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PART I: THE MULTIPLICATIVE ERGODIC THEORY

by

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ABSTRACT

We consider a class of stochastic linear functional differential systems driven by semimartingales with stationary ergodic increments. We allow smooth convolution-type dependence of the noise terms on the history of the state. Using a stochastic variational technique we construct a compactifying stochastic semiflow on the state space. A multiplicative Ruelle-Oseledec ergodic theorem then gives the existence of a discrete Lyapunov spectrum and a saddle-point property in the hyperbolic case.

Running Title:

LYAPUNOV EXPONENTS OF LINEAR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

Key Words:

Stochastic f.d.e., measure-preserving flow, semimartingale, stationary increments, measure-valued process, quadratic variation, multiplicative cocycle, compact cocycle, Lyapunov spectrum, multiplicative ergodic theorem, exponential dichotomy.

§1. Introduction.

In [20] the first author developed a multiplicative ergodic theory for a class of n-dimensional stochastic linear functional differential equations

$$dx(t) = H(x(t-d),x(t),x_t)dt + g(x(t))dw(t) \qquad (*)$$
$$x_t(s) := x(t+s),\ -r \le s \le 0,\ t \ge 0,\ 0 \le d \le r$$

with state space $M_2 := \mathbb{R}^n \times \mathbb{L}^2([-r,0],\mathbb{R}^n)$. The analysis in [20] depended crucially on the fact that the diffusion term $g(x(t))$ does *not look into the (past) history* $x(s)$, $s < t$, of the state. The present article is an attempt to relax this limitation. (Note, however, the pathological example in Mohammed [19], pp. 144-148.) Indeed we wish to extend the results of [20] in two directions:

(i) We allow "smooth" convolution-type dependence on the history $x(s)$, $t-r \le s \le t$, in the noise coefficient.

(ii) The driving noise processes consist of a large class of *semimartingales with cadlag paths and jointly stationary (ergodic) increments*. Within this context, our results appear to be new *even in the non-delay case* $r = 0$.

More specifically we look at a linear stochastic functional differential equation

$$dx(t) = \left\{\int_{[-r,0]} \mu(t)(ds)x(t+s)\right\}dt + dN(t)\int_{-r}^{0} K(t)(s)x(t+s)ds + dL(t)x(t-) \qquad (I)$$
$$(x(0),x_0) = (v,\eta) \in M_2.$$

In the above stochastic f.d.e. (s.f.d.e.), μ is a stationary measure-valued process such that each $\mu(t,\omega)$ is an $n\times n$-matrix-valued measure on $[-r,0]$. The random field $K(t)(s)$ is a.s. C^1 in (t,s) and stationary in t. The process N is a general n×n-matrix valued semimartingale (Métivier [17], Definition 23.7, p. 153) with jointly stationary increments. The second noise process L is also n×n-matrix-valued, has jointly stationary increments but admitting a representation as a *continuous* local martingale plus a right continuous process of locally bounded variation. The Itô stochastic differentials dN and dL are to be understood in the spirit of the French school e.g. Dellacherie and Meyer [4], Meyer [18], and Métivier and Pellaumail [16]. Assuming that (μ,K,dN,dL) form an ergodic process and satisfy fairly general moment conditions, we show that (I) has an almost sure Lyapunov spectrum

$$\lim_{t\to\infty} \frac{1}{t} \log\|(x(t),x_t)\|_{M_2}$$

consisting of a discrete non-random set of <u>Lyapunov exponents</u> $\{\lambda_i\}_{i=1}^{\infty} \subset \mathbb{R} \cup \{-\infty\}$. If none of the Lyapunov exponents λ_i is zero, we obtain a flow-invariant exponential dichotomy for the stochastic flow X on M_2 associated with the trajectories $\{(x(t),x_t): t \geq 0, (x(0),x_0) = (v,\eta) \in M_2\}$ of (I).

In order to construct a sufficiently robust version of the flow X of (I), the key idea is to show that the s.f.d.e. (I) is equivalent to a *random* integral equation (viz. equation (IV) of §4). Both the cocycle property (Theorem (4.2)(vii)) and Ruelle-Oseledec integrability condition for the stochastic flow X (Theorem (5.1)) are then read off from the integral equation. This method of construction of the flow is different from the one used by Mohammed in [20]. It has the added advantage of being conceptually simpler and perhaps more efficient. This technique also points the way towards possible applications to certain types of stochastic linear P.D.E.s.

Once the regular version of the flow X is constructed, the existence of the Lyapunov spectrum (Theorem (5.2)) and the stable-manifold theorem (Theorem (5.3)) are established using Ruelle's infinite-dimensional multiplicative ergodic theorem (Ruelle [23],[22]). This part of the analysis is closely parallel to the one used by Mohammed in [20].

In order to outline the scope of the theory we indicate below examples of linear stochastic differential equations which are covered by the theorems in this article. The reader may formulate the appropriate conditions under which these results apply to the examples listed below. Note that in all of these examples the state x(t) is a multidimensional process.

Example 1: Linear o.d.e.'s driven by white noise

$$dx(t) = a(t)x(t)dt + \sum_{i=1}^{p} \sigma_i(t)x(t)dW_i(t) \tag{1}$$

The matrix-valued processes a(t), $\sigma_i(t)$ are stationary ergodic and non-anticipating; the Brownian motions W_i, $1 \leq i \leq p$, are independent and one-dimensional. The case of constant coefficients $a(t) \equiv a$, $\sigma_i(t) \equiv \sigma_i$, $1 \leq i \leq p$, has been studied by several authors e.g. Arnold, Kliemann & Oeljeklaus [2], Has'minskii [7], and Baxendale [3]. The Lyapunov spectrum of (1) has been discussed by Arnold & Kliemann [1] when a(t), $\sigma_i(t)$, i = 1,...,p, are stationary ergodic processes which are *independent* of W_i, i = 1,...,p. Note that our results do not necessarily require that a(t), $\sigma_i(t)$, i = 1,...,p, be independent of the noises W_i, i = 1,...,p.

Example 2: Random delay equations driven by white noise

$$dx(t) = \sum_{i=1}^{m} a_i(t)x(t-d_i(t))dt + \sum_{i=1}^{p} \sigma_i(t)x(t)dW_i(t) \tag{2}$$

The coefficients a_i, σ_i are matrices (possibly stationary) and the delays d_i are non-anticipating stationary bounded processes with non-negative values. The equation is driven by several Wiener processes W_i. The dynamics of (2) was studied in (Mohammed [19] VI §3, pp. 167-186) within the context of Markov processes on the state space $C([-r,0],\mathbb{R}^n)$ and under the condition that each d_i is fixed in t and is independent of $(W_1,\ldots,W_p)$. A sufficient condition is given in ([19], Corollary 3.1.2, p. 184) which guarantees asymptotic stability in distribution of the trajectory $\{x_t: t \geq 0\}$ of (2). Cf. also Lidskii ([14]) in the case $\sigma_i \equiv 0$ and d_i Markovian. Observe that (2) reduces to (1) when $d_i \equiv 0$, $1 \leq i \leq m$.

Example 3: Diffusions with distributed memory

$$dx(t) = \sum_{i=1}^{p} \sigma_i(t)\int_{-r}^{0} K(s)x(t+s)ds \; dW_i(t) \tag{3}$$

The matrix-valued processes $\sigma_i(t)$, $1 \leq i \leq p$, are stationary (ergodic) while $K(s)$ is just a deterministic matrix-valued function. The Brownian motions W_i, $1 \leq i \leq p$, are one-dimensional. Although equations like (3) fall under the class studied by Itô and Nisio ([10]) and Mohammed [19], so far little is known regarding the *almost sure* asymptotic behavior of the trajectory $(x(t),x_t)$ as $t \to \infty$.

Example 4: Linear o.d.e.'s driven by Poisson noise

$$dx(t) = a(t)x(t)dt + \sum_{i=1}^{p} \sigma_i(t)x(t-)dN_i(t) \tag{4}$$

The driving noises $N_i(t)$ are one-dimensional Poisson processes and the coefficients $a(t)$, $\sigma_i(t)$ are stationary ergodic matrix-valued, for $1 \leq i \leq p$. For constant coefficients, $a(t) \equiv a$, $\sigma_i(t) \equiv \sigma_i$ a.s. for all $t \geq 0$, $1 \leq i \leq p$, the Lyapunov exponents of (4) were studied by Li & Blankenship ([12]) using classical results on random matrix products.

Example 5: Linear functional differential equations driven by Poisson noise

$$dx(t) = \left\{\int_{[-r,0]} \mu(t)(ds)x(t+s)\right\}dt + \sum_{i=1}^{p} \sigma_i(t)x(t-)dN_i(t) \tag{5}$$

Here μ is a measure-valued process as in (I), $\sigma_i(t)$ are stationary matrices and $N_i(t)$ Poisson processes, $i = 1,\dots,p$. Under suitable conditions on the coefficients μ, σ_i, unique solutions to (5) are known to exist (Doléans-Dade [5], Métivier & Pellaumail [16], Protter [21]). However, to our knowledge, issues of almost sure asymptotic stability for solutions of (5) have hitherto not been explored.

Example 6: Linear f.d.e.'s driven by white noise

$$dx(t) = H(t,\cdot,x(t),x_t)dt + \sum_{i=1}^{p} g_i(t,\cdot,x(t))dW_i(t) \qquad (6)$$

The coefficients $H(t,\cdot,\cdot,\cdot)$, $g(t,\cdot,\cdot)$ are stationary ergodic processes with values in $L(M_2,\mathbb{R}^n)$ and $L(\mathbb{R}^n)$ respectively. The Brownian motion $W(t) = (W_1(t),\dots,W_p(t))$ is p-dimensional. The case of constant coefficients corresponds to equations like (*) whose Lyapunov exponents were studied in the article [20] referred to earlier.

It is evident that the s.f.d.e. (I) also includes as special cases various (finite) "linear combinations" of all the examples mentioned above.

§2. Basic Setting and Hypotheses

We wish to formulate the basic set-up and hypotheses on the stochastic f.d.e.

$$\left.\begin{aligned} dx(t) &= \left\{\int_{[-r,0]} \mu(t)(ds)x(t+s)\right\}dt + dN(t)\int_{-r}^{0} K(t)(s)x(t+s)ds + dL(t)x(t-),\ t>0 \\ x(0) &= v \in \mathbb{R}^n, \quad x(s) = \eta(s), \quad -r \le s \le 0; \quad r \ge 0 \end{aligned}\right\} \qquad (I)$$

which will be needed in the sequel.

Suppose $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ is a filtered probability space satisfying the "usual conditions" (Métivier & Pellaumail [16], Dellacherie & Meyer [4], Protter [21]). Let θ: $\mathbb{R}\times\Omega \to \Omega$ be a measurable flow preserving the probability measure P viz:

(i) For each $t \geq 0$, $\theta(t,\cdot): (\Omega,\mathcal{F}_t) \to (\Omega,\mathcal{F}_0)$ is measurable;

(ii) For each $t \in \mathbb{R}$, $P\circ\theta(t,\cdot)^{-1} = P$.

(iii) For every $t_1,t_2 \in \mathbb{R}$, $\theta(t_2,\cdot)\circ\theta(t_1,\cdot) = \theta(t_1+t_2,\cdot)$.

We shall impose two sets of hypotheses on the coefficients of (I). The first set of hypotheses, denoted by (C_i), $i = 1,2,\ldots$, guarantees the existence of a continuous linear stochastic flow on the state space $M_2 := \mathbb{R}^n\times L^2([-r,0],\mathbb{R}^n)$ with the Hilbert norm

$$\|(v,\eta)\|^2_{M_2} := |v|^2 + \int_{-r}^{0}|\eta(s)|^2 ds \qquad v \in \mathbb{R}^n,\ \eta \in L^2([-r,0],\mathbb{R}^n). \tag{7}$$

Observe that $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^n$. The second set of hypotheses (I_j), $j = 1,2,\ldots$, pertain to moment-type restrictions which are designed in order for the stochastic flow to satisfy Ruelle-Oseledec integrability condition (Theorem (5.1), §5). These integrability hypotheses are spelled out in §5.

The space of all real $n\times n$ matrices is denoted by $\mathbb{R}^{n\times n}$ and is usually given the Euclidean norm

$$\|A\|^2 := \sum_{i,j=1}^{n} a_{ij}^2, \qquad A = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n\times n}.$$

The symbol $\mathcal{M}([-r,0],\mathbb{R}^{n\times n})$ shall stand for the space of all $n\times n$-matrix valued Borel measures on $[-r,0]$ (or $\mathbb{R}^{n\times n}$-valued functions of bounded variation on $[-r,0]$). This space will be given the σ-algebra generated by all evaluations.

A <u>solution</u> of the stochastic f.d.e. (I) is a stochastic process $x: [-r,\infty)\times\Omega \to \mathbb{R}^n$ such that $x|\mathbb{R}^+\times\Omega$ has cadlag paths, is $(\mathcal{F}_t)_{t\geq 0}$-adapted and satisfies the stochastic integral equation

$$x(t) = \begin{cases} v+\int_0^t \int_{[-r,0]} \mu(u)(ds)x(u+s)du+\int_0^t dN(u)\int_{-r}^0 K(u)(s)x(u+s)ds+\int_0^t dL(u)x(u-), & t\geq 0 \\ \eta(t), & -r < t < 0 \end{cases} \qquad \text{(II)}$$

almost surely. Note that in (I) and (II) all n-vectors are column vectors and the products are to be understood in the sense of matrix multiplication. Throughout the article we shall adopt the following terminology regarding a stochastic process $\{y(t,\omega): t \in \mathbb{R}, \omega \in \Omega\}$ defined on the whole line. We shall say that y is $(\mathcal{F}_t)_{t\geq 0}$-*adapted* if $y(t,\cdot)$ is $\mathcal{F}_0$-measurable for all $t \leq 0$ and $\mathcal{F}_t$-measurable for all $t > 0$. Similarly, we say y is an $(\mathcal{F}_t)_{t\geq 0}$-*semimartingale* (local martingale, etc.) if the restriction $y|\mathbb{R}^+\times\Omega$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale (local martingale, respectively).

<u>Hypotheses (C)</u>

(C_1) The process $\mu: \mathbb{R}\times\Omega \to \mathcal{M}([-r,0],\mathbb{R}^{n\times n})$ is measurable $(\mathcal{F}_t)_{t\geq 0}$-adapted and stationary with the representation

$$\mu(t,\omega) = \mu(0,\theta(t,\omega)) \qquad t \in \mathbb{R}, \quad \omega \in \Omega.$$

(C_2) For each $\omega \in \Omega$ and $t \geq 0$ let $\bar{\mu}(t,\omega)$ be the positive measure on $[-r,\infty)$ defined by

$$\bar{\mu}(t,\omega)(A) := |\mu|(t,\omega)\{(A-t) \cap [-r,0]\}$$

for all Borel subsets A of $[-r,\infty)$. Note that $|\mu|$ denotes the total variation measure of μ with respect to the norm on $\mathbb{R}^{n\times n}$. It is easy to check that, for each $\omega \in \Omega$,

$$\nu(\omega)(\cdot) := \int_0^\infty \bar{\mu}(t,\omega)(\cdot)dt$$

is also a positive measure on $[-r,\infty)$. Suppose that $\nu(\omega)$ has a density $\frac{d\nu(\omega)}{ds}$ with respect to Lebesgue measure which is locally essentially bounded for each fixed $\omega \in \Omega$. This implies that for each $t > 0$ and $\omega \in \Omega$ the measure

$$\nu(t,\omega)(\cdot) := \int_0^t \bar{\mu}(u,\omega)(\cdot)du$$

has a locally bounded density $\frac{d\nu(t,\omega)}{ds}$. We suppose further that the map

$$[0,\infty) \to \mathbb{L}^2([-r,0],\mathbb{R})$$
$$t \mapsto \frac{d\nu(t,\omega)}{ds}\Big|[-r,0]$$

is continuous for each $\omega \in \Omega$. Note that the last condition is satisfied in the deterministic case $\mu(t,\omega) = \mu^0$, $t \geq 0$, $\omega \in \Omega$, for a fixed $\mu^0 \in \mathcal{M}([-r,0],\mathbb{R}^{n\times n})$.

(C_3) Let $C^1([-r,0],\mathbb{R}^{n\times n})$ be the space of all C^1 maps $[-r,0] \to \mathbb{R}^{n\times n}$ given the σ-algebra generated by all evaluations. Assume that

$$K: \mathbb{R}\times\Omega \to C^1([-r,0],\mathbb{R}^{n\times n})$$

is a measurable $(\mathcal{F}_t)_{t\geq 0}$-adapted (stationary) process such that

$$K(t,\omega) = K(0,\theta(t,\omega))$$

for all $t \in \mathbb{R}$ and all $\omega \in \Omega$.

Suppose also that for a.a. ω, $K(t,\omega)(s)$ is jointly C^1 in $(t,s) \in \mathbb{R}^+\times[-r,0]$.

(C_4) The process $N: \mathbb{R}\times\Omega \to \mathbb{R}^{n\times n}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale with $N(0,\omega) = 0$ and admitting the following <u>additive cocycle property</u>

$$N(t+h,\omega) - N(t,\omega) = N(h,\theta(t,\omega))$$

for all $t,h \in \mathbb{R}$ and $\omega \in \Omega$ (Métivier [17], Definition 23.7, p. 153).

(C_5) The process $L: \mathbb{R}\times\Omega \to \mathbb{R}^{n\times n}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale admitting a representation

$$L = M + V$$

where M is a *continuous* $(\mathcal{F}_t)_{t\geq 0}$-local martingale and V is an $(\mathcal{F}_t)_{t\geq 0}$-adapted process whose sample paths are all right continuous and of bounded variation on compact sets of $\mathbb{R}^+$. Suppose L and M are such that $M(0) = 0$, $L(0) = 0$ and are additive cocycles:

$$\left.\begin{aligned} L(t+h,\omega) - L(t,\omega) &= L(h,\theta(t,\omega)) \\ M(t+h,\omega) - M(t,\omega) &= M(h,\theta(t,\omega)), \qquad t,h \in \mathbb{R}, \quad \omega \in \Omega. \end{aligned}\right\} \tag{8}$$

<u>Remark</u>:

Conditions (C_4), (C_5) imply that N and L have jointly stationary increments. In fact (μ,K,dN,dL) is a stationary process. As an alternative to the above setting we could have started with a stationary process (μ,K,dN,dL) defined on some probability space and having the required regularity properties. We then form the underlying product path space and define a shift θ on it by

$$\theta(t,\omega)(s) = (\omega_1(t+s),\omega_2(t+s)-\omega_2(t))$$

where $\omega = (\omega_1,\omega_2)$ and the suffixes 1,2 refer to the components corresponding to (μ,K) and (N,L) respectively. It then follows that on the product path space $\theta(t,\cdot)$ is measure-preserving and the canonical processes will automatically satisfy Hypotheses (C_1), (C_3), (C_4), (C_5).

§3. Some Preliminaries

Our strategy for a sample-wise analysis of the s.f.d.e. (I) is to *free the equation of stochastic differentials and replace it by an equivalent random family of integral equations*. In order to construct these random integral equations we shall require some preliminaries. These are discussed below.

Recall that the driving cadlag process L splits up in the form $L = M + V$ where M is a continuous local martingale satisfying the additive cocycle property in Hypothesis (C_5). Our first result in this section says that the additive cocycle behavior of M induces a multiplicative cocycle property for the solution $\varphi: \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n\times n}$ of the linear s.d.e.

$$\left.\begin{aligned} d\varphi(t) &= dM(t)\varphi(t), \quad t > 0 \\ \varphi(0) &= I. \end{aligned}\right\} \tag{III}$$

(See assertion (iii) of Theorem (3.1)). In order to prove the cocycle property for the solution of φ of the above equation we approximate the local martingale M by a family of C^1 processes $\{M^k\}_{k=1}^{\infty}$ defined by

$$M^k(t) := k \int_{t-\frac{1}{k}}^{t} M(u)du, \quad t \in \mathbb{R}. \tag{9}$$

It is clear that each M^k is $(\mathcal{F}_t)_{t\geq 0}$-adapted and has a.a. sample paths C^1 with $(M^k)'(t) = k\{M(t)-M(t-\frac{1}{k})\}$ for all $t \in \mathbb{R}$. Furthermore Hypothesis (C_5) implies that each M^k has the property

$$M^k(t+h,\omega) - M(t,\omega) = M^k(h,\theta(t,\omega))$$

for all $t,h \in \mathbb{R}$, $\omega \in \Omega$. Now for each $k \geq 1$, let $\varphi^k: \mathbb{R}\times\Omega \to \mathbb{R}^{n\times n}$ be the unique solution of the random family of o.d.e.'s

$$\begin{aligned} d\varphi^k(t,\cdot) &= (M^k)'(t)\varphi^k(t,\cdot)dt - \tfrac{1}{2}d\langle M\rangle(t,\cdot)\varphi^k(t,\cdot), \quad t \in \mathbb{R} \\ \varphi^k(0) &= I \in \mathbb{R}^{n\times n} \end{aligned} \tag{III_k}$$

where $\langle M\rangle$ denotes the $\mathbb{R}^{n\times n}$-valued quadratic variation process of M defined by

$$\langle M\rangle_{ij} = \sum_{m=1}^{n} \langle M_{im}, M_{mj}\rangle, \quad M = (M_{ij})_{i,j=1}^{n} \tag{10}$$

(Leandre [13]). Observe also that for each t, ω, k, $\varphi^k(t,\omega)$ is invertible ([13]).

We begin by showing that $\langle M\rangle$ is an additive cocycle whenever M is.

Lemma (3.1):

Let $N_i\colon \mathbb{R}\times\Omega \to \mathbb{R}$, $i = 1,2$, *be real* $(\mathcal{F}_t)_{t\geq 0}$*-semimartingales such that* $N_i(0) = 0$ *and*

$$N_i(t+h,\omega) - N_i(t,\omega) = N_i(h,\theta(t,\omega)) \tag{11}$$

for all $t,h \in \mathbb{R}$, $\omega \in \Omega$, $i = 1,2$. *Then there is a set* $\Omega_1 \in \mathcal{F}$ *of full P-measure and a measurable version of the mutual quadratic variation denoted by* $[N_1,N_2]$*:* $\mathbb{R}^+\times\Omega \to \mathbb{R}$ *which satisfy*

(*i*) $P(\Omega_1) = 1$; $\theta(t,\cdot)(\Omega_1) \subseteq \Omega_1$ *for all* $t \geq 0$;

(*ii*) $$[N_1,N_2](t+h,\omega) - [N_1,N_2](t,\omega) = [N_1,N_2](h,\theta(t,\omega)) \tag{12}$$

for all $h,t \in \mathbb{R}^+$ *and* $\omega \in \Omega_1$.

Proof:

The relation (11) implies that the semimartingales N_1+N_2, N_1-N_2 also satisfy an additive cocycle property. So in view of the identity

$$[N_1,N_2] = \tfrac{1}{4}[N_1+N_2] - \tfrac{1}{4}[N_1-N_2]$$

it is sufficient to prove the lemma for a single semimartingale $Q\colon \mathbb{R}\times\Omega \to \mathbb{R}$ satisfying (11). Write

$$[Q](t) = Q(t)^2 - 2\int_0^t Q(s-)dQ(s), \quad t \geq 0, \text{ a.s.} \tag{13}$$

(Métivier [17], p. 184). Approximate the process $\{Q(s-)\colon s \geq 0\}$ by the sequence

$$Q^k(s) = k\int_{s-\frac{1}{k}}^{s} Q(u)du, \quad s \geq 0. \tag{14}$$

Then, for each k, Q^k is predictable and for all $\omega \in \Omega$, $\lim_{k\to\infty} Q^k(s,\omega) = Q(s-,\omega)$ for every $s \in \mathbb{R}^+$. Furthermore, it is easy to see that for every $\omega \in \Omega$ and $s \in \mathbb{R}^+$ we have

$$|Q^k(s,\omega)| \leq \sup_{0\leq u\leq s} |Q(u,\omega)|, \quad k \geq 1$$

i.e. $\{Q^k\}_{k=1}^{\infty}$ is dominated by the increasing process $\phi(s,\omega) := \sup_{0\le u\le s} |Q(u,\omega)|$, $s \geq 0$. Define the sequence of processes

$$[Q]^k(t) := Q(t)^2 - 2\int_0^t Q^k(s)dQ(s), \quad t \geq 0, \quad k \geq 1. \tag{15}$$

Then by the dominated convergence theorem for stochastic integrals (Métivier [17], Theorem 24.2, p. 171) it follows that, for each $t \geq 0$, $\{[Q]^k(t)\}_{k=1}^{\infty}$ converges in probability to $[Q](t)$. In fact there is a subsequence $\{[Q]^{k_j}\}_{j=1}^{\infty}$ of $\{[Q]^k\}_{k=1}^{\infty}$ such that for a.a. $\omega \in \Omega$ the sequence of paths $\{[Q]^{k_j}(\cdot,\omega)\}_{j=1}^{\infty}$ converges uniformly on compact subsets of $\mathbb{R}^+$ to a limit

$$[Q](t,\omega) := \lim_{j\to\infty} [Q]^{k_j}(t,\omega), \quad t \in \mathbb{R}^+ \tag{16}$$

(Métivier [17], Theorem 24.2, p. 171). We now show that each $[Q]^k$, $k \geq 1$, is an additive cocycle over θ. Using integration by parts we can replace the right hand side of (15) by a version of $[Q]^k$, denoted by the same symbol and satisfying

$$[Q]^k(t,\omega) = Q(t,\omega)^2 - 2Q^k(t,\omega)Q(t,\omega) + 2\int_0^t Q(s,\omega)dQ^k(s,\omega) \tag{17}$$

for all $\omega \in \Omega$, $t \in \mathbb{R}^+$. Fix $t,h \in \mathbb{R}^+$, $\omega \in \Omega$ and consider

$$\begin{aligned}
[Q]^k(h,\theta(t,\omega)) &= Q(h,\theta(t,\omega))^2 - 2Q^k(h,\theta(t,\omega))Q(h,\theta(t,\omega)) \\
&\quad + 2\int_0^h Q(s,\theta(t,\omega))dQ^k(s,\theta(t,\omega)) \\
&= \{Q(t+h,\omega)-Q(t,\omega)\}^2 - 2\{Q^k(t+h,\omega)-Q(t,\omega)\}\{Q(t+h,\omega)-Q(t,\omega)\} \\
&\quad + 2\int_0^h \{Q(s+t,\omega) - Q(t,\omega)\}dQ^k(s+t,\omega) \\
&= Q(t+h,\omega)^2 - 2Q(t+h,\omega)Q(t,\omega) + Q(t,\omega)^2 - 2Q^k(t+h,\omega)Q(t+h,\omega) \\
&\quad + 2Q(t,\omega)Q(t+h,\omega) + 2Q^k(t+h,\omega)Q(t,\omega) - 2Q(t,\omega)^2 \\
&\quad + 2\int_t^{t+h} Q(s,\omega)dQ^k(s,\omega) - 2Q(t,\omega)\{Q^k(t+h,\omega)-Q^k(t,\omega)\}
\end{aligned}$$

$$= Q(t+h,\omega)^2 - 2Q^k(t+h,\omega)Q(t+h,\omega) + 2\int_0^{t+h} Q(s,\omega)dQ^k(s,\omega)$$
$$-\left\{Q(t,\omega)^2 - 2Q^k(t,\omega)Q(t,\omega) + 2\int_0^t Q(s,\omega)dQ^k(s,\omega)\right\}$$
$$= [Q]^k(t+h,\omega) - [Q]^k(t,\omega). \tag{18}$$

Define the set $\Omega_1 \in \mathcal{F}$ of full P-measure by

$$\Omega_1 := \{\omega\colon \omega \in \Omega, \lim_{j\to\infty} [Q]^{k_j}(t,\omega) \text{ exists uniformly for } t \text{ in bounded subsets of } \mathbb{R}^+\}.$$

To prove that Ω_1 is $\theta(t,\cdot)$-invariant, fix $\omega \in \Omega_1$ and $t \in \mathbb{R}^+$. Then it follows from (18) that

$$\lim_{j\to\infty} [Q]^{k_j}(h,\theta(t,\omega)) = \lim_{j\to\infty} [Q]^{k_j}(t+h,\omega) - \lim_{j\to\infty} [Q]^{k_j}(t,\omega)$$

exists uniformly for h in bounded subsets of $\mathbb{R}^+$. This says that $\theta(t,\omega) \in \Omega_1$. Thus $\theta(t,\cdot)(\Omega_1) \subseteq \Omega_1$ for all $t \geq 0$. If we set $k = k_j$ in (18) and let $j \to \infty$ we immediately get

$$[Q](h,\theta(t,\omega)) = [Q](t+h,\omega) - [Q](t,\omega), \quad t,h \geq 0, \quad \omega \in \Omega_1.$$

This completes the proof of the lemma. □

<u>Remark</u>:

In Lemma (3.1) the semimartingales N_i, $i = 1,2$, need not have continuous sample paths. However if both N_1,N_2 are sample continuous, it is easy to modify Ω_1 so that for each $\omega \in \Omega_1$, $[N_1,N_2](\cdot,\omega)$ is continuous and assertions (i), (ii) of the lemma hold.

The next lemma shows that under Hypothesis (C_5) each (φ^k,θ) is a multiplicative cocycle.

<u>Lemma (3.2)</u>:

Suppose M satisfies Hypothesis (C_5). Then there is a set $\Omega_2 \in \mathcal{F}$ of full P-measure such that $\theta(t,\cdot)(\Omega_2) \subseteq \Omega_2$ for all $t \geq 0$ and

$$\varphi^k(t_1+t_2,\omega) = \varphi^k(t_2,\theta(t_1,\omega))\varphi^k(t_1,\omega) \tag{19}$$

for all $k \geq 1$, $t_1,t_2 \in \mathbb{R}^+$, $\omega \in \Omega_2$.

<u>Proof</u>:

Since M satisfies Hypothesis (C_5), it follows from equation (10), Lemma (3.1) and the remark following it, that there is a version $\langle M\rangle$ of the quadratic variation of M and a set $\Omega_2 \in \mathcal{F}$ of full P-measure such that $\theta(t,\cdot)(\Omega_2) \subseteq \Omega_2$ for every $t \geq 0$,

$$\langle M\rangle(t+h,\omega) - \langle M\rangle(t,\omega) = \langle M\rangle(h,\theta(t,\omega)) \tag{20}$$

for all $\omega \in \Omega_2$, $t,h \in \mathbb{R}^+$, and $\langle M\rangle(\cdot,\omega)$ is continuous for every $\omega \in \Omega_2$.

To prove the multiplicative cocycle property (19) let us fix $t_1 \geq 0$, $k \geq 1$ and $\omega \in \Omega_2$. We shall consider the two $\mathbb{R}^{n\times n}$-valued paths

$$z_1(t) := \varphi^k(t+t_1,\omega), \quad z_2(t) := \varphi^k(t,\theta(t_1,\omega))\varphi^k(t_1,\omega), \quad t \geq 0. \tag{21}$$

Note that $z_1(0) = z_2(0) = \varphi^k(t_1,\omega)$ and from (III_k) it follows that

$$\begin{aligned} dz_1(t) &= (M^k)'(t+t_1,\omega)z_1(t)dt - \tfrac{1}{2}d\langle M\rangle(t+t_1,\omega)z_1(t) \\ &= (M^k)'(t,\theta(t_1,\omega))z_1(t)dt - \tfrac{1}{2}d\langle M\rangle(t,\theta(t_1,\omega))z_1(t), \quad t \geq 0. \end{aligned}$$

Also

$$dz_2(t) = (M^k)'(t,\theta(t_1,\omega))z_2(t)dt - \tfrac{1}{2}d\langle M\rangle(t,\theta(t_1,\omega))z_2(t), \quad t \geq 0.$$

Therefore by uniqueness of solutions to (III_k) we have $z_1(t) = z_2(t)$ for all $t \geq 0$ and (19) holds. □

The following theorem shows that (φ,θ) is a multiplicative cocycle:

Theorem (3.1):

Suppose M satisfies Hypothesis (C_5). Then there is an $(\mathcal{F}_t)_{t\geq 0}$-adapted version φ: $\mathbb{R}^+\times\Omega \to \mathbb{R}^{n\times n}$ of the solution of (III) and a set $\Omega_3 \subset \Omega$ such that

(*i*) $\Omega_3 \in \mathcal{F}$, $P(\Omega_3) = 1$;

(*ii*) $\theta(t,\cdot)(\Omega_3) \subseteq \Omega_3$ *for all* $t \geq 0$;

(*iii*) $\varphi(t_1+t_2,\omega) = \varphi(t_2,\theta(t_1,\omega))\varphi(t_1,\omega)$ (24)
for all $t_1,t_2 \geq 0$ *and every* $\omega \in \Omega_3$;

(*iv*) $\varphi(\cdot,\omega)$ *is continuous for every* $\omega \in \Omega_3$.

Proof:

The idea is to show that the sequence of multiplicative cocycles $\{(\varphi^k,\theta)\}_{k=1}^{\infty}$ (Lemma (3.2)) has a subsequence $\{(\varphi^{k'},\theta)\}_{k'=1}^{\infty}$ which converges almost surely uniformly on compacta to the required cocycle (φ,θ). We break the proof up into three steps:

Step 1:

We use a smooth partition of unity to approximate the identity map $id_{\mathbb{R}^{n\times n}}$: $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by a sequence of bounded C^∞ maps f_m: $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $m = 1,2,3,\ldots$, satisfying

$$f_m(A) = A$$

whenever $\|A\| \leq m$; and f_m, together with all its derivatives are globally bounded on $\mathbb{R}^{n\times n}$, for each $m \geq 1$. For each $k,m \geq 1$ let φ_m^k: $\mathbb{R}^+\times\Omega \to \mathbb{R}^{n\times n}$ be the unique sample continuous solution of the d.e.

$$\left.\begin{aligned} d\varphi_m^k(t) &= (M^k)'(t)f_m(\varphi_m^k(t))dt - \tfrac{1}{2}d\langle M\rangle(t)f_m(\varphi_m^k(t)), \quad t \in \mathbb{R} \\ \varphi_m^k(0) &= I \in \mathbb{R}^{n\times n}. \end{aligned}\right\} \qquad (III_k^m)$$

Following Mackevičius ([15], Example 1, Corollary 1) we see that there is a sequence of a.s. finite stopping times $\{T_j\}_{j=1}^{\infty}$ such that $T_j \uparrow \infty$ a.s. as $j \to \infty$ and

$$\lim_{k\to\infty} E \sup_{0\le t\le T_j} |\mathbf{M}^k(t)-\mathbf{M}(t)|^2 = 0 \tag{25}$$

for j = 1,2,3,... It is easy to see that (III_k^m) can be localized on each stochastic interval $[0,T_j]$ to give

$$\left.\begin{aligned} d\varphi_m^k(t\wedge T_j) &= d(\mathbf{M}^k)(t\wedge T_j)f_m(\varphi_m^k(t\wedge T_j)) - \tfrac{1}{2}d\langle\mathbf{M}\rangle(t\wedge T_j)f_m(\varphi_m^k(t\wedge T_j)) \\ \varphi_m^k(0\wedge T_j) &= I. \end{aligned}\right. \tag{$\mathrm{III}_{k,j}^m$}$$

Since each f_m is C^∞ with bounded derivatives and $\{\mathbf{M}^k(\cdot\wedge T_j)\}_{k=1}^\infty$ is a sequence of S^2 symmetric approximations to the continuous local martingale $\mathbf{M}(\cdot\wedge T_j)$, it follows from Theorem 1 of (Mackevičius [15]) that

$$\lim_{k\to\infty} E \sup_{0\le t\le T_j} \|\varphi_m^k(t)-\varphi_m(t)\|^2 = 0, \quad m,j = 1,2,..., \tag{26}$$

where φ_m: $\mathbb{R}^+\times\Omega \to \mathbb{R}^{n\times n}$ is the unique sample-continuous solution of the Stratonovich s.d.e.

$$\left.\begin{aligned} d\varphi_m(t) &= \circ d\mathbf{M}(t)f_m(\varphi_m(t)) - \tfrac{1}{2}d\langle\mathbf{M}\rangle(t)f_m(\varphi_m(t)) \\ \varphi_m(0) &= I. \end{aligned}\right\} \tag{III^m}$$

Step 2:

We shall prove that $\{\varphi^k\}_{k=1}^\infty$ converges to φ locally uniformly in probability i.e. given $\epsilon,\delta > 0$, $j = 1,2,...$, there exists k_0 $(= k_0(\epsilon,\delta,j)) \ge 1$ such that

$$P\left[\sup_{0\le t\le T_j} |\varphi^k(t)-\varphi(t)| > \epsilon\right] < \delta \tag{27}$$

for all $k \ge k_0$. To this end we fix $\epsilon,\delta > 0$ and $j \ge 1$ till further notice. Since

$$\sup_{0\le t\le T_j} |\varphi(t)| < \infty$$

a.s., we may choose and *fix* $m \ge 1$ sufficiently large such that

$$P\left[\sup_{0\le t\le T_j} |\varphi(t)| \ge m - \frac{\epsilon}{3}\right] < \frac{\delta}{4}\,. \tag{28}$$

Now

$$P\left[\sup_{0\le t\le T_j} |\varphi^k(t)-\varphi(t)| > \epsilon\right] \le P_1 + P_2 + P_3 \tag{29}$$

where

$$P_1 := P\left[\sup_{0\le t\le T_j} |\varphi^k(t)-\varphi^k_m(t)| > \frac{\epsilon}{3}\right]$$

$$P_2 := P\left[\sup_{0\le t\le T_j} |\varphi^k_m(t)-\varphi_m(t)| > \frac{\epsilon}{3}\right]$$

and

$$P_3 := P\left[\sup_{0\le t\le T_j} |\varphi_m(t)-\varphi(t)| > \frac{\epsilon}{3}\right].$$

By (26) there exists $k_0 = k_0(\epsilon,\delta,j,m) > 0$ such that

$$P_2 \le \frac{\delta}{4} \tag{30}$$

for all $k \ge k_0$. To estimate P_3 observe that the coefficients of (III^m) and (III) agree on the ball $B_m = \{A: A \in \mathbb{R}^{n\times n}, \|A\| \le m\}$ of radius m. Hence φ_m and φ satisfy the same s.d.e. whenever $\varphi_m(t), \varphi(t) \in B_m$. So by uniqueness of solutions the following estimate easily follows

$$P_3 \le P\left[\sup_{0\le t\le T_j} |\varphi(t)| \ge m - \frac{\epsilon}{3}\right] < \frac{\delta}{4} \tag{31}$$

because of (28). Similarly, $\varphi^k_m(t) = \varphi^k(t)$ whenever $\varphi^k_m(t)$ or $\varphi^k(t) \in B_m$ and it follows from (30) and (28) that

$$\begin{aligned} P_1 &\le P\left[\sup_{0\le t\le T_j} |\varphi^k_m(t)| \ge m\right] \\ &\le P\left[\sup_{0\le t\le T_j} |\varphi^k_m(t)-\varphi_m(t)| > \frac{\epsilon}{3}\right] + P\left[\sup_{0\le t\le T_j} |\varphi_m(t)| > m - \frac{\epsilon}{3}\right] \\ &\le \frac{\delta}{4} + P\left[\sup_{0\le t\le T_j} |\varphi(t)| > m - \frac{\epsilon}{3}\right] \\ &< \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}. \end{aligned} \tag{32}$$

Combining (29), (30), (31) and (32) immediately gives (27).

Step 3:

For each $j = 1,2,\ldots$, the sequence $y^{k,j} := \sup_{0\le t\le T_j} |\varphi^k(t)-\varphi(t)|$, $k \ge 1$ converges to zero in probability as $k \to \infty$ because of (27). Hence by induction on j, there exist subsequences $\{\varphi^{k,j}\}_{k=1}^{\infty}$, $j = 1,2,\ldots$, of $\{\varphi^k\}_{k=1}^{\infty}$ with the following properties

(a) $\{\varphi^{k,j}\}_{k=1}^{\infty}$ is a subsequence of $\{\varphi^{k,j-1}\}_{k=1}^{\infty}$ for each $j > 1$;

(b) for a.a. $\omega \in \Omega$, $\lim_{k\to\infty} \varphi^{k,j}(t,\omega) = \varphi(t,\omega)$ uniformly for $t \in [0,T_j(\omega)]$.

The diagonal subsequence $\{\varphi^{k'} := \varphi^{k',k'}\}_{k'=1}^{\infty}$ of $\{\varphi^k\}_{k=1}^{\infty}$ converges to φ a.s. uniformly on $[0,T]$ for every $0 < T < \infty$ (cf. Protter [21], Lemma (2.3)). Define the set

$$\Omega_3 := \{\omega\colon \omega \in \Omega_2, \lim_{k'\to\infty} \varphi^{k'}(t,\omega) \text{ exists uniformly for } t \in [0,T] \text{ for every } 0 < T < \infty\}.$$

Clearly $\Omega_3 \in \mathcal{F}$ and $P(\Omega_3) = 1$. To see that Ω_3 is shift-invariant, let $\omega \in \Omega_3$, $t_1,T > 0$. Then the limit

$$\lim_{k'\to\infty} \varphi^{k'}(t,\theta(t_1,\omega)) = \lim_{k'\to\infty} \varphi^{k'}(t+t_1,\omega) \lim_{k'\to\infty} \varphi^{k'}(t_1,\omega)^{-1}$$

exists uniformly for $t \in [0,T]$. Note that in the above equality we have used Lemma (3.2). Hence $\theta(t_1,\omega) \in \Omega_3$. This proves assertions (i), (ii) of the theorem. To complete the proof of the cocycle property (24) pick a version of φ, also denoted by the same symbol, such that

$$\varphi(t,\omega) := \lim_{k'\to\infty} \varphi^{k'}(t,\omega), \quad \omega \in \Omega_3, \quad t \in \mathbb{R}^+$$

and pass to the limit as $k' \to \infty$ in the identity

$$\varphi^{k'}(t_1+t_2,\omega) = \varphi^{k'}(t_2,\theta(t_1,\omega))\varphi^{k'}(t_1,\omega)$$

$\omega \in \Omega_3$, $t_1,t_2 \in \mathbb{R}^+$. □

By [11] and [13] we know that $\varphi^{-1}(u)$ exists a.s. for all $u \geq 0$ and is sample continuous in u. We may then consider the stochastic integral $\int_0^t \varphi^{-1}(u)dN(u)$. The next result gives a version of this integral which satisfies the additive property (33) below. This fact will be needed in the construction of the flow of the s.f.d.e. (I).

Theorem (3.2):

Assume Hypotheses (C_4) *and* (C_5). *Then there is a set* $\Omega_5 \subseteq \Omega_3 \in \mathcal{F}$ *of full P-measure and a cadlag* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process* $Z: \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n\times n}$ *such that*

(*i*) $\theta(t,\cdot)(\Omega_5) \subseteq \Omega_5$ *for all* $t \geq 0$;

(*ii*) $Z(t,\cdot) = \int_0^t \varphi^{-1}(u)dN(u)$ *for each* $t \geq 0$ a.s.;

(*iii*) $Z(t_1+t_2,\omega) - Z(t_1,\omega) = \varphi^{-1}(t_1,\omega)Z(t_2,\theta(t_1,\omega))$ (33)

for all $t_1,t_2 \in \mathbb{R}^+$ *and every* $\omega \in \Omega_5$.

N.B.

The version φ above is the one given by Theorem (3.1).

Proof:

The idea of the proof is to approximate the stochastic integral $\int_0^t \varphi^{-1}(u)dN(u)$ by $\int_0^t (\varphi^k)^{-1}(u)dN(u)$, use integration by parts to read off the additive property (33) and then pass to the limit as $k \to \infty$.

For each $k \geq 1$ define the cadlag process $Z_k: \mathbb{R}^+ \times \Omega \to \mathbb{R}^{n\times n}$ by

$$Z_k(t,\omega) := (\varphi^k)^{-1}(t,\omega)N(t,\omega) - \int_0^t \frac{d(\varphi^k)^{-1}(u,\omega)}{du} N(u,\omega)du, \quad t \in \mathbb{R}^+, \ \omega \in \Omega. \quad (34)$$

Since $(\varphi^k)^{-1}$ has paths locally of bounded variation, then integration by parts gives

$$Z_k(t,\cdot) = \int_0^t (\varphi^k)^{-1}(u)dN(u), \quad t \in \mathbb{R}^+, \text{ a.s.}, \ k \geq 1. \quad (35)$$

Now, since M satisfies Hypothesis (C_5), it follows from Lemma (3.2) that

$$\frac{d}{du}(\varphi^k)^{-1}(u,\theta(t_1,\omega)) = \varphi^k(t_1,\omega)\frac{d}{du}(\varphi^k)^{-1}(u+t_1,\omega) \tag{36}$$

for $t_1, u \in \mathbb{R}^+$, $\omega \in \Omega_2$.

Using (34), (36) and the additive cocycle property of (N,θ), a simple computation shows that

$$Z_k(t_1+t_2,\omega) - Z_k(t_1,\omega) = (\varphi^k)^{-1}(t_1,\omega)Z_k(t_2,\theta(t_1,\omega)) \tag{37}$$

for $t_1, t_2 \in \mathbb{R}^+$, $\omega \in \Omega_2$, $k \geq 1$.

For a.a. $\omega \in \Omega$, $\varphi(t,\omega)$ is invertible for all $t \in \mathbb{R}^+$ (Jacod [11], Leandre [13]); so it follows immediately from the cocycle property (24) that the set of full P-measure

$$\Omega_4 := \{\omega\colon \omega \in \Omega_3,\ \varphi(t,\omega) \text{ is invertible for all } t \in \mathbb{R}^+\}$$

is $\theta(t,\cdot)$-invariant for every $t \in \mathbb{R}^+$. Let $\{\varphi^{k'}\}_{k'=1}^{\infty}$ be the subsequence of $\{\varphi^k\}_{k=1}^{\infty}$ constructed in Step 3 of the proof of Theorem (3.1). Then, for each $\omega \in \Omega_3$,

$$\varphi(t,\omega)^{-1} = \lim_{k'\to\infty} (\varphi^{k'})^{-1}(t,\omega), \quad t \in \mathbb{R}^+ \tag{38}$$

uniformly for $t \in [0,T]$, $0 < T < \infty$.

Now use the dominated convergence theorem (Métivier [17], Theorem 26.3, p. 183) to find a subsequence $\{\varphi^{k''}\}_{k''=1}^{\infty}$ of $\{\varphi^{k'}\}_{k'=1}^{\infty}$ such that for a.a. $\omega \in \Omega$ the limit

$$Z(t,\omega) := \lim_{k''\to\infty} (\omega)\int_0^t (\varphi^{k''})^{-1}(u)dN(u) = (\omega)\int_0^t \varphi^{-1}(u)dN(u) \tag{39}$$

exists uniformly for $t \in [0,T]$, $0 < T < \infty$. Let Ω_5 be the set of all $\omega \in \Omega_4$ such that the above uniform convergence (39) holds. Therefore for each $\omega \in \Omega_5$, $t_1 \in \mathbb{R}^+$, the sequence

$$Z_{k'}(t,\theta(t_1,\omega)) = \varphi^{k'}(t_1,\omega)[Z_{k'}(t+t_1,\omega)-Z_{k'}(t_1,\omega)]$$

converges uniformly for $t \in [0,T]$ as $k' \to \infty$. Hence Ω_5 is $\theta(t_1,\cdot)$-invariant. The proof of the theorem is completed by passing to the limit as $k' \to \infty$ in the above relation. □

§4. The Random Integral Equation

We are now in a position to formulate the random integral equation which we advertised in §1. We shall first show that this integral equation is pathwise equivalent to our s.f.d.e. (I). We then establish the existence of a unique solution to the integral equation which depends *linearly* and *continuously* on the initial data $(v,\eta) \in M_2$. The cocycle property for the trajectory $X(t) := (x(t),x_t)$, $t \geq 0$, then follows directly from uniqueness of the solution to the integral equation.

Throughout this section we assume Hypotheses (C_i) $i = 1,2,3,4,5$ and take φ: $\mathbb{R}^+\times\Omega \to \mathbb{R}^{n\times n}$, Z: $\mathbb{R}^+\times\Omega \to \mathbb{R}^{n\times n}$ to be the processes constructed in Theorems (3.1), (3.2) of the last section.

Let $[M,N]$ denote the $\mathbb{R}^{n\times n}$-valued mutual variation process of M and N viz. $[M,N] = ([M,N]_{ij})$ where

$$[M,N]_{ij} := \sum_{m=1}^{n} [M_{im},N_{mj}],$$

$M = (M_{ij})_{i,j=1}^n$, $N = (N_{ij})_{i,j=1}^n$. From Hypothesis (C_4), (C_5) and Lemma (3.1), it follows that there is a version $[M,N]$ of the mutual variation of M and N and a set $\tilde{\Omega}_1 \in \mathcal{F}$ of full P-measure such that $\theta(t,\cdot)(\tilde{\Omega}_1) \subseteq \tilde{\Omega}_1$ for all $t \in \mathbb{R}^+$ and $\{([M,N](t,\omega),\theta(t,\omega))\colon t \in \mathbb{R}^+,\ \omega \in \tilde{\Omega}_1\}$ is an additive cocycle.

Let $\Omega_5 \subset \Omega$ be as in Theorem (3.2) and $\Omega_6 := \Omega_5 \cap \tilde{\Omega}_1$. Denote by $\mathcal{E}$ the vector space of all Borel-measurable maps $g\colon [-r,\infty) \to \mathbb{R}^n$ such that $g|[-r,0]$ belongs to

$\mathbb{L}^2([-r,0],\mathbb{R}^n)$ and $g|[0,\infty)$ is cadlag. For each $\omega \in \Omega_6$ define the linear map $I(\omega)\colon \mathcal{E} \to \mathcal{E}$ as follows: For any $g \in \mathcal{E}$ set

$$I(\omega)(g)(t) := g(t) \qquad \text{a.e. } t \in [-r,0] \tag{40}$$

and

$$\begin{aligned} I(\omega)(g)(t) := \varphi(t,\omega)\Big[g(0) &- \int_0^t Z(u,\omega)\{K(u,\omega)(0)g(u) - K(u,\omega)(-r)g(u-r) \\ &+ \int_{u-r}^{u} \frac{\partial}{\partial u}(K(u,\omega)(s-u))g(s)ds\}du + Z(t,\omega)\int_{-r}^{0} K(t,\omega)(s)g(t+s)ds \\ &+ \int_0^t \varphi^{-1}(u,\omega) \int_{[-r,0]} \mu(u,\omega)(ds)g(u+s)du + \int_0^t \varphi^{-1}(u,\omega)dV(u,\omega)g(u-) \\ &- \int_0^t \varphi^{-1}(u,\omega)d[M,N](u,\omega) \int_{-r}^{0} K(u,\omega)(s)g(u+s)ds\Big] \end{aligned} \tag{41}$$

for $t \in \mathbb{R}^+$.

Our first result in this section (Theorem (4.1) below) shows that the random family of integral equations

$$\begin{aligned} x &= I(\omega)(x), \qquad x \in \mathcal{E}, \quad \omega \in \Omega_6 \\ (x(0),x_0) &= (v,\eta) \in M_2 \end{aligned} \tag{IV}$$

is equivalent to our s.f.d.e. (I).

Note that, if we fix and suppress $\omega \in \Omega_6$ and use the cocycle property for (φ,θ), then (IV) reads

$$\begin{aligned} x(t) = \varphi(t)\Big[v &- \int_0^t Z(u)\{K(u)(0)x(u) - K(u)(-r)x(u-r) + \int_{u-r}^{u} \frac{\partial}{\partial u}(K(u)(s-u))x(s)ds\}du\Big] \\ &+ \varphi(t)Z(t) \int_{-r}^{0} K(t)(s)x(t+s)ds + \int_0^t \varphi(t-u,\theta(u,\cdot)) \int_{[-r,0]} \mu(u)(ds)x(u+s)du \\ &+ \int_0^t \varphi(t-u,\theta(u,\cdot))dV(u)x(u-) - \int_0^t \varphi(t-u,\theta(u,\cdot))d[M,N](u)\int_{-r}^{0} K(u)(s)x(u+s)ds, \\ &\qquad t \in \mathbb{R}^+ \end{aligned} \tag{IV}$$

$$x(t) = \eta(t) \qquad \text{a.e. } -r \le t < 0.$$

The existence of a unique (cadlag $(\mathcal{F}_t)_{t\geq 0}$-adapted) solution to the above integral equation will be established in Theorem (4.2). We now prove:

Theorem (4.1):

The s.f.d.e. (I) and the random integral equation (IV) are equivalent: Every cadlag $(\mathcal{F}_t)_{t\geq 0}$-adapted solution of (IV) is a solution of (I). Conversely, every solution of (I) has a version which satisfies (IV).

Proof:

Fix $(v,\eta) \in \mathbf{M}_2$ and let $x\colon [-r,\infty)\times\Omega \to \mathbb{R}^n$ be a solution of the s.f.d.e. (I) starting off at (v,η).

The $\mathbb{R}^n$-valued process

$$H(t) := v + \int_0^t \int_{[-r,0]} \mu(u)(ds)x(u+s)du + \int_0^t dV(u)x(u-) + \int_0^t dN(u) \int_{-r}^0 K(u)(s)x(u+s)ds, \quad t \in \mathbb{R}^+ \tag{42}$$

is clearly an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale because x is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Denote by $[M,H]$ the B.V. process

$$[M,H] := ([M,H]_i)_{i=1}^n, \quad [M,H]_i(t) := \sum_{j=1}^n [M_{ij},H_j](t), \quad t \in \mathbb{R}^+.$$

Applying the integration by parts formula (Métivier [17], Equation (26.9.3), p. 185) to the process

$$\tilde{x}(t) := \varphi(t)\left\{v + \int_0^t \varphi^{-1}(u)dH(u) - \int_0^t \varphi^{-1}(u)d[M,H](u)\right\}, \quad t \in \mathbb{R}^+,$$

it is easy to see that $\tilde{x}$ satisfies the linear s.d.e.

$$\left.\begin{aligned} d\tilde{x}(t) &= dH(t) + dM(t)\tilde{x}(t), \quad t \in \mathbb{R}^+ \\ \tilde{x}(0) &= v \end{aligned}\right\} \tag{V}$$

a.s.

Now our s.f.d.e. (I) says that x also satisfies the above s.d.e. (V). So by uniqueness of solutions to (III) we get that

$$\tilde{x}(t) = x(t) = \varphi(t)\left\{v + \int_0^t \varphi^{-1}(u)dH(u) - \int_0^t \varphi^{-1}(u)d[M,H](u)\right\} \tag{43}$$

for all $t \in \mathbb{R}^+$, a.s. (see also Jacod [11], Theorem 2).

Inserting H from (42) into (43) and using the definition of Z (Theorem (3.2)) we get

$$\begin{aligned} x(t) = \varphi(t)\Big\{v &+ \int_0^t \varphi^{-1}(u) \int_{[-r,0]} \mu(u)(ds)x(u+s)du + \int_0^t \varphi^{-1}(u)dV(u)x(u-) \\ &+ \int_0^t dZ(u) \int_{-r}^0 K(u)(s)x(u+s)ds \\ &- \int_0^t \varphi^{-1}(u)d[M,N](u) \int_{-r}^0 K(u)(s)x(u+s)ds\Big\}, \quad t \in \mathbb{R}^+. \end{aligned} \tag{44}$$

Note that the last term in the above relation is obtained via the equality

$$\left[M,\int_0^{(\cdot)} dN(u) \int_{-r}^0 K(u)(s)x(u+s)ds\right](t) = \int_0^t d[M,N](u)\int_{-r}^0 K(u)(s)x(u+s)ds, \ t \in \mathbb{R}^+. \tag{45}$$

Now in (44) integration by parts (Métivier [17], p. 192) and Hypothesis (C_3) yield:

$$\begin{aligned} \int_0^t dZ(u)\int_{-r}^0 K(u)(s)x(u+s)ds &= Z(t)\int_{-r}^0 K(t)(s)x(t+s)ds - \int_0^t Z(u)\frac{d}{du} \int_{-r}^0 K(u)(s)x(u+s)ds\, du \\ &= Z(t) \int_{-r}^0 K(t)(s)x(t+s)ds - \int_0^t Z(u)\frac{d}{du} \int_{u-r}^u K(u)(s'-u)x(s')ds'du \\ &= Z(t)\int_{-r}^0 K(t)(s)x(t+s)ds - \int_0^t Z(u)\{K(u)(0)x(u) - K(u)(-r)x(u-r)\}du \\ &\quad - \int_0^t Z(u) \int_{u-r}^u \frac{\partial}{\partial u}(K(u)(s'-u))x(s')ds'du, \quad t \in \mathbb{R}^+. \end{aligned} \tag{46}$$

Substituting the above relation into (44) implies that x satisfies the integral equation (IV) a.s. for all $t \in \mathbb{R}^+$.

Conversely, let x be a cadlag $(\mathcal{F}_t)_{t\geq 0}$-adapted process which solves the integral equation (IV). Using (46) it is easy to see that x satisfies (44). If we define H by (42) as before, then (43) holds. The latter relation implies that x fulfills (V) and is therefore a solution of our s.f.d.e. (I). This completes the proof of the theorem. □

The following result is the main theorem of this section. It is crucial for the existence of the Lyapunov spectrum of our s.f.d.e. (I). Basically it says that the random integral equation (IV) has a unique solution which yields a robust version of the trajectory $(x(t),x_t)$ of (I).

Theorem (4.2)

Let Hypotheses (C) be satisfied. Then for each $\omega \in \Omega_6$ *and* $(v,\eta) \in M_2$, *the integral equation (IV) has a unique cadlag solution* $x(\cdot,\omega,(v,\eta)): [-r,\infty) \to \mathbb{R}^n$. *Define the map* $X: \mathbb{R}^+\times\Omega\times M_2 \to M_2$ *by*

$$X(t,\omega,(v,\eta)) := (x(t,\omega,(v,\eta)),\ x_t(\cdot,\omega,(v,\eta))) \tag{47}$$

for $t \in \mathbb{R}^+$, $\omega \in \Omega_6$, $(v,\eta) \in M_2$. *Then the following is true:*

(*i*) *For each* $(v,\eta) \in M_2$, $\{X(t,\cdot,(v,\eta)): t \in \mathbb{R}^+\}$ *is the unique* $(\mathcal{F}_t)_{t\geq 0}$-*adapted trajectory of the s.f.d.e. (I) starting off at* (v,η).

(*ii*) *For every* $\omega \in \Omega_6$ and $(v,\eta) \in M_2$, *the path* $X(\cdot,\omega,(v,\eta)): \mathbb{R}^+ \to M_2$ *is cadlag.*

(*iii*) *The map* $X(t,\omega,\cdot): M_2 \to M_2$ *is continuous linear for all* $t \in \mathbb{R}^+$ *and* $\omega \in \Omega_6$.

(*iv*) *For each* $\omega \in \Omega_6$, *the map*

$$\mathbb{R}^+ \to L(M_2)$$
$$t \mapsto X(t,\omega,\cdot)$$

is Borel-measurable and locally bounded with respect to the uniform operator norm on $L(M_2)$;

(*v*) *For each* $t \geq r$ *and* $\omega \in \Omega_6$, $I(t,\omega,\cdot)\colon M_2 \to M_2$ *is compact;*

(*vi*) *The event* Ω_6 *has full P-measure and is shift-invariant:* $\theta(t,\cdot)(\Omega_6) \subseteq \Omega_6$ *for all* $t \in \mathbb{R}^+$.

(*vii*)
$$I(t_2,\theta(t_1,\omega),\cdot) \circ I(t_1,\omega,\cdot) = I(t_1+t_2,\omega,\cdot) \tag{48}$$
for all $\omega \in \Omega_6$ *and* $t_1,t_2 \in \mathbb{R}^+$.

<u>Proof</u>:

We establish a unique cadlag solution $x(\cdot,\cdot,(v,\eta))\colon [-r,\infty)\times\Omega \to \mathbb{R}^n$ for the integral equation (IV) using the classical technique of successive approximations.

Fix $\omega \in \Omega_6$ and $0 < T < \infty$ till further notice. Define a sequence of successive approximations
$$\{x^k(t,\omega,(v,\eta))\colon t \in [-r,\infty),\ (v,\eta)) \in M_2\}, \quad k = 1,2,\ldots$$
as follows
$$x^1(t,\omega,(v,\eta)) := \begin{cases} \eta(t) & \text{a.e. } t \in [-r,0) \\ v & t \geq 0 \end{cases} \tag{49}$$
and
$$x^{k+1}(t,\omega,(v,\eta)) = I(\omega)(x^k(\cdot,\omega,(v,\eta)))(t), \quad t \geq -r, \tag{50}$$
$(v,\eta) \in M_2$, $k \geq 1$. It is clearly seen, by induction on k, that $x^k(\cdot,\omega,(v,\eta)) \in \mathcal{E}$ for $k \geq 1$ and $(v,\eta) \in M_2$.

In order to examine the convergence of the sequence $\{x^k(t,\omega,(v,\eta))\}_{k=1}^{\infty}$ we first contend that there are positive numbers C_1, C_2, C_3 (depending on ω, T, μ, N and L only) such that
$$|I(\omega)(g)(t)| \leq C_1|g(0)| + C_2\int_{-r}^{t}|g(u)|du + C_3\int_0^t|g(u-)|d|V|(u) \tag{51}$$
for all $g \in \mathcal{E}$ and $0 \leq t \leq T$.

The proof of (51) goes by estimating separately each of the eight terms on the right hand side of (41). We shall only indicate here how to treat the two terms

$$J_1(g) := \varphi(t,\omega) \int_0^t \varphi^{-1}(u,\omega) \int_{[-r,0]} \mu(u,\omega)(ds)g(u+s)du \tag{52}$$

$$J_2(g) := \varphi(t,\omega) \int_0^t \varphi^{-1}(u,\omega)d[M,N](u,\omega) \int_{-r}^0 K(u,\omega)(s)g(u+s)ds. \tag{53}$$

Changing the variable s to s-u in (52) and recalling Hypothesis (C_2) we see that

$$\begin{aligned} |J_1(g)| &\le \Big[\sup_{0\le t\le T} \|\varphi(t,\omega)\|\Big]\int_0^t \|\varphi^{-1}(u,\omega)\| \int_{[u-r,u]} |g(s)|\,|\mu|(u,\omega)(-u+ds)du \\ &\le \|\varphi(\cdot,\omega)\|_\infty \|\varphi^{-1}(\cdot,\omega)\|_\infty \int_{-r}^t |g(s)| \Big[\int_0^t \bar{\mu}(u,\omega)(\cdot)du\Big](ds) \\ &\le \|\varphi(\cdot,\omega)\|_\infty \|\varphi^{-1}(\cdot,\omega)\|_\infty \int_{-r}^t |g(s)|\nu(\omega)(ds) \\ &\le \|\varphi(\cdot,\omega)\|_\infty \|\varphi^{-1}(\cdot,\omega)\|_\infty \cdot \sup_{0\le s\le T} \Big|\frac{d\nu(\omega)}{ds}(s)\Big| \int_{-r}^t |g(s)|ds, \quad 0 \le t \le T \end{aligned} \tag{54}$$

where

$$\|\varphi(\cdot,\omega)\|_\infty := \sup_{0\le t\le T} \|\varphi(t,\omega)\|, \quad \|\varphi^{-1}(\cdot,\omega)\|_\infty := \sup_{0\le t\le T} \|\varphi^{-1}(t,\omega)\|$$

and $\frac{d\nu}{ds}(\omega)$ is the locally bounded Radon-Nikodym derivative of ν with respect to Lebesgue measure on $[-r,\infty)$. Similarly, using Hypothesis (C_3), we have

$$\begin{aligned} |J_2(g)| &\le \|\varphi(\cdot,\omega)\|_\infty \Big|\int_0^t \varphi^{-1}(u,\omega)d[M,N](u,\omega) \int_{u-r}^u K(u,\omega)(s-u)g(s)ds\Big| \\ &\le \|\varphi(\cdot,\omega)\|_\infty \|K(\cdot,\omega)(\cdot)\|_\infty \int_{-r}^t \Big\{\int_0^t \|\varphi^{-1}(u,\omega)\|d|[M,N]|(u,\omega)\Big\}|g(s)|ds \\ &\le \|\varphi(\cdot,\omega)\|_\infty \|K(\cdot,\omega)(\cdot)\|_\infty \Big\{\int_0^T \|\varphi^{-1}(u,\omega)\|d|[M,N]|(u,\omega)\Big\}\int_{-r}^t |g(s)|ds, \\ &\qquad 0 \le t \le T \end{aligned} \tag{55}$$

where $\|K(\cdot,\omega)(\cdot)\|_\infty = \sup_{\substack{-r\le s\le 0 \\ 0\le t\le T}} \|K(t,\omega)(s)\| < \infty$, by Hypothesis (C_3). The reader may check that all other six terms on the right hand side of (41) satisfy similar estimates to (54) and (55).

Furthermore, for each $g \in \mathcal{E}$ let

$$\Delta I(\omega)(g)(t) := I(\omega)(g)(t) - I(\omega)(g)(t-)$$

denote the jump of $I(\omega)(g)$ at $t \in [0,\infty)$. Noting that

$$\sup_{0\le t\le T} \|\Delta N(t,\omega)\| < \infty, \qquad \sup_{0\le t\le T} \|\Delta V(t,\omega)\| < \infty \text{ a.s.},$$

it follows from (41) that

$$\begin{aligned} |\Delta I(\omega)(g)(t)| &\le C_4|\Delta Z(t,\omega)|\cdot\int_{-r}^{t}|g(u)|du + |g(t-)|\cdot|\Delta V(t)| \\ &\le C_5\Big\{\int_{-r}^{t}|g(u)|du + |g(t-)|\Big\}, \qquad 0 \le t \le T, \end{aligned} \tag{55$'$}$$

for positive constants C_4, C_5 depending on ω, T, K, N, L.

For the time being let us suppress $\omega \in \Omega_6$ and $(v,\eta) \in M_2$; so we write $x^k(t) := x^k(t,\omega,v,\eta)$, $k \ge 1$ and let $\alpha: \mathbb{R}^+ \to \mathbb{R}^+$ stand for the non-negative increasing cadlag path

$$\alpha(t) := t + |V|(t,\omega), \qquad t \ge 0.$$

We shall show that there are positive constants $C_i := C_i(\omega,T,\mu,N,L)$, $i = 6,7,8,\ldots$ independent of k, (v,η) such that

$$|x^{k+1}(t-)-x^k(t-)| \le \frac{C_6^{k-1}\alpha(t-)^{k-1}}{(k-1)!} \sup_{0\le t\le T} |x^2(t)-x^1(t)|, \quad k \ge 1 \tag{56}$$

$$|\Delta x^{k+1}(t)-\Delta x^k(t)| \le C_7 \frac{C_6^{k-2}\alpha(t)^{k-2}}{(k-2)!} \sup_{0\le t\le T} |x^2(t)-x^1(t)|, \quad k \ge 2 \tag{57}$$

and

$$|x^{k+1}(t)-x^k(t)| \le C_8 \frac{C_6^{k-2}\alpha(t)^{k-2}}{(k-2)!} \sup_{0\le t\le T} |x^2(t)-x^1(t)|, \quad k \ge 2 \tag{58}$$

for all $0 \le t \le T$. To prove (56) we use induction on $k \ge 1$. Note first that it holds trivially for $k = 1$. Suppose now that (56) is true for some $k \ge 1$. Then it follows immediately from (51) and (56) that there is a positive constant C_6

such that

$$|x^{k+2}(t-)-x^{k+1}(t-)| \le C_6 \int_0^{t-} |x^{k+1}(u-)-x^k(u-)|d\alpha(u)$$

$$\le C_6 \frac{C_6^{k-1}}{(k-1)!} \sup_{0\le t\le T} |x^2(t)-x^1(t)| \int_0^{t-} \alpha(u-)^{k-1}d\alpha(u)$$

$$\le C_6^k \frac{\alpha(t-)^k}{k!} \sup_{0\le t\le T} |x^2(t)-x^1(t)| \qquad (59)$$

where we have used the inequality

$$\int_0^t \alpha(u-)^{k-1}d\alpha(u) \le \frac{1}{k}\,\alpha(t)^k, \quad t \ge 0, \quad k \ge 1. \qquad (60)$$

Note that the above inequality (60) is easily checked by using successive integrations by parts and the fact that α is non-negative and non-decreasing. This proves (56).

To prove (57) we replace g in (55)′ by $x^k - x^{k-1}$ and use (56) and (60) to obtain

$$|\Delta x^{k+1}(t)-\Delta x^k(t)| \le C_5\Big\{\int_0^t |x^k(u-)-x^{k-1}(u-)|du + |x^k(t-)-x^{k-1}(t-)|\Big\}$$

$$\le C_5\Big\{\frac{C_6^{k-2}}{(k-2)!}\int_0^t \alpha(u-)^{k-2}d\alpha(u) + \frac{C_6^{k-2}\ \alpha(t-)^{k-2}}{(k-2)!}\Big\} \sup_{0\le t\le T} |x^2(t)-x^1(t)|$$

$$\le C_7 \frac{C_6^{k-2}\ \alpha(t-)^{k-2}}{(k-2)!} \sup_{0\le t\le T}|x^2(t)-x^1(t)|, \ 0 \le t \le T$$

where

$$C_7 := C_5[\alpha(T)+1].$$

The inequality (58) now follows directly from (56), (57) and the obvious inequality

$$|x^{k+1}(t)-x^k(t)| \le |x^{k+1}(t-)-x^k(t-)| + |\Delta x^{k+1}(t)-\Delta x^k(t)|, \quad 0 \le t \le T, \quad k \ge 1.$$

Now let $B := \{(v,\eta): (v,\eta) \in \mathbf{M}_2, \|(v,\eta)\|_{\mathbf{M}_2} \le 1\}$ be the closed unit ball in $\mathbf{M}_2$. We no longer suppress $\omega \in \Omega_6$, t, (v,η), but rather think of x^k as a function

$x^k(\cdot,\omega,\cdot)$ of $(t,(v,\eta)) \in \mathbb{R}^+ \times \mathbf{M}_2$ into $\mathbb{R}^n$. It follows easily by induction from (50), (49) and (51) that for each $k \geq 1$ we have

$$\|x^k\|_\infty := \sup_{0\leq t\leq T} \sup_{(v,\eta)\in B} |x^k(t,\omega,v,\eta)| < \infty. \tag{61}$$

In particular for fixed $k \geq 1$ and $t \geq 0$, each $x^k(t,\omega,\cdot)\colon \mathbf{M}_2 \to \mathbb{R}^n$ is a continuous linear map.

Let E be the space of all bounded maps $f\colon [0,T]\times B \to \mathbb{R}^n$ such that for each $(v,\eta) \in B$, $f(\cdot,v,\eta)$ is cadlag, and for each $t \in [0,T]$, $f(t,\cdot)$ is continuous on B. We equip E with the Banach norm

$$\|f\|_E = \sup_{0\leq t\leq T} \sup_{(v,\eta)\in B} |f(t,v,\eta)|.$$

Then E is a real Banach space. It follows directly from (58) that the series $\sum_{k=2}^{\infty} \{x^{k+1}(\cdot,\omega,\cdot)-x^k(\cdot,\omega,\cdot)\}$ may be compared with the convergent exponential series

$$\sum_{k=2}^{\infty} C_8 C_6^{k-2} \frac{\alpha(T,\omega)^{k-2}}{(k-2)!} \sup_{0\leq t\leq T} \sup_{(v,\eta)\in B} |x^2(t,\omega,v,\eta)-x^1(t,\omega,v,\eta)|.$$

Hence the series $\sum_{k=1}^{\infty} \{x^{k+1}(\cdot,\omega,\cdot)-x^k(\cdot,\omega,\cdot)\}$ is absolutely convergent and so the sequence $\{x^k(\cdot,\omega,\cdot)\}_{k=1}^{\infty}$ converges to a limit $x(\cdot,\omega,\cdot) \in E$. This limit extends by linearity to a map $x(\cdot,\omega,\cdot)\colon \mathbb{R}^+\times\mathbf{M}_2 \to \mathbb{R}^n$ such that for each $t \in \mathbb{R}^+$, $x(t,\omega,\cdot)\colon \mathbf{M}_2 \to \mathbb{R}^n$ is continuous linear; and for each $(v,\eta) \in \mathbf{M}_2$, $x(\cdot,\omega,v,\eta)\colon \mathbb{R}^+ \to \mathbb{R}^n$ is cadlag. Furthermore,

$$\sup_{0\leq t\leq T} \sup_{(v,\eta)\in B} |x(t,\omega,v,\eta)| < \infty. \tag{62}$$

Let us momentarily suppress ω and fix $(v,\eta) \in \mathbf{M}_2$. Since all the processes appearing on the right-hand side of (41) are $(\mathcal{F}_t)_{t\geq 0}$-adapted (Hypotheses C), we can easily see by induction on k that each $x^k(t,\cdot,v,\eta)$ is $\mathcal{F}_t$-measurable for $t \geq 0$. Letting $k \to \infty$ we get that $x(\cdot,\cdot,v,\eta)$ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted solution of the integral equation (IV). This implies by Theorem (4.1) that it is also a solution of the s.f.d.e. (I).

To prove uniqueness of the solution of the integral equation (IV) for each $\omega \in \Omega_6$, let $y(\cdot,\omega)$: $[-r,\infty) \to \mathbb{R}^n$ be any solution in $\mathcal{E}$ of (IV) with $(y(0),y_0) = (v,\eta) = (x(0),x_0)$. Then using (51), (55) and the same argument underlying the proofs of (56), (57) and (58) the reader may check that

$$|x(t-,\omega)-y(t-,\omega)| \leq \frac{C_6^{k-1}\ \alpha(t-,\omega)^{k-1}}{(k-1)!} \sup_{0\leq t\leq T} |x(t,\omega)-y(t,\omega)| \tag{63}$$

$$|\Delta x(t,\omega)-\Delta y(t,\omega)| \leq C_7 \frac{C_6^{k-1}\ \alpha(t,\omega)^{k-1}}{(k-1)!} \sup_{0\leq t\leq T} |x(t,\omega)-y(t,\omega)| \tag{64}$$

and

$$|x(t,\omega)-y(t,\omega)| \leq C_8 \frac{C_6^{k-1}\ \alpha(t,\omega)^{k-1}}{(k-1)!} \sup_{0\leq t\leq T} |x(t,\omega)-y(t,\omega)| \tag{65}$$

for all $0 \leq t \leq T$ and *every* integer $k \geq 1$. It follows from (65) (e.g. let $k \to \infty$) that $x(t,\omega) = y(t,\omega)$ for all $0 \leq t \leq T$ and pathwise uniqueness follows.

Let the flow X: $\mathbb{R}^+ \times \Omega \times \mathbb{M}_2 \to \mathbb{M}_2$ be defined by (47). Then it follows from (62) that for each $\omega \in \Omega_6$ we have

$$\sup_{0\leq t\leq T}\ \sup_{(v,\eta)\in B} \|X(t,\omega,v,\eta)\|_{\mathbb{M}_2}^2 = \sup_{0\leq t\leq T}\ \sup_{(v,\eta)\in B} \left(|x(t,\omega,v,\eta)|^2 + \int_{t-r}^{t} |x(u,\omega,v,\eta)|^2 du\right)$$
$$< \infty \tag{66}$$

The Borel measurability of X follows easily from the corresponding property for x. Hence (iii) and (iv) are proved.

To prove (v) we fix $t_0 \geq r$ and $\omega \in \Omega_6$. If we define X^k: $\mathbb{R}^+ \times \Omega \times \mathbb{M}_2 \to \mathbb{M}_2$ by

$$X^k(u,\omega,(v,\eta)) := (x^k(u,\omega,v,\eta), x_u^k(\cdot,\omega,v,\eta))$$

for $u \in \mathbb{R}^+$, $(v,\eta) \in \mathbb{M}_2$, then it is clear that each $X^k(u,\omega,\cdot)$: $\mathbb{M}_2 \to \mathbb{M}_2$ is continuous linear and

$$\lim_{k\to\infty} \sup_{0\leq u\leq T} \|X^k(u,\omega,\cdot)-X(u,\omega,\cdot)\|_{L(M_2)} = 0. \tag{67}$$

Hence it suffices to show that $X^k(t_0,\omega,\cdot)$ is compact for every $k \geq 1$ as the compactness of $X(t_0,\omega,\cdot)$ will then follow from (67). In order to do so we pick a sequence $\{(v_i,\eta_i)\}_{i=1}^{\infty} \subset B$ and show by induction on k that there is a subsequence $\{(v_{i'},\eta_{i'})\}_{i'=1}^{\infty}$ of $\{(v_i,\eta_i)\}_{i=1}^{\infty}$ such that $\{x^k(u,\omega,v_{i'},\eta_{i'})\}_{i'=1}^{\infty}$ converges uniformly for $u \in [0,T]$. This will obviously imply that $\{X^k(t_0,\omega,v_{i'},\eta_{i'})\}_{i'=1}^{\infty}$ converges in M_2 because uniform convergence implies convergence in M_2. Therefore it remains to justify our inductive hypothesis. First note that $\{x^1(u,\omega,v_i,\eta_i) = v_i\}_{i=1}^{\infty}$ clearly has a subsequence uniformly convergent for $u \in [0,T]$, because $|v_i| \leq 1$ for $i = 1,2,\dots$. Suppose next that the inductive hypothesis holds for some $k \geq 1$. In the subsequent computation we shall denote all subsequences of a given sequence by the same symbol in order to simplify notation. So we pick a uniformly convergent subsequence $\{x^k(u,\omega,v_i,\eta_i)\}_{i=1}^{\infty}$ of $\{x^k(u,\omega,v_i,\eta_i)\}_{i=1}^{\infty}$. Note that the choice of such a subsequence may depend on k, ω but not on $u \in [0,T]$. Now write

$$x^{k+1}(\cdot,\omega,v_i,\eta_i) = I(\omega)(x^k(\cdot,\omega,v_i,\eta_i)), \quad i \geq 1, \tag{68}$$

$$I(\omega)(g) = \sum_{j=1}^{8} I_j(g)(t), \quad 0 \leq t \leq T, \tag{69}$$

where

$$I_1(g)(t) := \varphi(t)g(0) \tag{70}$$

$$I_2(g)(t) := \varphi(t) \int_0^t dA_1(u)g(u-) \tag{71}$$

$$I_3(g)(t) := \varphi(t) \int_{-r}^0 A_2(u)g(u)du \tag{72}$$

$$I_4(g)(t) := \varphi(t) \int_0^{t-r} A_2(u)g(u)du \tag{73}$$

$$I_5(g)(t) := \varphi(t) \int_0^t\int_{u-r}^u A_3(u,s)g(s)ds\, du \tag{74}$$

$$I_6(g)(t) := \psi(t) \int_{t-r}^{t} A_4(t,u)g(u)du \tag{75}$$

$$I_7(g)(t) := \varphi(t) \int_0^t A_5(u) \int_{[u-r,u]} \mu(u)(-u+ds)g(s)du \tag{76}$$

$$I_8(g)(t) := \varphi(t) \int_0^t dA_6(u) \int_{u-r}^{u} A_4(u,s)g(s)ds. \tag{77}$$

Observe that in the defining relations (70)-(77), $g \in \mathcal{E}$, $\omega \in \Omega_6$ is suppressed, A_2, A_3, A_4, A_5, φ are continuous matrix-valued processes and ψ, A_1, A_6 are cadlag matrix-valued processes of bounded variation. Our strategy is to show that each sequence $\{I_j(x^k(\cdot,\omega,v_i,\eta_i))(t)\}_{i=1}^{\infty}$, $j = 1,2,\ldots,8$, has a subsequence converging uniformly on $[0,T]$. We shall only discuss the cases $j = 6, 7$. The remaining cases may be treated similarly. Consider first the sequence $\{I_6(x^k(\cdot,\omega,v_i,\eta_i))(t)\}_{i=1}^{\infty}$ for $0 \leq t \leq r$ and write

$$I_6(x^k(\cdot,\omega,v_i,\eta_i))(t)=\psi(t)\left\{\int_{t-r}^{0} A_4(t,u)\eta_i(u)du+\int_0^t A_4(t,u)x^k(u,\omega,v_i,\eta_i)du\right\}, \tag{78}$$

$$J_1^i(t) := \int_{t-r}^{0} A_4(t,u)\eta_i(u)du, \quad J_2^i(t) := \int_0^t A_4(t,u)x^k(u,\omega,v_i,\eta_i)du. \tag{79}$$

Then by continuity of A_4 it follows that J_1^i, J_2^i are continuous functions $[0,r] \to \mathbb{R}^n$ and

$$\sup_{\substack{0\leq t\leq r\\ i\geq 1}} |J_1^i(t)| \leq \|A_4\|_\infty \sup_{i\geq 1} \int_{-r}^{0} |\eta_i(u)|du$$

$$\leq r^{1/2}\|A_4\|_\infty < \infty \tag{80}$$

because $\|\eta_i\|_{\mathcal{L}^2} \leq 1$ for all $i \geq 1$. Similarly by (61) we see that

$$\sup_{\substack{0\leq t\leq r\\ i\geq 1}} |J_2^i(t)| < \infty. \tag{81}$$

We now show that the families $\{J_1^i\colon i \geq 1\}$, $\{J_2^i\colon i \geq 1\}$ are equicontinuous. Let $\epsilon > 0$ be given. Then by (uniform) joint continuity of A_4, there is a

$\delta \in (0,\epsilon)$ such that

$$\|A_4(t_2,u) - A_4(t_1,u)\| < \epsilon \tag{82}$$

for all $u \in [0,T]$ whenever $|t_1-t_2| < \delta$, $t_1,t_2 \in [0,T]$. So suppose $t_1,t_2 \in [0,r]$ are such that $t_1 < t_2$ and $t_2-t_1 < \delta$. Then

$$\begin{aligned}|J_1^i(t_2) - J_1^i(t_1)| &\le \int_{t_1-r}^{t_2-r} \|A_4(t_2,u)\|\,|\eta_i(u)|du \\ &\quad + \int_{t_1-r}^{0} \|A_4(t_2,u)-A_4(t_1,u)\|\,|\eta_i(u)|du \\ &\le \|\eta_i\|_{\mathcal{L}^2}\|A_4\|_\infty|t_1-t_2|^{1/2} + \epsilon\|\eta_i\|_{\mathcal{L}^2}\, r^{1/2} \\ &< \|A_4\|_\infty\epsilon^{1/2} + \epsilon r^{1/2}.\end{aligned}$$

Also by (61) we have

$$\begin{aligned}|J_2^i(t_2) - J_2^i(t_1)| &\le \int_{t_1}^{t_2}\|A_4(t_2,u)\|\,|x^k(u,\omega,v_i,\eta_i)|du \\ &\quad + \int_0^{t_1}\|A_4(t_2,u)-A_4(t_1,u)\|\,|x^k(u,\omega,v_i,\eta_i)|du \\ &\le \|A_4\|_\infty\|x^k\|_\infty|t_1-t_2| + \epsilon r\|x^k\|_\infty < \epsilon\|x^k\|_\infty(\|A_4\|_\infty+r)\end{aligned}$$

where

$$\|x^k\|_\infty := \sup_{0\le u\le T}\ \sup_{(v,\eta)\in B} |x^k(u,\omega,v,\eta)| < \infty. \tag{61}$$

Hence by Ascoli's theorem there is a subsequence of $\{J_1^i + J_2^i\}_{i=1}^\infty$, denoted also by the same symbol, such that $\{J_1^i(t) + J_2^i(t)\}_{i=1}^\infty$ converges uniformly on $[0,r]$. Therefore since ψ is locally bounded it follows from (78) that the corresponding subsequence $\{I_6(x^k(\cdot,\omega,v_i,\eta_i))(t)\}_{i=1}^\infty$ converges uniformly for $0 \le t \le r$. A similar argument to the above yields the uniform convergence for $r \le t \le T$ of a further subsequence of $\{I_6(x^k(\cdot,\omega,v_i,\eta_i))(t)\}$. Hence the above subsequence converges uniformly on $[0,T]$.

Next let us look at the sequence $\{I_7(x^k(\cdot,\omega,v_i,\eta_i))(t)\}_{i=1}^\infty$ for $0 \le t \le r$. From (76) we obtain

$$I_7(x^k(\cdot,\omega,v_i,\eta_i))(t) = \varphi(t)\{J_3^i(t) + J_4^i(t)\} \tag{83}$$

where

$$J_3^i(t) := \int_0^t A_5(u) \int_{[u-r,0]} \mu(u)(-u+ds)\eta_i(s)du \tag{84}$$

$$J_4^i(t) := \int_0^t A_5(u) \int_{[0,u]} \mu(u)(-u+ds)x^k(s,\omega,v_i,\eta_i)du \tag{85}$$

for $0 \le t \le r$. To prove that $\{J_4^i(t)\}_{i=1}^{\infty}$ has a uniformly convergent subsequence we pick a uniformly convergent subsequence of $\{x^k(s,\omega,v_i,\eta_i)\}_{i=1}^{\infty}$ and use Hypothesis (C_2) to see that for the corresponding subsequence of $\{J_4^i(t)\}$ we have

$$\begin{aligned}|J_4^i(t)-J_4^j(t)| &\le \left\{\int_0^t\int_0^t |x^k(s,\omega,v_i,\eta_i)-x^k(s,\omega,v_j,\eta_j)|\bar{\mu}(u)(ds)du\right\}\|A_5\|_\infty \\ &\le \sup_{0\le s\le r} |x^k(s,\omega,v_i,\eta_i)-x^k(s,\omega,v_j,\eta_j)|(\nu(r)[0,r])\|A_5\|_\infty\end{aligned}$$

which goes to zero as $i,j \to \infty$. Thus $\{J_4^i(t)\}$ has a uniformly convergent subsequence. To prove that $\{J_3^i(t)\}_{i=1}^{\infty}$ is uniformly bounded we use (84) and Hypotheses (C_2), (C_5) to get

$$\begin{aligned}|J_3^i(t)| &\le \int_0^t \left\{\int_{[-r,0]} \|A_5(u)\| |1_{[u-r,0]}(s)| |\eta_i(s)| |\mu(u)(-u+\cdot)|(ds)\right\}du \\ &\le \|A_5\|_\infty \int_{-r}^0 |\eta_i(s)| \left\{\int_0^t \bar{\mu}(u)(\cdot)du\right\}(ds) \\ &= \|A_5\|_\infty \int_{-r}^0 \left|\frac{d\nu(t)}{ds}(s)\right| |\eta_i(s)|ds \\ &\le \|A_5\| \|\eta_i\|_{\mathbb{L}^2} \left\{\int_{-r}^0 \left|\frac{d\nu(t)}{ds}(s)\right|^2 ds\right\}^{1/2} \\ &\le \|A_5\|_\infty \left\|\frac{d\nu(t)}{ds}\right\|_\infty r^{1/2} \le \|A_5\|_\infty \left\|\frac{d\nu(r)}{ds}\right\|_\infty r^{1/2}, \quad i \ge 1, \quad 0 \le t \le r\end{aligned}$$

where

$$\left\|\frac{d\nu(r)}{ds}\right\|_\infty = \sup_{-r\le s\le 0}\left|\frac{d\nu(r)}{ds}(s)\right|, \quad \|A_5\|_\infty = \sup_{0\le u\le r}\|A_5(u)\|.$$

A similar computation shows that

$$|J_3^i(t_2)-J_3^i(t_1)| \le \|A_5\|_\infty \int_{-r}^{0}\left|\frac{d\nu(t_2)}{ds}(s) - \frac{d\nu(t_1)}{ds}(s)\right| |\eta_i(s)|ds$$
$$\le \|A_5\|_\infty \left\|\frac{d\nu(t_2)}{ds} - \frac{d\nu(t_1)}{ds}\right\|_{\mathbb{L}^2} \tag{86}$$

for $0 \le t_1, t_2 \le r$ and $i \ge 1$. Because of the (uniform) continuity of the map $[0,r] \ni t \mapsto \frac{d\nu(t)}{ds} \in \mathbb{L}^2([-r,0],\mathbb{R})$ (Hypothesis (C_2)), the above inequality implies that $\{J_3^i(t)\}$ is equicontinuous for $0 \le t \le r$. Therefore $\{J_3^i(t) + J_4^i(t)\}$ has a uniformly convergent subsequence for $0 \le t \le r$. Thus (83) implies that $\{I_7(x^k(\cdot,\omega,v_i,\eta_i))(t)\}_{i=1}^\infty$ has a uniformly convergent subsequence on $[0,r]$.

Using the inductive hypothesis and the relation

$$I_7(x^k(\cdot,\omega,v_i,\eta_i))(t) = \varphi(t)\{J_3^i(r)+J_4^i(r)\} +$$
$$+ \varphi(t) \int_r^t A_5(u) \int_{[u-r,u]} \mu(u)(-u+ds)x^k(s,\omega,v_i,\eta_i)du,$$

for $r \le t \le T$, it is easy to see that $\{I_7(x^k(\cdot,\omega,v_i,\eta_i))(t)\}_{i=1}^\infty$ has a subsequence which converges uniformly for $0 \le t \le T$.

In view of (70), (71), (72), (73), (74), (77) the reader may check that the remaining six sequences $\{I_j(x^k(\cdot,\omega,v_i,\eta_i))(t)\}_{i=1}^\infty$, $j = 1,2,3,4,5,8$ have subsequences which converge uniformly for $0 \le t \le T$. Therefore the inductive hypothesis is valid for $k+1$ and $X^k(t_0,\omega,\cdot)$ is compact for every $k \ge 1$. This completes the proof of assertion (v) of the theorem.

Assertion (vi) holds trivially because $\Omega_6 = \Omega_5 \cap \tilde{\Omega}_1$ and Ω_5, $\tilde{\Omega}_1$ are of full P-measure and are $\theta(t,\cdot)$-invariant (Theorem 3.2).

Finally we prove the cocycle property (vii) for (X,θ). Fix $t_1 \ge 0$, $\omega \in \Omega_6$ and $(v,\eta) \in \mathbb{M}_2$. Let $y^i(\cdot,\omega)$: $[-r,\infty) \to \mathbb{R}^n$, $i = 1,2$ denote the paths

$$y^1(t,\omega) := x(t,\theta(t_1,\omega),X(t_1,\omega,v,\eta)), \quad t \ge -r \tag{87}$$
$$y^2(t,\omega) := x(t+t_1,\omega,v,\eta), \quad t \ge -r. \tag{88}$$

Note that the cocycle property (48) will follow immediately if we show that

$$y^1(t,\omega) = y^2(t,\omega), \quad t \geq -r. \tag{89}$$

To prove the above relation (89), first observe that

$$(y^1(0,\omega), y^1_0(\cdot,\omega)) = X(t_1,\omega,v,\eta) = (y^2(0,\omega), y^2_0(\cdot,\omega)). \tag{90}$$

We shall next prove that y^2 satisfies the integral equation (IV) with ω replaced by $\theta(t_1,\omega)$ viz:

$$y^2(t,\omega) = I(\theta(t_1,\omega))(y^2(\cdot,\omega))(t), \quad t \geq 0. \tag{91}$$

Since y^1 also satisfies (91) with the same initial condition $X(t_1,\omega,v,\eta)$, uniqueness of the solution to the integral equation will give (89) and hence (48). Because of the relation

$$y^2(t,\omega) = I(\omega)(x(\cdot,\omega,v,\eta))(t+t_1), \quad t \geq 0$$

(91) will follow from

$$I(\theta(t_1,\omega))(y^2(\cdot,\omega))(t) = I(\omega)(x(\cdot,\omega,v,\eta))(t+t_1), \quad t \geq 0. \tag{92}$$

Therefore it remains to prove the above relation (92). To do so we outline the following rather lengthy computation. Start with the left hand side of (92) and use the definition of $I(\theta(t_1,\omega))$, Theorem (3.2)(iii), Theorem (3.1)(iii), hypotheses (C_1), (C_3), (C_5) and the additive cocycle property of $[M,N]$ to get

$$\begin{aligned} I(\theta(t_1,\omega))(y^2(\cdot,\omega))(t) &= \varphi(t,\theta(t_1,\omega))I(\omega)(x(\cdot,\omega,v,\eta))(t_1) \\ &- \varphi(t+t_1,\omega)\int_0^t \{Z(u+t_1,\omega)-Z(t_1,\omega)\} \\ &\times [K(u+t_1,\omega)(0)x(u+t_1,\omega,v,\eta)-K(u+t_1,\omega)(-r)x(u+t_1-r,\omega,v,\eta) \\ &+ \int_{u+t_1-r}^{u+t_1} \frac{\partial}{\partial u}(K(u+t_1,\omega)(s-(u+t_1)))x(s,\omega,v,\eta)ds]du \\ &+ \varphi(t+t_1,\omega)\{Z(t+t_1,\omega)-Z(t_1,\omega)\}\int_{-r}^0 K(t+t_1,\omega)(s)x(t+t_1+s,\omega,v,\eta)ds \end{aligned}$$

$$+ \varphi(t+t_1,\omega)\int_0^t \varphi^{-1}(u+t_1,\omega) \int_{[-r,0]} \mu(u+t_1,\omega)(ds)x(u+t_1+s,\omega,v,\eta)du$$

$$+ \varphi(t+t_1,\omega)\int_0^t \varphi^{-1}(u+t_1,\omega)dV(u+t_1,\omega)x((u+t_1)-,\omega,v,\eta)$$

$$-\varphi(t+t_1,\omega)\int_0^t \varphi^{-1}(u+t_1,\omega)d[M,N](u+t_1,\omega)\int_{-r}^0 K(u+t_1,\omega)(s)x(u+t_1+s,\omega,v,\eta)ds. \quad (93)$$

In the above relation we change the variable u to $u-t_1$ and substitute for $I(\omega)(x(\cdot,\omega,v,\eta))(t_1)$ from (41) to obtain

$$\begin{aligned} I(\theta(t_1,\omega))(y^2(\cdot,\omega))(t) &= I(\omega)(x(\cdot,\omega,v,\eta))(t+t_1) \\ &+ \varphi(t+t_1,\omega)Z(t_1,\omega)\int_{t_1}^{t+t_1}[K(u,\omega)(0)x(u,\omega,v,\eta) \\ &- K(u,\omega)(-r)x(u-r,\omega,v,\eta) + \\ &+ \int_{u-r}^{u} \frac{\partial}{\partial u}(K(u,\omega)(s-u))x(s,\omega,v,\eta)ds]du \\ &+ \varphi(t+t_1,\omega)Z(t_1,\omega)\int_{-r}^0 K(t_1,\omega)(s)x(t_1+s,\omega,v,\eta)ds. \quad (94) \end{aligned}$$

The equality (92) now follows from (94) by integrating the relation

$$\begin{aligned} \frac{d}{du}\int_{-r}^0 K(u,\omega)(s)x(u+s,\omega,v,\eta)ds &= K(u,\omega)(0)x(u,\omega,v,\eta) - K(u,\omega)(-r)x(u-r,\omega,v,\eta) \\ &+ \int_{u-r}^{u} \frac{\partial}{\partial u}(K(u,\omega)(s-u))x(s,\omega,v,\eta)ds \end{aligned}$$

with respect to u between t_1 and t_1+t. This completes the proof of the cocycle property (48). □

<u>Remark</u>:

The continuity of $X(t,\omega,\cdot)$: $M_2 \to M_2$ in the norm $\|\cdot\|_{M_2}$ is guaranteed by Hypothesis (C_2). On the other hand, if the state space M_2 is replaced by the space $D := D([-r,0],\mathbb{R}^n)$ of all cadlag paths η: $[-r,0] \to \mathbb{R}^n$ with the supremum norm

$\|\eta\|_\infty := \sup_{-r\le s\le 0} |\eta(s)|$, then Hypothesis (C_2) may be dropped and Theorem (4.2) will hold with $\mathbb{M}_2$ replaced by $\mathbb{D}$.

§5. Lyapunov Exponents

In this section we prove the existence of a countable set of Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,\cdot,v(\cdot),\eta(\cdot)\|_{\mathbb{M}_2} \tag{95}$$

for the stochastic flow of the s.f.d.e. (I) which we constructed in §4 (Theorem 4.2). Such a Lyapunov spectrum corresponds to almost sure exponential growth rates for trajectories $\{(x(t),x_t): t \ge 0,\ (x(0),x_0) = (v,\eta)\}$ of (I) starting off at possibly random initial states $(v,\eta) \in \mathbb{L}^2(\Omega,\mathbb{M}_2;\mathcal{F}_0)$. The existence of the Lyapunov spectrum is achieved using Ruelle's infinite-dimensional discrete version of Oseledec's multiplicative ergodic theorem (Ruelle [23], [22]). In the hyperbolic case, when all the Lyapunov exponents are non-zero, we establish an exponential dichotomy for the flow which is invariant under the cocycle (X,θ). The *continuous-time* limit (95) is shown to exist by noting the compactness of $X(r,\omega,\cdot)$ (Theorem (4.2)) and then discretizing (95) using multiples of the delay r:

$$\lim_{k\to\infty} \frac{1}{kr} \log \|X(kr,\cdot,v,\eta)\|_{\mathbb{M}_2}. \tag{96}$$

A key step in identifying the limits (95) and (96) is to establish the integrability property

$$E \sup_{0\le t_1,t_2\le r} \log^+\|X(t_1,\theta(t_2,\cdot),\cdot)\|_{L(\mathbb{M}_2)} < \infty \tag{97}$$

where $\|\cdot\|_{L(M_2)}$ is the uniform operator norm on $L(M_2)$ (cf. Lemma 4, §4 in [20]). Much of the work in this section is directed towards realizing the above integrability property. To begin with we shall use the following moment hypotheses on the driving processes in the s.f.d.e. (I):

Hypotheses (I):

(I_1) If ν is the measure defined in Hypothesis (C_2), suppose that

$$E \sup_{-r\leq s\leq 2r} \left|\frac{d\nu(s)}{ds}\right|^3 < \infty ;$$

(I_2) $E \sup_{\substack{0\leq t\leq 2r \\ -r\leq s\leq 0}} \|K(t,\cdot)(s)\|^4 < \infty;$

$$E \sup_{\substack{0\leq t\leq 2r \\ -r\leq s\leq 0}} \left\{\|\frac{\partial}{\partial t} K(t,\cdot)(s)\|^4 + \|\frac{\partial}{\partial s} K(t,\cdot)(s)\|^4\right\} < \infty, \quad E\{|V|(2r,\cdot)\}^4 < \infty;$$

$\|\cdot\|$ denotes the (Euclidean) norm on $\mathbb{R}^{n\times n}$.

(I_3) Write the semimartingale N in the form $N = N^0+V^0$ where the local $(\mathcal{F}_t)_{t\geq 0}$-martingale $N^0 = (N^0_{ij})^n_{i,j=1}$ and the bounded variation process $V^0 = (V^0_{ij})^n_{i,j=1}$ satisfy

$$E\{[N^0_{ij}](2r,\cdot)\}^4 < \infty,$$

$$E\{[V^0_{ij}](2r,\cdot)\}^4 = E\left[\sum_{0\leq s\leq 2r} |\Delta V^0_{ij}(s)|^2\right]^4 < \infty,$$

$$E\{|V^0_{ij}|(2r,\cdot)\}^8 < \infty,$$

for all $1 \leq i,j \leq n$. Note that $|V^0_{ij}|(2r,\cdot)$ is the total variation of V^0_{ij} over $[0,2r]$ and $\Delta V^0_{ij}(s)$ is its jump at s.

(I_4) Write $M = (M_{ij})^n_{i,j=1}$ and suppose that

$$E\{\langle M_{ij}\rangle(2r,\cdot)\}^4 < \infty$$

for all $1 \leq i,j \leq n$.

(I_5) Suppose there exists a *non-random* time $t_0 > 0$ such that

$$\sum_{i,j,k,\ell} |\langle M_{ij}, M_{k\ell}\rangle|(t_0,\cdot) \in \mathbb{L}^\infty(\Omega,\mathbb{R}).$$

Our first goal is to establish the integrability property (97) under Hypotheses (C) and (I). To do this we stress the dependence on $\omega \in \Omega$ of the "constants" C_i, $i = 1,2,\dots,8$ appearing in (51), (55)′, (56), (57), (58). From the proof of Theorem (4.2) ((58)) it follows that for each $\omega \in \Omega_6$ we have

$$\begin{aligned}
&\sup_{\substack{0\le t_1,t_2\le r\\ \|(v,\eta)\|\le 1}} |x(t_1,\theta(t_2,\omega),(v,\eta))| \\
&\qquad \le \sup_{0\le t_2\le r} C_8(\theta(t_2,\omega))\exp\Big\{\sup_{0\le t_2\le r} C_6(\theta(t_2,\omega))\cdot \sup_{0\le t_2\le r} a(r,\theta(t_2,\omega))\Big\} \times \\
&\qquad\quad \times \sup_{\substack{0\le t_1,t_2\le r\\ \|(v,\eta)\|\le 1}} |x^2(t_1,\theta(t_2,\omega),v,\eta)-x^1(t_1,\theta(t_2,\omega),v,\eta)| \\
&\qquad\quad + \sup_{\substack{0\le t_1,t_2\le r\\ \|(v,\eta)\|\le 1}} |x^2(t_1,\theta(t_2,\omega),v,\eta)|
\end{aligned} \tag{98}$$

$$C_6(\omega) = C_2(\omega) + C_3(\omega) \tag{99}$$

$$C_8(\omega) = C_6(\omega)a(r,\omega) + C_7(\omega) \tag{100}$$

$$C_7(\omega) = C_5(\omega)[a(r,\omega)+1] = C_5(\omega)[|V|(r,\omega)+r+1] \tag{101}$$

$$\begin{aligned}
C_5(\omega) = &\sup_{0\le t\le r} \|\varphi(t,\omega)\| \cdot \sup_{\substack{0\le t\le r\\ s\in[-r,0]}} \|K(t,\omega)(s)\| \cdot \sup_{0\le t\le r} \|\Delta Z(t,\omega)\| \\
&+ \sup_{0\le t\le r} \|\Delta V(t,\omega)\|.
\end{aligned} \tag{102}$$

Note that the integrability property (97) is implied by

$$\int_\Omega \log^+ \sup_{\substack{0\le t_1,t_2\le r\\ \|(v,\eta)\|\le 1}} |x(t_1,\theta(t_2,\omega),v,\eta)|\,dP(\omega) < \infty. \tag{103}$$

Taking $\log^+$ in (98) and using the elementary inequality

$$\log^+(a+b) \le \log^+ a + \log^+ b + \log 2, \qquad a,b \in \mathbb{R}^+ \tag{104}$$

we see that

$$\begin{aligned}
\log^+ \sup_{\substack{0\le t_1,t_2\le r\\ \|(v,\eta)\|\le 1}} &|x(t_1,\theta(t_2,\omega),v,\eta)| \\
&\le \log^+ \sup_{0\le t_2\le r} C_8(\theta(t_2,\omega)) + \sup_{0\le t_2\le r} C_6(\theta(t_2,\omega))\cdot \sup_{0\le t_2\le r} a(r,\theta(t_2,\omega)) \\
&\quad + \log^+ \sup_{\substack{0\le t_1,t_2\le r\\ \|(v,\eta)\|\le 1}} |x^2(t_1,\theta(t_2,\omega),v,\eta) - x^1(t_1,\theta(t_2,\omega),v,\eta))| \\
&\quad + \log^+ \sup_{\substack{0\le t_1,t_2\le r\\ \|(v,\eta)\|\le 1}} |x^2(t_1,\theta(t_2,\omega),v,\eta)| + \log 2.
\end{aligned} \tag{105}$$

In view of the above inequality we establish (103) by showing the existence of appropriate higher order moments for each of the random variables appearing on the right hand side of (105). This is done in the following sequence of lemmas under Hypotheses (I) and (C).

Our first lemma asserts that Hypotheses (I_5) and (C_5) are sufficient to guarantee the existence of all higher order moments for the stochastic flows $\{\varphi(t): t \ge 0\}$, $\{\varphi^{-1}(t): t \ge 0\}$ (Theorem 3.1).

<u>Lemma (5.1)</u>:

Let $\mathbb{M}$ satisfy Hypotheses (C_5) and (I_5). Then for each $0 < T < \infty$ *and every integer* $p \ge 1$,

$$E \sup_{0\le t\le T} |\varphi(t,\cdot)\|^{2p} < \infty, \tag{106}$$

$$E \sup_{0\le t\le T} \|\varphi^{-1}(t,\cdot)\|^{2p} < \infty, \tag{107}$$

and

$$E \sup_{0\le t_1,t_2\le T} \|\varphi(t_1,\theta(t_2,\cdot),\cdot)\|^{2p} < \infty . \tag{108}$$

Proof:

For each $t \in \mathbb{R}^+$ we write

$$\varphi(t) = (\varphi_{ij}(t))_{i,j=1}^{n}$$

where

$$\varphi_{ij}(t) = \delta_{ij} + \sum_{s=1}^{n} \int_0^t \varphi_{sj}(u)dM_{is}(u), \quad i,j = 1,2,\dots,n \qquad \text{a.s.}$$

Recall that $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{n\times n}$ and so the function $\|\cdot\|^{2p}$ is smooth on $\mathbb{R}^{n\times n}$ for any integer $p \geq 1$. Hence by Itô's formula (Métivier [17], Theorem 27.1, pp. 188-189) we obtain a.s.

$$\begin{aligned}\|\varphi(t)\|^{2p} &= 1 + 2p \sum_{i,j,s=1}^{n} \int_0^t \|\varphi(u)\|^{2(p-1)} \varphi_{ij}(u)\varphi_{sj}(u)dM_{is}(u) + \\ &+ 2p(p-1) \cdot \sum_{i,j,k,\ell,q,s=1}^{n} \int_0^t \|\varphi(u)\|^{2(p-2)} \varphi_{k\ell}(u)\varphi_{ij}(u)\varphi_{sj}(u)\varphi_{q\ell}(u)d\langle M_{is},M_{kq}\rangle(u) \\ &+ p \cdot \sum_{i,j,q,s=1}^{n} \int_0^t \|\varphi(u)\|^{2(p-1)} \varphi_{sj}(u)\varphi_{qj}(u)d\langle M_{is},M_{iq}\rangle(u), \quad t \geq 0. \qquad (109)\end{aligned}$$

Define the continuous increasing process

$$\beta(t) := \sum_{i,k,s,q=1}^{n} |\langle M_{is},M_{kq}\rangle|(t), \quad t > 0 \qquad (110)$$

and a sequence of stopping times

$$\tau_m := \inf\{t: t > 0, \|\varphi(t)\| > m\}, \quad m = 1,2,3,\dots .$$

Set

$$\varphi_m(t) := \varphi(t \wedge \tau_m), \quad t > 0, m = 1,2,\dots . \qquad (111)$$

Since M is a martingale, (109) implies that

$$\begin{aligned}E\|\varphi_m(t)\|^{2p} &\leq 1 + 2p(p-1)n^6 E \int_0^{t\wedge\tau_m} \|\varphi_m(u)\|^{2p} d\beta(u) \\ &+ pn^4 E \int_0^{t\wedge\tau_m} \|\varphi_m(u)\|^{2p} d\beta(u).\end{aligned}$$

Hence there is a deterministic constant $C_9 = C_9(n,p) > 0$ independent of m and t such that

$$E\|\varphi_m(t)\|^{2p} \le 1 + C_9 E\int_0^t \|\varphi_m(u)\|^{2p} d\gamma(u), \quad t > 0 \tag{112}$$

where γ is the continuous predictable increasing process

$$\gamma(t) := \beta(t) + t, \quad t \ge 0. \tag{113}$$

For a.a. $\omega \in \Omega$ we denote by $\gamma^{-1}(\cdot,\omega)$ the inverse of $\gamma(\cdot,\omega)$. Then for each $t \ge 0$, $\gamma^{-1}(t,\cdot)$ is an $(\mathcal{F}_t)_{t\ge 0}$-stopping time. Thus in (109) we can replace t by $\gamma^{-1}(t,\cdot)\wedge\tau_m$ and take expectations to get

$$\begin{aligned} E\|\varphi_m(\gamma^{-1}(t))\|^{2p} &\le 1 + C_9 E\int_0^{\gamma^{-1}(t)} \|\varphi_m(u)\|^{2p} d\gamma(u) \\ &= 1 + C_9 \int_0^t E\|\varphi_m(\gamma^{-1}(u))\|^{2p} du, \quad t \ge 0. \end{aligned} \tag{114}$$

Applying Gronwall's lemma to (114) gives

$$E\|\varphi_m(\gamma^{-1}(t))\|^{2p} \le e^{C_9 t}, \quad t \ge 0 \tag{115}$$

for all integers $m \ge 1$. Now from the definition of φ_m ((111)) it is clear that a.s.

$$\lim_{m\to\infty} \varphi_m(\gamma^{-1}(t)) = \varphi(\gamma^{-1}(t))$$

for all $t \ge 0$. Since the right hand side of (115) is independent of m, it follows from Fatou's lemma that

$$E\|\varphi(\gamma^{-1}(t))\|^{2p} \le \liminf_{m\to\infty} E\|\varphi_m(\gamma^{-1}(t))\|^{2p} \le e^{C_9 t} < \infty. \tag{116}$$

Now let $k \ge 1$. Then by similar reasoning we may replace $\gamma^{-1}(t)$ in (116) by $\gamma^{-1}(t) \wedge k$ and get

$$E\|\varphi(\gamma^{-1}(t)\wedge k)\|^{2p} \le e^{C_9 t}, \quad t \ge 0. \tag{117}$$

Since φ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale and $\gamma^{-1}(t) \wedge k$ is a bounded stopping time, it follows from Doob's optional sampling theorem that $\{\varphi(\gamma^{-1}(t)\wedge k): t \geq 0\}$ is an $(\mathcal{F}_{\gamma^{-1}(t)\wedge k})_{t\geq 0}$-martingale and

$$E \sup_{0\leq t\leq T} \|\varphi(\gamma^{-1}(t) \wedge k)\|^{2p} \leq C_{10} E\|\varphi(\gamma^{-1}(T) \wedge k)\|^{2p} \leq C_{10} e^{C_9 T} \tag{118}$$

(Ikeda & Watanabe [9], p. 34). The constant C_{10} does not depend on k or T. Taking liminf as $k \to \infty$ in (118) and using Fatou's lemma once more gives

$$E \sup_{0\leq t\leq T} \|\varphi(\gamma^{-1}(t))\|^{2p} \leq C_{10} \liminf_{k\to\infty} E\|\varphi(\gamma^{-1}(T) \wedge k)\|^{2p} \leq C_{10} e^{C_9 T} < \infty.$$

Since $\gamma^{-1}([0,T],\cdot) = [0,\gamma^{-1}(T,\cdot)]$ a.s., the above inequality now reads:

$$E \sup_{0\leq u\leq \gamma^{-1}(T)} \|\varphi(u)\|^{2p} \leq C_{10} e^{C_9 T} \tag{119}$$

for every $T > 0$.

By Hypothesis (I_5) we can define the deterministic time

$$T_0 := \operatorname*{essup}_{\omega} \beta(t_0,\omega).$$

Then $T_0+t_0 \geq \gamma(t_0)$ a.s. and so $t_0 \leq \gamma^{-1}(T_0+t_0)$ a.s. Replacing T by T_0+t_0 in (119) we see that

$$E \sup_{0\leq u\leq t_0} \|\varphi(u)\|^{2p} \leq C_{10} e^{C_9(T_0+t_0)} < \infty \tag{120}$$

for all $p \geq 1$.

Now invoke the cocycle property for φ, Hölder's inequality and the measure-preserving property of $\theta(t_0,\cdot)$ to get

$$\begin{aligned} E \sup_{0\leq u\leq 2t_0} \|\varphi(u,\cdot)\|^{2p} &\leq \Big[E \sup_{0\leq u\leq t_0} \|\varphi(u,\theta(t_0,\cdot))\|^{4p}\Big]^{1/2} \cdot \Big[E\|\varphi(t_0,\cdot)\|^{4p}\Big]^{1/2} \\ &= \Big[E \sup_{0\leq u\leq t_0} \|\varphi(u,\cdot))\|^{4p}\Big]^{1/2} \cdot \Big[E\|\varphi(t_0,\cdot)\|^{4p}\Big]^{1/2} \\ &< \infty. \end{aligned}$$

Hence by induction it follows that

$$E \sup_{0\le u\le mt_0} \|\varphi(u,\cdot)\|^{2p} < \infty$$

for every integer $m \ge 1$ and (106) holds for every $T > 0$.

In order to prove assertion (107) of the lemma, note first that a simple application of the product rule shows that φ^{-1} is the unique solution of the matrix s.d.e.

$$\left.\begin{aligned} d\varphi^{-1}(t) &= -\varphi^{-1}(t)dM(t) + \varphi^{-1}(t)d\langle M\rangle(t), \quad t > 0 \\ \varphi^{-1}(0) &= I \end{aligned}\right\} \qquad \text{(VI)}$$

where $\langle M\rangle_{ij} := \sum_{k=1}^{n} \langle M_{ik}, M_{kj}\rangle$ (cf. Leandre [13], p. 273).

Write $\varphi^{-1}(t) := (\varphi_{ij}^{-1}(t))_{i,j=1}^{n}$ and apply Itô's formula as before (cf. (109)) to get

$$\begin{aligned} \|\varphi^{-1}(t)\|^{2p} &\le 1 + C_{11} \cdot \sum_{i,j,s=1}^{n} \left|\int_0^t \|\varphi^{-1}(u)\|^{2(p-1)} \varphi_{ij}^{-1}(u)\varphi_{is}^{-1}(u)dM_{sj}(u)\right| + \\ &+ C_{12} \cdot \sum_{i,j,s=1}^{n} \int_0^t \|\varphi^{-1}(u)\|^{2(p-1)} |\varphi_{ij}^{-1}(u)||\varphi_{is}^{-1}(u)|d|\langle M,M\rangle_{sj}|(u) + \\ &+ C_{13} \cdot \sum_{i,j,k,\ell,s,r=1}^{n} \int_0^t \Big\{\|\varphi^{-1}(u)\|^{2(p-2)} |\varphi_{k\ell}^{-1}(u)||\varphi_{ij}^{-1}(u)||\varphi_{kr}^{-1}(u)||\varphi_{is}^{-1}(u)| \\ &\qquad + \|\varphi^{-1}(u)\|^{2(p-1)} |\varphi_{is}^{-1}(u)||\varphi_{kr}^{-1}(u)|\Big\} d|\langle M_{sj}, M_{r\ell}\rangle|(u) \end{aligned} \qquad (121)$$

a.s. for all $t \ge 0$. The constants C_{11}, C_{12}, C_{13} depend only on p. For each $m \ge 1$ define

$$\tau_m' := \inf\{t: t > 0, \sup_{0\le s\le t} \|\varphi^{-1}(\gamma^{-1}(s))\| > m\}$$

and

$$\varphi_m^{-1}(t) := \varphi^{-1}(t \wedge \gamma^{-1}(\tau_m')), \quad t \ge 0.$$

It is clear that a.s. $\|\varphi_m^{-1}(t)\| \le m$ for all $t \ge 0$. In (121) we replace t by $s \wedge \gamma^{-1}(\tau_m')$, square both sides and take $E \sup_{0\le s\le \gamma^{-1}(t)}$ to get

$$E \sup_{0\le s\le\gamma^{-1}(t)} \|\varphi_m^{-1}(s)\|^{4p} \le 1 + C_{14}\, E \int_0^{\gamma^{-1}(t)} \|\varphi_m^{-1}(u)\|^{4p} d\beta(u)$$

$$+ C_{15} E\{\beta(\gamma^{-1}(t)) \int_0^{\gamma^{-1}(t)} \|\varphi_m^{-1}(u)\|^{4p} d\beta(u)\},\ t \ge 0, \qquad (122)$$

where the constants C_{14}, C_{15} depend only on p and n.

Now $\beta(\gamma^{-1}(t)) \le t$ a.s., so (122) implies

$$E \sup_{0\le s\le\gamma^{-1}(t)} \|\varphi_m^{-1}(s)\|^{4p} \le 1 + C_{16}\, E \int_0^{\gamma^{-1}(t)} \sup_{0\le s\le u} \|\varphi_m^{-1}(s)\|^{4p} d\gamma(u), \quad 0 \le t \le T$$

$$= 1 + C_{16} \int_0^t E \sup_{0\le s\le\gamma^{-1}(u)} \|\varphi_m^{-1}(s)\|^{4p} du, \quad 0 \le t \le T,$$

with $C_{16} = C_{16}(n,p,T) > 0$ and independent of $m \ge 1$. Hence by Gronwall's lemma,

$$E \sup_{0\le s\le\gamma^{-1}(t)} \|\varphi_m^{-1}(s)\|^{4p} \le e^{C_{16}t}, \quad 0 \le t \le T, \quad m \ge 1. \qquad (123)$$

If $m > \sup_{0\le s\le T} \|\varphi^{-1}(\gamma^{-1}(s))\|$, then

$$\sup_{0\le s\le\gamma^{-1}(t)} \|\varphi_m^{-1}(s)\| = \sup_{0\le s\le\gamma^{-1}(t)} \|\varphi^{-1}(s)\|, \quad \text{a.s.},\ 0 \le t \le T.$$

Thus it follows from (123) and Fatou's lemma that

$$E \sup_{0\le s\le\gamma^{-1}(t)} \|\varphi^{-1}(s)\|^{4p} \le e^{C_{16}t} < \infty, \quad 0 \le t \le T.$$

Now using the above inequality, Hypothesis (I_5) and the cocycle property for φ^{-1} one easily obtains the assertion (107) of the lemma.

Finally note that (108) follows immediately from (106), (107), the cocycle property for φ and Hölder's inequality. This completes the proof of the lemma. □

Lemma (5.2)

Suppose $E\{|V|(2r,\cdot)\}^p < \infty$ *for a fixed* $p \geq 1$. *Then*

$$E \sup_{0\leq t_2\leq r} \{a(r,\theta(t_2,\cdot))\}^p < \infty. \tag{124}$$

Proof:

The lemma follows directly from

$$a(r,\theta(t_2,\omega)) = r + |V|(r,\theta(t_2,\omega))$$

and the fact that

$$|V|(r,\theta(t_2,\omega)) \leq |V|(r+t_2,\omega) + |V|(t_2,\omega) \quad \text{(Hypothesis } (C_5)). \qquad \square$$

Lemma (5.3):

Let M,N *satisfy Hypotheses* (C_5), (C_4), *and let* $p \geq 1$ *be such that*

$$E\{\langle M_{ij}\rangle(2r,\cdot)\}^p < \infty \tag{125}$$

$$E\{[N^0_{ij}](2r,\cdot)\}^p < \infty \tag{126}$$

and

$$E\{[V^0_{ij}](2r,\cdot)\}^p = E\Big[\sum_{0\leq s\leq 2r} |\Delta V^0_{ij}(s)|^2\Big]^p < \infty \quad \textit{for all } 1 \leq i,j \leq n. \tag{127}$$

Then

$$E \sup_{0\leq t_2\leq r} \{|[M,N]|(r,\theta(t_2,\cdot))\}^p < \infty. \tag{128}$$

Proof:

First observe that the given hypotheses on N^0 and V^0 imply that

$$E\{[N_{ij}](2r,\cdot)\}^p < \infty, \quad i,j = 1,2,\ldots,n. \tag{129}$$

To see this, we write

$$\begin{aligned}[N_{ij}](2r,\cdot) &= [N^0_{ij}](2r,\cdot) + \sum_{0\leq s\leq 2r} |\Delta V^0_{ij}(s)|^2 + 2[N^0_{ij},V^0_{ij}](2r,\cdot) \\ &\leq [N^0_{ij}](2r,\cdot) + [V^0_{ij}](2r,\cdot) + \{N^0_{ij}(2r,\cdot)\}^{1/2}\cdot\{[V^0_{ij}](2r,\cdot)\}^{1/2}\end{aligned} \tag{130}$$

a.s. for $i,j = 1,2,\dots,n$, by the Kunita-Watanabe inequality (Elliott [6], p. 126). Hence by Hölder's inequality

$$E\{[N_{ij}](2r,\cdot)\}^p \le 3^pE\{N^0_{ij}(2r,\cdot)\}^p + 3^pE\{[V^0_{ij}](2r,\cdot)\}^p$$
$$+ 3^p(E\{N^0_{ij}(2r,\cdot)\}^p)^{1/2}\cdot(E\{[V^0_{ij}](2r,\cdot)\}^p)^{1/2}$$
$$< \infty, \qquad i,j = 1,2,\dots,n$$

and (129) holds.

To prove (128) we recall that $|[M,N]|(r,\cdot)$ is the total variation of the $\mathbb{R}^{n\times n}$-valued process

$$[M,N] = \left[[M,N]_{ij} = \sum_{k=1}^{n}[M_{ik},N_{kj}]\right]_{i,j=1}^{n}$$

measured with respect to the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$. Then by the Kunita-Watanabe inequality we have a.s.

$$|[M,N]|(r,\theta(t_2,\cdot)) \le n\sum_{i,j=1}^{n}\sum_{k=1}^{n}|[M_{ik},N_{kj}]|(r,\theta(t_2,\cdot))$$
$$\le n\sum_{i,j=1}^{n}\sum_{k=1}^{n}\{[M_{ik}](2r,\cdot)\}^{1/2}\{[N_{kj}](2r,\cdot)\}^{1/2}$$

for all $0 \le t_2 \le r$. Taking supremum over t_2 and expectations we see from Hölder's inequality that

$$E\sup_{0\le t_2\le r}\{|[M,N]|(r,\theta(t_2,\cdot))\}^p \le C_{17}\sum_{i,j=1}^{n}\sum_{k=1}^{n}(E\{[M_{ik}](2r,\cdot)\}^p)^{1/2}(E\{[N_{kj}](2r,\cdot)\}^p)^{1/2}$$
$$\le C_{17}\sum_{i,j=1}^{n}\left[\sum_{k=1}^{n}E\{[M_{ik}](2r,\cdot)\}^p\right]^{1/2}\cdot\left[\sum_{k=1}^{n}E\{[N_{kj}](2r,\cdot)\}^p\right]^{1/2}$$

which is finite by (125) and (129). This completes the proof of the lemma. □

The following lemma gives an integrability property for the process Z defined in Theorem (3.2):

Lemma (5.4):

Suppose M satisfies Hypotheses (C_5) and (I_5). Let N^0, V^0 be such that

$$E\{[N^0_{ij}](2r,\cdot)\}^{2p} < \infty \tag{131}$$

$$E\{|V^0_{ij}|(2r,\cdot)\}^{4p} < \infty \tag{132}$$

for all $1 \leq i,j \leq n$ *and a given* $p \geq 1$. *Then*

$$E \sup_{0\leq t_1,t_2\leq r} \|Z(t_1,\theta(t_2,\cdot))\|^p < \infty. \tag{133}$$

Proof:

From Theorem (3.2)

$$Z(t_1,\theta(t_2,\cdot)) = \varphi(t_2,\cdot)\{Z(t_2+t_1,\cdot)-Z(t_2,\cdot)\} \quad \text{a.s.,} \quad t_1,t_2 \geq 0. \tag{134}$$

Therefore

$$E \sup_{0\leq t_1,t_2\leq r} \|Z(t_1,\theta(t_2,\cdot))\|^p \leq 2^p\Big\{E \sup_{0\leq t_2\leq r} \|\varphi(t_2,\cdot)\|^{2p}\Big\}^{1/2}\cdot\Big\{E \sup_{0\leq t\leq 2r} \|Z(t,\cdot)\|^{2p}\Big\}^{1/2}. \tag{135}$$

In view of this and (106) of Lemma (5.1), it is sufficient to prove that

$$E \sup_{0\leq t\leq 2r} \|Z(t,\cdot)\|^{2p} < \infty. \tag{136}$$

Now write $Z(t) = (Z_{ij}(t))^n_{i,j=1}$ where

$$Z_{ij}(t) = I^1_{ij}(t) + I^2_{ij}(t), \tag{137}$$

$$I^1_{ij}(t) := \sum_{k=1}^n \int_0^t \varphi^{-1}_{ik}(u)dN^0_{kj}(u), \tag{138}$$

$$I^2_{ij}(t) := \sum_{\ell=1}^n \int_0^t \varphi^{-1}_{i\ell}(u)dV^0_{\ell j}(u), \tag{139}$$

for $t \geq 0$, $1 \leq i,j \leq n$, a.s.

Therefore (136) will follow from

$$E \sup_{0\le t\le 2r} |I^m_{ij}(t)|^{2p} < \infty, \quad m = 1,2, \quad 1 \le i,j \le n. \tag{140}$$

It remains to prove (140). Using the martingale property of the N^0_{kj} and standard estimates on the stochastic integral in (138) (Métivier [17], E. 3, p. 212), there is a deterministic constant $C_{18} = C_{18}(n,p) > 0$ such that

$$\begin{aligned}
E \sup_{0\le t\le 2r} |I^1_{ij}(t)|^{2p} &\le C_{18} \sum_{k=1}^{n} E\left|\int_0^{2r} \varphi^{-1}_{ik}(u)dN^0_{kj}(u)\right|^{2p} \\
&\le C_{18} \sum_{k=1}^{n} E\left|\int_0^{2r} \{\varphi^{-1}_{ik}(u)\}^2 d[N^0_{kj}](u)\right|^{p} \\
&\le C_{18} \sum_{k=1}^{n} E\left\{\sup_{0\le u\le 2r} |\varphi^{-1}_{ik}(u)|^{2p}[N^0_{kj}](2r,\cdot)^p\right\} \\
&\le C_{18} \left\{\sum_{k=1}^{n} E \sup_{0\le u\le 2r} |\varphi^{-1}_{ik}(u)|^{4p}\right\}^{1/2} \cdot \left\{\sum_{k=1}^{n} E([N^0_{kj}](2r,\cdot))^{2p}\right\}^{1/2}
\end{aligned}$$

which is finite by (107) and (131).

The estimate

$$E \sup_{0\le t\le 2r} |I^2_{ij}(t)|^{2p} < \infty, \quad i,j = 1,2,\ldots,n$$

is obtained in a similar way from (139) by using (107) and (132). This proves (140) and the lemma. □

The next lemma establishes the integrability of the second term on the right hand side of (105).

<u>Lemma (5.5)</u>:

Assume Hypotheses (C) and (I). Then

$$E\{\sup_{0\le t_2\le r} C_6(\theta(t_2,\cdot)) \sup_{0\le t_2\le r} a(r,\theta(t_2,\cdot))\} < \infty. \tag{141}$$

Proof:

Let $\omega \in \Omega_6$. Then, by (99),

$$C_6(\theta(t_2,\omega)) = C_2(\theta(t_2,\omega)) + C_3(\theta(t_2,\omega)), \quad t_2 \geq 0 \tag{142}$$

where $C_2(\cdot)$ and $C_3(\cdot)$ are the random variables appearing in (51). Recall that these were arrived at by estimating each of the eight terms on the right hand side of (41). In particular, replacing ω by $\theta(t_2,\omega)$, we get

$$\sup_{0\leq t_2\leq r} C_3(\theta(t_2,\omega)) \leq \sup_{0\leq t_1,t_2\leq 2r} \|\varphi(t_1,\theta(t_2,\omega))\|. \tag{143}$$

Furthermore, substituting $\theta(t_2,\omega)$ for ω in (41) and using the cocycle property for φ (Theorem 3.1), the stationarity of μ,K (Hypotheses (C_1), (C_3)) and Hypothesis (C_2), the reader may check that

$$\sup_{0\leq t_2\leq r} C_2(\theta(t_2,\omega)) \leq \sum_{i=1}^{4} C_2^i(\omega) \tag{144}$$

where

$$C_2^1(\omega) := 6 \sup_{0\leq t_1,t_2\leq r} \|\varphi(t_1,\theta(t_2,\omega))\|^2 \sup_{0\leq t\leq 2r} \|Z(t,\omega)\| \cdot \sup_{\substack{0\leq t\leq 2r\\ -r\leq s\leq 0}} \|K(t,\omega)(s)\| \tag{145}$$

$$C_2^2(\omega) := 2r \sup_{0\leq t_1,t_2\leq r} \|\varphi(t_1,\theta(t_2,\omega))\|^2 \sup_{0\leq t\leq 2r} \|Z(t,\omega)\| \times$$

$$\times \left\{ \sup_{\substack{0\leq t\leq 2r\\ -r\leq s\leq 0}} \|\frac{\partial}{\partial t}K(t,\omega)(s)\| + \sup_{\substack{0\leq t\leq 2r\\ -r\leq s\leq 0}} \|\frac{\partial}{\partial s}K(t,\omega)(s)\| \right\} \tag{146}$$

$$C_2^3(\omega) := \sup_{0\leq t_1,t_2\leq 2r} \|\varphi(t_1,\theta(t_2,\omega))\| \cdot \sup_{-r\leq s\leq 2r} \left|\frac{d\nu(\omega)}{ds}(s)\right| \tag{147}$$

$$C_2^4(\omega) := \sup_{0\leq t_1,t_2\leq 2r} \|\varphi(t_1,\theta(t_2,\omega))\| \cdot \sup_{\substack{0\leq t\leq 2r\\ -r\leq s\leq 0}} \|K(t,\omega)(s)\| \cdot \sup_{0\leq t_2\leq r} |[M,N]|(r,\theta(t_2,\omega)). \tag{148}$$

Now multiply (145) by $a(r,\theta(t_2,\omega))$, take supremum over $t_2 \in [0,r]$ and note that all the factors involved belong to $\mathbb{L}^4(\Omega,\mathbb{R};P)$ by virtue of Lemma (5.1) (p = 4), Lemma (5.4) (p = 4), Hypothesis (I_2) and Lemma (5.2) (p = 4). Therefore by a simple application of Hölder's inequality (Hewitt and Ross [8], p. 138), we see that

$$E \sup_{0\le t_2\le r} C_2^1(\cdot)a(r,\theta(t_2,\cdot)) < \infty. \tag{149}$$

By similar reasoning the random variables $\sup_{0\le t_2\le r} C_2^i(\cdot)a(r,\theta(t_2,\cdot))$, $i = 2,3,4$, have finite expectations because of Hypotheses (I_2), (I_1) and Lemma (5.3) (p = 4). Since the left hand side of (143) is square-integrable (Lemma (5.1)), the assertion (141) of the lemma is proved. □

The integrability of the first term in the right hand side of (105) is given by

Lemma (5.6):

Assume Hypotheses (C) and (I). Then

$$E \sup_{0\le t_2\le r} C_8(\theta(t_2,\cdot)) < \infty.$$

Proof:

From (100), (101) and (102) we see that

$$\begin{aligned}\sup_{0\le t_2\le r} C_8(\theta(t_2,\omega)) &\le \sup_{0\le t_2\le r} C_6(\theta(t_2,\omega))a(r,\theta(t_2,\omega)) + \\ &+ \sup_{0\le t_1,t_2\le r} \|\varphi(t_1,\theta(t_2,\omega))\|\cdot \sup_{\substack{0\le t\le 2r\\ -r\le s\le 0}} \|K(t,\omega)(s)\| \cdot \sup_{0\le t_1,t_2\le r} \|\Delta Z(t_1,\theta(t_2,\omega))\| \\ &\times \Big[\sup_{0\le t_2\le r} a(r,\theta(t_2,\omega)) + 1\Big] + \\ &+ \sup_{0\le t_1,t_2\le r} \|\Delta V(t_1,\theta(t_2,\omega))\| \Big[\sup_{0\le t_2\le r} a(r,\theta(t_2,\omega)) + 1\Big], \quad \omega \in \Omega_6. \end{aligned} \tag{150}$$

But, by Theorem (3.2) (iii),

$$\sup_{0\le t_1,t_2\le r} \|\Delta Z(t_1,\theta(t_2,\omega))\| \le \sup_{0\le t_2\le r} \|\varphi(t_2,\omega)\|\cdot \sup_{0\le t_1,t_2\le r} \|\Delta Z(t_1+t_2,\omega))\|$$
$$\le 2 \sup_{0\le t_2\le r} \|\varphi(t_2,\omega)\| \sup_{0\le t\le 2r} \|Z(t,\omega)\|. \quad (151)$$

Also by Hypothesis (C_5) we have

$$\sup_{0\le t_1,t_2\le r} \|\Delta V(t_1,\theta(t_2,\omega))\| = \sup_{0\le t_1,t_2\le r} \|\Delta V(t_1+t_2,\omega))\|$$
$$\le |V|(2r,\omega). \quad (152)$$

Combining (150), (151) and (152), the Hypotheses (C) and (I) easily imply that

$$E \sup_{0\le t_2\le r} C_8(\theta(t_2,\cdot)) < \infty$$

(cf. Lemma (5.5) and its proof). □

We are now ready to prove the basic integrability property for the cocycle (X,θ):

Theorem (5.1):

Assume Hypotheses (C) and (I). Then

$$E \sup_{0\le t_1,t_2\le r} \log^+\|X(t_1,\theta(t_2,\cdot),\cdot)\|_{L(M_2)} < \infty. \quad (97)$$

Proof:

As noted at the beginning of this section, it is sufficient to prove (103). In view of (105) and Lemmas (5.6), (5.5), we need only show that

$$E \log^+ \sup_{\substack{0\le t_1,t_2\le r \\ \|(v,\eta)\|\le 1}} |x^2(t_1,\theta(t_2,\cdot),(v,\eta)) - x^1(t_1,\theta(t_2,\cdot),(v,\eta))| < \infty \quad (153)$$

and

$$E \log^+ \sup_{\substack{0\le t_1,t_2\le r \\ \|(v,\eta)\|\le 1}} |x^2(t_1,\theta(t_2,\cdot),(v,\eta))| < \infty. \tag{154}$$

Observe first that

$$x^1(t_1,\theta(t_2,\omega),(v,\eta)) = v, \qquad t_1,t_2 \ge 0,\ \omega \in \Omega_6,\ (v,\eta) \in M_2$$

and so (154) implies (153). The prove (154) we use the definition of x^2 (relation (50)) and the estimate (51) to obtain

$$\begin{aligned}|x^2(t_1,\theta(t_2,\omega),(v,\eta))| &= |I(\theta(t_2,\omega))(x^1(\cdot,\theta(t_2,\omega),(v,\eta)))(t_1)| \\ &\le C_1(\theta(t_2,\omega))|v| + C_2(\theta(t_2,\omega))\int_{-r}^{t}|x^1(u,\theta(t_2,\omega),(v,\eta))|du + \\ &\quad + C_3(\theta(t_2,\omega))\int_0^t |x^1(u-,\theta(t_2,\omega),(v,\eta))|d|V|(u,\theta(t_2,\omega)) \\ &\le C_1(\theta(t_2,\omega))|v|+r^{1/2}C_2(\theta(t_2,\omega))\|\eta\| + C_6(\theta(t_2,\omega))|v|a(r,\theta(t_2,\omega))\end{aligned} \tag{155}$$

for all $\omega \in \Omega_6$, $0 \le t_1,t_2 \le r$, $(v,\eta) \in M_2$. Taking suprema over $0 \le t_1$, $t_2 \le r$, $(v,\eta) \in M_2$ with $\|(v,\eta)\| \le 1$ and expectations in (155) immediately gives (154) because of Lemmas (5.1) and (5.5). This completes the proof of the theorem. □

Once the integrability property (97) is established we can now state the following multiplicative ergodic theorem for the stochastic flow (X,θ) of (I). The proof of the theorem is analogous to that of Theorem 4 in Mohammed ([20], §4) for the white noise case $L = W$, $N = 0$. In the case when θ is ergodic the theorem gives a *discrete* set of *non-random* Lyapunov exponents for X. The reader may supply the details of the argument by consulting the proof of Theorem 4 in [20] (pp. 117-122). See also Lemmas 6 and 7 ([20] pp. 113-117).

Theorem (5.2):

Suppose θ is ergodic and let the stochastic f.d.e. (I) satisfy Hypotheses (C) and (I). Then there exist

(*a*) *a set* $\Omega^* \in \mathcal{F}$ *such that* $P(\Omega^*) = 1$ *and* $\theta(t,\cdot)(\Omega^*) \subseteq \Omega^*$ *for all* $t \in \mathbb{R}^+$,

(*b*) *a fixed (i.e. non-random) sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *of real numbers,*

(*c*) *a random family* $\{E_i(\omega): i \geq 1, \omega \in \Omega^*\}$ *of closed finite-codimensional subspaces of* $\mathbb{M}_2$,

satisfying the following properties:

(*i*) *if the* *Lyapunov spectrum* $\{\lambda_i\}_{i=1}^{\infty}$ *is infinite, then* $\lambda_{i+1} < \lambda_i$ *for all* $i \geq 1$ *and* $\lim_{i\to\infty} \lambda_i = -\infty$; *otherwise the spectrum is a finite set* $\{\lambda_i\}_{i=1}^{N}$ *with* $N > 1$ *a non-random integer and* $\lambda_N = -\infty < \lambda_{N-1} < \ldots < \lambda_2 < \lambda_1$.

(*ii*) *for each* $\omega \in \Omega^*$,

$$E_{i+1}(\omega) \subset E_i(\omega) \subset \ldots \subset E_2(\omega) = E_1(\omega) := \mathbb{M}_2, \quad i \geq 1.$$

(*iii*) *for each* $\omega \in \Omega^*$ *and* $(v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega)$,

$$\lim_{t\to\infty} \frac{1}{t} \log\|X(t,\omega,(v,\eta))\|_{\mathbb{M}_2} = \lambda_i$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log\|X(t,\omega,\cdot)\|_{L(\mathbb{M}_2)} = \lambda_1,$$

(*iv*) *for each* $i \geq 1$, *the family* $\{E_i(\omega): \omega \in \Omega^*\}$ *is* $\mathcal{F}$*-measurable into the Grassmannian of* $\mathbb{M}_2$ *and is invariant under the cocycle* (X,θ) *i.e.*

$$X(t,\omega,\cdot)(E_i(\omega)) \subseteq E_i(\theta(t,\omega)), \quad \omega \in \Omega^*, \ t \geq 0,$$

(*v*) *for each* $i \geq 1$, *codim* $E_i(\omega)$ *is fixed independently of* $\omega \in \Omega^*$.

As in [20] we say that the s.f.d.e. (I) is hyperbolic if its Lyapunov spectrum does not contain 0. By a straightforward adaptation of the argument in

Corollary 2 of [20] (pp. 126-130) we get the following version of the stable-manifold theorem (viz. an exponential dichotomy) in the hyperbolic case:

<u>Theorem (5.3)</u>: (Exponential Dichotomy)

Let Hypotheses (C) and (I) hold and θ *be ergodic. Assume that the s.f.d.e. (I) is hyperbolic. Then there exist*

(*a*) *a set* $\hat{\Omega}^* \in \mathcal{F}$ *such that* $P(\hat{\Omega}^*) = 1$ *and* $\theta(t,\cdot)(\hat{\Omega}^*) = \hat{\Omega}^*$ *for all* $t \in \mathbb{R}$,

(*b*) *a measurable splitting*

$$\mathbb{M}_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \quad \omega \in \hat{\Omega}^*$$

with the following properties:

(*i*) $\mathcal{U}(\omega)$, $\mathcal{S}(\omega)$, $\omega \in \hat{\Omega}^*$, *are closed linear subspaces of* $\mathbb{M}_2$, $\dim \mathcal{U}(\omega)$ *is finite and fixed independently of* $\omega \in \hat{\Omega}^*$.

(*ii*) *The maps* $\omega \mapsto \mathcal{U}(\omega)$, $\omega \mapsto \mathcal{S}(\omega)$ *are* $\mathcal{F}$*-measurable into the Grassmannian of* $\mathbb{M}_2$.

(*iii*) *For each* $\omega \in \hat{\Omega}^*$ *and* $(v,\eta) \in \mathcal{U}(\omega)$, *there exist* $t_1 = t_1(\omega,v,\eta) > 0$ *and a positive* δ_1, *independent of* (ω,v,η), *such that*

$$\|X(t,\omega,(v,\eta))\|_{\mathbb{M}_2} \geq \|(v,\eta)\|_{\mathbb{M}_2} e^{\delta_1 t}, \quad t \geq t_1.$$

(*iv*) *For each* $\omega \in \hat{\Omega}^*$ *and* $(v,\eta) \in \mathcal{S}(\omega)$, *there exist* $t_2 = t_2(\omega,v,\eta) > 0$ *and a positive* δ_2, *independent of* (ω,v,η), *such that*

$$\|X(t,\omega,(v,\eta))\|_{\mathbb{M}_2} \leq \|(v,\eta)\|_{\mathbb{M}_2} e^{-\delta_2 t}, \quad t \geq t_2.$$

(*v*) *For each* $t \geq 0$ *and* $\omega \in \hat{\Omega}^*$,

$$X(t,\omega,\cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),$$

$$X(t,\omega,\cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)).$$

Remark:

Under the hypotheses of Theorem (5.2), the Lyapunov spectrum of (I) does not change if the state space $\mathbb{M}_2$ is replaced by $D := D([-r,0],\mathbb{R}^n)$ with the supremum norm $\|\cdot\|_\infty$. In fact the existence of the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,\omega,(v,\eta))\|_{\mathbb{M}_2}, \quad \omega \in \Omega^*$$

implies the existence of

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,\omega,(v,\eta))\|_\infty, \quad \omega \in \Omega^*$$

and both limits agree for $(v,\eta) \in \mathbb{M}_2$. To see this the reader may note the inequalities:

$$\|X(t,\omega,(v,\eta))\|_{\mathbb{M}_2} \le (r+1)^{1/2}\|X(t,\omega,(v,\eta))\|_\infty, \quad t \ge 0,$$

$$\|X(t,\omega,(v,\eta))\|_\infty \le \sup_{-r\le s\le 0} \|X(t+s,\omega,(v,\eta))\|_{\mathbb{M}_2}, \quad t \ge r,$$

for $\omega \in \Omega^*$ and $(v,\eta) \in \mathbb{M}_2$.

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