14.10) that

$$f_2(i,p:1,2) < 0, \ \ f_2(j,p:1,2) > 0. \tag{14.12}$$

By writing (14.12) in the form

$$\mathcal{A}_{p1} - \frac{\mathcal{A}_{p2}}{\mathcal{A}_{i2}}\mathcal{A}_{i1} < 0, \ \ \mathcal{A}_{p1} - \frac{\mathcal{A}_{p2}}{\mathcal{A}_{j2}}\mathcal{A}_{j1} > 0, \tag{14.13}$$

we have

$$f_2\,(i,j:1,2) > 0. \tag{14.14}$$

Similarly, it is a consequence of (14.7), (14.10) and (14.8), (14.9) that

$$f_2\,(i,j:1,4) < 0, \ \ f_2\,(i,j:2,4) > 0. \tag{14.15}$$

After interchanging the i^{th} and first constraints and updating, we have the coefficient matrix

$$\frac{1}{f_1\,(i:1)}\begin{pmatrix} 1 & -f_1\,(i,2) & -f_1\,(i:3) & -f_1\,(i:4) \\ f_1\,(j,1) & f_2\,(i,j:1,2) & f_2\,(i,j:1,3) & f_2\,(i,j:1,4) \\ f_1\,(p:1) & f_2\,(i,p:1,2) & f_2\,(i,p:1,3) & f_2\,(i,p:1,4) \end{pmatrix} \tag{14.16}$$

It follows from the imposed signs (14.6) - (14.15) that the matrix (14.16) has the sign configuration

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ i & - & - & + & + \\ j & + & - & \pm & + \\ p & - & + & \pm & - \end{array} \tag{14.17}$$

If the coefficient indexed by $p,3$ were negative the third basic constraint would be redundant. Therefore, we impose the sign

$$f\,(i,p:1,3) < 0 \tag{14.18}$$

leaving the configuration

$$\begin{pmatrix} - & - & + & + \\ + & - & \pm & + \\ - & + & + & - \end{pmatrix} \tag{14.19}$$

with only the $1,3$ element having an arbitrary sign. In any case the interchange of the j^{th} and second constraints is admissible. After this interchange, we have the matrix with $D_2 = f_2\,(i,j:1,2) > 0$

$$\frac{1}{D_2}\begin{pmatrix} f_1\,(j:2) & -f_1\,(i:2) & f_2\,(i,j:2,3) & f_2\,(i,j:2,4) \\ -f_1\,(j:1) & f_1\,(i:1) & -f_2\,(i,j:1,3) & -f_2\,(i,j,1,4) \\ -f_2\,(j,p:1,2) & f_2\,(i,p:1,2) & f_3\,(i,j,p:1,2,3) & f_3\,(i,j,p:1,2,4) \end{pmatrix} \tag{14.20}$$

Now let the coefficients of the p^{th} constraint be denoted by $\mathcal{A}''_{pj}$. It follows from (14.12) that

$$\mathcal{A}''_{p1} < 0 \text{ and } \mathcal{A}''_{p2} < 0. \tag{14.21}$$

Hence, if $\mathcal{A}''_{p4} < 0$ the set is either empty or there is a redundant constraint. If $\mathcal{A}''_{p4} > 0$, this configuration does not contribute to the promoted version of $\mathcal{T}_k^+$. Of course, this statement does not apply if the interchange is made with respect to some other constraint. Let us now examine the other admissible exchanges within the present matrix. From (14.19) it appears that the interchange of the i^{th} and second variables is one such possibility. But this follows the interchange of the i^{th} and the first. But this is, by Theorem 12.2, the interchange of the i^{th} and second followed by a permutation.

From (14.19) we see that the only other admissible interchange is the interchange of the j^{th} and third constraints under the condition

$$f_{32}(i,j:1,3) > 0. \tag{14.22}$$

This interchange gives the matrix

$$\frac{1}{D_2}\begin{pmatrix} f_1(j:3) & -f_1(i:3) & f_2(i,j:3,2) & f_2(i,j:3,4) \\ -f_1(j:1) & f_1(i:1) & -f_2(i,j:1,2) & -f_2(i,j:1,4) \\ -f_2(j,p:1,3) & f_2(i,p:1,3) & f_3(i,j,p:1,3,2) & F_2(i,j,p:1,3,4) \end{pmatrix} \tag{14.23}$$

with $D_2 = f_2(i,j:1,3)$ which by (14.22) is positive. By (14.18) we have $f_2(i,p:1,3) < 0$. This configuration appears to have insufficient information to resolve the sign of $f_2(j,p:1,3)$. However, if the assumption (14.22) leads to a legitimate simplex step it does impose the additional sign

$$f_2(i,j:3,4) > 0. \tag{14.24}$$

In any case, the previous configuration was sufficient to resolve the case of the constraints in three variables. When there are more constraints the additional condition (14.24) may be helpful in analyzing the interaction of various sets of three non-basic constraints combined with the basic constraints.

We remark, also, that if the same constraints i and j solve the maximum problem determining the next simplex step for two steps in a row, the analysis of (14.20) is sufficient to produce either a complete simplex step or to find a redundant constraint. That this be the case when both maximums are achieved by the i^{th} constraint would require the interchange $i-3$. By (14.19) this is impossible since both the $1,3$ and $3,3$ elements are positive.

Finally, we consider the sign configuration

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ i & \ominus & - & + & + \\ j & + & \ominus & + & + \\ p & + & + & - & - \end{array} \tag{14.25}$$

The interchange of the i^{th} and first constraints leads to

$$\begin{array}{cccc} - & - & + & + \\ - & - & + & + \\ - & + & + & - \end{array} \tag{14.26}$$

instead of (14.19). Some of these signs are determined as before and the others are consequences of Theorem 10.2. Now we notice that the interchange of the j^{th} constraint with the second is the only admissible simplex interchange. Now to apply the preceding analysis to (14.20), we need only (14.21). This is again a consequence of Theorem 10.2.

15. The case of Six Constraints in Three Variables.

The analysis of the preceding section yields the following Theorem.

Theorem 15.1. *Let us consider a set of Six Constraints in Three Variables which is in Canonical form and with only one constraint not satisfying the basic vertex. If completing a simplex step or finding a redundant constraint requires more than three steps then we may assume the configuration of the non-basic constraints*

$$\begin{array}{cccccc} i & : & \ominus & + & \pm & + \\ j & : & + & \ominus & \pm & + \\ p & : & + & + & - & - \end{array} \tag{15.1}$$

We leave open the question of whether the number of steps can be reduced from three to two by starting with the configuration

$$+++- \tag{15.2}$$

for the p^{th} contraint.

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