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FUZZY LOGIC AND CATEGORIES OF FUZZY SETS

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Introduction

This paper deals with three topics:

1. Fuzzy logic,
2. Categories of fuzzy sets,
3. Logic of fuzzy subsets.

While much of its contents can be found in my book [17] and in the existing literature, it also includes new results and a large amount of unpublished folklore. There is also a section discusssing a formal language for fuzzy logic, with interpretations of formulas. Thus I believe that this coherent and not too technical survey of fuzzy logic and categories of fuzzy sets is useful. I have tried to make the paper reasonably self-contained, except that I use the basic language of categories freely. Proofs in Sections 1 and 2 are mostly omitted; they are usually straightforward or can be found in the given references.

When L. ZADEH [18] introduced fuzzy sets, he regarded fuzzy sets essentially as "crisp" sets with a $[0,1]$-valued membership degree function. Membership degrees were soon perceived as truth-values, and this called for a fuzzy logic, with truth-values in the unit interval. ZADEH used Lukasiewicz logic for propositional connectives, without saying why, and most users of fuzzy logic have followed his example. There have been claims in the literature that Lukasiewicz logic *must* be used in certain applications, but these claims do not stand up to scrutiny.

As J. GOGUEN [6] soon pointed out, there is no mathematical need to use the real unit interval as set of truth-values, or to use Lukasiewicz logic. For technical reasons, truth-values must form a complete lattice, and preferably a complete Heyting algebra. This puts intuitionistic logic at our disposal, but it does not exclude non-intuitionistic propositional connectives and logics. We note that every order-complete chain, and in particular the real unit interval $[0,1]$, is a complete Heyting algebra. Open sets of a topological space form a complete Heyting algebra, and every finite Heyting algebra is of this type, up to an isomorphism of Heyting algebras.

In some respects, logic and set theory are siamese twins. Set operations are based on logical connectives and quantifiers, and these connectives and quantifiers can be retrieved from the set operations. We cannot lay a foundation for fuzzy sets without fuzzy logic, and a language for fuzzy logic is based on a category of fuzzy sets.

Everybody working with fuzzy sets seems to agree that fuzzy sets are crisp sets with additional structure, and that this additional structure includes a degree of membership function with values in a complete lattice H. For the categorically minded, fuzzy sets are the objects of a category, and this category should be as set-like as possible. J. GOGUEN [6] was the first, but by no means the last author to present such a category, and it seems likely that there is no single category of fuzzy sets which satisfies all needs. There are basic questions which have not yet been answered to everybody's satisfaction. Two of these questions ask: how fuzzy should things be? Should equality be fuzzy or crisp? And should morphisms be crisp maps, extensional maps, or fuzzy functions? Another important question: what should the underlying fuzzy logic of our set theory be?

We do not try to answer the first two of these three questions; thus Section 2 presents six categories of fuzzy sets, each equipped with fuzzy logic, from which the reader can choose. On the other hand, we do not leave the choice of logics open; we base our categories on intuitionistic logic. This needs some discussion.

When we choose a lattice H of truth-values for membership degrees, then meets and joins in H define standard conjunction and disjunction. If H is complete, then infima and suprema in H define universal and existential quantifiers which generalize standard conjunction and disjunction. By basic principles, implication should be right adjoint to conjunction, and this right adjoint exists if H is a Heyting algebra. The standard logic thus obtained is intuitionistic, it is always there, and it agrees with the basic categorical constructions. Other logics can also be obtained and used, depending on the choice of H, but basing a category of fuzzy sets on a non-intuitionistic logic seems to lead to complications. There may be gains justifying these complications, but I do not see them at this time. Another point is that categories with intuitionistic fuzzy logic may serve as models for categories with a non-standard fuzzy logic.

A sufficiently set-like category has an internal logic. This logic is intuitionistic when it exists, but it may not be what we want and need. For example, the internal logic of Goguen's category of H-valued fuzzy sets is always crisp, *i.e.* classical with just two truth-values. L.N. STOUT [16] has shown a way out of this seeming contradiction. A generalization of Stout's theory, based on a notion of fuzzy subset, was obtained in [17]. For the six categories constructed in Section 2, we present this theory in Section 3.

In Section 2, we justify or motivate axioms for fuzzy sets and maps by translating them informally into a first-order language. Section 4 tries to undergird this by introducing an appropriate first-order language, modelled on the MITCHELL–BÉNABOU language of topos theory, with interpretations in an underlying category of fuzzy sets. The description of such a language is of course just a starting point. Syntax and semantics of the language should be worked out, with particular attention to validity of statements, but this is beyond the scope of the present paper.

We use standard notations as much as possible, but a few remarks on notations may be in order.

For a product $A \times B$ in a category $\mathbf{C}$, and for morphisms $f : X \to A$ and $g : X \to B$, we denote by $\langle f, g \rangle$ the unique morphism of $\mathbf{C}$ characterized by

$$p \circ \langle f,g \rangle = f, \qquad q \circ \langle f,g \rangle = g,$$

for the projections $A \xleftarrow{p} A \times B \xrightarrow{q} B$ of the product.

We use the notation $S \subset T$, as introduced by G. PEANO [12], for subset inclusion, including the case $S \subset S$. There is usually no need for a "proper subset" notation.

Notations like $f(A)$ and $f^{-1}(B)$ for direct and inverse images are too ambigouous to be acceptable, and we replace them by the following. For a function $f : S \to T$ between sets, and for subsets A of S and B of T, we denote by

$$f^{\rightarrow}(A) = \{f(x) \mid x \in A\} \qquad \text{and} \qquad f^{\leftarrow}(B) = \{x \in A \mid f(x) \in B\}$$

the direct image of A and the inverse image of B by f. This defines functions $f^{\rightarrow} : \mathsf{P}S \to \mathsf{P}T$ and $f^{\leftarrow} : \mathsf{P}T \to \mathsf{P}S$ between the powersets $\mathsf{P}S$ and $\mathsf{P}T$.

Parentheses in function-value notation $f(x)$ are often superfluous, and it can be convenient to omit them. We shall always omit unnecessary parentheses for arguments of functors, and we usually omit the parentheses for direct images $f^{\rightarrow}A$ and inverse images $f^{\leftarrow}B$.

1. Fuzzy Propositional Connectives and Quantifiers

1.1. Generalities. Fuzzy logic deals with statements which have truth-values in a complete lattice H, and preferably in a complete Heyting algebra. The standard choice for H is the real unit interval $[0,1]$. Nothing in the theory of fuzzy sets prevents the use of another lattice of truth-values; this may in fact be desirable for certain applications. Products $[0,1]^n$ are an example, with lists $(t_1, \cdots, t_n)$ of real numbers $0 \leq t_i \leq 1$ as truth-values. Open sets of a topological space form a complete Heyting algebra, and Boolean algebras are special Heyting algebras. Decreasing subsets of a preordered set are open sets for a topology; this special case delivers all finite distributive lattices.

We do not deal with modal logic in this paper. Thus if $\diamond$ is a binary propositional connective, then the truth-value of a statement $\varphi \diamond \psi$ will depend only on the truth-values of φ and ψ. This generalizes to other connectives, and it means that we can introduce propositional connectives as operations on the set H of truth-values.

1.2. Standard connectives. We recall that a lattice is an ordered set, or "poset", in which every finite subset has an infimum, also called its *meet*, and a supremum or *join*. Thus every lattice H has a greatest element (meet of the empty subset), which we denote by $\top$, and a least element $\bot$. These elements represent nullary propositional connectives *"true"* and *"false"*. Any two elements p, q of H have a meet $p \wedge q$ and a join $p \vee q$. Viewed as propositional connectives, $\vee$ is *standard disjunction*, and $\wedge$ *standard conjunction.*

A lattice H is called *complete* if every subset of H has a supremum in H, and a *Heyting algebra* if we can define *standard implication* $p \rightarrow q$ for all p, q in H by requiring

$$t \leq p \rightarrow q \qquad \text{iff} \qquad t \wedge q \leq q\,,$$

for all t in H. This leads to *standard negation*, defined by

$$\neg p \;=\; p \rightarrow \bot$$

for $p \in H$. A complete lattice H is a (complete) Heyting algebra if and only if H satisfies the infinite distributive law

$$p \wedge (\bigvee_i q_i) \;=\; \bigvee_i (p \wedge q_i)\,.$$

1.3. Other connectives. There is a general agreement that a fuzzy propositional connective, restricted to the crisp subalgebra $\{\bot, \top\}$ of H, should reduce to the corresponding crisp or "classical" connective. With this in mind; we define a *fuzzy conjunction* as a commutative and associative order preserving binary operation & on H which satisfies $\top \,\&\, p = p = p \,\&\, \top$ for all $p \in H$.

There is also general agreement that *fuzzy implication* $\Rightarrow$ should be right adjoint to fuzzy conjunction &, *i.e.* that for p, q in H, we must have

$$t \leq p \Rightarrow q \qquad \text{iff} \qquad t \,\&\, p \leq p\,,$$

for all t in H. This determines $p \Rightarrow q$ uniquely. It is well known that a fuzzy conjunction & admits a right adjoint implication $p \Rightarrow q$, defined for all p, q in H, if and only if & satisfies the infinite distributive law

$$p \,\&\, (\bigvee_i q_i) \;=\; \bigvee_i (p \,\&\, q_i)\,, \tag{1}$$

for all p in H and all families (q_i) of elements in H. A fuzzy conjunction which satisfies this law is also called a t-*norm*.

Fuzzy negation $\sim p$ is always defined by

$$\sim p = p \Rightarrow \bot .$$

Thus $q \leq \sim p$ for p, q in H iff $p \,\&\, q = \bot$.

There is less agreement on how to connect a fuzzy disjunction with a fuzzy conjunction &. We shall not need fuzzy disjunction; thus we do not discuss it here.

1.4. Examples and comments. For the unit interval $[0,1]$, standard conjunction and implication are given by

$$(1) \qquad p \wedge q = \min(p,q) \quad \text{and} \quad p \rightarrow q = \begin{cases} 1 & \text{if } p \leq q, \\ q & \text{otherwise.} \end{cases}$$

Lukasiewicz conjunction and implication are given by:

$$(2) \quad p \,\&\, q = \max(p+q-1, 0) \quad \text{and} \quad p \Rightarrow q = \min(1, q-p+1),$$

and a third example is given by:

$$(3) \qquad p \,\&\, q = pq \quad \text{and} \quad p \Rightarrow q = \begin{cases} 1 & \text{if } p \leq q, \\ q/p & \text{otherwise.} \end{cases}$$

In these three examples, conjunction is continuous, but implication is only continuous for Lukasiewicz logic. It is a reasonable conjecture that Lukasiewicz conjunction is the only t-norm in $[0,1]$ for which both conjunction and implication are continuous.

This raises the question: how important is continuity of $[0,1]$-valued propositional connectives? We observe that every t-norm is continuous from below by 1.3.(1). All implications satisfy the laws

$$(\bigvee_i p_i) \Rightarrow q = \bigwedge_i (p_i \Rightarrow q) \quad \text{and} \quad p \Rightarrow (\bigwedge_i q_i) = \bigwedge_i (p \Rightarrow q_i).$$

Thus $p \Rightarrow q$ is continuous from below in p, and continuous from above in q. We observe also that continuity only makes sense in the special case $H = [0,1]$, where it is based on order. In the general theory, only order matters.

Standard conjunction is idempotent: $p \wedge p = p$ for $p \in H$. Non-standard conjunctions and t-norms usually are not idempotent, but this does not seem to be tremendously important. Non-commutative conjunctions and t-norms have also been proposed, but the ensuing complications in fuzzy logic have not been fully investigated. We would need two implications, one right adjoint to functors $p \,\&\, —$, and one right adjoint to functors $— \,\&\, p$, and the troubles just begin with this.

1.5. Quantifiers. F.W. LAWVERE [9] observed, and practitioners of fuzzy logic agree, that existential quantification is left adjoint, and universal quantification right adjoint, to substitution. This determines existential and universal quantifiers $\exists_f$ and $\forall_f$ for morphisms, not just for variables.

In the setting of fuzzy logic, mappings $S \to H$ for a set S form a complete Heyting algebra H^S, with order and operations defined point-wise. For a mapping $f : S \to T$, substituting $f(x)$ with $x \in S$ for $y \in T$ means that we replace $h : B \to H$ by $hf : A \to H$. This defines an operator from H^T to H^S, denoted by $f^{\leftarrow} : h \mapsto hf$, which preserves order, suprema and infima, and all propositional connectives defined pointwise.

We assign to $f : S \to T$ quantification maps

$$\exists_f : H^S \to H^T \qquad \text{and} \qquad \forall_f : H^S \to H^T ,$$

left and right adjoint to $f^{\leftarrow}$, by requiring

$$\exists_f \alpha \leq \beta \iff \alpha \leq f^{\leftarrow}\beta \qquad \text{and} \qquad \beta \leq \forall_f \alpha \iff f^{\leftarrow}\beta \leq \alpha ,$$

for $\alpha : S \to H$ and $\beta : T \to H$. As $\alpha \leq f^{\leftarrow}\beta$ iff always $\alpha(x) \leq \beta(y)$ for $y = f(x)$, we get

$$(1) \qquad (\exists_f \alpha)(y) = \bigvee_{f(x)=y} \alpha(x)$$

for $y \in T$. A similar argument shows that

$$(2) \qquad (\forall_f \alpha)(y) = \bigwedge_{f(x)=y} \alpha(x)$$

for $y \in T$.

We note that in particular

$$(\exists_q \alpha)(y) = \bigvee_{x \in S} \alpha(x, y) \qquad \text{and} \qquad (\forall_q \alpha)(y) = \bigwedge_{x \in S} \alpha(x, y) ,$$

for $y \in T$ and the projection $q : S \times T \to T$. Thus $\exists_q$ and $\forall_q$ are quantifieres $(\exists x)$ and $(\forall x)$.

1.6. Formal laws. With fuzzy implication $\Rightarrow$ right adjoint to fuzzy conjunction &, every formal law for & produces a formal law for $\Rightarrow$. From $p \,\&\, \top = p = \top \,\&\, p$, we get

1.6.1. *$\top \Rightarrow p = p$, and $p \Rightarrow q = \top$ iff $p \leq q$.*

Since $p \,\&\, q$ is monotone increasing in p amd q, we have

1.6.2. *$p \,\&\, q \leq p \wedge q$, and $p \to q \leq p \Rightarrow q$.*

The associative law for & is equivalent to

1.6.3. $(p \mathbin{\&} q) \Rightarrow r \;=\; p \Rightarrow (q \Rightarrow r)$.

The idempotent law $p \mathbin{\&} p = p$, which is usually not satisfied for t-norms, states that $p \leq p \Rightarrow q$ iff $p \leq q$, but there is no similar simple translation for the commutative law for &. This fact has led to the consideration of non-commutative t-norms. However, the commutative law for & is often used, as *e.g.* for the following formal laws.

1.6.4. $p \leq (p \Rightarrow q) \Rightarrow q$; *in particular* $p \leq \sim\sim p$.

PROOF. We claim that $p \mathbin{\&} (p \Rightarrow q) \leq q$, and we observe that

$$(p \Rightarrow q) \mathbin{\&} p \leq q \qquad \text{iff} \qquad p \Rightarrow q \leq p \Rightarrow q\,,$$

which is always true.

1.6.5. $p \mathbin{\&} (p \Rightarrow q) \;\leq\; q$, *and* $(p \Rightarrow q) \mathbin{\&} (q \Rightarrow r) \;\leq\; p \Rightarrow r$.

Fuzzy equivalence is defined by

$$p \Leftrightarrow q \;=\; (p \Rightarrow q) \mathbin{\&} (q \Rightarrow p)\,.$$

We note some basic properties of this connective.

1.6.6. (i) $p \Leftrightarrow q = \top$ *iff* $p = q$.
(ii) $p \Leftrightarrow q \;=\; q \Leftrightarrow p$.
(iii) $(p \Leftrightarrow q) \mathbin{\&} (q \Leftrightarrow r) \;\leq p \Leftrightarrow r$.

We note that there are two kinds of logical equivalence for fuzzy logic. A statement "φ iff ψ" means that φ is valid iff ψ is valid. Validity of a statement $\varphi \Leftrightarrow \psi$ is stronger; it means that φ and ψ always have the same truth-value. We also note that validity of statements $\varphi \Rightarrow \psi$ or $\varphi \Leftrightarrow \psi$ depends only on the truth-values of φ and ψ; and not on the particular t-norm & used for $\Rightarrow$ and $\Leftrightarrow$.

We need standard connectives for two useful laws.

1.6.7. (i) $\sim(p \vee q) \;=\; \sim p \wedge \sim q$.
(ii) $p \wedge (p \rightarrow q) \;=\; p \wedge (p \leftrightarrow q) \;=\; p \wedge q$.

The infinite distributive laws for & and $\Rightarrow$ can be stated as equivalences:

1.6.8. *If x does not occur in a statement φ, then*
(i) $\varphi \mathbin{\&} (\exists x)\psi \;\Leftrightarrow\; (\exists x)(\varphi \mathbin{\&} \psi)$,
(ii) $(\exists x)\psi \Rightarrow \varphi \;\Leftrightarrow\; (\forall x)(\psi \Rightarrow \varphi)$,
(iii) $\varphi \Rightarrow (\forall x)\psi \;\Leftrightarrow\; (\forall x)(\varphi \Rightarrow \psi)$,
for any statement ψ.

1.7. Theorem. *For every pullback square*

$$\begin{array}{ccc} P & \xrightarrow{v} & T \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle g} \\ S & \xrightarrow{f} & U \end{array}$$

in **Set**, *we have* $g^{\leftarrow}\exists_f = \exists_v u^{\leftarrow}$ *and* $g^{\leftarrow}\forall_f = \forall_v u^{\leftarrow}$.

Proof. Let $\alpha : S \to H$. By the definitions,

$$(g^{\leftarrow}\exists_f \alpha)(y) = \bigvee_{fx=gy} \alpha(x), \qquad \text{and} \qquad (\exists_v u^{\leftarrow}\alpha)(y) = \bigvee_{vz=y} \alpha(uz),$$

for $y \in T$. Now $u(z) = x$ induces a bijection between $z \in P$ with $v(z) = y$, and $x \in S$ with $f(x) = g(y)$. Thus the two suprema range over the same values, proving the first assertion. Taking right adjoints, we get $f^{\leftarrow}\forall_g = \forall_u v^{\leftarrow}$. This is the second equation with f and g, and u and v, interchanged.

2. Categories of Fuzzy Sets

2.1. Fuzzy and totally fuzzy sets. Throughout this section, we use fuzzy logic informally, with truth-values in a complete Heyting algebra H. A *fuzzy set* A will have fuzzy membership degrees $\varepsilon_A(x)$, and a *totally fuzzy set* A will have fuzzy equality $\delta_A(x, y)$.

Thus we define an H-*valued fuzzy set* as a pair $A = (|A|, \varepsilon_A)$, consisting of a (crisp) set $|A|$ and a membership degree mapping $\varepsilon_A : |A| \to H$, with no further conditions. We regard $\varepsilon_A(x)$, for $x \in |A|$, as truth-value of a statement $x \in A$. An H-valued *totally fuzzy set* is defined as a pair $A = (|A|, \delta_A)$, consisting of a (crisp) set $|A|$ and a fuzzy equality mapping $\delta_A : |A| \times |A| \to H$, subject to the two conditions of symmetry and transitivity:

2.1.1. $\delta_A(x, y) = \delta_A(y, x)$,

2.1.2. $\delta_A(x, y) \wedge \delta_A(y, z) \leq \delta_A(x, z)$,

for all x, y, z in $|A|$.

With $\delta_A(x, y)$ regarded as truth-value of a statement $x =_A y$, these axioms say that the statements

$$x =_A y \iff y =_A x$$

and

$$x =_A y \wedge y =_A z \implies x =_A z$$

are valid for variables x, y, z of type A.

2.2. H-sets and discrete H-sets. A totally fuzzy set A has a fuzzy set structure ε_A, given by putting

$$\varepsilon_A(x) = \delta_A(x,x)$$

for $x \in |A|$, or in other words by requiring that

$$x \in A \iff x =_A x,$$

is valid for a variable x of type A. It follows easily that

$$\delta_A(x,y) \leq \varepsilon_A(x) \wedge \varepsilon_A(y),$$

for x, y in $|A|$, or in other words that

$$x =_A y \Longrightarrow x \in A \wedge y \in B,$$

is valid for variables x, y of type A.

H-valued totally fuzzy sets will also be called *H-sets.*

On the other hand, we can regard fuzzy sets as totally fuzzy sets, with

$$\delta_A(x,y) = \begin{cases} \varepsilon_A(x) & \text{if } x = y, \\ \bot & \text{otherwise}. \end{cases}$$

We shall call H-sets with this property *discrete* H-sets.

2.3. The category $\mathbf{Set}_{dc}H$. This is the category introduced by J. Goguen [6], with crisp equality and crisp morphisms. Its objects are discrete fuzzy sets, and a morphism $f : A \to B$ of $\mathbf{Set}_{dc}H$ is a mapping $f : |A| \to |B|$ of the underlying crisp sets with

$$\varepsilon_A(x) \leq \varepsilon_B(f(x)),$$

for all $x \in |A|$, or in other words such that

$$x \in A \Longrightarrow f(x) \in B,$$

is valid for a variable x of type A. Composition of morphisms is composition of the underlying mappings.

It is easily seen that $\mathbf{Set}_{dc}H$ is a topological category over sets in the sense of [1], with small fibres. Thus $\mathbf{Set}_{dc}H$ is complete and cocomplete, with limits and colimits lifted from sets. The forgetful functor from $\mathbf{Set}_{dc}H$ to sets has a concrete left adjoint V and a concrete right adjoint C, assigning to every set S the *void* fuzzy set VS with $\varepsilon_{VS}(x) = \bot$, and the *crisp* set CS with $\varepsilon_{CS}(x) = \top$, for $x \in S$.

It is well known, and proved *e.g.* in [17], that $\mathbf{Set}_{dc}H$ is a quasitopos. This fact does not help us much for fuzzy logic since $\mathbf{Set}_{dc}H$ shares with all topological categories over sets the property that its internal logic is crisp. Overcoming this handicap has been a strong motivation for Stout's theory of fuzzy subsets.

2.4. $\mathbf{Set}_{tc}H$: totally fuzzy sets and crisp maps. We define a *map* or *crisp map* $f : A \to B$ of H-sets as a mapping $f : |A| \to |B|$ such that

$$\delta_A(x, x') \leq \delta_B(fx, fx')$$

for x, x' in $|A|$, or in other words such that

$$x =_A x' \implies f(x) =_B f(x')$$

is valid for variables x, x' of type A. It follows easily that

$$\varepsilon_A(x) \leq \varepsilon_B(fx), \qquad \text{or} \qquad x \in A \implies f(x) \in B,$$

for x in $|A|$.

With composition of underlying mappings as composition, H-sets and their maps form a concrete category over sets which we denote by $\mathbf{Set}_{tc}H$. This is a topological category over sets, not previously discussed in the literature, with $\mathbf{Set}_{dc}H$ as a full concrete subcategory. The full embedding $I : \mathbf{Set}_{dc}H \to \mathbf{Set}_{tc}H$ has a concrete right adjoint J, with JA for an H-set A obtained by

$$\delta_{JA}(x, x') = \begin{cases} \varepsilon_A(x) & \text{if } x = x', \\ \bot & \text{otherwise.} \end{cases}$$

The full embedding I also preserves all collectively injective initial sources, and hence all categorical limits. Thus I also has a left adjoint (and left inverse) $K : \mathbf{Set}_{tc}H \to \mathbf{Set}_{dc}H$. If $\equiv$ is the equivalence relation on $|A|$ generated by the pairs (x, x') with $\delta_A(x, x') \neq \bot$, then the unit $\eta_A : A \to IKA$ is the quotient mapping $|A| \to |A|/\equiv$, with

$$\varepsilon_{KA}(y) = \bigvee_{\eta_A(x)=y} \varepsilon_A(x)$$

for $y \in |KA|$.

2.5. Finite products. Every categogy of fuzzy sets has a terminal object, or empty product, which we denote by $\mathbf{1}$, with $|\mathbf{1}|$ a singleton $\{\star\}$, with fuzzy membership $\varepsilon_A(\star) = \top$.

The product $A \times B$ of H-sets A and B is given by

$$|A \times B| = |A| \times |B|,$$

and $$\delta_{A\times B}(\langle x, y\rangle, \langle x', y'\rangle) = \delta_A(x, x') \wedge \delta_B(y, y')$$

for x, x' in $|A|$ and y, y' in $|B|$. Projections of this product are the projections of $|A| \times |B|$.

It is easily seen that this defines products $A \times B$ in the categorical sense, for all six categories of fuzzy sets considered in this paper; see 2.16 below.

2.6. Fuzzy relations. We define a *fuzzy relation* on an H-set A as a mapping $\alpha : |A| \to H$ which satisfies the following inequalities:

2.6.1. $\alpha(x) \leq \varepsilon_A(x)$ for $x \in |A|$, and

2.6.2. $\alpha(x) \wedge \delta_A(x, x') \leq \alpha(x')$ for x, x' in $|A|$.

This means that we require the validity of

$$\alpha(x) \implies x \in A, \qquad \text{and} \qquad \alpha(x) \wedge (x =_A x') \implies \alpha(x'),$$

for variables x, x' of type A We note that 2.6.2 is equivalent to

$$\alpha(x) \wedge \delta_A(x, x') = \alpha(x') \wedge \delta_A(x, x'),$$

for x, x' in $|A|$.

With pointwise order, $\alpha \leq \beta$ iff $\alpha(x) \leq \beta(x)$ for every $x \in |A|$, fuzzy relations on A form a complete lattice which we denote by H^A. Arbitrary suprema $\bigvee_i \alpha_i$, and nonempty infima $\bigwedge_i \alpha_i$, in H^A are obtained pointwise, and ε_A is the top element of H^A. It follows easily that H^A is a complete Heyting algebra, with implication given by

$$(\alpha \to \beta)(x) = \varepsilon_A(x) \wedge (\alpha(x) \to \beta(x)),$$

for every $x \in |A|$. We shall discuss this further in 3.8 and 3.9.

2.7. Binary fuzzy relations. For H-sets A and B, a *binary fuzzy relation* $\rho : A \to B$ is defined as a fuzzy relation ρ on $A \times B$. We regard $\rho(x, y)$ as the truth-value of a statement $x\rho y$. The defining inequalities for relations then become statements

$$x\rho y \implies x \in A \wedge y \in B,$$

and

$$x\rho y \wedge x =_A x' \wedge y =_B y' \implies x'\rho y',$$

valid for variables x, x' of type A and y, y' of type B.

Fuzzy relations $\rho : A \to B$ and $\sigma : B \to C$ can be composed, with

$$(\sigma \circ \rho)(x, z) = \bigvee_{y \in |B|} (\rho(x, y) \wedge \sigma(y, z))$$

for $x \in |A|$ and $z \in |C|$. This means that

$$x(\sigma \circ \rho)z \iff (\exists y)(x\rho y \wedge y\sigma z)$$

is valid, for variables x, y, z of types A, B, C. Using the distributivity of $\wedge$ over $\bigvee$ in the complete Heyting algebra H, one sees easily that this defines a binary relation $\sigma \circ \rho : A \to C$. Composition is associative, and fuzzy equalities are identity relations $\delta_A : A \to A$. Thus we have a category of H-sets and fuzzy relations.

Every fuzzy relation $\rho : A \to B$ has a *dual fuzzy relation* $\rho^{\text{op}} : B \to A$, given by

$$\rho^{\text{op}}(y,x) = \rho(x,y),$$

for $x \in |A|$ and $y \in |B|$. Dual relations satisfy $(\rho^{\text{op}})^{\text{op}} = \rho$, and

$$(\sigma \circ \rho)^{\text{op}} = \rho^{\text{op}} \circ \sigma^{\text{op}}$$

if either composition is defined.

Fuzzy relations on an H-set B can also be regarded as binary fuzzy relations $B \to \mathbf{1}$. With this convention, we have a useful lemma.

2.7.1. Lemma. *If $\rho, \sigma : A \to B$ are binary fuzzy relations such that $\beta \circ \rho = \beta \circ \sigma$ for all β in H^B, then $\rho = \sigma$.*

PROOF. For $\beta = \delta_B(y, —)$ with $y \in |B|$, the compositions are $\rho(—, y)$ and $\sigma(—, y)$.

2.8. Special fuzzy relations. For H-sets A and B, binary fuzzy relations $\rho : A \to B$ form a complete Heyting algebra $H^{A \times B}$, with order and suprema defined point-wise. It is easily seen that composition of relations satisfies infinite distributive laws

$$\sigma \circ (\bigvee_i \rho_i) = \bigvee_i (\sigma \circ \rho_i) \qquad \text{and} \qquad (\bigvee_i \sigma_i) \circ \rho = \bigvee_i (\sigma_i \circ \rho),$$

valid whenever either side is defined.

We say that $\rho : A \to B$ is:
single-valued if always $\rho(x,y) \wedge \rho(x,y') \leq \delta_B(y,y')$,
injective if always $\rho(x,y) \wedge \rho(x',y) \leq \delta_A(x,x')$,
total if always $\bigvee_y \rho(x,y) = \varepsilon_A(x)$,
surjective if always $\bigvee_x \rho(x,y) = \varepsilon_B(y)$.
These properties are clearly dual in pairs. We note that a fuzzy relation $\rho : A \to B$ is
single-valued iff $\rho \circ \rho^{\text{op}} \leq \delta_B$,
injective iff $\rho^{\text{op}} \circ \rho \leq \delta_A$,
total iff $\rho^{\text{op}} \circ \rho \geq \delta_A$,
surjective iff $\rho \circ \rho^{\text{op}} \geq \delta_B$.
The first two inequalities are almost immediate; the other two inequalities take a bit longer to prove.

It follows easily that all four classes of fuzzy relations are closed under composition and include all equality relation. We note that $\rho \circ \rho^{\text{op}} = \delta_B$ iff ρ is single-valued and surjective, and that $\rho^{\text{op}} \circ \rho = \delta_A$ iff ρ is injective and total.

2.9. $\mathbf{Set}_{tf}H$: totally fuzzy sets and fuzzy functions. For H-sets A and B, we define a *fuzzy function* $\rho : A \to B$ as a single-valued and total fuzzy relation. Identity relations are fuzzy functions, and the relational composition of fuzzy functions $\rho : A \to B$ and $\sigma : B \to C$ is a fuzzy function $\sigma\rho : A \to C$. Thus H-sets and fuzzy functions define a category which we denote by $\mathbf{Set}_{tf}\, H$. This category was introduced by D. HIGGS in a widely circulated but never published preprint [7], and denoted by $\mathbf{Set}\, H$. It has also been studied in detail by M. FOURMAN and D. SCOTT in [4].

2.10. We note two useful special properties of fuzzy functions.

Proposition. (i) *If $\rho : A \to B$ is a total and $\sigma : A \to B$ a single-valued fuzzy relation, with $\rho \leq \sigma$, then $\rho = \sigma$.*

(ii) *If $\sigma \circ \rho = \delta_A$ and $\rho \circ \sigma = \delta_B$, with $\rho : A \to B$ and $\sigma : B \to A$ both single-valued or both total fuzzy relations, then $\sigma = \rho^{\mathrm{op}}$, and ρ and σ are surjective and injective fuzzy functions.*

PROOF. For (i), we have

$$\sigma \leq \sigma \circ \rho^{\mathrm{op}} \circ \rho \leq \sigma \circ \sigma^{\mathrm{op}} \circ \rho \leq \rho .$$

For (ii) with ρ and σ total, we have

$$\sigma \leq \rho^{\mathrm{op}} \circ \rho \circ \sigma = \rho^{\mathrm{op}} , \qquad \text{and} \qquad \rho^{\mathrm{op}} \leq \rho^{\mathrm{op}} \circ \sigma^{\mathrm{op}} \circ \sigma = \sigma .$$

The proof for ρ and σ single-valued is similar.

2.11. $\mathbf{Set}_{te}\, H$: Totally fuzzy sets and extensional maps. One trouble with maps of H-sets is that they are not always extensional, *i.e.* the statement

$$(\forall x)(x \in A \Longrightarrow f(x) =_B g(x))$$

may be valid for distinct maps $f, g : A \to B$. We say that crisp maps $f, g : A \to B$ are *extensionally equal* if the displayed statement is valid, *i.e.* if

$$\varepsilon_A(x) \leq \delta_B(f(x), g(x))$$

for all $x \in |A|$. Extensional equality clearly is an equivalence relation; we define an *extensional map* $[f] : A \to B$ as an equivalence class of a map $f : A \to B$ for this relation. We note that extensional maps were called crisp in [17]. With composition $[g][f] = [gf]$ if gf is defined, totally fuzzy sets and extensional maps form a category which we denote by $\mathbf{Set}_{te}\, H$.

This category has been constructed in two different ways. From a map $f : A \to B$, we can construct the set R_f of all pairs (x, y) in $|A| \times |B|$ with $\varepsilon_A(x) \leq \delta_B(f(x), y)$. D. PONASSE [14, 15] characterized sets R of this form by the following three properties:

(i) For x in $|A|$, there is always y in $|B|$ with $(x, y) \in R$,

(ii) If $(x,y) \in R$ and $(x',y') \in R$, then $\delta_A(x,x') \leq \delta_B(y,y')$,

(iii) If $(x,y) \in R$ and $\varepsilon_A(x) \leq \delta_B(y,y')$, then $(x,y') \in R$,

and used them as morphisms $R : A \to B$ of totally fuzzy sets. A set R satisfying these conditions always contains graphs of mappings $f : |A| \to |B|$. These mappings are crisp maps $f : A \to B$, with $R = R_f$, and $R_f = R_{f'}$ iff f and f' are extensionally equal. Composition is given by $R_g R_f = R_{gf}$, for maps $f : A \to B$ and $g : B \to C$.

PONASSE called $\mathbf{Set}_{te} J$, with J denoting the complete Heyting algebra of truth-values, a category of totally fuzzy sets (ensembles totalement flous) and denoted it by JTF.

2.12. Fuzzy functions induced by crisp maps. The second construction is due to G.P. MONRO [11]. From a mapping $f : |A| \to |B|$, we construct $\langle f \rangle : |A| \times |B| \to H$ by

$$\langle f \rangle(x,y) = \varepsilon_A(x) \wedge \delta_B(f(x),y),$$

for $(x,y) \in |A| \times |B|$. We say that $\langle f \rangle$ is *induced* by f, and note the following result.

Lemma. *$\langle f \rangle$ is a fuzzy function from A to B if and only if f is a map from A to B, and $\langle f \rangle = \langle f' \rangle$, for maps $f, f' : A \to B$, iff f and f' are extensionally equal. Moreover, id_A induces δ_A, and $\langle g \rangle \langle f \rangle = \langle gf \rangle$ if a composition gf of maps is defined.*

MONRO defined the category $\mathbf{Mod} H$ of *H-valued models* as the subcategory of $\mathbf{Set} H = \mathbf{Set}_{tf} H$ with non-empty H-sets as objects, and with fuzzy functions $\langle f \rangle$ induced by crisp maps as morphisms. In view of the Lemma just stated, we replace extensional maps by induced fuzzy functions, replacing $[f] : A \to B$ by $\langle f \rangle : A \to B$ for a map $f : A \to B$. With these replacements, $\mathbf{Set}_{te} H$ without the empty set becomes MONRO's subcategory $\mathbf{Mod} H$ of $\mathbf{Set} H$. We note that

$$(\sigma \circ \langle f \rangle)(x,z) = \varepsilon_A(x) \wedge \sigma(f(x),z),$$

for $(x,z) \in |A| \times |C|$, if $f : A \to B$ is a crisp map and $\sigma : B \to C$ a fuzzy function.

2.13. Singletons. We say that a fuzzy relation β on a totally fuzzy set B is a *singleton* on B if β, considered as a binary fuzzy relation $B \to \mathbf{1}$, is injective, *i.e.* if

$$\beta(y) \wedge \beta(y') \leq \delta_B(y,y')$$

for all y, y' in $|B|$. With $\varepsilon_{SB}(\sigma) = \bigvee_{y \in |B|} \sigma(y)$, and

$$\delta_{SB}(\sigma,\tau) = \bigvee_{y \in |B|} (\sigma(y) \wedge \tau(y)) = \varepsilon_{SB}(\sigma) \wedge \bigwedge_{y \in |B|} (\sigma(y) \leftrightarrow \tau(y)),$$

for singletons σ, τ on B, singletons on B form an H-set. We denote this H-set by SB.

Regarded as fuzzy relations $\beta : \mathbf{1} \to B$, singletons on B are partial maps: single-valued but not necessarily total.

Every mapping $\delta_B(y, —) : |B| \to H$ is a singleton on B, and

$$s : y \mapsto \delta_B(y, —) : B \to SB$$

defines a map $s : B \to SB$, with

$$\delta_{SB}(s(y), \sigma) = \sigma(y) = \langle s \rangle (y, \sigma)$$

for $y \in |B|$, a singleton σ on B, and the induced fuzzy function $\langle s \rangle$ of s. It follows that $\langle s \rangle$ is injective and surjective, and thus an isomorphism of $\mathbf{Set}_{tf}\, H$ with inverse $\langle s \rangle^{\mathrm{op}}$, but not necessarily an isomorphism in $\mathbf{Set}_{te}\, H$.

For a fuzzy function $\rho : A \to B$, the mappings $\rho(x, —) : |B| \to H$, for $x \in |A|$ are singletons on B. It is easily seen that this defines a map

$$r : x \mapsto \rho(x, —) : A \to SB,$$

with $\langle r \rangle (x, s(y)) = \rho(x, y)$ for (x, y) in $|A| \times |B|$. Conversely, if $\varphi : A \to SB$ in $\mathbf{Set}_{tf}\, H$, then $\varphi = \langle s \rangle \circ \rho$ for a unique $\rho : A \to B$, and hence $\varphi = \langle r \rangle$ for the map $r : A \to SB$ obtained above from ρ. Thus every morphism $\varphi : A \to SB$ in $\mathbf{Set}_{te}\, H$ is induced by a map, and composition by $\langle s \rangle$ defines a bijection between morphisms $\rho : A \to B$ in $\mathbf{Set}_{tf}\, H$ and morphisms $\langle r \rangle : A \to SB$ in $\mathbf{Set}_{te}\, H$. This bijection is clearly natural in A; thus H-sets of singletons define a right adjoint of the embedding $\mathbf{Set}_{te}\, H \to \mathbf{Set}_{tf}\, H$.

2.14. Six categories of fuzzy sets. In addition to the four categories already described, we have a category $\mathbf{Set}_{de}\, H$ of discrete H-sets and extensional maps, and a category $\mathbf{Set}_{df}\, H$ of discrete H-sets and fuzzy functions. The latter category was introduced by M. EYTAN [3], who denoted it by $\mathbf{Fuz}\, H$.

The six categories can be arranged in a commutative diagram

$$\begin{array}{ccccc} \mathbf{Set}_{dc}\, H & \longrightarrow & \mathbf{Set}_{de}\, H & \longrightarrow & \mathbf{Set}_{df}\, H \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Set}_{tc}\, H & \longrightarrow & \mathbf{Set}_{te}\, H & \longrightarrow & \mathbf{Set}_{tf}\, H \end{array} \tag{1}$$

of categories and functors. In this diagram, the vertical arrows are full embeddings, and the horizontal arrows are bijective on objects. The horizontal arrows at left are full, but not faithful, and the horizontal arrows at right are embeddings, but not full embeddings.

2.15. Void fuzzy sets. We say that an H-set A is *void* if $\varepsilon_A(x) = \bot$ for all $|x| \in A$. A void H-set is necessarily discrete, and there is a void set A with $|A| = S$ for every crisp set S. The empty H-set is void.

For a void H-set A and an H-set B, there is exactly one fuzzy relation $\zeta : A \to B$, with $\zeta(x,y) = \bot$ for all (x,y) in $|A| \times |B|$, This relation is an injective fuzzy function, and surjective iff B is void. Every mapping $f : |A| \to |B|$ is a map $f : A \to B$ with $\langle f \rangle = \zeta$. It follows that all void sets are isomorphic in $\mathbf{Set}_{tf}\, H$ and in $\mathbf{Set}_{df}\, H$, and initial objects of these categories.

If $\rho : A \to B$ is a fuzzy function with B void, then A is void, and ρ an isomorphism in $\mathbf{Set}_{df}\, H$ and $\mathbf{Set}_{tf}\, H$.

The situation is more complicated for $\mathbf{Set}_{te}\, H$ and $\mathbf{Set}_{de}\, H$. In these categories, only the empty set is an initial object. All non-empty void sets are isomorphic, with exactly one morphism to any non-empty H-set, but there is no morphism from a non-empty object to the empty fuzzy set.

We conclude from this discussion that it is always safe, and in the extensional case definitely a simplification, to remove the empty fuzzy set from categories of fuzzy sets with extensional maps or fuzzy functions as morphisms. If we do so, we must modify the functors in 2.14 from categories with crisp maps to categories with extensional maps or fuzzy functions, by $\emptyset \mapsto \mathbf{0}$ for a specified non-empty void H-set $\mathbf{0}$.

2.16. Adjunctions and finite limits. We have already described in 2.4 the left and right adjoint functors of the full embedding $\mathbf{Set}_{dc}\, H \to \mathbf{Set}_{tc}\, H$, and in 2.13 the right adjoint S of the embedding $\mathbf{Set}_{te}\, H \to \mathbf{Set}_{tf}\, H$.

Two crisp maps $f, g : A \to B$ of discrete H-sets are extensionally equal if and only if $f(x) = g(x)$ for all $x \in |A|$ with $\varepsilon_A(x) \neq \bot$. It follows easily that the functor $\mathbf{Set}_{dc}\, H \to \mathbf{Set}_{de}\, H$ has a left adjoint R, *if the empty set is removed from* $\mathbf{Set}_{de}\, H$, with RA obtained for a non-empty discrete fuzzy set A by removing all x in $|A|$ with $\varepsilon_A(x) = \bot$, with $\varepsilon_{RA}(x) = \varepsilon_A(x)$ for the remaining elements of $|A|$. For an extensional map $\langle f \rangle : A \to B$, the map $R\langle f \rangle$ is the restriction of $f : |A| \to |B|$ to $|RA|$ and $|RB|$.

For special Heyting algebras H, there may be other adjunctions related to the diagram of 2.14, but we shall not discuss them.

Finite products are constructed in the same way in all six categories of 2.14, and hence preserved by all functors in the diagram of 2.14. For fuzzy functions $\rho : C \to A$ and $\sigma : C \to B$, we get $\langle \rho, \sigma \rangle : C \to A \times B$ by putting

$$\langle \rho, \sigma \rangle (z, x, y) \;=\; \rho(z, x) \wedge \sigma(z, y),$$

for z, x, y in $|C| \times |A| \times |B|$. It follows immediately that $\langle \rho, \sigma \rangle$ is induced by the map $\langle f, g \rangle$ if ρ and σ are induced by maps $f : C \to A$ and $g : C \to B$. We shall see in 3.12 that equalizers, and hence also pullbacks, are constructed in the same way in the four categories with extensional maps or fuzzy functions as morphisms; thus the embedding functors between these categories preserve all finite limits. The embeddings $\mathbf{Set}_{dc}H \to \mathbf{Set}_{tc}H$ and $\mathbf{Set}_{dc}H \to \mathbf{Set}_{de}H$ have left adjoints and thus preserves all categorical limits. The following example shows that the functor $f \mapsto \langle f \rangle : \mathbf{Set}_{tc}H \to \mathbf{Set}_{te}H$ does not preserve equalizers or pullbacks.

2.16.1. Example. Let B be a fuzzy set with two elements $0, 1$, with $\varepsilon_B(0) = \varepsilon_B(1) = \top$ and $\delta_B(0,1) \neq \bot$. There are two maps $f, g : \mathbf{1} \to B$, with an empty equalizer in $\mathbf{Set}_{tc}H$. By 3.12, the equalizer of $\langle f \rangle$ and $\langle g \rangle$ in $\mathbf{Set}_{te}H$ is a non-void singleton; thus the functor $f \mapsto \langle f \rangle$ does not preserve the equalizer of f and g.

2.17. Topoi and quasitopoi of fuzzy sets. GOGUEN's category $\mathbf{Set}_{dc}H$ is a topological universe, a topological quasitopos over sets. Unfortunately, this makes the internal logic of $\mathbf{Set}_{dc}H$ crisp, hence useless for purposes of fuzzy logic. The category $\mathbf{Set}_{tc}H$ of H-sets and crisp maps is cartesian closed, but not a quasitopos.

As is well-known (see *e.g.* [7, 4, 17]), the category $\mathbf{Set}_{tf}H$ of H-sets and fuzzy functions is a topos, with H-valued internal logic and with very pleasing set-theoretic constructions. However, most practitioners of fuzzy logic prefer extensional maps to fuzzy functions, which brings us to $\mathbf{Set}_{te}H$. This is a quasitopos, with H-valued internal logic and equally pleasing set-theoretic constructions, and equivalent to the full subcategory of $\mathbf{Set}_{tf}H$ given by separated objects, in the topos-theoretic sense.

The other categories in 2.14.(1) are not topoi or quasitopoi, except for special Heyting algebras H (see *e.g.* [13] or [2]).

2.18. Two special cases. If H is a singleton, then H-sets and discrete H-sets just are sets, and maps just are mappings. Any two mappings $f : S \to T$ become extensionally equal; thus we have exactly one extensional map $f : S \to T$, except for S non-empty and T empty. For fuzzy functions, even this distinction disappears; we have exactly one $\rho : S \to T$ for any two sets S and T.

The case $H = \{\bot, \top\}$ is a bit more interesting. In this case, an H-set A is a triple $(|A|, S_A, \alpha_A)$, consisting of a set $|A|$, a subset S_A of $|A|$, and an equivalence relation α_A on S_A, given by $x \in S_A$ iff $\varepsilon_A(x) = \top$, and $x \alpha_A y$ iff $\delta_A(x, y) = \top$. For discrete H-sets, α_A is the identity relation on S_A. Crisp maps $f : A \to B$ map S_A into S_B and preserve equivalence. A map $f : A \to B$ induces a mapping $[f] : S_A/\alpha_A \to S_B/\alpha_B$ of equivalence classes, and maps $f, g : A \to B$ are extensionally equal iff $[f] = [g]$. Fuzzy functions with non-empty codomain are induced by maps, and in addition there is a unique fuzzy function $Z \to \emptyset$ for every void H-set Z.

3. Fuzzy Subsets and their Logic

3.1. Subsets and insertions. Fuzzy relations on an H-set A can also be viewed as fuzzy subset structures of A. We assign to every fuzzy relation α on A a *fuzzy subset* $A\lceil\alpha$ of A, with

(i) $|A\lceil\alpha| = |A|$, and

(ii) $\delta_{A\lceil\alpha}(x,x') = \alpha(x) \wedge \delta_A(x,x') = \alpha(x') \wedge \delta_A(x,x')$ for x,x' in $|A|$.

This equality relation is clearly symmetric and transitive, with

$$\varepsilon_{A\lceil\alpha}(x) \;=\; \alpha(x)$$

for $x \in |A|$, and $\mathrm{id}_{|A|}$ lifts to a map

$$j_\alpha : A\lceil\alpha \rightarrow A$$

which we call the *insertion* of $A\lceil\alpha$ into A.

This works equally well for all six categories $\mathbf{Set}_{xy}\,H$ introduced in Section 2, and we shall see that many properties of fuzzy subsets are shared by the six categories. We note that a fuzzy subset of a discrete H-set is again discrete.

3.2. Categories $\mathcal{M}A$. As already noted in 2.6, fuzzy subsets structures of an H-set A form a complete Heyting algebra H^A. The top element of H^A is ε_A, with $A\lceil\varepsilon_A = A$ and insertion id_A, and $A\lceil\bot$ is void for the bottom element. If $\alpha \le \beta$ in H^A, then $A\lceil\alpha$ is a fuzzy subset of $A\lceil\beta$, with $j_\alpha = j_\beta j_{\alpha,\beta}$ for the insertion $j_{\alpha,\beta} : A\lceil\alpha \rightarrow A\lceil\beta$. Thus fuzzy subsets of an H-set A and their subset insertions form a category, isomorphic to H^A regarded as a category. We denote this category by $\mathcal{M}A$.

For the two categories with crisp maps, fuzzy subset insertions are monomorphic and epimorphic. Regarded as extensional maps or fuzzy functions, fuzzy subset insertions are injective (2.8), and hence monomorphic. Thus categories $\mathcal{M}A$ are full subcategories of the slice category $(\mathbf{Set}_{xy}\,H)/A$, in each of the six categories $\mathbf{Set}_{xy}\,H$.

In the three categories of discrete H-sets, every map $\mathrm{id}_{|A|} : C \rightarrow A$ is (or induces) a subset insertion. This is not the case for totally fuzzy sets, where $\langle \mathrm{id}_{|A|} \rangle$ need not be injective, and may be injective but not a subset insertion.

3.3. Direct and inverse images. For a map $f : A \rightarrow B$ and fuzzy subset structures α of A and β of B, we define the *image* $f^{\rightarrow}\alpha$ of α by f by putting

$$(f^{\rightarrow}\alpha)(y) \;=\; \bigvee_{x\in|A|} (\alpha(x) \wedge \delta_B(f(x),y))$$

for $x \in |A|$, and the *inverse image* $f^{\leftarrow}\beta$ of β by f by

$$(f^{\leftarrow}\beta)(x) \;=\; \varepsilon_A(x) \wedge \beta(f(x))\,,$$

for $x \in |B|$. It is easily verified that $f^{\rightarrow}\alpha$ is a fuzzy subset structure of B, and $f^{\leftarrow}\beta$ a fuzzy subset structure of A. We note that

$$(f^{\rightarrow}\alpha)(y) = \bigvee_{f(x)=y} \alpha(x)$$

for a map $f : A \to B$ of discrete H-sets.

For a fuzzy function $\rho : A \to B$, we define the image $\rho^{\rightarrow}\alpha$ and the inverse image $\rho^{\leftarrow}\beta$ by putting

$$(1) \qquad (\rho^{\rightarrow}\alpha)(y) = \bigvee_{x\in|A|} (\alpha(x) \wedge \rho(x,y)),$$

$$(2) \qquad (\rho^{\leftarrow}\beta)(x) = \bigvee_{y\in|B|} (\rho(x,y) \wedge \beta(y)),$$

for $y \in |B|$ and $x \in |A|$ respectively. These again are fuzzy subset structures as desired.

For a map $f : A \to B$, the maps $f^{\rightarrow}$ and $f^{\leftarrow}$ clearly satisfy

$$(3) \qquad f^{\rightarrow} = \langle f\rangle^{\rightarrow} \qquad \text{and} \qquad f^{\leftarrow} = \langle f\rangle^{\leftarrow};$$

thus $f^{\rightarrow}$ and $f^{\leftarrow}$ are well defined for extensional maps. For a fuzzy function $\rho : A \to B$, the maps $\rho^{\rightarrow}\alpha$ amd $\rho^{\leftarrow}\beta$ clearly preserve order and thus define functors

$$\rho^{\rightarrow} : \mathcal{M}A \to \mathcal{M}B \qquad \text{and} \qquad \rho^{\leftarrow} : \mathcal{M}B \to \mathcal{M}A.$$

If we regard α and β as binary fuzzy relations $\alpha : A \to \mathbf{1}$ and $\beta : B \to \mathbf{1}$, then clearly

$$(4) \qquad \rho^{\rightarrow}\alpha = \alpha \circ \rho^{\mathrm{op}} \qquad \text{and} \qquad \rho^{\leftarrow}\beta = \beta \circ \rho.$$

It follows that $\rho^{\rightarrow}$ and $\rho^{\leftarrow}$ are functorial, with

$$(\mathrm{id}_A)^{\rightarrow} = \mathrm{Id}\,\mathcal{M}A = (\mathrm{id}_A)^{\leftarrow}$$

for a fuzzy set A, and

$$(\sigma \circ \rho)^{\rightarrow} = \sigma^{\rightarrow} \circ \rho^{\rightarrow} \qquad \text{and} \qquad (\sigma \circ \rho)^{\leftarrow} = \rho^{\leftarrow} \circ \sigma^{\leftarrow}$$

if a composition $\sigma \circ \rho$ of fuzzy functions is defined.

By (3), the properties obtained in the preceding paragraph for fuzzy functions are also valid for crisp maps and extensional maps.

3.4. Theorem. *For a fuzzy function $\rho : A \to B$ and fuzzy subset structures α of A and β of B, the following are equivalent.*

(i) $\alpha(x) \wedge \rho(x,y) \le \beta(y)$ *for all $x \in |A|$ and $y \in |B|$.*

(ii) $\alpha \leq \rho^{\leftarrow}\beta$.
(iii) $\rho^{\rightarrow}\alpha \leq \beta$.
(iv) $\rho \circ j_\alpha = j_\beta \circ \sigma$ *for a fuzzy function* $\sigma : A\lceil\alpha \to B\lceil\beta$.

If ρ *is induced by a map* $f : A \to B$, *then* σ *in* (iv) *is induced by the map* $f : A\lceil\alpha \to B\lceil\beta$.

By 3.3.(3), this result applies to all six categories of fuzzy sets.

PROOF. (i) $\iff$ (iii) is immediate from the definition of $\rho^{\rightarrow}\alpha$.

If (iii) holds, then $\alpha\rho^{\mathrm{op}} \leq \beta\rho\rho^{\mathrm{op}} \leq \beta$ by 2.8, and (ii) holds. Conversely, if (ii) holds, then $\alpha \leq \alpha\rho^{\mathrm{op}}\rho \leq \beta\rho$, and (iii) is valid.

From the definitions, we have

$$\begin{aligned}(\rho \circ j_\alpha)(x,y) &= \bigvee_{x' \in |A|} (\alpha(x) \wedge \delta_A(x,x') \wedge \rho(x',y)) \\ &= \alpha(x) \wedge \bigvee_{x' \in |A|} (\delta_A(x,x') \wedge \rho(x',y)) = \alpha(x) \wedge \rho(x,y),\end{aligned}$$

and similarly $(j_\beta \circ \sigma)(x,y) = \sigma(x,y) \leq \beta(y)$ if σ in (iv) exists. Thus we have (i) $\iff$ (iv).

If ρ is induced by $f : A \to B$, then

$$\sigma(x,y) = \alpha(x) \wedge \beta(y) \wedge \delta_B(fx,y)$$

in (iv); thus σ is induced by $f : A\lceil\alpha \to B\lceil\beta$ if (i)–(iv) are valid.

3.5. Theorem. *For a map* $f : A \to B$ *or a fuzzy function* $\rho : A \to B$, *and for fuzzy subset structures* β *of* B *and* $\alpha = \rho^{\leftarrow}\beta$ *of* A, *the commutative square*

$$(1) \qquad \begin{array}{ccc} A\lceil\alpha & \xrightarrow{f} & B\lceil\beta \\ \downarrow{\scriptstyle j_\alpha} & & \downarrow{\scriptstyle j_\beta} \\ A & \xrightarrow{f} & B \end{array} \qquad \text{or} \qquad \begin{array}{ccc} A\lceil\alpha & \xrightarrow{\sigma} & B\lceil\beta \\ \downarrow{\scriptstyle j_\alpha} & & \downarrow{\scriptstyle j_\beta} \\ A & \xrightarrow{\rho} & B \end{array}$$

of 3.4.(iv) *is a pullback square.*

PROOF. For maps, with ρ induced by $f : A \to B$, (1) lifts a pullback in **Set**; thus (1) is a pullback for $f : C \to B$ with $|C| = |A|$ and

$$\delta_C(x,x') = \delta_A(x,x') \wedge \beta(fx) \wedge \delta_B(fx,fx').$$

Since $\delta_A(x,x') \leq \delta_B(fx,fx')$, this means that $C = A\lceil f^{\leftarrow}\beta$.

For fuzzy functions $\varphi : C \to A$ and $\psi : C \to B\lceil\beta$, with $\rho \circ \varphi = j_\beta \circ \psi$, we have

$$\varphi(z,x) \wedge \rho(x,y) \leq \psi(z,x) \leq \beta(y)$$

for $x \in |A|$, $y \in |B|$, $z \in |C|$, and hence

$$\varphi(z,x) \wedge \rho(x,y) \ \le\ \rho(x,y) \wedge \beta(y)\,.$$

Taking suprema $\bigvee_y$ and using totality of ρ, we get

$$\varphi(z,x) \wedge \varepsilon_A(x) \ =\ \varphi(z,x) \ \le\ (\rho^{\leftarrow}\beta)(x)\,.$$

Thus $\varphi = j_\alpha \circ \chi$ for a fuzzy function $\chi : C \longrightarrow A\lceil\alpha$, by 3.4 for id_C and j_α. Then also $\psi = \sigma \circ \chi$ since j_β is monomorphic; thus the Theorem is valid for fuzzy functions.

If φ is induced by a map, then so is χ by 3.4, thus the Theorem is also valid for extensional maps.

3.6. Covers. Using a concept of [5], we say that a morphism $f : A \longrightarrow B$ is a *cover* if $f^{\rightarrow}\varepsilon_A = \varepsilon_B$.

In $\mathbf{Set}_{tf}H$ and its subcategories $\mathbf{Set}_{te}H$, $\mathbf{Set}_{df}H$, $\mathbf{Set}_{de}H$, covers are the same as surjective morphisms. It follows that covers in these categories are epimorphic. Conversely, it can be proved that epimorphisms in $\mathbf{Set}_{tf}H$ and in $\mathbf{Set}_{te}H$ are covers. Epimorphisms in $\mathbf{Set}_{de}H$ and in $\mathbf{Set}_{df}H$ need not be covers.

Covers in $\mathbf{Set}_{dc}H$ and in $\mathbf{Set}_{tc}H$ can have any underlying mapping and thus need not be epimorphic. In $\mathbf{Set}_{dc}H$ covers are the same as final maps $f : A \longrightarrow B$, *i.e.* maps with the following property: if a map $h : A \longrightarrow C$ factors $h = gf$ at the set level, then $g : |B| \longrightarrow |C|$ always lifts to a map $g : B \longrightarrow C$. This is not the case for $\mathbf{Set}_{tc}H$. For example, if A is an H-set, and if C is the discrete H-set with $|C| = |A|$, and $\varepsilon_C(x) = \varepsilon_A(x)$ for $x \in |A|$, then the map $\mathrm{id}_{|A|} : C \longrightarrow A$ is always a cover, but final only if A is discrete, and $C = A$.

3.7. The following result shows that covers and fuzzy subset insertions define a factorization structure in the sense of [1], or a diagonal polarity in the sense of [17], for each of the six categories considered in Section 2.

Theorem. (i) *Every morphism $\rho : A \longrightarrow B$ has a unique factorization $\rho = j_\beta \circ e$ into a cover e followed by a fuzzy subset insertion. If ρ is a fuzzy function induced by a map $f : A \longrightarrow B$, then e is induced by $f : A \longrightarrow B\lceil\beta$.*

(ii) *Every commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ e\ } & C \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\sigma} \\ B\lceil\beta & \xrightarrow{\ j_\beta\ } & B \end{array}\ ,$$

with e a cover and j_β a subset insertion, has a unique diagonal $\tau : C \to B\lceil\beta$ with $\tau \circ e = \rho$ and $j_\beta \circ \tau = \sigma$. If σ is a fuzzy function induced by a map $g : C \to B$. then τ is induced by $g : C \to B\lceil\beta$.

PROOF. For (i), we must put $\beta = \rho^{\rightarrow}\varepsilon_A$, and then ρ factors as claimed, by 3.4.

For (ii), we have $\varepsilon_C = e^{\rightarrow}\varepsilon_A$, and hence

$$\sigma^{\rightarrow}e_C = \rho^{\rightarrow}\varepsilon_A \le \beta .$$

But then σ factors $\sigma = j_\beta \circ \tau$ by 3.4, with $\tau e = \rho$ since j_β is monomorphic. If σ is a fuzzy function induced by a map g, then τ is induced by g.

3.7.1. Remark. In the factorization $\rho j_\alpha = j_\beta \sigma$ of 3.4, the morphism σ is a cover iff $\beta = \rho^{\rightarrow}\alpha$. Thus the left adjoint $f^{\rightarrow}$ or $\rho^{\rightarrow}$ of the pullback functor $f^{\leftarrow}$ or $\rho^{\leftarrow}$ is obtained by (cover, insertion) factorizations. This is a special case of a general result for factorization structures; see [1] or [17].

3.8. Propositional connectives. The top element of a Heyting algebra H^A is ε_A, corresponding to id_A in $\mathcal{M}A$, and the bottom element has constant value $\perp$, corresponding in $\mathcal{M}A$ to the insertion of the void fuzzy subset of A into A.

Unary and binary propositional connectives can be carried out pointwise in complete Heyting algebras $H^{|A|}$, and the result is natural in A in the sense that

$$f^{\leftarrow}(\beta \diamond \beta') = (f^{\leftarrow}\beta) \diamond (f^{\leftarrow}\beta'), \tag{1}$$

for β, β' in $H^{|B|}$, a mapping $f : |A| \to |B|$ and a binary connective $\diamond$, with a similar formula for a unary connective. This follows immediately from the fact that $(f^{\leftarrow}\beta)(x) = \beta(fx)$ for $x \in |A|$.

Algebras H^A need not be closed under pointwise connectives; thus a further step is needed. Since H^A is closed under suprema in $H^{|A|}$, the embedding $I_A : H^A \to H^{|A|}$ has a right adjoint $M_A : H^{|A|} \to H^A$, called *modification*, with

$$M_A\alpha = \bigvee\{\gamma \in H^A \mid \gamma \le \alpha\}$$

for $\alpha \in H^{|A|}$. Now

$$\alpha \diamond \alpha' = M_A(I_A\alpha \diamond I_A\alpha'), \tag{2}$$

with the pointwise connective at right, is the best we can do. However, the naturality expressed by (1) is usually lost by this process.

For standard connectives, we can do better.

3.9. Proposition. *Standard conjunction and disjunction in H^A are given by pointwise evaluation, and implication by*

$$(1) \qquad (\alpha \rightarrow \alpha')(x) = \varepsilon_A(x) \wedge (\alpha(x) \rightarrow \alpha'(x)),$$

for $x \in |A|$. All standard connectives in algebras H^A are preserved by inverse image functors.

PROOF. The first part is clear for $\wedge$, and follows for $\vee$ immediately from the $(\vee, \wedge)$ distributivity of H^A. Now if

$$t \leq \delta_A(x, x') \wedge (\alpha(x) \rightarrow \alpha'(x)),$$

then we have

$$t \wedge \alpha(x') = t \wedge \alpha(x') \wedge \delta_A(x, x') = t \wedge \alpha(x) \wedge \delta_A(x, x') \\ \leq \alpha'(x) \wedge \delta_A(x, x') \leq \alpha'(x').$$

Thus $t \leq \alpha(x') \rightarrow \alpha'(x')$, so that $\alpha \rightarrow \alpha'$ defined by (1) is in H^A.

By 3.3.(3), it suffices for the second part to consider $\rho^{\leftarrow} : H^B \rightarrow H^A$ for a fuzzy function $\rho : A \rightarrow B$. Since $\rho^{\leftarrow}$ has a left adjoint and a right adjoint, by 3.11, it preserves top and bottom of H^A. For $\beta \wedge \beta'$ and $x \in |A|$, we have

$$\begin{aligned}(\rho^{\leftarrow}\beta)(x) \wedge (\rho^{\leftarrow}\beta')(x) &= \bigvee_{y \in |B|} (\rho(x, y) \wedge \beta(y)) \wedge \bigvee_{y' \in |B|} (\rho(x, y') \wedge \beta(y')) \\ &= \bigvee_{y, y'} (\rho(x, y) \wedge \rho(x, y') \wedge \beta(y) \wedge \beta'(y')) \\ &= \bigvee_{y, y'} (\rho(x, y) \wedge \rho(x, y') \wedge \delta_B(y, y') \wedge \beta(y) \wedge \beta'(y')) \\ &= \bigvee_{y, y'} (\rho(x, y) \wedge \beta(y) \wedge \beta'(y) \wedge \delta_B(y, y')) \\ &= \bigvee_{y} (\rho(x, y) \wedge \beta(y) \wedge \beta'(y)) = (\rho^{\leftarrow}(\beta \wedge \beta'))(x).\end{aligned}$$

using single-valuedness of ρ. The proof for $\beta \vee \beta'$ is similar, but simpler.

For implication, we have

$$\alpha \leq \rho^{\leftarrow}\beta \rightarrow \rho^{\leftarrow}\beta' \iff \alpha \wedge \rho^{\leftarrow}\beta \leq \rho^{\leftarrow}\beta' \iff \rho^{\rightarrow}(\alpha \wedge \rho^{\leftarrow}\beta) \leq \beta',$$

and

$$\alpha \leq \rho^{\leftarrow}(\beta \rightarrow \beta') \iff \rho^{\rightarrow}\alpha \leq \beta \rightarrow \beta' \iff \rho^{\rightarrow}\alpha \wedge \beta \leq \beta'.$$

Now the following Lemma, which is of independent interest, completes the proof.

3.10. Lemma. *For α in H^A, β in H^B, and $\rho : A \rightarrow B$, we have:*

$$\rho^{\rightarrow}(\alpha \wedge \rho^{\leftarrow}\beta) = \rho^{\rightarrow}\alpha \wedge \beta.$$

PROOF. Starting with the value of the lefthand side at $y \in |B|$, we have

$$\begin{aligned}
&\bigvee_{x\in|A|} (\alpha(x) \wedge \rho(x,y) \wedge \bigvee_{y'\in|B|} (\rho(x,y') \wedge \beta(y'))) \\
&\quad = \bigvee_{x,y'} (\alpha(x) \wedge \beta(y') \wedge \rho(x,y) \wedge \rho(x,y')) \\
&\quad = \bigvee_{x,y'} (\alpha(x) \wedge \beta(y') \wedge \rho(x,y) \wedge \rho(x,y') \wedge \delta_B(y,y')) \\
&\quad = \bigvee_{x} (\alpha(x) \wedge \rho(x,y) \wedge \beta(y)) = \beta(y) \wedge \bigvee_{x} (\alpha(x) \wedge \rho(x,y)),
\end{aligned}$$

which is $(\rho^{\rightarrow}\alpha \wedge \beta)(y)$.

3.11. Quantifiers. It is universally agreed, following [9], that universal quantifiers are right adjoint, and existential quantifiers left adjoint, to substitution, and it is also generally agreed that substitution is represented in categories by pullbacks of subobjects.

This means in our present context that substitution is given by pullback functors $f^{\leftarrow} : \mathcal{M}B \to \mathcal{M}A$, for morphisms $f : A \to B$, and that existential quantifiers are given by factorization functors

$$\exists_f = f^{\rightarrow} : \mathcal{M}A \to \mathcal{M}B .$$

For a fuzzy function $\rho : A \to B$, and for fuzzy subset structures α of A and β of B, we want

$$\beta \leq \forall_\rho \alpha \quad \Longleftrightarrow \quad \rho^{\leftarrow}\beta \leq \alpha ,$$

and this is the case iff

$$\rho(f,y) \wedge \beta(y) \leq \alpha(x) ,$$

for all $x \in |A|$ and $y \in |B|$. Thus we must put

$$(1) \qquad (\forall_\rho \alpha)(y) = \varepsilon_B(y) \wedge \bigwedge_{x\in|A|} (\rho(x,y) \to \alpha(x)) ,$$

for all $y \in |B|$. This defines the universal quantifier functor $\forall_\rho$.

Since $f^{\leftarrow} = \langle f \rangle^{\leftarrow}$ for a map f, we put

$$\forall_f = \forall_{\langle f \rangle}$$

for a map or extensional map f. We note that for a map $f : A \to B$ of discrete H-sets, this specializes to

$$(\forall_f \alpha)(y) = \varepsilon_B(y) \wedge \bigwedge_{fx=y} (\varepsilon_A(x) \to \alpha(x)) .$$

By 3.3, the assignments $A \mapsto \mathcal{M}A$ and $f \mapsto f^{\leftarrow}$, or $\rho \mapsto \rho^{\leftarrow}$, define contravariant functors on the six categories of 2.14. It follows by adjunction that existential and universal quantifiers define covariant functors on these six categories.

3.12. Equalizers. For the topological categories $\mathbf{Set}_{dc}H$ and $\mathbf{Set}_{tc}H$, equalizers are embeddings and lifted from **Set**. For the other four categories of 2.14, the following result describes equalizers.

Proposition. *For fuzzy functions $\rho : A \to B$ and $\sigma : A \to B$, the subset embedding $j_\mu : A\lceil\mu \to A$ with*

$$\mu(x) = \bigvee_{y \in |B|} (\rho(x,y) \wedge \sigma(x,y)),$$

for $x \in |A|$, is an equalizer of ρ and σ.

If ρ and σ are induced by maps f and g, then

$$\mu(x) = \varepsilon_A(x) \wedge \delta_B(fx, gx)$$

for $x \in |A|$, and j_μ is an equalizer of $\langle f \rangle$ and $\langle g \rangle$ for extensional maps.

PROOF. If $\rho\varphi = \sigma\varphi$ for $\varphi : C \to A$, then

$$\begin{aligned}\varphi(z,x) \wedge \rho(x,y) &\le \varphi(z,x) \wedge \bigvee_{x' \in |A|} (\varphi(z,x') \wedge \sigma(x',y)) \\ &= \bigvee_{x'} (\varphi(z,x) \wedge (\varphi(z,x') \wedge \sigma(x',y)) \\ &\le \bigvee_{x'} (\delta_A(x,x') \wedge \sigma(x',y)) = \sigma(x,y)\end{aligned}$$

for (x,y,z) in $|A| \times |B| \times |C|$, and

$$\varphi(z,x) \wedge \rho(x,y) = \varphi(z,x) \wedge \rho(x,y) \wedge \sigma(x,y)$$

follows. Taking suprema $\bigvee_y$ and using the totaltiy of ρ, we get $\varphi(z,x) \le \mu(x)$. Thus φ factors ψj_μ by the proof of 3.4. Similar computations show that

$$\rho(x,y) \wedge j_\mu(x) = \sigma(x,y) \wedge j_\mu(x)$$

for (x,y) in $|A| \times |B|$, so that $\rho j_\mu = \sigma j_\mu$.

For the second part, if $\rho = \langle f \rangle$ and $\psi = \langle g \rangle$, then

$$\mu(x) = \varepsilon_A(x) \wedge \bigvee_{y \in |B|} (\delta_B(fx,y) \wedge \delta_B(gx,y)) = \varepsilon_A(x) \wedge \delta_B(fx,gx)$$

for $x \in |A|$, and if φ in the preceding paragraph is induced by $h : C \to A$, then ψ is induced by $h : C \to A\lceil\mu$, by 3.4.

3.13. Theorem. *Let* **FS** *be one of the categories of* 2.14. *For a pullback square*

(1)
$$\begin{array}{ccc} P & \xrightarrow{v} & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & B \end{array}$$

in **FS**, *the following are equivalent.*

(i) $g^{\leftarrow} \circ \exists_f = \exists_v \circ u^{\leftarrow}$. (ii) $g^{\leftarrow} \circ \forall_f = \forall_v \circ u^{\leftarrow}$.

(iii) $f^{\leftarrow} \circ \exists_g = \exists_u \circ v^{\leftarrow}$. (iv) $f^{\leftarrow} \circ \forall_g = \forall_u \circ v^{\leftarrow}$.

(v) *The functor* $\mathbf{FS} \to \mathbf{Set}_{tf} H$ *in* 2.14 *preserves the pullback* (1).

Except for $\mathbf{FS} = \mathbf{Set}_{tc} H$, *these conditions are satisfied for every pullback square in* **FS**.

The maps f and g of 2.16.1 provide an example of a pullback in $\mathbf{Set}_{tc} H$ for which (i)–(v) are not satisfied.

Proof. As in the proof of 1.7, (i) $\iff$ (iv) and (ii) $\iff$ (iii) by adjunction. It will be convenient to extend the notation $\langle f \rangle$ from maps to fuzzy function, putting $\langle f \rangle = f$ in this case. Then (iii) is valid, by 3.3.(4) and 2.7.1, iff

$$\langle g \rangle^{\mathrm{op}} \circ \langle f \rangle = \langle v \rangle \circ \langle u \rangle^{\mathrm{op}}.$$

(i) requires equality for the dual relations; thus (i) $\iff$ (iii).

Now the relations $\langle g \rangle^{\mathrm{op}} \circ \langle f \rangle$ and $\langle v \rangle \circ \langle u \rangle^{\mathrm{op}}$ can be regarded as fuzzy subset structures μ and ν of $A \times B$. For fuzzy functions, we get

$$\mu(x, y) = \bigvee_{z \in |C|} (f(x, z) \wedge g(y, z))$$

for (x, y) in $|A| \times |B|$; this becomes

$$\mu(x, y) = \varepsilon_A(x) \wedge \varepsilon_B(y) \wedge \delta_C(fx, gy)$$

for crisp or extensional maps.

The map $\langle u, v \rangle : P \to A \times B$ in (1) is the equalizer of fp and gq for the projections p and q of $A \times B$. For $\mathbf{Set}_{tf} H$ and its subcategories, this equalizer is given by the subset structure

$$\bigvee_z ((fp)(x, y, z) \wedge (gq)(x, y, z)) = \bigvee_z (f(x, z) \wedge g(y, z));$$

this is the structure μ of the preceding paragraph. We have $u = pj_\mu$ and $v = qj_\mu$, and thus

$$(\langle v\rangle \circ \langle u\rangle^{\mathrm{op}})(x,y) = \bigvee_{x',y'} (\delta_A(x,x') \wedge \mu(x',y') \wedge \delta_B(y,y')) = \mu(x,y).$$

Thus (i)–(iv) are valid in these categories.

For $\mathbf{Set}_{tc}H$, the pullback P is lifted from sets, with

$$\delta_P(t,t') = \delta_A(ut,ut') \wedge \delta_B(vt,vt')$$

for $t \in P$. It follows for $\nu = \langle v\rangle\langle u\rangle^{\mathrm{op}}$ that

$$\nu(x,y) = \bigvee_{t\in|P|} (\delta_A(ut,x) \wedge \delta_B(vt,y)).$$

Now $h = \langle u,v\rangle$ defines an embedding $h : P \to A \times B$, with $\nu = h^{\to}\varepsilon_P$. The induced fuzzy function $\langle h\rangle$ is injective; thus $\langle h\rangle : P \to (A \times B)\lceil\nu$ is an isomorphism of $\mathbf{Set}_{tf}H$. This shows that $\nu = \mu$ iff the pullback (1) remains a pullback in $\mathbf{Set}_{tf}H$, proving (iii) $\Longleftrightarrow$ (v).

Except for $\mathbf{FS} = \mathbf{Set}_{tc}H$, the functor $f \mapsto \langle f\rangle : \mathbf{FS} \to \mathbf{Set}_{tf}H$ in 2.14 preserves all pullbacks, by 2.16.

3.14. Powerset objects. In category theory, powersets represent relations.

For a set T with powerset $\mathsf{P}T$, introduce a backward membership relation $\ni_T\colon \mathsf{P}T \to T$ by putting $B \ni_T y$, for $y \in T$ and $B \subset T$, iff $y \in B$. If $\rho : S \to T$ is a relation, then $\rho = \ni_T \circ h$ for a mapping $h : S \to \mathsf{P}T$ iff

$$y \in h(x) \iff x\rho y,$$

for $x \in S$ and $y \in T$. This mapping h is called the *characteristic mapping* of ρ.

This can be generalized to categories with relations, which includes our six categories of fuzzy sets. We say that a relation $\ni_B\colon \mathsf{P}B \to B$, in a category $\mathbf{C}$ with relations, *represents relations* with codomain B if every relation $\rho : A \to B$ has a unique factorization $\rho = \ni_B \circ \langle h\rangle$ with $h : A \to \mathsf{P}B$ in $\mathbf{C}$, and $\langle h\rangle$ the relation induced by h. In this situation, $\mathsf{P}B$ is called a *powerset object* of B, with *membership relation* $\ni_B$. If the factorization always exists, but is not unique, then we speak of a *weak powerset object.*

For topological spaces with continuous maps, and with relations $\rho : X \to Y$ given by subspaces (and hence by sub*sets*) of the product space $X \times Y$, a powerset object $\mathsf{P}Y$ is the powerset $\mathsf{P}|Y|$ of the underlying set $|Y|$ of the space Y, with the indiscrete topology. This generalizes to all topological categories over sets.

Categories with relations were dicussed in a paper by A. KLEIN [8], for categories with (cover,insertion) factorization structures, with epimorphic covers and monomorphic insertions. This setting excludes fuzzy sets or H-sets with crisp maps. An exposition of relations in categories which includes these examples is planned, but cannot be accommodated in this paper.

3.15. Fuzzy powersets. Let **FS** be one of the six categories of 2.14, and let $\langle f \rangle$ be the fuzzy function induced by a morphism f of **FS**, with $\langle f \rangle = f$ if the morphisms of **FS** are fuzzy functions. We define the *fuzzy powerset* $\mathsf{P}B$ of an object B of **FS** as the object with underlying set H^B of fuzzy subset structures of B, with $\varepsilon_{\mathsf{P}B}(\beta) = \top$ for $\beta \in H^B$. If the objects of **FS** are discrete H-sets, then $\mathsf{P}B$ is a crisp discrete H-set. If the objects of **FS** are totally fuzzy sets, then we define $\delta_{\mathsf{P}B}$ by

$$\delta_{\mathsf{P}B}(\beta, \beta') = \bigwedge_{y \in |B|} (\beta(y) \leftrightarrow \beta'(y)),$$

for β, β' in H^B. In other words, we require

$$\beta =_{\mathsf{P}B} \beta' \leftrightarrow (\forall y)(\beta(y) \leftrightarrow \beta'(y)),$$

with y a variable of type B.

For all six categories **FS**, we put

$$\ni_B (\beta, y) = \beta(y),$$

for $\beta \in H^B$ and $y \in |B|$.

We skip the proof that this always works, *i.e.* that $\delta_{\mathsf{P}B}$ always is a fuzzy equality, and $\ni_B \colon \mathsf{P}B \to B$ a fuzzy relation.

Theorem. *The fuzzy powersets just defined are powerset objects for* $\mathbf{Set}_{te} H$ *and* $\mathbf{Set}_{tf} H$, *and weak powerset objects for the other four categories of* 2.14.

PROOF. For a fuzzy relation $\rho : A \to B$ and $x \in |A|$, put

$$(\chi\rho)(x) = \rho(x, -) : |B| \to H.$$

This is in H^B, and it is easily verified that we have defined a map $\chi\rho : A \to \mathsf{P}B$. Moreover by the definitions,

$$\ni_B ((\chi\rho)(x), y) = \rho(x, y)$$

for $(x, y) \in |A| \times |B|$. Thus $\ni_B \circ \langle \chi\rho \rangle = \rho$, and $\mathsf{P}B$ is a weak powerset object in each of the six categories.

If $\rho = \ni_B \circ \varphi$ for a fuzzy function $\varphi : A \to \mathsf{P}B$ in $\mathbf{Set}_{tf} H$, then

$$\rho(x, y) = \bigvee_{\beta} (\varphi(x, \beta) \wedge \beta(y))$$

for (x, y) in $|A| \times |B|$. Thus $\varphi(x, y) \wedge \beta(y) \leq \rho(xy)$ for β in H^B, and

$$\begin{aligned}\varphi(x,\beta) \wedge \rho(x,y) &= \bigvee_{\beta'}(\varphi(x,\beta) \wedge \varphi(x,\beta') \wedge \beta'(y)) \\ &= \bigvee_{\beta'}(\varphi(x,\beta) \wedge \delta_{\mathsf{P}B}(\beta,\beta') \wedge \beta'(y)) \;=\; \varphi(x,\beta) \wedge \beta(y)\,.\end{aligned}$$

It follows that

$$\varphi(x,\beta) \leq \varepsilon_A(x) \wedge \bigwedge_y (\rho(x,y) \wedge \beta(y)) = \langle \chi\rho \rangle (x,\beta)\,,$$

for (x, β) in $|A| \times H^B$. But then $\varphi = \langle \chi\rho \rangle$ by 2.10.(i), and $\mathsf{P}B$ is a powerset object in $\mathbf{Set}_{tf}\,H$ and in $\mathbf{Set}_{te}\,H$.

4. A Language for Fuzzy Logic

4.1. Outline. We shall describe a language for fuzzy logic, consisting of typed terms and statements, and based on a category **FS** of H-valued fuzzy sets or totally fuzzy sets, where H is a complete Heyting algebra of truth-values. We do not specify the requirements for **FS**, but all six categories introduced in Section 2 qualify. Types of terms will be objects of **FS**, and statements form a type Ω which need not be an object of **FS**. We shall also describe models of the language obtained from interpretations of its formulas, where formulas of the language are its terms and statements, but we shall not discuss validity of statements for these models.

The language to be described is a variant of the MITCHELL–BÉNABOU language of topos theory, as described *e.g.* in [17], and in a slightly different form in [10].

4.2. Terms. Terms are recursively defined as follow.

4.2.1. Every variable is a term, and there is a sufficiently large supply of variables of type A for every object A of **FS**.

4.2.2. There is a term $\star$ of type $\mathbf{1}$, with $\mathbf{1}$ a fixed terminal object of **FS**.

4.2.3. For terms s of type S and t of type T, there is a term $\langle s, t\rangle$ of type $S \times T$.

Generalizing 4.2.3 recursively, we get a term $\langle t_1, \cdots, t_n\rangle$ of type $\prod_{i=1}^n A_i$ for terms t_i of types A_i, with $\langle\rangle = \star$ for $n = 0$.

4.2.4. For a term t of type A and a morphism $f : A \to B$ in **FS**, there is a term $f(t)$ of type B.

This last requirement may have to be modified if interpretations of statements with non-standard propositional connectives are desired.

4.3. Statements. Statements are recursively defined as follows.

4.3.1. For terms s and t of type A, there are atomic statements

$$(t) \in A \qquad \text{and} \qquad (s) =_A (t).$$

4.3.2. For a term t of type A and a fuzzy subset structure α of A, there is an atomic statement $\alpha(t)$.

4.3.3. The elements $\top$ and $\bot$ of H are atomic statements.

If $\square$ is a unary propositional connective and φ a statement, then $\square(\varphi)$ is a statement.

For statements φ, ψ and a binary propositional connective $\diamond$, there is a statement $(\varphi) \diamond (\psi)$.

4.3.4. If φ is a statement and x a variable, then $(\forall x)(\varphi)$ and $(\exists x)(\varphi)$ are statements in which all occurences of the variable x in φ are replaced by links to the quantifier. Thus the variable x does *not* occur in the quantified statements.

4.4. Parentheses and Substitution. In 4.2 and 4.3, we have put parentheses around all formulas which appear as building blocks of mor complex formulas. These parentheses can often be omitted, but we do not try to formulate rules for this. Of course, "Polish" notation would allow us to do away with parentheses altogether.

Formulas can be interpreted as rooted trees in the obvious way, with the operations introduced in 4.2 and 4.3 as nodes. Leaves of these trees are variables, links to quantifiers, the term $\star$, and nullary propositional connectives. Every node has a unique path to the root of the ree. If a leaf is a link to a quantifier $(\exists x)$ or $(\forall x)$, then the quantifier is in the path from the leaf to the root of the tree, and the first quantifier $(\exists x)$ or $(\forall x)$ in this path.

Our language allows unrestricted substitutions of terms of the same type for variables. Substituting a term Σ of the same type for an occurrence of a variable x in a formula F means replacing the leaf x in the tree for F by the tree for Σ. We denote by $F[\Sigma \leftarrow\!\!\dashv x]$ (read "Σ for x") the result of substituting the term Σ, of the same type as x, for all occurrences of the variable x in the formula F. We note the following obvious fact.

4.4.1. *If u is a variable of the same type as x, but not occurring in a formula F, then $(F[u \leftarrow\!\!\dashv x])[\Sigma \leftarrow\!\!\dashv u]$ is the same formula as $F[\Sigma \leftarrow\!\!\dashv x]$.*

4.5. Products of types. We denote by T_x the type of a variable x. For a finite set L of variables, we denote by P_L "the" product $\prod_{x \in L} T_x$, with projections $\pi_x^L : P_L \to T_x$. We put in particular $P_{\{x\}} = T_x$ and $P_\emptyset = \mathbf{1}$. If $L \subset L'$, then $\pi_x^L \circ \pi_L^{L'} = \pi_x^{L'}$, for $x \in L$, defines a projection $\pi_L^{L'} : P_{L'} \to P_L$. In particular, $\pi_\emptyset^L$ is the unique morphism $P_L \to \mathbf{1}$.

It will also be convenient to put $Lx = L \cup \{x\}$ if $x \notin L$. Then P_{Lx} is a product $P_L \times T_x$, with projections π_L^{Lx} and π_x^{Lx}, and every element v of $|P_{Lx}|$ can be regarded as a pair (t, u) in $|P_L| \times |T_x|$, with $t = \pi_L^{Lx}(v)$ and $u = \pi_x^{Lx}(v)$.

If variables x and y have the same type and do not occur in L, then

$$\pi_L^{Lx} = \pi_L^{Ly} \circ \pi_{Ly}^{Lx}, \qquad \pi_x^{Lx} = \pi_y^{Ly} \circ \pi_{Ly}^{Lx},$$

for a morphism $\pi_{Ly}^{Lx} : P_{Lx} \longrightarrow P_{Ly}$. This is obviously an isomorphism, with inverse π_{Lx}^{Ly}.

We note that always $\pi_L^{L''} = \pi_L^{L'} \circ \pi_{L'}^{L''}$ if the three projections are defined.

4.6. Interpretations of terms. Proceeding recursively in the obvious way, we assign to every formula F a finite set of all variables occuring in F. This set, called the *support* of F, may be empty, as *e.g.* for the term $\star$ or a statement $(\exists x)(x \in A)$. If F is a term of type A, and L a finite set of variables containing the support of F, then we define the interpretation $|F|_L$ as a mapping $|P_L| \longrightarrow |A|$, or as a crisp map $P_L \longrightarrow A$ in **FS**, by the following rules.

4.6.1. $|x|_L = \pi_x^L$ for a variable x in L.

4.6.2. $|\langle s,t \rangle|_L = \langle |s|_L, |t|_L \rangle : P_L \longrightarrow A \times B$ if the interpretations $|s|_L : P_L \longrightarrow A$ and $|t|_L : P_L \longrightarrow B$ are defined.

4.6.3. $|\star|_L = \pi_\emptyset^L$, the unique mapping or morphism $P_L \longrightarrow \mathbf{1}$.

4.6.4. $|f(t)|_L = f \cdot |t|_L$ if the righthand side is defined.

For a list $t_1, \cdots, t_n$ of terms of types A_i, it follows that

$$|\langle t_1, \cdots, t_n \rangle|_L = \langle |t_1|_L, \cdots . |t_n|_L \rangle : P_L \longrightarrow \prod_i A_i .$$

if the interpretations $|t_i|_L$ are defined.

4.7. Heyting algebras $[A, H]$. In order to interpret statements, we assign to every object A of **FS** a complete Heyting algebra $[A, H]$ of mappings $\alpha : |A| \longrightarrow H$, ordered pointwise and satisfying the following conditions.

4.7.1. $[A, H]$ is closed in $H^{|A|}$ under meets $\alpha \wedge \alpha'$, and under arbitrary suprema.

4.7.2. $[A, H]$ contains the algebra H^A of fuzzy subset structures of A.

4.7.3. For every map $f : A \longrightarrow B$ in **FS**, there is a composition map $f^{\leftarrow} : \beta \longmapsto \beta \cdot f$ from $[B, H]$ to $[A, H]$ which preserves infima and suprema. These maps are functorial, with

$$(\mathrm{id}_A)^{\leftarrow} = \mathrm{id}_{[A,H]}, \qquad \text{and} \qquad (gf)^{\leftarrow} = f^{\leftarrow} \cdot g^{\leftarrow}$$

if gf is defined in **FS**.

As in 3.8, there are modification functors $M_A : H^{|A|} \to [A, H]$, left adjoint to the embeddings $I_A : [A, H] \to H^{|A|}$; thus we can use (1) and (2) in 3.8 to obtain propositional connectives in the algebras $[A, H]$.

Examples of algebras $[A, H]$ are the algebras $H^{|A|}$ and the algebras H^A of 2.6. Other examples are the sets of mappings $\alpha : |A| \to H$ which satisfy just one of the conditions 2.6.1 and 2.6.2.

4.8. Interpretations of statements. An interpretation $|\varphi|_L$ of a statement φ will be an element of the complete Heyting algebra $[P_L, H]$, as follows.

4.8.1. For terms s, t of type A, with $|s|_L$ and $|t|_L$ defined, we put

$$|t \in A|_L = \varepsilon_A \cdot |t|_L, \quad \text{and} \quad |s =_A t|_L = \delta_A \cdot \langle |s|_L, |t|_L \rangle .$$

4.8.2. For a subset structure α of an object A of **FS**, and for a term t of type A with $|t|_L$ defined, we put

$$|\alpha(t)|_L = \alpha \cdot |t|_L .$$

4.8.3. $|\top|_L$ and $|\bot|_L$ are the top and bottom elements of $[P_L, H]$. For statements φ and ψ with $|\varphi|_L$ and $|\psi|_L$ defined, we put $|\square\, \varphi|_L = \square\, |\varphi|_L$ for a unary propositional connective $\square$, and $|\varphi \diamond \psi|_L = |\varphi|_L \diamond |\psi|_L$ for a binary propositional connective $\diamond$.

4.8.4. If x of type A is not in L and $|\varphi|_{Lx}$ is defined, then

$$|(\exists x)\varphi|_L = \exists_p |\varphi|_{Lx} \quad \text{and} \quad |(\forall x)\varphi|_L = \forall_p |\varphi|_{Lx} ,$$

for $p = \pi_L^{Lx}$, and we put

$$|(\exists x)\varphi|_{Lx} = |(\exists x)\varphi|_L \cdot \pi_L^{Lx} \quad \text{and} \quad |(\forall x)\varphi|_{Lx} = |(\forall x)\varphi|_L \cdot \pi_L^{Lx} .$$

4.9. Discussion. Our description of the formal language for fuzzy logic is not meant to be complete; other constructions can be added if they can be interpreted. Parentheses used in the description of 4.2 and 4.3 are often superfluous and should then be omitted.

For finite sets L and M of variables, with $|F|_L$ defined for a formula F of the language and $L \subset M$, we need

$$(1) \qquad |F|_M = |F|_L \cdot \pi_L^M .$$

We also want

$$(2) \qquad |F[\Sigma \leftarrow y]|_L = |F|_{Ly} \cdot \langle \mathrm{id}_{P_L}, |\Sigma|_L \rangle ,$$

if the substitution at left and the interpretations in the righthand side are defined, with y not in L, and not occurring in the term Σ.

These laws are proved recursively, by induction over the length of the tree corresponding to F. We omit the details, noting only three points.

The first point is that (2) is automatically valid if y does not occur in F, so that $F[\Sigma \leftarrow y]$ is F and $|F|_{Ly} = |F|_L \cdot \pi_L^{Ly}$.

For propositional connectives, we need naturality, such as

$$(\alpha \cdot \pi_L^M) \diamond (\beta \cdot \pi_L^M) = (\alpha \diamond \beta) \cdot \pi_L^M$$

for a binary connective $\diamond$, and α, β in $[P_L, H]$.

For quantifiers $(\exists x)$ and $(\forall x)$, we need 3.13.(i)–(iv) for a pullback square 3.13.(1) with $f = \pi_L^{Lx}$ or $g = \pi_L^{Lx}$. We note that such pullbacks are preserved by every functor which preserves finite products.

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