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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A CLASS OF TIME-DEPENDENT VOLTERRA EQUATIONS

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Research Report No. 89-52

July 1989

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Abstract. The asymptotic properties of solutions to a time–dependent nonlinear Volterra integral equation are studied in a general Banach space. The concept of completely positive kernel plays a crucial role in the analysis.

1. Introduction. The purpose of this paper is to discuss the asymptotic behavior as $t\to\infty$ of solutions to the abstract Volterra equation

$$u(t) + \int_0^t b(t-s)\,(Au(s) + g(s)\,u(s))\,ds \ni f(t),\ t \in R^+ = [0,+\infty), \qquad (V_{b,g,f})$$

in a real Banach space X. Here $b\colon R^+ \to R^+$ is a completely positive kernel, A is a nonlinear (possibly multivalued) m–accretive operator in X, $g\colon R^+ \to R^+$ is a given function, f maps R^+ into X, and the integral is taken in the sense of Bochner.

General existence, uniqueness and continuous dependence results for $(V_{b,g,f})$ have been established by Crandall and Nohel [8] and Gripenberg [10]. The asymptotic properties of solutions of $(V_{b,g,f})$ have primarily been studied in the case when $g \equiv 0$. See e.g. [2,4,5,13,16]. Recently, Kato, Kobayasi and Miyadera [15] have discussed the asymptotic behavior of solutions to a class of functional–differential equations related to $(V_{b,g,f})$. When applied to $(V_{b,g,f})$, their theory requires that $0 \in R(A)$ and $g \in L^1(R^+)$, being thereby restricted to bounded solutions.

The present work is mainly concerned with the "unbounded behavior", as $t\to\infty$, of solutions to $(V_{b,g,f})$, so that we generally assume that R(A) is zero free and $g \notin L^1(R^+)$. Our study can be viewed as an attempt to extend earlier results obtained by Israel and Reich [14], and Kobayasi [17] for $(V_{1,g,f})$ (that is, the case when $(V_{b,g,f})$ reduces to an evolution equation), as well as the asymptotic theory developed in [13,16] for $(V_{b,0,f})$. Although we consider $(V_{b,g,f})$ in a general Banach space, we emphasize that our results are new even in

Hilbert space. We also note that $(V_{b,g,f})$ is a special case of the more general equation

$$u(t) + \int_0^t b(t-s)\, A(s)u(s)ds \ni f(t), \quad t \in R^+, \tag{V}$$

where $\{A(t), t \in R^+\}$ denotes a family of m–accretive operators in X. An analysis of asymptotic properties of bounded solutions of (V) has recently been carried out in [1], under the assumption that X is a Hilbert space, and A(t) is cyclically maximal monotone for each $t \geq 0$.

The plan of the paper is as follows. In section 2 we recall for easy reference some basic facts about m–accretive operators and completely positive kernels, and we comment briefly on the existence and uniqueness of solutions to $(V_{b,g,f})$. The main asymptotic results are presented and proved in Sections 3 and 4, respectively. An application of physical interest is discussed in Section 5.

2. Preliminaries. Let X be a real Banach space of norm $\|\cdot\|$, and dual $(X^*, \|\cdot\|_*)$. The duality pairing between X and X^* will be denoted by $<\, ,\, >$. Let A be a set–valued operator in X with domain D(A) and range R(A). We say that A is accretive if $\| x_1 - x_2\| \leq \| x_1 - x_2 + \lambda (y_1 - y_2)\|$, for all $\lambda > 0$ and $y_i \in Ax_i$, $i = 1, 2$. A is called m–accretive, if it is accretive and $R(I + \lambda A) = X$, $\forall\, \lambda > 0$. (Here I stands for the identity on X). When A is m–accretive, one can define its Yosida approximation A_λ by $A_\lambda = \lambda^{-1}(I-J_\lambda)$, with $J_\lambda = (I + \lambda A)^{-1}$, $\lambda > 0$. It is easily seen that J_λ is nonexpansive on X, A_λ is Lipschitz continuous on X, and $A_\lambda x \in A J_\lambda x$, $x \in H$.

We will frequently use the following characterization of accretivity (cf.e.g. [7]). Let $[,]_\lambda : XxX \to R$ be defined for $\lambda \neq 0$ by

$$[y, x]_\lambda = (\| x + \lambda y \| - \| x \|) / \lambda, \ \forall x, y \in X,$$

and set :

$$[y, x]_+ = \lim_{\lambda \downarrow 0} [y, x]_\lambda = \inf_{\lambda > 0} [y, x]_\lambda,$$

$$[y, x]_- = \lim_{\lambda \uparrow 0} [y, x]_\lambda = \sup_{\lambda < 0} [y, x]_\lambda.$$

(Note that $\lambda \to \| x + \lambda y\|$ is convex, so that $[y, x]_\lambda$ is monotonically nondecreasing in λ.) Then A is accretive in X if and only if $[y_2 - y_1, x_2 - x_1]_+ \geq 0, \ \forall \ y_i \in A x_i, \ i = 1, 2$. Also recall that the Yosida approximation A_λ $(\lambda > 0)$ of an m–accretive operator A is strictly accretive, i.e. $[A_\lambda x - A_\lambda y, x-y]_- \geq 0, \forall x, y \in X$. Some of the basic properties of $[,]_\pm$ are summarized below.

<u>Propostion 2.1</u>. [7,9] Let $x, y, z \in X$ and $c \in R$. Then

(i) $|[y,x]_+| \leq \| y \|$,

(ii) $[cx,x]_\pm = c \| x \|$,

(iii) $[-y,x]_- = -[y,x]_+$,

(iv) $[y+z, x]_- \leq [y,x]_- + [z,x]_+$,

(v) $[,]_+ : XxX \to R$ is upper semicontinuous.

If, in addition $u: R^+ \to X$ is such that u, $\| u \|$ are differentiable at $t > 0$, then

(vi) $\frac{d}{dt} \| u(t) \| = [u'(t), u(t)]_\pm$ $(' = d/dt)$.

We assume throughout that A is an m–accretive operator on X, and consider equation $(V_{b,g,f})$ under the following minimal assumptions:

(H_b) $b \in AC_{loc}(R^+; R), b(0) = 1, b' \in BV_{loc}(R^+; R)$,

(H_g) $g \in C(R^+; R^+)$,

(H_f) $f \in W^{1,1}_{loc}(R^+; X)$, $f(0) \in \overline{D(A)}$.

Let $\lambda > 0$ and A_λ be the Yosida approximation of A. Since $A_\lambda: X \to X$ is Lipschitzian, and g is continuous, a simple contraction argument shows that the approximating equation

$$u_\lambda(t) + \int_0^t b(t-s)(A_\lambda u_\lambda(s) + g(s)\, u_\lambda(s))\, ds = f(t),\ 0 \le t < \infty, \tag{2.1}$$

has a unique solution $u_\lambda \in W^{1,1}_{loc}(R^+;X)$. Moreover (cf. [8]), equation (2.1) is equivalent to

$$\begin{aligned} &\frac{du_\lambda}{dt}(t) + \frac{d}{dt}(k * u_\lambda)(t) + A_\lambda u_\lambda(t) + g(t)u_\lambda(t) = k(t)\, f(0) + F(t), \\ &\qquad\qquad \text{a.e. on } R^+ \\ &u_\lambda(0) = f(0), \end{aligned} \tag{2.2}$$

where $*$ denotes the convolution, k satisfies

$$b(t) + k * b(t) = 1,\quad 0 \le t < \infty, \tag{2.3}$$

and f is given by

$$F(t) = f'(t) + k * f'(t),\ \text{a.e. } t \in R^+. \tag{2.4}$$

Note that (2.3) can be rewritten as $k + b' * k = -b'$, so that, by (H_b), k is uniquely determined in $BV_{loc}(R^+; R)$. It also follows (see (H_f)) that $F \in L^1_{loc}(R^+; X)$.

The next result is a direct consequence of [8, Theorems 3 and 4] (cf. also [10, Theorem 5]).

Proposition 2.2 Let (H_b), (H_g) and (H_f) hold. Then there exists a (unique) function $u \in C(R^+; X)$ such that $\lim_{\lambda \downarrow 0} u_\lambda = u$ in $C([0,T]; X)$ for any $0 < T < \infty$, where u_λ is the solution of (2.1) (equivalently, (2.2)).

Definition 2.1 The limit function u, introduced in Proposition 2.2 is called the generalized solution of $(V_{b,g,f})$.

To develop an asymptotic theory for generalized solutions of $(V_{b,g,f})$ we rely on the concept of completely positive kernel [5,6]. We confine ourselves to kernels satisfying (H_b).

Definition 2.2. Let (H_b) hold, and let k be defined by (2.3). Then b is said to be completely positive if k is nonnegative and nonincreasing on R^+.

We next collect several important properties of completely positive kernels.

Poposition 2.3. [5,19]. Assume that b is completely positive. Then $0 \leq b(t) \leq 1$ for all $t \geq 0$, and $\lim_{t\to\infty} b(t) = b(\infty)$ exists, with $b(\infty) = (1 + \int_0^\infty k(s)\,ds)^{-1}$ if $k \in L^1(R^+)$, and $b(\infty) = 0$ if $k \notin L^1(R^+)$.

Proposition 2.4. (cf. e.g. [13, 15]). Let b be completely positive, and $w \in W^{1,1}_{loc}(R^+; X)$. Then $k * w$ and $k * \|w\|$ are locally absolutely continuous and differentiable a.e. on R^+. Moreover

$$[\tfrac{d}{dt}(k * w)(t), w(t)]_+ \geq \tfrac{d}{dt}(k * \|w\|)(t), \tag{2.5}$$

for almost all $t > 0$.

Remark 2.1. Let b be completely positive. Then, according to Propositon 2.3, $b(\infty) > 0$ iff $k \in L^1(0,\infty)$. Also, in this case, $b \notin L^1(R^+)$.

3. Statement of Results. Let (H_b) and (H_g) hold, and let k be given by (2.3). For $0 \leq s \leq t < \infty$, set

$$a(t,s) = k(t-s) + g(s) \tag{3.1}$$

and define the associated resolvent kernel r(t,s) by

$$r(t,s) + \int_s^t a(t,\tau)\, r(\tau,s)d\tau = a\,(t,s). \tag{3.2}$$

Since $k \in L^\infty_{loc}(R^+)$ and g is continuous, equation (3.2) has a unique solution r, of class $L^2_{loc}(R^+ \times R^+)$ (at least). See [11, chap. 9] or [20, chap. IV]. (We will often extend a and r by 0 for $t < s$.) Define next

$$R(t,s) = 1 - \int_s^t r(t,\tau)d\tau,\ 0 \le s \le t < \infty. \tag{3.3}$$

We need the following generalization of [18, Lemma 1.3].

Lemma 3.1. Let (H_b) and (H_g) be satisfied and k,R be given by (2.3) and (3.1) – (3.3), respectively. If also b is completely positive, then

$$0 \le R(t,s) \le 1,\ \forall\, 0 \le s \le t < \infty. \tag{3.4}$$

Our first important result for solutions of $(V_{b,g,f})$ is:

Theorem 3.1. Let (H_b) and (H_g) hold. Suppose that $f(\hat{f})$ satisfy $(H_f)((H_{\hat{f}}))$, and that F $(\hat{F})$ are associated to $f(\hat{f})$ by (2.4). Let u and $\hat{u}$ be the generalized solutions of $(V_{b,g,f})$ and $(V_{b,g,\hat{f}})$, respectively. If, in addition, b is completely positive, then

$$\begin{aligned}\| u(t) - \hat{u}(t) \| \le \| u(0) - \hat{u}(0) \| \,(1 - \int_0^t R(t,s)g(s)ds)\\ + \int_0^t R(t,s)\,[F(s) - \hat{F}(s),\, u(s) - \hat{u}(s)]_+ ds,\end{aligned} \tag{3.5}$$

for all $t \ge 0$.

As an immediate consequence, we obtain

Corollary 3.1. Suppose that (H_b), (H_g) and (H_f) hold, and b is completely positive. Let u be the generalized solution of $(V_{b,g,f})$, and F be given by (2.4). Then

$$\| u(s)-y \| \le \| u(0)-y \| \Big(1 - \int_0^s R(s,\tau)g(\tau)d\tau\Big) + \int_0^s R(s,\tau)[F(\tau)-g(\tau)y-z,u(\tau)-y]_+ d\tau, \tag{3.6}$$

for all $z \in Ay$, and all $s \ge 0$. In addition, for any $s, t > 0$,

$$\| u(s) - J_t u_o \| \le \Big(1 - \frac{\int_0^s R(s,\tau)d\tau}{t} \Big) \| u_o - J_t u_o \| + \int_0^s R(s,\tau)\| F(\tau)\| d\tau + \frac{2}{t}\int_0^s R(s,\tau) \|u(\tau) - u_o\| d\tau + \Big(\int_0^s R(s,\tau)g(\tau)d\tau\Big) \| u_o\|, \tag{3.7}$$

where $u_o = u(0)$.

We are now in a position to state our main asymptotic result. Here and in the sequel, u denotes the generalized solution of $(V_{b,g,f})$.

Theorem 3.2. Let (H_b), (H_g) and (H_f) be satisfied. Also assume that b is completely positive with $b(\infty) > 0$, and F verifies

$$\lim_{t\to\infty} \frac{h(t)}{H(t)} \int_0^t \| F(s)\| \, ds = 0, \tag{3.8}$$

where

$$h(t) = \exp \Big(\int_0^t g(s)ds\Big), \quad H(t) = \int_0^t h(s) \, ds. \tag{3.9}$$

If either

$$g \in L^1(R^+), \tag{3.10}$$

or

$$g \notin L^1(R^+),\ g \text{ positive},\ \lim_{t\to\infty} g(t) = 0, \tag{3.11}$$

then there exists an element $\Theta \in S(X^*) = \{z \in X^* : \| z \|_* = 1\}$,

such that

$$\lim_{t\to\infty} \langle u(t), \Theta \rangle / \int_0^t R(t,s)ds = \lim_{t\to\infty} \| u(t) \| / \int_0^t R(t,s)ds = d(0,R(A)), \tag{3.12}$$

where $d(0,R(A))$ denotes the distance from 0 to $R(A)$.

A key tool in the proof of theorem 3.2 is

Lemma 3.2. Let the assumptions of Theorem 3.2 be satisfied. Then

$$\lim_{t\to\infty} \int_0^t R(t,\tau)\, d\tau = +\infty, \tag{3.13}$$

$$\lim_{t\to\infty} \frac{\int_0^t R(t,\tau)g(\tau)d\tau}{\int_0^t R(t,\tau)d\tau} = 0. \tag{3.14}$$

The following consequence of Theorem 3.2 can easily be deduced (see[17]).

Corollary 3.2. Let the assumptions of Theorem 3.2 hold.

(i) If X is reflexive and strictly convex, then

$$w - \lim_{t\to\infty} u(t) / \int_0^t R(t,s)ds = -v,$$

where $\| v \| = d(0,R(A))$ and w–lim stands for weak convergence.

(ii) If X^* has a Fréchet differentiable norm, then

$$\lim_{t\to\infty} u(t)/\int_0^t R(t,s)ds = -v,$$

where v is the unique point of least norm in $\overline{R(A)}$.

Remark 3.1. It is easily verified (see(3.1) – (3.3)) that $R(t,s) = b(t,s)$ if $g \equiv 0$, and $R(t,s) = \frac{h(s)}{h(t)}$ if $b \equiv 1$ (with h defined by (3.9)). Consequently, our Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 are natural generalizations of [13, Theorem 5.3], [16, Theorems 2.1, 2.4 and Corollaries 2.2, 2.5], as well as of [14, Corollary 5], [17, Theorem 2.1 and Corollaries 2.2, 2.3], corresponding to equations $(V_{b,0,f})$ and $(V_{1,g,f})$ respectively.

Remark 3.2. Necessary or sufficient conditions for the boundedness of u on R^+ can readily be derived from Theorem 3.2. or Corollary 3.1. If the assumptions of Theorem 3.2 hold, the boundedness of u necessarily implies $0 \in \overline{R(A)}$. On the other hand, if in addition to the assumptions of Corollary 3.1,$0 \in A\,0$ and $F \in L^1(0, \infty ; X)$, then u is bounded on R^+. When $g \in L^1(R^+)$, the condition $0 \in A\,0$ can be weakened to $0 \in R(A)$.

4. Proofs.

Proof of Lemma 3.1. Denote $\bar{r}(t,u) = \int_u^t r(t,s)ds, \forall\, 0 \leq u \leq t < \infty$. Integrating (3.2) over (u,t) and using Fubini's theorem, we get:

$$\bar{r}\,(t,u) + \int_0^t a(t,\tau)\bar{r}(\tau,u)d\tau = \bar{r}(t,u).$$

In view of (3.3), this yields

$$R(t+u,u) + \int_u^t a(t+\tau)R(\tau+u)d\tau = 1. \tag{4.1}$$

Replacing t by t + u in (4.1) leads to

$$R(t+u,u) + \int_0^t a(t+u,\ \xi+u)R(\xi+u,\ u)d\xi = 1. \tag{4.2}$$

Suppose $u \geq 0$ is fixed and denote $\bar{R}(t) = R(t+u,u)$, $t \geq 0$. Then (4.2) can be rewritten as (cf.(3.1))

$$\bar{R}(t) + \int_0^t [k(t-s)+g(s+u)]\ \bar{R}(s)ds = 1. \tag{4.3}$$

Clearly, (3.4) is equivalent to

$$0 \leq \bar{R}(t) \leq 1,\ \forall\, t \in [0,\infty). \tag{4.4}$$

Using the same approximation argument as Levin [18, Lemma 1.3], we see that it is sufficient to prove (4.4) for smooth k. Recalling (cf. Definition 2.2) that k is nonnegative and nonincreasing, we confine ourselves to the case when:

$$k \in C^1[0,\infty)\ ;\ k \geq 0,\ k' \leq 0 \quad (0 \leq t < \infty). \tag{4.5}$$

Then, from (4.3) it follows that $\bar{R} \in C^1[0,\infty)$. We are going to show that $0 < \bar{R}(t) \leq 1, \forall\, t \in [0,\infty)$. Assume that

$$0 < \bar{R}(t) \quad (0 \leq t < \infty) \tag{4.6}$$

does not hold. Then, since $\bar{R}(0) = 1$, there exists a unique $t_0 > 0$ such that

$$\bar{R}(t_0) = 0,\ \ 0 < \bar{R}(t) \ \text{ for } \ 0 \leq t < t_0. \tag{4.7}$$

This implies

$$\bar{R}'(t_0) \leq 0. \tag{4.8}$$

Differentiating (4.3) and setting $t = t_0$, yields

$$\bar{R}'(t_0) = -\int_0^{t_0} k'(t_0 - s)\ \bar{R}(s)ds. \tag{4.9}$$

By (4.5), (4.7), (4.9), we conclude that $\bar{R}'(t_0) > 0$, which contradicts (4.8), unless

$$k(t) \equiv k(0) \quad (0 \leq t \leq t_0). \tag{4.10}$$

But (4.3) and (4.10) lead to

$\bar{R}'(t) + (k(0) + g(t+u))\,\bar{R}(t) = 0 \quad (0 \le t \le t_0), \quad \bar{R}(0) = 1.$

It follows that $\bar{R}(t) = \exp(-(k(0)t + \int_0^t g(s+u)ds))$, $t \in [0,t_0]$, so that $\bar{R}(t_0) > 0$. This contradicts (4.7), and consequently (4.6) is established. Since $k \ge 0$ and (H_g) holds, we have $0 < \bar{R}(t) \le 1$ on $[0, +\infty)$. The proof is complete.

Proof of Theorem 3.1. Let u_λ be the solution of (2.1), and let $\bar{u}_\lambda$ satisfy the same equation where f is replaced by $\hat{f}$. In view of Proposition 2.1(v) and Proposition 2.2, it clearly suffices to show that (3.5) holds with u_λ, $\hat{u}_\lambda$ in place of u, $\hat{u}$, respectively. Using the equivalent form (2.2) of (2.1), we deduce that $u_\lambda - \hat{u}_\lambda$ satisfies

$$\frac{d}{dt}(u_\lambda - \hat{u}_\lambda)(t) + \frac{d}{dt}(k * (u_\lambda - \hat{u}_\lambda))(t) + A_\lambda u_\lambda(t) - A_\lambda \hat{u}_\lambda(t) + g(t)(u_\lambda(t) - \hat{u}_\lambda(t))$$
$$= k(t)(u_0 - \hat{u}_0) + F(t) - \hat{F}(t), \quad 0 < t < \infty, \tag{4.11}$$

where $u_0 = u_\lambda(0) = f(0)$, $\hat{u}_0 = \hat{u}_\lambda(0) = \hat{f}(0)$.

Recalling that A_λ is strictly accretive, and invoking Proposition 2.1, we infer from (4.11) that

$$\frac{d}{dt}\| u_\lambda - \hat{u}_\lambda \|(t) + [\frac{d}{dt}(k * (u_\lambda - \hat{u}_\lambda))(t), (u_\lambda - \hat{u}_\lambda)(t)]_+ + g(t)\| u_\lambda - \hat{u}_\lambda\|(t)$$
$$\le k(t)\| u_0 - \hat{u}_0\| + [F(t) - \hat{F}(t), u_\lambda(t) - \hat{u}_\lambda(t)]_+ .$$

Applying Proposition 2.4 (the inequality (2.5)) then yields

$$\frac{d}{dt}\| u_\lambda - \hat{u}_\lambda\|(t) + \frac{d}{dt}(k * \| u_\lambda - \hat{u}_\lambda\|)(t) + g(t)\| u_\lambda - \hat{u}_\lambda\|(t)$$
$$\le k(t)\| u_0 - \hat{u}_0\| + [F(t) - \hat{F}(t), u_\lambda(t) - \hat{u}_\lambda(t)]_+ . \tag{4.12}$$

Let (for a fixed $\lambda > 0$)

$$\| u_\lambda - \hat{u}_\lambda\|(t) - \| u_0 - \hat{u}_0\| = x(t),$$
$$[F(t) - \hat{F}(t), u_\lambda(t) - \hat{u}_\lambda(t)]_+ - g(t)\| u_0 - \hat{u}_0\| = \varphi(t). \tag{4.13}$$

Then (4.12) can be rewritten as

$$\frac{dx}{dt}(t) + \frac{d}{dt}(k * x)(t) + g(t)\,x(t) \le \varphi(t), \quad \text{a.e. } t > 0,$$
$$x(0) = 0. \tag{4.14}$$

If we denote

$$x(t) + k * x(t) + \int_0^t g(s)\, x(s)ds = \psi(t), \quad t \geq 0, \tag{4.15}$$

we see that (4.14) implies $\psi(0) = 0$ and

$$\psi'(t) \leq \varphi(t), \text{ a.e. } t > 0 \ . \tag{4.16}$$

Using (3.1), (3.2) we can solve (4.15) by means of the "variation of constants" formula [11, 20]:

$$x(t) = \psi(t) - \int_0^t r(t,s)\, \psi(s)ds, \quad 0 \leq t < \infty. \tag{4.17}$$

An integration by parts shows that (4.17) is equivalent to

$$x(t) = \int_0^t R(t,s)\, \psi'(s)ds, \quad t \geq 0. \tag{4.18}$$

where R is defined by (3.3). Since $R(t,s) \geq 0$ by Lemma 3.1, we deduce from (4.16) and (4.18) that

$$x(t) \leq \int_0^t R(t,s)\, \varphi(s)ds, \quad t \geq 0.$$

On account of (4.13) this yields

$$\| u_\lambda(t) - \hat{u}_\lambda(t) \| - \| u_0 - \hat{u}_0 \| \left(1 - \int_0^t R(t,s)g(s)ds\right)$$
$$\leq \int_0^t R(t,s)[F(s) - \hat{F}(s), u_\lambda(s) - \hat{u}_\lambda(s)]_+ ds,$$

and (3.5) follows.

Proof of Corollary 3.1. If $z \in Ay$, we obviously have

$$\frac{dy}{dt} + \frac{d}{dt}(k*y)(t) + z + g(t)y = k(t)y + z + g(t)y.$$

Applying (3.15) with $\hat{u}(t) \equiv y$, $\hat{F}(t) = g(t)y + z$, we get (3.6). Next take $y = J_t u_0$, $z = A_t u_0$ ($u_0 = u(0)$) in (3.6) and notice that

$$[-A_t u_o, u(\tau) - J_t u_o]_+ \le \frac{2\| u(\tau) - u_o \|}{t} - \frac{\|J_t u_o - u_o\|}{t},$$

$$[-g(\tau) J_t u_o, u(\tau) - J_t u_o]_+ \le g(\tau)\| J_t u_o - u_o\| + g(\tau)\| u_o\|.$$

The inequality (3.7) now follows easily.

Proof of Lemma 3.2. Let $p(t) = \int_0^t R(t,s)ds$, $t \ge 0$. Integrating (4.1) over (0,t) yields

$$p(t) + \int_0^t (k(t-s)+g(s))p(s)ds = t. \tag{4.19}$$

From (4.19) we conclude (cf.e.g. [10, Lemma 3.4]) that $p \in AC_{loc}(R^+; R)$; hence

$$\frac{dp}{dt}(t) + g(t)\,p(t) + \frac{d}{dt}(k*p)(t) = 1, \text{ a.e. } t > 0. \tag{4.20}$$

Recall now (see(3.9)) that $h' = hg \ge 0$; also, k is nonincreasing and $p \ge 0$. Consequently,

$$h(t)\frac{d}{dt}(k*p)(t) = h(t)(k(0)p(t) + \int_0^t p(t-s)dk(s))$$

$$\le \frac{d}{dt}(k*(hp))(t), \text{ a.e. } t > 0. \tag{4.21}$$

Multiplying (4.20) by h(t) and invoking (4.21) gives

$$\frac{d}{dt}(hp)(t) + \frac{d}{dt}(k*(hp))(t) \ge h(t), \text{ a.e. on } (0,+\infty). \tag{4.22}$$

Since $p(0) = 0$ and $h(0) = 1$, we may rewrite (4.22) as

$$(hp)'(t) + k*(hp)'(t) \ge h(t),\ t > 0. \tag{4.23}$$

Take the convolution of (4.23) with b, and use (2.3) and $b \ge 0$ to obtain

$$h(t)\,p(t) \ge b * h(t),\ 0 \le t < \infty. \tag{4.24}$$

On the other hand, we notice that $b*h$ is nondecreasing (since $(b*h)'(t) =$

$b(t)\,h(0) + \int_0^t h'(t-s)b(s)ds \ge 0$); this implies ($k \ge 0$)

$$(b*h) * k(t) \le (b*h)(t)\int_0^t k(s)ds,\ t \ge 0. \tag{4.25}$$

If we now take the convolution of (2.3) with h(t), we get on account of (4.25),

$$(b^*h)(t)\,(1+\int_0^t k(s)ds) \geq H(t). \tag{4.26}$$

Inasmuch as $b(\infty) > 0$ by hypothesis, we deduce from (4.26) (in view of Remark 2.1 and Proposition 2.3) that

$$b^*h(t) \geq b(\infty)\,H(t), \quad 0 \leq t < \infty. \tag{4.27}$$

From (4.24) and (4.27) it follows that

$$p(t) \geq b(\infty)\,\frac{H(t)}{h(t)} > 0, \quad 0 < t < \infty, \tag{4.28}$$

which implies (3.13). (In case when (3.10) is satisfied, $h(\infty) < +\infty$ and $H(t) \geq t$; if (3.11) holds, then $\lim_{t\to\infty} \frac{H(t)}{h(t)} = +\infty$.) To prove (3.14) when (3.10) is fulfilled, we simply remark (cf.(3.4)) that

$$0 \leq \frac{\int_0^t R(t,s)g(s)ds}{\int_0^t R(t,s)ds} \leq \frac{\int_0^\infty g(s)ds}{p(t)}.$$

In case when (3.11) holds, it is easily verified that (3.14) is a consequence of (3.4), (3.13) and $g(\infty) = 0$.

Proof of Theorem 3.2. By (3.6) and Proposition 2.1(i), we have

$$\|u(t)-y\| \leq \|u(0)-y\| + \int_0^t \|R(t,\tau)\|\, \|F(\tau)\| d\tau + \left(\int_0^t R(t,\tau)g(\tau)d\tau\right)\|y\| + \left(\int_0^t R(t,\tau)d\tau\right)\|\,z\,\|, \tag{4.29}$$

for any $[y,z] \in A$. Taking into account (3.4), (3.13), (3.14), (4.28) and assumption (3.8), we infer from (4.29) that

$$\limsup_{t\to\infty} \| u(t)\| \Big/ \int_0^t R(t,s)ds \le d(0,R(A)). \tag{4.30}$$

If $d(0,R(A)) = 0$, then (3.12) holds for any $\Theta \in S(X^*)$, so that we consider the case when $d(0,R(A)) > 0$. Following [16, Theorem 2.4] (cf. also [17, Theorem 2.1]), we choose for each $t > 0$, an element $\Theta_t \in S(X^*)$ with the property that $< J_t u_o - u_o, \Theta_t > = \| J_t u_o - u_o\|$. This together with (3.7) implies (recall that $p(s) = \int_0^s R(s,\tau)d\tau > 0$, $\forall$ $s > 0$, cf. (4.28))

$$< u(s)-u_o, \Theta_t > \Big/ \int_0^s R(s,\tau)d\tau \ge \frac{1}{t} \| u_o - J_t u_o \| - \frac{1}{\int_0^s R(s,\tau)d\tau} \int_0^s R(s,\tau)\| F(\tau)\| d\tau$$

$$- \frac{\| u_o \|}{\int_0^s R(s,\tau)d\tau} \left(\int_0^s R(s,\tau)g(\tau)d\tau\right) - \frac{2}{t\int_0^s R(s,\tau)d\tau} \int_0^s R(s,\tau)\|u(\tau)-u_o\|d\tau \quad (0<s<t< \infty),$$

which, by (3.4), (4.28) leads to

$$< u(s)-u_o, \Theta_t > \Big/ \int_0^t R(s,\tau)d\tau \ge \frac{1}{t} \| u_o-J_t u_o \| - \frac{1}{b(\infty)} \frac{h(s)}{H(s)} \int_0^s \| F(\tau) \| d\tau$$

$$\tag{4.31}$$

$$\frac{1}{\int_0^s R(s,\tau)d\tau} \left(\int_0^s R(s,\tau)g(\tau)d\tau\right) \| u_o \| - \frac{2}{t\ b(\infty)} \cdot \frac{h(s)}{H(s)} \int_0^s \| u(\tau) - u_o \|d\tau,$$

On the other hand, for $0 < s < t$, we have

$$< J_s u_o - u_o, \Theta_t > /s \ge \| J_t u_o - u_o \| /t \cdot \tag{4.32}$$

Also recall [22, Lemma 2.1] that

$$\lim_{t\to\infty} \| J_t u_o \| /t = d(0,R(A)) \cdot \tag{4.33}$$

Let $\Theta \in X^*$ be a weak–star cluster point of $\{\Theta_t\}$, as $t \to \infty$. Then from (4.31) – (4.33), we obtain

$$< u(s) - u_0, \Theta > / \int_0^t R(s,\tau)d\tau \geq d(0(R(A)) - \frac{1}{b(\infty)} \cdot \frac{h(s)}{H(s)} \int_0^s \| F(\tau)\| \, d\tau$$

$$- \frac{1}{\int_0^s R(s,\tau)d\tau} \left(\int_0^s R(s,\tau)g(\tau)d\tau\right) \| u_0 \|, \tag{4.34}$$

$$< J_s u_0 - u_0, \Theta > /s \geq d(0,R(A)). \tag{4.35}$$

Letting $s \to \infty$ in (4.34) yields (in view of (3.8), (3.13), (3.14))

$$\liminf_{s \to \infty} < u(s), \Theta > / \int_0^s R(s,\tau)d\tau \geq d(0,R(A)), \tag{4.36}$$

while (4.35) implies

$$\liminf_{s \to \infty} < J_s u_0, \Theta > /s \geq d(0,R(A)). \tag{4.37}$$

The conclusion of Theorem 3.2 now follows from (4.30), (4.33), (4.36) and (4.37).

5. An Example. In this section we suggest a special heat flow model to which our previous theory applies. Consider a homogeneous bar of unit length of a material with memory. Let $u(t,x)$, $e(t,x)$, $q(t,x)$ and $\mu(t,x)$ denote, respectively, the temperature, internal energy, heat flux, and external heat supply at time t and position x ($-\infty < t < \infty$, $0 \leq x \leq 1$). Let the ends of the bar at $x = 0$ and $x = 1$ be maintained at zero temperature, and for simplicity, let the history of u be prescribed as zero when $t < 0$ and $0 \leq x \leq 1$. According to the theory developed by e.g. Gurtin and Pipkin [12] and Nunziato [21] for heat flow in materials of fading memory type, we may assume that

$$e(t,x) = u(t,x) + \int_0^t \beta(t-s)\, u(s,x)ds + \int_0^t \alpha(t-s)g(s)\, u(s,x)ds, \tag{5.1}$$

$$q(t,x) = -\sigma(u_x(t,x)) + \int_0^t \gamma(t-s)\, \sigma(u_x(s,x))ds, \tag{5.2}$$

for $t \geq 0$ and $0 < x < 1$. Here β, γ: $[0,\infty) \to R$ are sufficiently smooth functions, $\alpha(t) = 1 - \int_0^t \gamma(s)ds$, $g \in C(R^+, R^+)$, and σ is a real function satisfying

$$\sigma \in C^1(R),\ \sigma(0) = 0,\ \sigma'(\xi) \geq c_0 > 0\ (\xi \in R), \text{ for some } c_0 > 0. \tag{5.3}$$

The balance of heat requires that the equation $e_t = -q_x + \mu$ should hold. If also $u(0,x) = u_0(x)$ $(0 < x < 1)$ is the initial temperature distribution, we obtain in view of (5.1), (5.2) and the assumption that the temperature at the ends of the rod is zero:

$$\frac{\partial}{\partial t}[u(t,x)+(\beta * u)(t,x)+(\alpha * gu)(t,x)]=\sigma(u_x(t,x))_x-(\gamma * \sigma(u_x)_x)(t,x)+\mu(t,x), \quad 0 < t < \infty,\ 0 < x < 1, \tag{5.4}$$

$$u(t,0) = u(t,1) = 0,\ t > 0,$$

$$u(0,x) = u_0(x),\ 0 < x < 1.$$

Following [5, Section 4] we transform the initial–boundary value problem (5.4) to a Volterra integral equation in the space $X=L^2(0,1)$. Let

$$G(t,x) = u_0(x) + \int_0^t \mu(s,x)ds,\ 0 \leq t < \infty,\ 0 < x < 1 \tag{5.5}$$

and remark that

$$\sigma(u_x)_x - \gamma * \sigma(u_x)_x = \frac{\partial}{\partial t}(\alpha * \sigma(u_x)_x).$$

Then (5.4) leads to the equation

$$u + \beta * u + \alpha * (Au + gu) = G,\ 0 \leq t < \infty,\ 0 < x < 1, \tag{5.6}$$

where A: $D(A) \subset X \to X$ is defined by $Au = -\sigma(u_x)_x$, with $D(A) = \{u \in H_0^1\ (0,1)\text{: } \sigma(u_x)_x \in X\}$. By (5.3), it is easily verified that A is maximal monotone (equivalently, m–accretive, cf[3]) in X, with $0 \in R(A)$. If $r(\beta)$ denotes the resolvent kernel of β (i.e. β satisfies $r(\beta) + \beta * r(\beta) = \beta$; $r(\beta) \in L^1_{loc}[0,\infty)$ if $\beta \in L^1_{loc}[0,\infty)$), and

$$b = \alpha - r(\beta) * \alpha, \tag{5.7}$$

$$f = G - r(\beta) * G, \tag{5.8}$$

then the variation of constants formula shows that (5.6) is equivalent to

$$u + b*(Au+gu) = f, \tag{5.9}$$

i.e. an equation of the standard form $(V_{b,g,f})$ in X. The next result is essentially [5, Lemma 4.2]:

Lemma 5.1. Let β be bounded, nonnegative, nonincreasing and convex on $[0,\infty)$. Let γ be positive, nonincreasing, log convex, and bounded on $[0,\infty]$. Suppose that

$$\alpha(\infty) = 1 - \int_0^\infty \gamma(s)ds > 0, \text{ and } \beta'(t) + \gamma(0)\beta(t) \leq 0, \text{ a.e. } t > 0.$$

Then b (given by (5.7)) satisfies (H_b) and is completely positive, with $b(\infty) > 0$.

We can now apply the theory developed in § § 3 and 4 to discuss the asymptotic behavior of the generalized solution of equation (5.9) (equivalent to the heat flow problem (5.4)). We assume that $u_0 \in L^2(0,1)$ and the forcing function $\mu \in L^1_{loc}([0,\infty); L^2(0,1))$. Then, by (5.5), (5.8) and $r(\beta) \in L^1_{loc}[0,\infty)$ (at least) it is easily seen that $f \in W^{1,1}_{loc}([0,\infty); L^2(0,1))$. Also remark that D(A) is dense in X, so that all of (H_f) is satisfied. As soon as (H_b), (H_g) hold, Proposition 2.2 implies that (5.9) has a unique generalized solution u on $[0,\infty)$. A direct application of Corollary 3.2, combined with Lemma 5.1 now yields

Theorem 5.1. Let the assumptions of Lemma 5.1 be satisfied. Let $u_0 \in L^2(0,1)$, $\mu \in L^1_{loc}([0,\infty); L^2(0,1))$, and b,k,f,F be defined by (5.7), (2.3), (5.8), (2.4), respectively. If also g satisfies (H_g) and (3.11), R(t,s) is given by (3.1)–(3.3), and (3.8), (3.9) hold ($\|\cdot\|$ stands for the norm in $L^2(0,1)$), then equation (5.9) has a unique generalized solution u, such that:

$$\lim_{t\to\infty} u(t) \Big/ \int_0^t R(t,s)ds = 0, \text{ strongly in } L^2(0,1).$$

Acknowledgment. The third author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund.

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