There is a maximal homogeneous family over ω

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Abstract

We prove that under CH there is a homogeneous family over ω which is maximal with respect to inclusion.

Introduction

Homogeneous families of sets were first studied in [GGK]. The class of homogeneous families over an infinite set is a proper sub-class of the class of independent families over the set.

The study of homogeneous families is related to set theory of the continuum, model theory and — as every homogeneous family is studied together with its automorphism group — also to permutation groups theory.

Interrelations between homogeneous families over ω and their automorphism groups were discussed in [KS], where it was also proved that there are $2^{2^{\aleph_0}}$ isomorphism types of such families. In this paper we address a problem raised in [KS]: does a maximal homogeneous family exist? As an increasing union of homogeneous families is not, in general, homogeneous itself, this is a non-trivial problem. The construction of a maximal homogeneous family of sets has to take into account the way the automorphism groups of one family and another containing it are related.

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We give a partial answer to this problem here by proving that CH implies the existence of a maximal homogeneous family over ω. The proof uses CH to diagonalize over all permutations of ω.

A simple variation on the proof gives $2^{2^{\aleph_0}} = 2^{\aleph_1}$ many isomorphism types of maximal homogeneous families over ω from CH.

Notation

We denote by ω the set of natural numbers. A natural number n is the set $\{0, 1, ..., n-1\}$ of smaller natural numbers. A subset $X \subseteq \omega$ is sometimes called a "real". $Sym\,\omega$ is the group of all permutations of ω. A function is a set of ordered pairs, in particular, $f_1 \subseteq f_2$ means that the function f_2 extends the function f.

We use Forcing terminology in a non-essential way: more for notational convenience then as a real mathematical tool. Let us specify all that is needed: the Cohen forcing for adding a single Cohen real is $P = \{p : p \text{ is a finite function from } \omega \text{ to } 2\}$. P is partially ordered by inclusion $(p_1 \leq p_2 \Leftrightarrow p_1 \subseteq p_2)$. A set $D \subseteq P$ is *dense* if $\forall p \in P\, \exists q \in D\, (p \leq q)$. $G \subseteq P$ is a filter if and only if G is downward closed and $\forall p_1, p_2 \in G\, \exists p_3 \in G\, (p_1 \leq p_3 \wedge p_2 \leq p_3)$. A filter G is *generic* for a countable transitive model N of set theory if and only if $G \cap D \neq \emptyset$ for every dense $D \subseteq P$ which belongs to N. For every countable transitive N there is a generic filter G for N. (N will be no more than a concise way to list $\aleph_0$ many relevant dense subsets of P). If $G \subseteq P$ is generic for N, let $r_G = r = \{n \in \omega : \exists p \in G\, (p(n) = 1)\}$ be a *Cohen real over* N. We say that a condition p *forces* some property φ of r if and only if φ holds for all $r = r_G$ with $p \in G$, and write $p \Vdash \varphi$.

Definition 0.1. *Suppose $\mathcal{F}$ is a family of subsets of ω.*

(0) $FF\,\mathcal{F} = \{\tau : \tau \text{ finite function from } \mathcal{F} \text{ to } \{-1, 1\}\}$

(1) If $A \subseteq \omega$ let $A^1 = A$ and $A^{-1} = \omega \backslash A = -A$. For $\tau \in FF\,\mathcal{F}$ let $\mathcal{F}^\tau = \bigcap_{A \in dom\,\tau} A^{\tau(A)}$. Denote $\mathcal{F}^\tau$ also as B_τ and call it a "boolean combination".
We say that A **participates** in $\mathcal{F}^\tau$ (or in B_τ) if $A \in dom\ \tau$.
Let $FI\,\mathcal{F} = \{B_\tau : \tau \in FF\,\mathcal{F}\}$.

(2) $\mathcal{F}$ is **independent** if and only if $\mathcal{F}^\tau$ is infinite for all $\tau \in FF\,\mathcal{F}$.

(3) $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is **dense** if and only if for all $\eta \in {}^{<\omega}2$ there is $X \in \mathcal{F}$ s.t. $x \in X$ if and only if $\eta(x) = 1$ for $x \in dom\,\eta$.

(4) Let $id_{\mathcal{F}} = \{X \subseteq \omega : (\forall \tau \in FF\,\mathcal{F})(\exists \tau' \in FF\,\mathcal{F})\ \tau' \supseteq \tau$ and $\left|\mathcal{F}^{\tau'} \cap X\right| < \aleph_0\}$. $id_{\mathcal{F}}$ is an ideal over ω and is proper if and only if $\mathcal{F}$ is independent. If $\mathcal{F}$ is dense then $X \in id_{\mathcal{F}}$ if and only if for all $\tau \in FF\,\mathcal{F}$ there is $\tau' \supseteq \tau$ in $FF\,X$ and $\mathcal{F}^{\tau'} \cap X = \emptyset$.

Definition 0.2.

(0) Let $\mathcal{F} \subseteq P(\omega)$ be a family of sets. Let $Aut\,\mathcal{F} = \{\sigma \in Sym\,\omega : \forall X \subseteq \omega\ X \in \mathcal{F} \Leftrightarrow \sigma[X] \in \mathcal{F}\}$ (where $\sigma[X] := \{\sigma(x) : x \in X\}$). $Aut\,\mathcal{F}$ is the *automorphism group* of $\mathcal{F}$. If $\sigma \in Sym\,\omega$, let $\bar{\sigma} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be defined by $\bar{\sigma}(x) = \sigma[X]$. If $\sigma \in Aut\,\mathcal{F}$ then $\bar{\sigma}\mathcal{F} \in Sym\,\mathcal{F}$.

(1) A **demand** on $\mathcal{F} \subseteq P(\omega)$ is a pair $d = \langle h, f\rangle$ s.t. $h : \omega \to \omega$ finite 1-1 function, $f : \mathcal{F} \to \mathcal{F}$ finite 1 - 1 function and $x \in X$ if and only if $h(x) \in f(X)$ for $x \in dom\,h$, $X \in dom\,f$. We say that an automorphism $\sigma \in Aut\,\mathcal{F}$ **satisfies** a demand d if and only if $h \subseteq \sigma$ and $f \subseteq \bar{\sigma}$.

(2) For a permutation $\sigma \in Sym\,\omega$, let $Supp\,\sigma = \{x \in \omega : \sigma(x) \neq x\}$ and $Fix\ \sigma = \{x \in \omega : \sigma(x) = x\}$. Let 1 denote the unit in $Sym\,\omega$.

(3) If G is a group of permutations and $r \subseteq \omega$ a real, then $G[r] = \{\sigma[r] : \sigma \in G\}$, the orbit of r under the action of G on $\mathcal{P}(\omega)$.

The following four examples of $\mathcal{F}$ satisfy $Aut\,\mathcal{F} = Sym\,\omega : \mathcal{F} = \{\emptyset\}$, $\mathcal{F} = \{\omega\}$, $\mathcal{F} = \{\{x\} : x \in \omega\}$, $\mathcal{F} = \{\omega \setminus \{x\} : x \in \omega\}$. Therefore, for each of these four families it is trivially true that every demand on $\mathcal{F}$ is satisfied.

Definition 0.3.

(1) $\mathcal{F} \subseteq P(\omega)$ is **homogeneous** if and only if every demand on $\mathcal{F}$ is satisfied by an automorphism of $\mathcal{F}$ and $Aut\,\mathcal{F} \neq Sym\,\omega$.

(2) A group of permutations $G \subseteq Sym\,\omega$ **acts homogeneously** on $\mathcal{F} \subseteq P(\omega)$ if and only if $G \subseteq Aut\,\mathcal{F}$ and every demand on $\mathcal{F}$ is satisfied by a member of G.

We quote some basic facts about homogeneous families.

Fact 1.2 ([GGK], § 1):

(0) Every homogeneous $\mathcal{F} \subseteq P(\omega)$ is independent.

(1) Every homogeneous family $\mathcal{F} \subseteq P(\omega)$ is dense.

(2) All countable homogeneous families over ω are isomorphic. Moreover, any countable dense independent family over ω is isomorphic to the countable homogeneous family over ω.

Lemma 0.4. *(See also GGK 2.2) If $\mathcal{F} \subseteq P(\omega)$ is homogeneous, then $Fix\ \sigma \in id_{\mathcal{F}}$ for all $id \neq \sigma \in Aut\ \mathcal{F}$. Consequently, for every finite list $\sigma_0, \sigma_2, ..., \sigma_{k-1}$ of distinct automorphisms of $\mathcal{F}$ and $\tau \in FF\ \mathcal{F}$ there is some $\tau \subseteq \tau' \in FF\ \mathcal{F}$ s.t. the points $\sigma_0(x), \sigma_1(x), ...\sigma_{k-1}(x)$ are distinct for all $x \in B_{\tau'}$.*

Proof: We repeat the proof here for completeness of presentation. The second part of the Lemma follows from the first by considering $Fix\ \left(\sigma_i\sigma_j^{-1}\right)$ for all $i < j < k$, the fact $id_{\mathcal{F}}$ is closed under finite unions, and the density of $\mathcal{F}_1$ which implies that if $X \in id_{\mathcal{F}}$ then $B_{\tau'} \cap X = \emptyset$ for some τ' extending a given τ.

Let us prove the first part. Suppose $id \neq \sigma \in Aut\ \mathcal{F}$ and let $\tau \in FF\ \mathcal{F}$ be given. Pick $x \in \omega$ such that $\sigma(x) \neq x$. By density of $\mathcal{F}$, there are infinitely many $X \in \mathcal{F}$ with $x \in X$ and $\sigma(x) \notin X$. Pick one such X so that X and $\bar{\sigma}(X)$ do not participate in B_τ. Clearly, $X \neq \bar{\sigma}(X) \in \mathcal{F}$ as $\sigma(x) \in \bar{\sigma}(X) \backslash X$, and consequently $X \cap -\bar{\sigma}(X) \in FI\ \mathcal{F}$. For all $x \in X \backslash \bar{\sigma}(X)$ we have $\sigma(x) \in \bar{\sigma}(X)$, so $\sigma(x) \neq x$ as $x \in X$. Let $\tau' \supseteq \tau$ be defined by $\tau' = \tau \cup \{\langle X, 1\rangle, \langle\bar{\sigma}(X), -1\rangle\}$. It follows that $\sigma(x) \neq X$ for all $x \in B_{\tau'} \subseteq B_\tau$.

We shall need the following generalization of Lemma 0.4:

Lemma 0.5. *Suppose $\mathcal{F}_0 \subseteq \mathcal{F}_1$, $\mathcal{F}_0$ homogeneous and $\mathcal{F}_1$ independent. If $\sigma \in Aut\ \mathcal{F}_0$, $f \in Aut\ \mathcal{F}_1$ and $\sigma \neq f$, then $\{x \in \omega : f(x) = \sigma(x)\} \in id_{\mathcal{F}_1}$.*

Proof: Find $x \in \omega$ s.t. $\sigma^{-1}(x) \neq f^{-1}(x)$ and let $y = \sigma f^{-1}(x)$. Clearly $y \neq x$. Any $A \in F_0$ for which $x \in A$ and $y \notin A$ satisfies that $f^{-1}(x) \in \bar{f}^{-1}(A)$ and $\sigma y = f^{-1}(x) \notin \bar{\sigma}^{-1}(A)$, and therefore that $\bar{\sigma}^{-1}(A) \neq \bar{f}^{-1}(A)$. As $\mathcal{F}_0$ is dense (Fact 1.2 above), there are infinitely many $A \in \mathcal{F}_0$ satisfying this requirement. Given $\tau \in FF\ \mathcal{F}$. Find such $A \in \mathcal{F}_0$ so that $A, B := \bar{\sigma}^{-1}(A)$ and $C := \bar{f}^{-1}(A)$

are distinct elements of $\mathcal{F}$, and do not participate in B_τ. This is possible by the above.

Let τ'' be defined by $dom\, \tau'' = \{B, C\}, \tau''(B) = 1$ and $\tau''(C) = -1$. Let $\tau' = \tau \cup \tau''$. So $\tau' \in FF\ \mathcal{F}_1$ and $B_{\tau'} \in FI\ \mathcal{F}$. If $x \in B_{\tau'}$ then $x \in B \backslash C$ and $\sigma(x) \in A,\ f(x) \in -A$. This proves the Lemma.

Lemma 0.6. *If $\mathcal{F}$ is countable homogeneous d, $G \subseteq Aut\ \mathcal{F}$ is countable and acts homogeneously on $\mathcal{F}$, and r is a Cohen real over a countable transitive model N with $\mathcal{F}, G \in N$, then*

(0) $G[r] \cap G[X] = \emptyset$ for all $X \in N$, $X \subseteq \omega$.

(1) $\mathcal{F} \cup G[r]$ is countable homogeneous.

(2) $G \subseteq Aut\, \mathcal{F} \cup G[r]$.

Proof: If $X \subseteq \omega$ and $X \in N$ then $Y = \bar{\sigma}^{-1}(X) \in N$ for all $\sigma \in G$. The set $D_Y = \{p \in P : \exists_n \in dom\, p\ (p(n) = 0 \Leftrightarrow n \in X)\}$ also belongs to N and is dense in P. If $p \in D_y$ then $p \Vdash$ "$X \neq r$", and there is one such p in the filter defining r by genericity.

As $\mathcal{F}$ is dense, so is $\mathcal{F} \cup G[r]$. It is obvious that $G \subseteq Aut\ \mathcal{F} \cup G[r]$. All that is left to show, then, is that $\mathcal{F} \cup G[r]$ is independent, because a countable dense independent family is homogeneous by Fact 1.2(3).

Let $\tau \in FF\ (\mathcal{F} \cup G[r])$ and break τ into two parts $\tau = \tau_1 \cup \tau_2$, $dom\ \tau_1 \subseteq \mathcal{F}$ and $dom\ \tau_2 \subseteq G[r]$. The two parts are disjoint because of (0).

Let p be a condition in the Cohen forcing and let $n \in \omega$ be arbitrary that for some $p' \geq p$. We show

$$p' \Vdash \text{“}B_\tau \backslash n \neq \emptyset\text{”}.$$

This implies that B_τ is infinite.

B_{τ_1} is certainly infinite. Let $\sigma_0, \sigma_1, ..., \sigma_{k-1}$ be the list of all automorphisms in G for which $\sigma[r_\alpha]$ participates in B_{τ_2}. Using Lemma 1.3 find $\tau_1' \supseteq \tau_1$, $\tau_1' \in FF\ \mathcal{F}$, such that for all $x \in B_{\tau_1}$, we have that $\sigma_i(x) \neq \sigma_j(x)$ for $i < j < k$. $B_{\tau_1'}$ is infinite. Pick $x \in B_{\tau_1'} \backslash n$ such that $\sigma_i^{-1}(x) \notin dom\, p$ for all $i < k$. This is possible, as $dom\, p$ is finite. Now $p' = p \cup p'$ forces that $x \in B_\tau$ if $dom\ p' = \left\{\sigma_1^{-1}(x) : i < k\right\}$ and $p'\left(\sigma_i^{-1}(x)\right) = \tau_2(\bar{\sigma}_i[r])$ for $i < k$.

Theorem 0.7. *(CH) There exists a homogeneous $\mathcal{F} \subseteq P(\omega)$ which is maximal with respect to inclusion in the class of all homogeneous families over ω.*

Proof: Fix an enumeration $\langle f_\alpha : \alpha < \omega \rangle$ of $Sym\,\omega \setminus \{1\}$. By induction on $\alpha < \omega_1$ we construct $\langle \mathcal{F}_\alpha, G_\alpha \rangle$ satisfying:

(0) $\mathcal{F}_\alpha \subseteq P(\omega)$ countable, $G_\alpha \subseteq Aut\ \mathcal{F}_\alpha$ countable and G_α acts homogeneously on $\mathcal{F}_\alpha$.

(1) $\alpha < \beta \Rightarrow \mathcal{F}_\alpha \subseteq F_\beta$ and $G_\alpha \subseteq G_\beta$ and if α is limit then $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$, $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$.

(2) $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup G_\alpha [s_\alpha]$ where $s_\alpha \subseteq \omega$. [$\mathcal{F}_{\alpha+1}$ is obtained from $\mathcal{F}_\alpha$ by adding the orbit $G_\alpha [s_\alpha]$ of a single real s_α under G_α].

(3) If $\mathcal{F} \supseteq \mathcal{F}_{\alpha+1}$ is an independent family and $f_\alpha \in Aut\ \mathcal{F}$ then $f_\alpha \in G_\alpha$.

Suppose first that this construction can be carried out. Let $\mathcal{F} = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$, $G = \bigcup_{\alpha<\omega_1} G_\alpha$. By (0) and (1) it is clear that $G \subseteq Aut\ \mathcal{F}$ and G acts homogeneously on $\mathcal{F}$. Therefore, $\mathcal{F}$ is homogeneous. We now argue that $\mathcal{F}$ is maximal with respect to inclusion among the homogeneous families over ω. Suppose to the contrary that $\mathcal{F}' \supset \mathcal{F}$ is homogeneous and $\mathcal{F}' \neq \mathcal{F}$. By Fact (1.2) $\mathcal{F}'$ is independent. Let $A \in \mathcal{F}' \setminus \mathcal{F}$ and let $B \in \mathcal{F}$. We show that no automorphism of $\mathcal{F}'$ carries B to A. Let $f = f_\alpha \in Aut\ \mathcal{F}'$ be any automorphism of $\mathcal{F}'$. By condition (3), and as $\mathcal{F}' \supseteq \mathcal{F}_{\alpha+1}$ is independent, $f_\alpha \in G_\alpha \subseteq G$. Therefore, $f_\alpha(B) \in \mathcal{F}$ and cannot equal A.

The proof will be complete once we prove:

Claim: The induction can be carried out.

We concentrate on successor stages, the limit stages presenting no problems. As $\mathcal{F}_0, G_0$ pick any countable homogeneous family and a countable group acting homogeneously on it.

Once $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup G_\alpha [s_\alpha]$ is defined, and shown to be independent, it follows by 1.2(3) that $\mathcal{F}_{\alpha+1}$ is homogeneous. Then $G_{\alpha+1}$ can be generated from $G_\alpha \subseteq Aut\,\mathcal{F}_{\alpha+1}$ by adding countably many automorphisms needed to satisfy all demands on $\mathcal{F}_{\alpha+1}$.Thus, we need only define $\mathcal{F}_{\alpha+1}$, show it is independent and see that condition (3) holds.

Suppose $\mathcal{F}_\alpha, G_\alpha$ are defined. Let r_α be a Cohen real over a countable transitive model M with $f_\alpha, G_\alpha, \mathcal{F}_\alpha \in M$. By Lemma 6 we know that $\mathcal{F}_\alpha \cup G_\alpha [r_\alpha]$ is independent. Now we distinguish two cases.

Case 0: $f_\alpha \in G_\alpha$ or $f_\alpha \notin Aut\,\mathcal{F}$ for all independent $\mathcal{F} \supseteq \mathcal{F}_\alpha \cup G_\alpha [r_\alpha]$.

In this case let $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup G_\alpha [r_\alpha]$. By Lemma 0.6 $\mathcal{F}_{\alpha+1}$ is homogeneous. $G_{\alpha+1}$ is readily chosen so that (0)–(3) hold.

Case 1: $f_\alpha \notin G_\alpha$ and for some independent family $\mathcal{F} \supseteq \mathcal{F}_\alpha \cup G_\alpha [r_\alpha]$ it holds that $f_\alpha \in Aut\, \mathcal{F}$.

Claim: $\mathcal{F}_\alpha, G_\alpha [r_\alpha]$ and $G_\alpha [f_\alpha^{-1} r_\alpha]$ are pairwise disjoint and $\mathcal{F}_\alpha \cup G_\alpha [r_\alpha] \cup G_\alpha [f_\alpha^{-1} r_\alpha]$ is independent.

Proof: Let $t_\alpha := f_\alpha^{-1} r_\alpha$. Clearly $r_\alpha \notin N$ being generic over N. Therefore $r_\alpha \notin \mathcal{F}_\alpha$. It follows that $G_\alpha [r_\alpha]$ and $\mathcal{F}_\alpha$ are two different orbits under the action of G_α on $\mathcal{P}(\omega)$, and are therefore disjoint. Similarly, $f_\alpha^{-1} r_\alpha \notin N$ and therefore $\mathcal{F}_\alpha$ and $G_\alpha [t_\alpha]$ are disjoint.

Finally, we check that $G_\alpha [r_\alpha] \cap G_\alpha [t_\alpha] = \emptyset$. Let $\sigma \in G_\alpha$ be arbitrary. As $\sigma \neq f_\alpha^{-1}$ and $f_\alpha^{-1} \in Aut\, \mathcal{F}$, Lemma 2.3 assures us that $\{x : \sigma(x) = f_\alpha^{-1}(x)\} \in id_{\mathcal{F}}$. Therefore, $A = \{x : \sigma(x) \neq f_\alpha^{-1}(x)\}$ is infinite (as $\mathcal{F}$ is independent). Also, $A \in N$.

Now if p is a condition in the Cohen forcing, find $x \in A$ s.t. x, $\sigma^{-1}(x)$ and $f_\alpha(x)$ are distinct and do not belong to $dom\, p$. Let p' extend p so that $p'(\sigma^{-1}(x)) = 1$ and $p'(f_\alpha(x)) = -1$. $p' \Vdash f_\alpha^{-1}[r_\alpha] \neq \sigma[r_\alpha]$ and now the claim follows.

Suppose now that $\mathcal{F}_\alpha \cup G_\alpha [r_\alpha] \cup G_\alpha [t_\alpha]$ is not independent, and we will show that $f_\alpha \in G_\alpha$, contrary to the assumption. Let p be a condition in the Cohen forcing, $\tau \in FF(\mathcal{F}_\alpha \cup G_0 [r_\alpha] \cup G_\alpha [t_\alpha])$ and

$$p \Vdash \text{“}B_\tau = \emptyset\text{”}.$$

partition $\tau = \tau_1 \cup \tau_2 \cup \tau_3$ such that $\tau_1 \in FF\ \mathcal{F}_\alpha$, $\tau_2 \in FF\ G[r_\alpha]$ and $\tau_3 \in FF\ G_\alpha [f_\alpha^{-1} r_\alpha]$.

Let $dom\ \tau_2 = \{\sigma_0 [r_\alpha], ..., \sigma_{k-1} [r_\alpha]\}$ and $dom\ \tau_3 = \{\sigma_k [f_\alpha^{-1} r_\alpha], ..., \sigma_{k+m-1} [f_\alpha^{-1} r_\alpha]\}$.

By Lemma 1 we may assume that $\mathbf{a}(x) = \left\{\sigma_i^{-1}(x) : i < k\right\}$ and $\mathbf{b}'(x) = \left\{\sigma_j^{-1}(x) : k \leq j < k + w\right\}$ are without repetition for all $x \in B_{\tau_1}$. Therefore, also $\mathbf{b}(x) = \left\langle f_\alpha^{-1} \sigma_j^{-1}(x) : k \leq j < k + m\right\rangle$ is without repetition.

This can be achieved by extending τ_1. By further extending τ_1, we may also assume that $B_{\tau_1} \cap \sigma_i^{-1} [dom\ p] = \emptyset$ for all $i < k + m$.

If for some $x \in B_\tau$, we had $\mathbf{a}(x) \cap \mathbf{b}(x) = \emptyset$, we could define p' with $dom\ p' = \mathbf{a}(x) \cup \mathbf{b}(x)$ and $p'\left(\sigma_i^{-1}(x)\right) = \tau_2(\sigma_i [r_\alpha])$ for $i < k$ and $p'\left(\sigma_j^{-1}(x)\right) = \tau_3(\sigma_j [f_\alpha^{-1} r_\alpha])$ for $k \leq j < k + m$.

In this case $p \cup p'$ is a condition, $p \cup p' \geq p$ and $p \cup p' \Vdash$ "$x \in B_\tau$" — a contradiction. Therefore, $\mathbf{a}(x) \cap \mathbf{b}(x) \neq \emptyset$ for all $x \in B_\alpha$. This means that for some $i(x) < k$ and $k \leq j(x) < k+m$ we have

$$\sigma_{i(x)}^{-1}(x) = f_\alpha^{-1} \sigma_{j(x)}^{-1}(x) \tag{0.1}$$

and therefore

$$f_\alpha^{-1}(x) = \sigma_{i(x)}^{-1} \sigma_{j(x)}(x) \tag{0.2}$$

By enumerating all possible $\sigma_{i(x)}^{-1}\sigma_{j(x)}$ for $x \in B_{\tau_1}$ in a list $\langle \sigma_\ell : \ell < \ell(x) \rangle$ we obtain for every $x \in B_\tau$:

$$\bigvee\nolimits_{\ell < \ell(*)} f_\alpha^{-1}(x) = \sigma_\ell(x). \tag{0.3}$$

If $f_\sigma^{-1} \neq \sigma_\ell$ for all $\ell < \ell(*)$, apply Lemma 2.3 to obtain $\tau' \supseteq \tau_1$, $\tau \in FF\ \mathcal{F}$ and $\forall x \in B_{\tau'} \forall_{\ell \leq \ell(*)} (f_\alpha^{-1}(x) \neq \sigma_\ell(x))$. This clearly contradicts (3) above. We conclude, then, that for some $\ell < \ell(*)$ we have $f_\alpha^{-1} = \sigma_\ell$, and therefore $f_\alpha \in G_\alpha$ — a contradiction to the assumption.

Let s_α be a Cohen real over a countable transitive model N with $\mathcal{F}_\alpha, G_\alpha, r_\alpha, f_\alpha \in N$. By Lemma 5, $\mathcal{F}_\alpha \cup G_\alpha[r_\alpha] \cup G_\alpha[t_\alpha] \cup G_\alpha[s_\alpha]$ is independent, and $G_\alpha[s_\alpha] \cap (\mathcal{F}_\alpha \cup G_\alpha[r_\alpha] \cap G_\alpha[t_\alpha]) = \emptyset$.

Let $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup G_\alpha[s_\alpha \cap r_\alpha] \cup G_\alpha[t_\alpha]$.

Claim: $\mathcal{F}_{\alpha+1}$ is independent.

Proof: Let $B = \bigcap_{i<k} A^{\tau(i)} \cap \bigcap_{k \leq j < k+m} \bar{\sigma}_j (s_\alpha \cap r_\alpha)^{\tau(j)} \cap \bigcap_{\ell < k+m+n} \rho_\ell(t_\alpha)^{\tau(\ell)}$ be a boolean combination. $B = \bigcap A_i^{\tau(i)} \cap \bigcap \bar{\sigma}_j(r_\alpha)^{r\tau(i)} \cap \bigcap \bar{\sigma}_j(s_\alpha)^{\tau(j)} \cap \bigcap \rho_\ell(t_\alpha)^{\tau(\ell)}$.

Because the orbits under G_α of $t_\alpha, s_\alpha, r_\alpha$ are distinct and $\mathcal{F}_\alpha \cup G_\alpha[r_\alpha] \cup G_\alpha[r_\alpha]$ is independent, we are done.

Now we claim that if $\mathcal{F} \supseteq \mathcal{F}_{\alpha+1}$ is independent, then $f \notin Aut\,\mathcal{F}$. Indeed, $f_\alpha^{-1}[s_\alpha \cap r_\alpha] \subseteq f_\alpha^{-1}[r_\alpha] = t_\alpha$ and $s_\alpha \cap r_\alpha$, $t_\alpha \in \mathcal{F}_{\alpha+1}$. As no two members of an independent family are contained in each other, necessarily $f_\alpha^{-1}(s_\alpha \cap r_\alpha) \notin \mathcal{F}$, and therefore $f_\alpha \notin Aut\,\mathcal{F}$.

Variations

We use the proof of Theorem 7 to obtain a few more results. First, let us see that there are $2^{2^{\aleph_1}}$ non-isomorphic maximal independent families over ω under CH. It is enough to construct $2^{\aleph_1}$ *different* maximal homogeneous families over ω, under CH, because dividing by isomorphism does not change this number (see also [GGK], [KS]).

We imitate here the proof in [GGK] §2, and construct $2^{2^{\aleph}}$ maximal homogeneous families under CH, one for every $\eta \in {}^{\omega_1}2$. The key observation is the following:

Claim: s_α in the definition of $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup G_\alpha[s_\alpha]$ in the proof of Theorem 7 can be chosen as one of two disjoint sets s_α^0, s_α^1.

Proof: s_α was either r_α (a Cohen real over N containing $\mathcal{F}_\alpha, G_\alpha, f_\alpha$) or $s_\alpha \cap r_\alpha$ with s_α Cohen real over M containing $\mathcal{F}_\alpha, G_\alpha, r_\alpha, f_\alpha$. In each case, the Cohen real can be replaced by its complement (by symmetry of the definition).

Corollary 0.8. *(CH) There are $2^{2^{\aleph}}$ non-isomorphic maximal homogeneous families over ω.*

Proof: For every $\eta \in {}^{\omega_1}2$ we construct a maximal homogeneous $\mathcal{F}_\eta \subseteq \mathcal{P}(\omega)$ by induction on $\alpha < \omega_1$, as in the proof of Theorem 7. At stage $\alpha+1$ we let $s_\alpha = s_\alpha^{\eta(\alpha)}$, when s_α^0, s_α^1 are two disjoint sets that satisfy (0) – (3) in the proof of Theorem 7. The proof gives that $\mathcal{F}_\eta$ is maximal homogeneous for every $\eta \in {}^{\omega_1}2$. Moreover, $\eta_1 \neq \eta_2$ and $\eta_1, \eta_2 \in {}^{\omega_1}2$ imply that for the minimal α s.t. $\eta_1(\alpha) \neq \eta_2(\alpha)$ we have $s_\alpha^i \in \mathcal{F}_{\eta_1} \Leftrightarrow s_\alpha^j \in \mathcal{F}_{\eta_2}$ for $i \neq j$ in $\{0,1\}$. Therefore, $\mathcal{F}_{\eta_1} \neq \mathcal{F}_{\eta_2}$ if $\eta_1 \neq \eta_2$. Dividing $\{\mathcal{F}_\eta : \eta \in {}^{\omega_1}2\}$ by isomorphism we obtain $2^{\aleph}$ isomorphism classes, as each class contains $2^{\aleph_0} = \omega_1$ many members of $\{\mathcal{F}_\eta : \eta \in {}^{\omega_1}2\}$.

Another observation we make if the following:

Claim: If $\mathcal{F} = \mathcal{F}_\eta$ is any of the families constructed above, then

(0) $Aut\, \mathcal{F} = G_{\omega_1} = \bigcup_{\alpha<\omega_1} G_\alpha$

(1) if $\mathcal{F}' \supseteq \mathcal{F}$ is independent, then $Aut\ \mathcal{F}' \subseteq Aut\ \mathcal{F}$.

Proof: Clear.

References

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