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Risk–Sensitive and Robust Escape Criteria

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Abstract. The problem of controlling a noisy process so as to prevent it from leaving a prescribed set has a number of interesting applications. In this paper, new criteria for this problem are considered. First, a risk–sensitive criterion for a stochastic diffusion process model is examined, and it is shown that the value is a classical solution of a related PDE. The qualitative properties of this criteria are favorably contrasted with those of existing criteria in the risk–averse limit. It is proved that in the risk–averse limit the value of the risk–sensitive criterion converges to a viscosity solution of a first–order PDE. It is then demonstrated that the value function of a deterministic differential game is also a viscosity solution to the PDE. This game represents a robust control problem which appears to be analogous to H^∞ control. In particular, the opposing player attempts to push the process out of the prescribed set, and suffers an L^2 cost for his efforts. Lower bounds on the escape time as a function of this cost are obtained.

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1 Introduction

In problems that cover a range of interesting applications, there is a stochastic process model for a system and the goal is to keep the process in a given fixed open set G. Let X_t denote the stochastic process model under consideration. In some applications there is an a priori fixed time interval $[0,T]$ during which one desires $X_t \in G$. However, in most cases the length of time is not precisely specified, and the exact time one needs to maintain $X_t \in G$ is vague.

In this paper we will consider such problems in the context of optimization and control. As is often the case in control theory, the selection of a cost criterion is in a certain sense subjective, i.e., the cost is usually selected to force the system to achieve some prescribed qualitative behavior. For example, in many problems the goal is to keep the controlled process near a desired operating point. In this case one may choose a criteria that will lead to controls that "stabilize" the process about this operating point. For our problem the situation is quite different, in that the goal is not so much to keep the process near a particular point as it is to keep it away from the "bad" set G^c, where the c denotes complement. For a problem to fit well into such a framework, it must be the case that entry into the set G^c is in a certain sense "catastrophic," and avoiding such an event is a high priority. Examples are the failure of a machine, loss of data in a communication network, loss of "lock" in an adaptive tracking device, and entry of bi-stable adaptive control algorithms, such as those using ALOHA-type protocols, into the "bad" region.

There are two criteria that are often associated with the problem described above. The first criterion is the probability of escape over some interval $[0,T]$:

$$P_x \{X_t \notin G \text{ for some } t \in [0,T]\},$$

where P_x denotes probability conditioned on $X_0 = x$. If the process is controlled, then obviously the goal is to choose the control to minimize this probability. The second criterion is the mean escape time: $E_x\tau$, where τ is the first time the process X_t escapes from G. Here we maximize when a control is available. As we discuss in detail in Section 3, both these cost criteria have some theoretical and computational shortcomings. One of the contributions of the present paper is the introduction of a new cost criteria for this problem. To have a concrete model for the purposes of discussion, consider the special case of a Markov process described by the stochastic differential equation

$$dX_t^\varepsilon = f(X_t^\varepsilon)dt + \varepsilon^{1/2}\sigma(X_t^\varepsilon)dB_t,\ X_0^\varepsilon = x,$$

where the dimensions of the Wiener process B_t, f and σ are compatible. The quantity $\varepsilon > 0$ is a parameter whose role will be explained shortly.

In the simplest setting, the new criteria we consider in this paper takes the form

$$E_x \exp[-\theta\tau^\varepsilon/\varepsilon], \tag{1.1}$$

where θ is a positive design parameter and τ^ε is the first time the process X_t^ε exits G. If the process is controlled, we choose the control to minimize this quantity.

We will consider the qualitative properties of this cost from two different perspectives. The first perspective is that of control of "small noise" systems. In this setting, the term small noise essentially means that escape from G is a relatively rare event. For the particular model under consideration, and under appropriate conditions on f and σ, this corresponds to $\varepsilon > 0$ being small. Many problems of interest, such as the control of communication devices, fall into this "small noise" category, since design tolerances are quite strict and a system in which escape was common would not even be worth considering. As we discuss in Section 3, the quantity (1.1) has desirable properties when compared to standard criteria for problems of this type.

The second perspective is one in which (1.1) is viewed as a "risk-sensitive" analogue of a more standard criteria. We are also interested in the connection between this risk-sensitive problem and a robust control problem that is analogous to H^∞ control. In this setting, we do not necessarily assume that the true system is "small noise." Although here we are interested in the control problem for noise that is not necessarily small, we consider the same limit as in the small noise perspective: $\varepsilon \to 0$. The reason for this is discussed in Section 3.

Hence in both cases we are interested in the limit problem obtained when $\varepsilon \to 0$ in the cost (1.1). In order to obtain a well defined limit, it is necessary to work with the scaled quantity

$$-\varepsilon \log E_x \exp\left[-\theta\tau^\varepsilon/\varepsilon\right]. \tag{1.2}$$

Due to the presence of an additional minus sign, in the case with control we will seek to maximize this quantity. Depending on the means by which the designer may influence the system, we distinguish three classes of progressively more difficult optimization problems. The first case is that of "performance analysis." Here the designer cannot really influence the system at all. Instead, the designer would be interested in approximating (1.2) or its limit as $\varepsilon \to 0$ so that it could be used to compare two or more system designs. The second case is one we call "parametric optimization." Here the designer has control over a collection of parameters that determine the dynamics of the system. In this case the designer might use the limit of (1.2) as a convenient criteria when optimizing with respect to these parameters. In both of these first two cases, the limit of (1.2) as $\varepsilon \to 0$ can be characterized as the solution to a minimal cost deterministic optimal control problem. The last case is that in which the designer can choose an active, state-dependent control. In this case the limit of the supremum over the controls of (1.2) as $\varepsilon \to 0$ can be characterized as the solution to a deterministic differential game. The control that seeks to maximize the limit of (1.2) will be opposed by a minimizing control. In order to distinguish the prelimit control problem of this case from the limit control problem obtained in the previous two, we refer to it as the case of "maximizing control." Because the limit problem is much more involved in the case of maximizing control, in a number of places we will provide a separate statement of the results for this case, even though it automatically includes the simpler cases. Examples of all three sorts of problems are given in Section 2.

A summary of the contents of the paper is as follows. In order to illustrate some of the main issues and also to suggest some of the applications, we describe a number of examples in Section 2. In Section 3 we turn our attention to possible design criteria. We describe shortcomings of the standard criteria from the small noise perspective, and show how these shortcomings are avoided by considering the limit of (1.2) as the design criterion. The "risk-sensitive" interpretation of (1.2) is also given in this section, and we also discuss generalizations of (1.1). In Section 4 we characterize the limit of (1.2) for the various cases when $\varepsilon \to 0$, and state the convergence theorem. Owing to previously established connections between risk-sensitive control, risk-averse limits, and robust control, one might expect that the limit problem for the case of maximizing control would define a control that has an interpretation as a robust control. This is in fact the case, and the precise interpretation is stated in Section 5. We close in Section 6 with proofs of results that are stated in Sections 4 and 5.

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2 Examples

In this section we describe a number of examples that fit into the framework of the last section. The examples are intended only to be illustrative and to motivate the criteria we will consider later. Some of the examples fall outside the class of diffusion processes, and hence indicate interesting extensions that are not covered by the theory developed in this paper.

Problems for which an escape time criteria is appropriate fit into one of two categories. In the first category, exit from the region of desired operation essentially causes the system to shut down, and the system is more or less "off-line" until the state of the system can be steered back into the good operating region. For example, when an ALOHA-type system exits its stable operating region the entire system is shut down and then restarted. In Example 1 below exit from the operating region means that the pair of communicating satellites must suspend data transmission and initiate the procedure to regain "lock." In the second category exit from the domain is not fatal, but still an event to be avoided because of a drastic decline in performance when outside the good region. An example in this category is the queueing example given below, in which escape from G corresponds to a non-negligible fraction of incoming customers being turned away.

For other examples the reader can consult the references in [27].

EXAMPLE 1 (SATELLITE LASER COMMUNICATION PROBLEM). In space-based laser communication an essential role is played by the tracking and pointing subsystems of the satellites that are involved. In particular, data may be transmitted between two satellites by a laser communication crosslink. In order that the communication links not be broken, the pointing system of each satellite must keep the laser focused on the detector of the other. Since the communication beams are very narrow, the pointing requirements are rather stringent [25]. In this example the set G is defined in terms of an angular cone of allowable orientations of the gimbal-mounted optics used to control the beam direction. At any given time one of the satellites is allowed to transmit data at a high rate while the other transmits a "beacon" signal at a low rate, the purpose of which is to help maintain the "lock." Owing to "noises" (such as internal vibrations) the operating state of the system can be driven from G, at which time there will be a communication interruption and lock between the transmitter and receiver will have to be re-established via a separate acquisition algorithm. It is desirable to have the time between losses of lock be quite large (e.g., several months).

For such a system linear methods automatically make little sense, since the effective state space in which the system can operate is not itself a linear space. This is true even if the system dynamics are linear. The criteria we have described in the Introduction are more natural for this problem, since it focuses on the event that is actually of interest, escape from G. For systems such as this where reliable behavior is critical, it is important to work with the best design criteria possible. In this problem one may be interested in parametric optimization or feedback control.

EXAMPLE 2 (TRACKING PROBLEM). Many synchronization systems in advanced communication systems are digital. One might model such a system as

$$X_{i+1}^n = X_i^n + \frac{1}{n} b(X_i^n, \xi_i).$$

In this equation ξ_i is a random sequence composed of noise and the inputs to the system. The discrete time parameter i is used because time is "slotted," and data are communicated only at discrete times. The factor $1/n$ reflects the fact that the state X_i^n of the system changes slowly as a function of each new piece of data ξ^i, although the data rates are very high. In order that the system operate properly, it is crucial that the transmitter and receiver both be on the same "clock," so that it is clear when a discrete time interval begins and ends. In a synchronization system, one of the components of X_i^n (say $(X_i^n)_1$) will be the difference between an estimate of a phase timing indicator and the true value. For accurate communication or tracking one needs a very good estimate. Here the set G will be of the form $\{x : -a_1 \leq x_1 \leq a_2\}$. As part of the design procedure "stabilizing" dynamics are always built into the system in order to keep X_i^n in the acceptable region G. However, owing to the presence of noise, the difference between the estimate of the phase timing indicator and the true value is eventually driven from G. A risk–sensitive escape criteria is natural in this context. For these problems, one is typically interested only in performance analysis (for purposes of comparing competing designs), or at most in parametric optimization.

This example is illustrative of a large class of similar problems from statistics and adaptive stochastic algorithms [4]. The large deviation analysis, numerical computations, and simulations for a related analog device known as a phase locked loop appear in [7,9].

EXAMPLE 3 (QUEUEING PROBLEMS). Numerous problems involving the design and optimization of queues can be cast in terms of the risk–sensitive criteria. Such criteria will be suitable whenever the main purpose of the design is to reduce the possibility of very large buffers. A number of examples and references to additional examples are given in the book [29]. Although there is at the present time no theory of "robust control" for queueing systems, one would expect by analogy with the case for diffusions that controls designed on the basis of a risk–sensitive criteria would enjoy the features one would desire of such a "robust" control.

Escape criteria are suitable whenever buffer overflow constitutes a critical event. Notable among these are the problems associated with the analysis and design of high-speed data networks. These networks carry many types of data in digitized form: e.g., voice, computer, and video data. Each of these classes has its own requirements and characteristics. In addition, for some classes of data there may be contractually agreed upon requirements regarding network performance. These requirements will be stringent (e.g., probabilities of data loss in the 10^{-9} range). Data loss occurs when buffer capacities are exceeded, and thus corresponds to an "escape." There are numerous difficult and interesting design and control problems that are associated with such networks.

EXAMPLE 4 (POWER SYSTEM STABILITY). In this example a diffusion model is believed to be appropriate for describing the time evolution of the system (cf. [6] and the references therein). The state of the system is a vector whose components consist of various generator frequencies, voltage phase angles and voltage magnitudes. The set G is defined to be a region in which the "security" of the system is acceptable. Exit from the region may require substantial intervention to return the state of the system to the stable region, and hence is an event to be avoided. Coefficient matrices in the system of equations that describe operation provide the system parameters over which optimization can be performed. In addition, feedback control in various forms can also be possible.

3 Comparison and Interpretation of the Cost Criterion

In this section we study the optimization criterion

$$E_x \exp\left[-\theta\tau^\varepsilon/\varepsilon\right] \tag{3.1}$$

from a qualitative point of view. We first consider the cost in the small noise setting, and compare its properties with those of more standard cost criteria. After this, we consider (3.1) from the risk–sensitive point of view. At the end of the section, we comment briefly on generalizations.

3.1 The Small Noise Setting

In many of the problems for which a criterion based on escape times is appropriate, it is useful to assume that the stochastic process model is in some sense a "small noise" model. Indeed, if this were not the case then (essentially regardless of system design) escapes from the acceptable operating region would be common, and for many problems the models might not be worth considering.

As remarked in the introduction, there are two criteria that are often associated with such problems, namely, the probability of escape over some interval $[0,T]$:

$$P_x\left\{X_t^\varepsilon \notin G \text{ for some } t \in [0,T]\right\},$$

and the mean escape time: $E_x\tau^\varepsilon$, where $\tau^\varepsilon = \inf\{t : X_t^\varepsilon \notin G\}$. The parameter $\varepsilon > 0$ indicates the "strength" of the noise, with zero noise in the limit $\varepsilon \to 0$.

As we now discuss, both of these criteria have shortcomings. We begin with the escape probability. While the escape probability might be acceptable for problems for which the duration T is fixed and known, it is probably not appropriate otherwise. In many problems one has a rough idea of the interval of interest, but not much more than that. The escape probability criterion can be inconvenient to work with even when the interval of interest is known. For example, in the control setting it typically results in controls that are not stationary. A second difficulty is of a computational nature. In cases where the control space is large, there can be numerical difficulties due to the fact that the controls become somewhat singular near the final time T. A final problem is greater sensitivity (relative to more well-behaved functionals) with regard to the parameter that measures the strength of the noise. Hence escape probabilities can be more difficult to approximate than smoother functionals of a process (such as an expectation of a smooth function of the process). Because of this, the behavior of systems designed on such a criterion is less reliably predicted by the asymptotic theory.

While the mean escape time criterion has the advantage of yielding time independent feedback controls, it too can be problematic, especially in the desired situation where the probability of escape over $O(1)$ time intervals is rare. In this setting, computation and approximation of the mean escape time, even for the case of a fixed control, can be difficult. In order to make this statement precise, we return to the small noise diffusion model

$$dX_t^\varepsilon = f(X_t^\varepsilon)dt + \varepsilon^{1/2}\sigma(X_t^\varepsilon)dB_t,\ X_0^\varepsilon = x, \tag{3.2}$$

It is well known (under some assumptions) that $W^\varepsilon(x) \doteq E_x\tau^\varepsilon$ satisfies a second order PDE

$$\mathcal{L}^\varepsilon W^\varepsilon(x) = -1, x \in G^0,\ \ W^\varepsilon(x) = 0, x \in \partial G, \tag{3.3}$$

where

$$\mathcal{L}^\varepsilon g(x) \doteq \frac{\varepsilon}{2}\mathrm{tr}\,[g_{xx}(x)a(x)] + \langle f(x), g_x(x)\rangle, \quad a(x) = \sigma(x)\sigma^T(x),$$

and tr$B$ denotes the trace of the square matrix B. The case with control involves obvious modifications of this equation. The smallness of the noise translates into the small ε coefficient in front of the second derivative term. This produces a solution that is essentially flat over much of G^0 but with steep gradients close to ∂G. Because of these properties, it is very difficult to accurately approximate the solution numerically. This is rather unsatisfactory, since the very conditions that are required for escapes to be a rare event lead to difficult computational problems.

A problem that may be even more important than the one just discussed is that in the small noise setting the mean escape time (and by extension any controls that are designed using it as the performance criterion) may focus on events that are not of any practical interest, since the dominant contribution to this criterion comes from paths that take a very long time to escape. Under certain technical conditions on f, σ, and their relation to ∂G, one can show [19] there exists a constant $C^* > 0$ such that the limit

$$\lim_{\varepsilon\to 0} \varepsilon \log E_x \tau^\varepsilon = C^*$$

holds uniformly for x in compact subsets of G. For any $\delta \in (0, C^*)$, the scaling in ε of this quantity implies that when $\varepsilon > 0$ is sufficiently small the dominant contribution to $E_x\tau^\varepsilon$ is due to sample paths that take at least $\exp([C^* - \delta]/\varepsilon)$ units of time to escape. While this may be a moot point if one is absolutely certain that the mean escape time is the criterion of interest, it is an important point if this is not the case. Consider for example a telecommunication problem involving routing of data through a network. For obvious reasons, the true process representing loading of the network is nonstationary, with cyclical variations (e.g., periods of 1 day). For this problem the mean escape criterion, even if it were computable, would be an inappropriate basis on which to design controls if the dominant contribution is due to sample paths that take longer than 1 day to escape. Such a criterion is clearly not desirable, since controls designed on it may allow a large number of escapes over a short time interval as long this is balanced by relatively few sample paths which take the very long time $\exp([C^* - \delta]/\varepsilon)$ to escape. Moreover, these paths would not even contribute to the overall performance of the true system, owing to its nonstationarity.

As we will see, the escape time criteria (3.1) and controls based on its asymptotic behavior avoid these difficulties. The controls will automatically be independent of time, and by appropriately choosing the design parameter θ, one can "tune" the system to focus on escapes over different $O(1)$ time intervals. By working with the logarithmic transform of (3.1), we obtain a Hamilton–Jacobi equation that is much more well behaved than (3.3), especially with regard to computational approximations. Finally, we note that in comparison with the escape probability, the relative smoothness of the functional mapping $X^\varepsilon_\cdot \to \tau^\varepsilon$ under the distribution of $X^\varepsilon_\cdot$ suggests that (3.1) [or rather the logarithmic transform of (3.1)] should be more reliably predicted by the asymptotic theory when $\varepsilon \to 0$. Although a proof of such a result is lacking, numerical evidence suggests that this is indeed the case.

3.2 The Risk–Sensitive Interpretation

The theory of risk–sensitive control investigates the effect of modifications of the cost structures upon the associated optimal policies. For example, one might be interested in the effect of so-called "risk–sensitizing transformations." At an intuitive level, the goal of such a transformation is to amplify the effect that certain outcomes have in determining the overall cost, and thereby force the optimal control (or any nearly optimal control) to "pay more attention" to these more heavily weighted outcomes. In particular, in the risk-averse case, "bad" events are weighed more heavily, and the control becomes more conservative with regard to allowing these events to occur.

From this perspective, for each fixed $\varepsilon > 0$ one may view the criterion (3.1) as a risk-sensitive version of the mean escape time criterion $E_x\tau^\varepsilon$. The effect of the nonlinear mapping $\tau \rightarrow \exp -\theta\tau/\varepsilon$ is to shift the attention towards paths that escape in a relatively short time, at the expense of optimizing the mean escape time. As noted in the previous section, this makes sense for many problems, especially those that may involve some type of non-stationarity. The design parameter $\theta > 0$ controls the degree to which these short time escapes are emphasized, with larger values of θ focusing attention more heavily on such escapes. On the other hand, if we fix $\varepsilon > 0$ and take the limit $\theta \rightarrow 0$, one can show (under suitable uniform integrability conditions) that the design criterion (3.1) becomes equivalent in this limit to the mean escape time.

For various reasons, including computational simplicity, model simplification, and because of the connection with robust control design, one may also be interested in taking limits with respect to a family of risk-sensitizing transformations. This is in fact the second motivation for the asymptotic analysis to be carried out in the next few sections. The interpretation of the limit in terms of robust control will be discussed further in Section 5. As the reader can easily check, unless the dynamics of the process model are modified in an ε-dependent way such as that of (3.2), then it will be difficult to normalize the quantity (3.1) so as to obtain a well-defined limit as $\varepsilon \rightarrow 0$. Because of this dependence, it is not guaranteed that controls that are optimal or nearly optimal for the limit problem will be useful for the risk-sensitive problem for a given value of ε. The issue is roughly the same as determining the range of validity of the large deviation approximations described in the last subsection. In any case, the robust interpretation of the limit control has a meaning that is independent of the asymptotic analysis.

It should be noted that the connection between risk-sensitive control and robust control in the nonlinear case has been examined before in different contexts. It was shown in [13,14,22] that the values of certain risk-sensitive infinite time horizon control problems converge to the values of H^∞ disturbance attenuation problems. In [13,26] the connection between finite time horizon risk-sensitive control and robust finite time horizon control was made. Finally, in [28], the connection was examined in the Markov chain case.

3.3 Generalizations of the Cost

Depending on the problem, one might consider various generalizations of the cost criterion

$$E_x \exp[-\theta\tau^\varepsilon/\varepsilon].$$

For example, one can measure "time until escape" in a state-dependent way by considering a cost of the form

$$E_x \exp\left[-\int_0^{\tau^\varepsilon} \theta(X_s^\varepsilon)ds/\varepsilon\right],$$

where $\theta(\cdot)$ is a continuous function on $\bar{G}$. As long as $\inf_{x\in\bar{G}}\theta(x) > 0$, the analysis for this case is essentially the same as for the case $\theta(\cdot) \equiv \theta$ that is considered in Sections 4–6.

For maximizing control problems where the control space is potentially unbounded, one might consider a cost of the form

$$E_x \exp\left[-\int_0^{\tau^\varepsilon} [\theta - \langle u_s, Au_s\rangle]\, ds/\varepsilon\right],$$

where A is a positive definite matrix of appropriate dimensions. Because of the lack of compactness in the control variable, the analysis in this case is more complicated than that given in Sections 4–6.

4 Asymptotic Analysis

In this section, the results for the small noise/risk–sensitive escape problem are developed. Since the proofs for the case with a maximizing control are not significantly more complex than those for the case without control, all the results in this section will be stated only for the case with control. For the case without control, the modifications are obvious.

For the remainder of the paper we will restrict our attention to controlled and uncontrolled diffusion processes, and further assume that the control only affects the drift. Hence the dynamics of the controlled process are given by

$$dX_t^\varepsilon = f(X_t^\varepsilon, u_t)\,dt + \varepsilon^{1/2}\,\sigma(X_t^\varepsilon)\,dB_t, \tag{4.1}$$

with the obvious modification for the uncontrolled case. $B.$ is a Brownian motion with sample space Ω, filtration F_t, and measure P. It will be assumed that $u.$ will be F_t–progressively measurable (see [17]) and take values in a compact set U.

The following assumptions will hold throughout the remainder of the paper; no additional assumptions will appear. Recall that $G \subseteq \Re^n$ denotes the set one wishes to keep the process in, where G is open and $\overline{G}$ is compact.

Condition 4.1 *The set $\overline{G}$ satisfies a uniform exterior sphere condition. That is, there exists $r > 0$ such that for any x_0 on the boundary of G, ∂G, there exists $y \in \overline{G}^c$ such that*

$$B_r(y) \cap \overline{G} = \{x_0\}$$

where $|x_0 - y| = r$.

Condition 4.2

$$f \in C(\overline{G}, U)$$

and furthermore f is uniformly Lipschitz in x in the sense that there exists $K_f < \infty$ such that

$$|f(x,u) - f(y,u)| \le K_f |x - y| \qquad \forall\, x, y \in \bar{G},\ \forall\, u \in U.$$

Condition 4.3

$$\sigma \in C^1(\overline{G}),$$

(which in particular implies that it is Lipschitz on $\overline{G}$ with a constant K_σ). In addition, there exists $\mu > 0$ such that

$$\xi^T \sigma(x)\sigma^T(x)\xi \ge \mu|\xi|^2 \qquad \forall\, x \in \overline{G},\ \forall\, \xi \in \Re^n.$$

Let $\mathcal{U}_\nu$ be the set of F_t–progressively measurable control processes with values in U with respect to the reference probability system $\nu = (\Omega, \{F_t\}, P, B.)$ (see [17]). Although X^ε and τ^ε depend on $u \in \mathcal{U}_\nu$, we omit this dependence from the notation. Define the function

$$\Phi^\varepsilon(x) \doteq \inf_{u \in \mathcal{U}_\nu} E_x \exp\left[-\theta\tau^\varepsilon/\varepsilon\right]$$

where θ is a positive constant, τ^ε is the exit time for the above diffusion process, and E_x indicates expectation conditioned on an initial state x. As is well known (and easy to prove using a large deviation calculation), for each fixed control the quantity

$$E_x \exp\left[-\theta\tau^\varepsilon/\varepsilon\right]$$

scales exponentially in ε as $\varepsilon \to 0$, in the sense that for each $x \in G$ there exists $c(x) \in I\!R$ such that

$$V^\varepsilon(x) = -\varepsilon \log \Phi^\varepsilon(x) = -\varepsilon \log E_x \exp\left[-\theta\tau^\varepsilon/\varepsilon\right] \to c(x)$$

as $\varepsilon \to 0$.

Thus it is natural to consider the logarithmic transform (see, for instance, [17,12]); that is, we consider a criterion of the form

$$V^\varepsilon(x) = -\varepsilon \log \Phi^\varepsilon(x) = \sup_{u \in \mathcal{U}_\nu} -\varepsilon \log E_x \exp\left[-\theta\tau^\varepsilon/\varepsilon\right]. \tag{4.2}$$

(As noted in Section 3, the cost in (4.2) is similar to $E_x\{\theta\tau\}$ but with the insertion of a risk–sensitizing transformation.)

By starting with the quasilinear PDE satisfied by Φ^ε, taking the log transform, and then multiplying by $-\varepsilon$, one formally obtains the PDE

$$\begin{aligned} 0 &= \theta + \frac{\varepsilon}{2}\mathrm{tr}\left[a(x)V_{xx}(x)\right] + H(x, V_x(x)) \quad && x \in G \\ V(x) &= 0 && x \in \partial G \end{aligned} \tag{4.3}$$

where

$$H(x,p) = \max_{v \in U}\langle f(x,v), p\rangle - \frac{1}{2}\langle p, a(x)p\rangle, \tag{4.4}$$

and where $a(x) = \sigma(x)\sigma^T(x)$. Criterion (4.2) is, of course, analogous to the risk–sensitive criterion being applied in many stochastic control problems (see [13,14,15,22,26,28], among others). The following results support this interpretation.

The first lemma is standard (see, for instance, [17, Theorem 15.18]).

Lemma 4.1 *There exists a solution $\tilde{V}^\varepsilon \in C^2(G) \cap C^0(\overline{G})$ to (4.3).*

Define the cost for a fixed control $u \in \mathcal{U}_\nu$ by

$$J^\varepsilon(x,u) \doteq -\varepsilon \log E_x \exp\left[-\theta\tau^\varepsilon/\varepsilon\right].$$

Theorem 4.2 *For all $x \in \overline{G}$ and all $u. \in \mathcal{U}_\nu$, we have the inequality*

$$\tilde{V}^\varepsilon(x) \geq J^\varepsilon(x,u),$$

where $X^\varepsilon_\cdot$ is given by (4.1) with initial condition $X^\varepsilon_0 = x$.

Let $\bar{u}$ be a Borel measurable function such that

$$\bar{u}(x) \in \mathit{argmax}_{v \in U}\langle f(x,v), \tilde{V}^\varepsilon_x(x)\rangle.$$

Let $\bar{X}^\varepsilon_\cdot$ be a solution of (4.1) with $u_t = \bar{u}(\bar{X}^\varepsilon_t)$ [31]. Then $u_t \doteq \bar{u}(\bar{X}^\varepsilon_t)$ is in $\mathcal{U}_\nu$, and

$$\tilde{V}^\varepsilon(x) = J^\varepsilon(x,u).$$

Consequently $V^\varepsilon = \tilde{V}^\varepsilon$.

Although the proof of this assertion is delayed until Section 6, we note here that Theorem 4.2 follows from Girsanov's Theorem and an application of Ito's rule.

As discussed in Section 3, in the context of the small noise problem it is useful to determine the limit as $\varepsilon \downarrow 0$ for this problem. Two common techniques for proving convergence are based on large deviations ideas (see [8], among others) and viscosity solutions (see [18], among others). The first is a probabilistic method, while the second is a PDE approach. The viscosity solution approach is used here. Since it is being assumed that the prelimit ($\varepsilon > 0$) problem is uniformly non-degenerate, the viscosity solution approach is relatively easy to apply. First, it is necessary to obtain bounds on the behavior of V^ε which are uniform in $\varepsilon > 0$. These are supplied by the following two lemmas. Their proofs, which are given in Section 6, are standard.

Lemma 4.3 *There exists $M_1 < \infty$ such that $0 \le V^\varepsilon(x) \le M_1$ for all $x \in \overline{G}$ and for all $\varepsilon > 0$.*

Lemma 4.4 *Given any $\varepsilon_0 < \infty$, there exists $M_2 < \infty$ such that $|V^\varepsilon(x) - V^\varepsilon(y)| \le M_2|x-y|$ for all $x \in \partial G$, all $y \in \overline{G}$, and all $\varepsilon \in (0, \varepsilon_0)$.*

As $\varepsilon \downarrow 0$, one formally obtains from (4.3) the limit PDE problem

$$\begin{aligned} &0 = -\theta - H(x, V_x(x)) && x \in G \\ &V(x) = 0 && x \in \partial G, \end{aligned} \tag{4.5}$$

where H is given by (4.4). (The choice of sign in (4.5) is a consequence of the definition of viscosity solutions that will be used in Section 5.)

It will be shown in Section 5 that there exists a unique continuous viscosity solution to (4.5) (meeting the boundary condition pointwise). Let this solution be denoted by W. The method of Barles and Perthame [2,17] will be used to show that

$$\lim_{\varepsilon \downarrow 0} V^\varepsilon(x) = W(x) \qquad \forall x \in \overline{G}.$$

This method eliminates the need to obtain gradient bounds which are uniform in $\varepsilon > 0$ and $x \in \overline{G}$, and here such bounds are only needed on the boundary (Lemma 4.4). Define

$$\begin{aligned} V^*(x) &\doteq \limsup\{V^\varepsilon(y) : \ y \to x, \ \varepsilon \downarrow 0, \ y \in \overline{G}\} \\ V_*(x) &\doteq \liminf\{V^\varepsilon(y) : \ y \to x, \ \varepsilon \downarrow 0, \ y \in \overline{G}\}. \end{aligned} \tag{4.6}$$

Note that by Lemma 4.4

$$V^*(x) = 0 = V_*(x) \qquad \forall x \in \partial G,$$

and since $W(x) = 0$ for all $x \in \partial G$, one has

$$V^*(x) = V_*(x) = W(x) \qquad \forall x \in \partial G. \tag{4.7}$$

By construction one automatically has

$$V^*(x) \ge V_*(x) \qquad \forall x \in G.$$

Therefore, all one needs to prove is that

$$V^*(x) \le W(x) \le V_*(x) \qquad \forall x \in \overline{G}.$$

The proof of this last statement is delayed to Section 6. However it immediately implies the following.

Theorem 4.5 *$V^\epsilon(x) \to W(x)$ uniformly on $\overline{G}$.*

Thus one has the characterization of the limit of the value function as a continuous viscosity solution to (4.5). Further, the comparison result at the heart of the proof of Theorem 4.5 also implies uniqueness of the solution of (4.5) among the class of continuous viscosity solutions. In Section 5 it will be shown that W is the value of a deterministic game corresponding to the associated robust control problem. (In the case where there is no control in the risk–sensitive problem, W is simply the value of a deterministic minimizing control problem rather than a game.) This game serves two roles. In the small noise problem it provides a convenient starting point for the analysis and construction of controls for the prelimit problem. It will also serve as the starting point for the interpretation of the maximizing control in the game as a robust control.

5 The Robust Limit Problem

In this section we consider the robust problem corresponding to the limit $\varepsilon \to 0$. The term robust is being used here to denote a system where the effect on the system output of a disturbance is bounded by a function of the power of that disturbance. This notion may take different forms in different contexts. The most well-known example is H^∞ control. In the state-space formulation, a system is said to satisfy an H^∞ bound if there exists a bound on the L^2 norm of an output in the form of a product of a disturbance attenuation constant and the L^2 norm of the disturbance. This H^∞ disturbance attenuation control problem may be formulated as a deterministic differential game. The robust (maximizing) control escape time problem will also be formulated as a game. In particular, the maximizing player in the game will correspond to the original maximizing control, and an opposing player will be introduced who will be attempting to minimize the same payoff. In a sense, now the noise will be a process chosen by this new antagonistic player as opposed to being a stochastic process. The noise will attempt to drive the process from the set G, but will pay a quadratic cost.

For the case without maximizing control (i.e. the performance evaluation or parameter optimization problems discussed earlier), the robust formulation will be a control problem rather than a game. The controller in the problem will be a minimizing controller corresponding to the player that represented the noise in the game.

In this section, the case with no control is significantly easier to analyze than the case with a maximizing controller due to the fact that the latter results in a game. Therefore, two separate and independent analyses will be provided so that the reader who is interested in only one of these two problems will be able to skip the irrelevant material.

5.1 The Case Without Control

Consider the following deterministic dynamics

$$\frac{dY_t}{dt} = f(Y_t) + \sigma(Y_t)w_t, \; Y_0 = x \tag{5.1}$$

for $t \in [0,\tau]$, where τ is the time of the first exit from G. The functions f and σ are those introduced in Section 4. Note that the disturbance, $w_\cdot$, is now a deterministic process. Assume $w \in L^2([0,T];\Re^m)$ for all $T \in (0,\infty)$, and denote this set of controls by $\mathcal{W}^0$.

Consider also a cost criterion of the form

$$J(x,w) = \int_0^\tau \left[\theta + \frac{1}{2}|w_t|^2\right] dt \tag{5.2}$$

where again $\theta > 0$ is a constant. The control problem which will yield the robust value, $\widetilde{W}$, is

$$\widetilde{W}(x) = \inf_{w \in \mathcal{W}^0} J(x,w). \tag{5.3}$$

The HJB equation corresponding to control problem (5.1)–(5.3) is

$$0 = -\theta - H_{\mathrm{nc}}(x, W_x(x)), \; x \in G, \qquad W(x) = 0 \,, x \in \partial G \,, \tag{5.4}$$

where

$$H_{\mathrm{nc}}(x,p) = \langle f(x), p\rangle + \min_{w \in \Re^m}\left[\langle \sigma(x)w, p\rangle + \frac{1}{2}|w|^2\right].$$

Evaluation of the minimum yields

$$H_{\rm nc}(x,p) = \langle f(x),p\rangle - \frac{1}{2}\langle p,a(x)p\rangle.$$

Hence (5.4) is the same as (4.5) for the case with no (maximizing) control. For easy reference, the definition of a continuous viscosity solution follows. $W \in C(\overline{G})$ is a continuous viscosity subsolution of (5.4) if

$$-\theta - H_{\rm nc}(x_0, \phi_x(x_0)) \le 0$$

provided $\phi \in C^1(G)$ and $W - \phi$ attains a maximum at $x_0 \in G$. $W \in C(\overline{G})$ is a continuous viscosity supersolution of (5.4) if

$$-\theta - H_{\rm nc}(x_0, \phi_x(x_0)) \ge 0$$

provided $\phi \in C^1(G)$ and $W - \phi$ attains a minimum at $x_0 \in G$. If W is both a subsolution and a supersolution, then it is a solution. Due to the nondegeneracy (Condition 4.3), it is not necessary to formulate the boundary conditions in the viscosity framework here; only solutions satisfying the boundary conditions pointwise will be considered. In particular, all assertions of uniqueness are within the class of continuous viscosity solutions satisfying the boundary conditions pointwise. (For modifications relevant to extending the comparison approach used in the proof of Theorem 4.5 to viscosity solutions satisfying the weaker viscosity form of the boundary conditions, see [17, Section 7.8].)

Theorem 5.1 *$\widetilde{W}$ is the unique continuous viscosity solution of (5.4).*

Proof. Since Theorem 5.1 is just a variant of what are now standard results. only an outline of the proof will be provided. The argument is as follows.

First note that there exists $M_3 < \infty$ such that

$$0 \le \inf_{w \in \mathcal{W}^0} J(x,w) \le M_3.$$

The proof of this statement is nearly identical to the proof of Lemma 5.2 which appears in Section 6, and simply involves constructing a piece-wise constant control u which guarantees exit prior to some fixed time T, where T is independent of x and ϕ. By the nature of the cost criterion (5.2), this implies that there exists $C < \infty$ such that $\int_0^T |w_s|^2 ds \le C$ for all w such that $J(x,w) \le \widetilde{W}(x) + 1$. Thus,

$$\widetilde{W}(x) = \inf_{w \in \mathcal{W}_b^0} J(x,w)$$

where

$$\mathcal{W}_b^0 = \left\{ w \in \mathcal{W}^0 : \|w\|_{L^2(0,\infty)} \le C \right\}.$$

Further, one sees that given $\delta > 0$, there exists $\eta > 0$ such that

$$|Y_t - x| \le \delta \qquad \forall t \in [0,\eta], \ \forall w \in \mathcal{W}_b^0, \tag{5.5}$$

and that for all $w \in \mathcal{W}_b^0$, $\tau = \tau(x,w)$ is bounded from below by some function $h(x)$ which is strictly positive on G.

Dynamic programming principles for $\widetilde{W}$ are easily obtained, and in fact are contained as special cases of the corresponding results in the next subsection. In particular, one has

$$\widetilde{W}(x) = \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{\sigma\wedge\tau} \left(\theta + \frac{1}{2}|w_t|^2 \right) dt + \widetilde{W}(Y_{\sigma\wedge\tau}) \right]$$

for all $x \in G$ and $\sigma \in [0, \infty)$, and

$$\widetilde{W}(x) \leq \inf_{w \in \widehat{\mathcal{W}}_m^0} \left[\int_0^{\sigma\wedge\tau} \left(\theta + \frac{1}{2}|w_t|^2 \right) dt + \widetilde{W}(Y_{\sigma\wedge\tau}) \right]$$

for all $x \in G$, $\sigma \in [0, \infty)$ and $m > 0$ where

$$\widehat{\mathcal{W}}_m^0 = \left\{ w \in \mathcal{W}^0 : |w_t| \leq m \ \forall t \right\}.$$

The last two paragraphs provide all that is needed for the proof of Theorem 5.1. The continuity of $\widetilde{W}$ follows from the nondegeneracy of σ and the boundedness of f. These properties imply that given $\delta > 0$ there exists $\varepsilon > 0$ such that given any point y within ε of x, one can construct a control that moves Y_t from x to y with cost less than δ and in time less than δ. This in turn implies the continuity. Uniqueness (among the class of continuous viscosity solutions satisfying the boundary conditions pointwise) follows from the comparison principle used in the proof of Theorem 4.5. The proof of the theorem is completed by showing that $\widetilde{W}$ is both a viscosity subsolution and supersolution of (5.4).

We first prove that $\widetilde{W}$ is a viscosity supersolution to (5.4). Suppose there exists $g \in C^1(\overline{G})$ and that $\widetilde{W} - g$ has a local minimum at $x_0 \in G$. To prove that $\widetilde{W}$ is a viscosity supersolution, it must be shown that

$$-\theta - \bar{H}_{\text{nc}}(x_0, g_x(x_0)) \geq 0.$$

If this inequality is not valid, then there exists $\alpha > 0$ such that

$$\theta + \bar{H}_{\text{nc}}(x_0, g_x(x_0)) > \alpha.$$

By (5.5) and the uniform (in w) bound from below on $\tau = \tau(x, w)$, it can be seen that there exists $\eta_0 \in (0, \tau)$ such that for all $\eta \in (0, \eta_0)$ and $w \in \mathcal{W}^0$

$$\int_0^\eta \left\{ \theta + \frac{1}{2}|w_t|^2 + \langle f(Y_t) + \sigma(Y_t)w_t, g_x(Y_t) \rangle \right\} dt \geq \frac{\eta\alpha}{2} > 0, \tag{5.6}$$

where Y_t is given by (5.1) with initial condition x_0. However, because $\widetilde{W} - g$ has a local minimum at x_0, for sufficiently small $\eta \in (0, \eta_0)$

$$\widetilde{W}(x_0) - g(x_0) \leq \widetilde{W}(Y_t) - g(Y_t) \qquad \forall t \in [0, \eta].$$

By the first dynamic programming principle above, this implies

$$\inf_{w \in \mathcal{W}_b^0} \left\{ \int_0^\eta \left[\theta + \frac{1}{2}|w_t|^2 \right] dt + g(Y_\eta) - g(x_0) \right\} \leq 0. \tag{5.7}$$

But (5.6) and (5.7) form a contradiction, and consequently $\widetilde{W}$ is a viscosity supersolution. The proof that $\widetilde{W}$ is a viscosity subsolution is analogous and employs the second dynamic programming principle above. Therefore $\widetilde{W}$ is a continuous viscosity solution.

The reader uninterested in the case with maximizing control may wish to skip to Subsection 5.3.

5.2 Case With Control

Consider the following deterministic differential game. For $t \in [0, \tau]$ the dynamics are given by

$$\frac{dY_t}{dt} = f(Y_t, u_t) + \sigma(Y_t) w_t,\ Y_0 = x \tag{5.8}$$

where τ is the time of first escape from G. The function $u.$ is the (deterministic) measurable control for the maximizing player, which takes values in U. Let this set of controls be denoted by $\mathcal{U}^0$. The new term, $w.$, is the deterministic control for the minimizing player, and we assume $w \in L^2([0,T]; \Re^m)$ for all $T \in (0, \infty)$. Let this set of controls be denoted by $\mathcal{W}^0$.

The Elliott–Kalton [10] definition of the game will be used, and consequently the set of strategies for each player must be defined. A strategy for the maximizing player is a mapping, ϕ, from $\mathcal{W}^0$ into $\mathcal{U}^0$ which is non-anticipating in the following sense. For each $t > 0$ and $w, \tilde{w} \in \mathcal{W}^0$ such that $w_r = \tilde{w}_r$ for a.e. $r \in [0,t]$, one has $\phi[w]_r = \phi[\tilde{w}]_r$ for a.e. $r \in [0,t]$. Let this set of strategies be denoted by Φ. Similarly, a strategy for the minimizing player is a mapping, λ, from $\mathcal{U}^0$ into $\mathcal{W}^0$ which is non-anticipating in the analogous sense. Let Λ denote the set of strategies for the minimizing player. The payoff for the game will be

$$J(x, u, w) = \int_0^{\tau} \left[\theta + \frac{1}{2}|w_t|^2\right] dt. \tag{5.9}$$

The upper and lower values in the Elliott–Kalton sense are given by

$$\widetilde{W}(x) = \sup_{\phi \in \Phi} \inf_{w \in \mathcal{W}^0} J(x, \phi[w], w) \text{ and } \widehat{W}(x) = \inf_{\lambda \in \Lambda} \sup_{u \in \mathcal{U}^0} J(x, u, \lambda[u]). \tag{5.10}$$

If $\widetilde{W} = \widehat{W}$, then we say the game has value.

The Isaacs equation corresponding to this game is given by

$$0 = -\theta - H(x, W_x(x)),\ x \in G, \qquad W(x) = 0,\ x \in \partial G, \tag{5.11}$$

where

$$H(x,p) = \max_{v \in U} \langle f(x,v), p \rangle + \min_{w \in \Re^m} \left[\langle \sigma(x) w, p \rangle + \frac{1}{2}|w|^2\right].$$

Evaluation of the minimum gives

$$H(x,p) = \max_{v \in U} \langle f(x,v), p \rangle - \frac{1}{2}\langle p, a(x) p \rangle.$$

It will now be shown that the upper value, $\widetilde{W}$, is a continuous viscosity solution of (5.11). The same result holds for the lower value. For a fixed finite time horizon problem under stronger assumptions, Evans–Souganidis [11] showed that a class of deterministic differential games had value and that this common value function was a viscosity solution of the corresponding Isaacs equations. The method of proof was to first obtain the dynamic programming principle, and then combine this relation with some arguments regarding the continuity of the state trajectories to prove that the value was the viscosity solution. In adapting this approach, some technical difficulties arise due to the unbounded controls for the minimizing player, the weaker assumptions on the dynamics and payoff, and the fact that this is not a fixed finite time horizon problem. Thus, some preliminary lemmas are necessary as well as some variations on the method of proof. In the statement and proofs of these lemmas, Y_t will denote the solution to (5.6) for the given controls and τ will be $\inf\{t : Y_t \notin G\}$.

Lemma 5.2 *There exists $M_3 < \infty$ such that*

$$0 \le \inf_{w \in \mathcal{W}^0} J(x, \phi[w], w) \le M_3 \qquad \forall \phi \in \Phi, \ \forall x \in \overline{G}.$$

The proof of Lemma 5.2 will be delayed until Section 6. The method is relatively standard, and involves constructing a piece–wise constant control w. which guarantees exit prior to some fixed time, T, independent of x and ϕ. Since the control so constructed will also be bounded, the result will follow. We note that a similar approach is used in [5].

Lemma 5.3 *Let $\varepsilon_0 > 0$ be fixed, and let M_3 satisfy the conclusion of Lemma 5.2. Then for all $x \in \overline{G}$, $\phi \in \Phi$, and any $\tilde{w}$ that satisfies*

$$J(x, \phi[\tilde{w}], \tilde{w}) \le \inf_{w \in \mathcal{W}^0} J(x, \phi[w], w) + \epsilon_0, \tag{5.12}$$

we have the bound

$$\int_0^\tau |\tilde{w}_t|^2 \, dt \le 2(M_3 + \epsilon_0).$$

Proof. By (5.12) and Lemma 5.2

$$J(x, \phi[\tilde{w}], \tilde{w}) \le M_3 + \varepsilon_0$$

which implies the result. □

Now let

$$\mathcal{W}_b^0 = \left\{ w \in \mathcal{W}^0 : \ \|w\|_{L^2(0,\infty)} \le 2(M_3 + \epsilon_0) \right\},$$

and note that

$$\widetilde{W}(x) = \sup_{\phi \in \Phi} \inf_{w \in \mathcal{W}_b^0} J(x, \phi[w], w).$$

Lemma 5.4 *Let $w \in \mathcal{W}_b^0$, $u \in \mathcal{U}^0$. Then, there exist $B_1, B_2 < \infty$ such that*

$$|Y_t - x| \le B_1 t + B_2 \sqrt{t} \qquad \forall t \in [0, \tau].$$

Proof. By (5.8)

$$|Y_t - x| \le \int_0^t |f(Y_r, u_r)| \, dr + \int_0^t |\sigma(Y_r)||w_r| \, dr.$$

By Conditions 4.1 and 4.2, there exist $C_f, C_\sigma < \infty$ such that the right hand side of ths inequality is less than or equal to

$$C_f t + C_\sigma \int_0^t |w_r| \, dr.$$

According to Lemma 5.3 this can be bounded above by

$$C_f t + C_\sigma [2(M_3 + \varepsilon_0)]^{\frac{1}{2}} \sqrt{t}. \ \square$$

From Lemma 5.4, one immediately obtains the following result.

Lemma 5.5 *Let $x \in G$, $w \in \mathcal{W}_b^0$ and $u \in \mathcal{U}^0$. Then*

$$B_1\tau + B_2\sqrt{\tau} \geq d(x, G^c).$$

The Lemmas 5.2–5.5 serve to bound the controls for the minimizing player, and demonstrate continuity of the state with respect to time. Next, dynamic programming principles will be obtained.

Theorem 5.6

$$\widetilde{W}(x) = \sup_{\phi \in \Phi} \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{\sigma \wedge \tau} \left(\theta + \frac{1}{2}|w_t|^2 \right) dt + \widetilde{W}(Y_{\sigma \wedge \tau}) \right]$$

for all $x \in G$ and $\sigma \in [0, \infty)$.

The proof of this theorem is standard but tedious; it will be delayed until Section 6. The proof of the following variation on this theorem involves only simple modifications of proof of Theorem 5.6.

Theorem 5.7

$$\widetilde{W}(x) \leq \sup_{\phi \in \Phi} \inf_{w \in \widehat{\mathcal{W}}_m^0} \left[\int_0^{\sigma \wedge \tau} \left(\theta + \frac{1}{2}|w_t|^2 \right) dt + \widetilde{W}(Y_{\sigma \wedge \tau}) \right]$$

for all $x \in G$, $\sigma \in [0, \infty)$ and $m > 0$ where

$$\widehat{\mathcal{W}}_m^0 = \left\{ w \in \mathcal{W}^0 : |w_t| \leq m \ \forall t \right\}.$$

Given the validity of these dynamic programming principles, one can prove that the upper value $\widetilde{W}$ is a continuous viscosity solution of the Isaacs equation (5.11). The next two lemmas imply certain semicontinuity properties of $\widetilde{W}(Y_\cdot)$.

Lemma 5.8 *Let $g \in C^1(\overline{G})$ satisfy*

$$0 > -\alpha \geq -\theta - H(x_0, g_x(x_0))$$

for some $x_0 \in G$ and $\alpha > 0$. Then there exists $\phi \in \Phi$ and $\eta_0 > 0$ such that for all $w \in \mathcal{W}_b^0$ and all $\eta \in (0, \eta_0)$

$$\int_0^{\eta \wedge \tau} \left\{ \theta + \frac{1}{2}|w_t|^2 + \langle f(Y_t, \phi[w]_t) + \sigma(Y_t) w_t, g_x(Y_t) \rangle \right\} dt \geq \frac{\eta\alpha}{2}$$

where Y_t is given by (5.8) with initial condition x_0.

Proof. Define

$$F(x, u, w) \doteq \theta + \frac{1}{2}|w|^2 + \langle f(x, u) + \sigma(x) w, g_x(x) \rangle, \tag{5.13}$$

and let $u_0 \in \operatorname{argmax} F(x_0, u, w)$. Note that

$$\theta + H(x, g_x(x)) = \max_{u \in U} \min_{w \in \mathbb{R}^m} F(x, u, w),$$

and that u_0 is independent of w since the Isaacs condition is satisfied. Then, by the assumption of the lemma,

$$F(x_0, u_0, w) \geq \theta + H(x_0, g_x(x_0)) \geq \alpha \qquad \forall w \in \Re^m.$$

Let $w \in \mathcal{W}_b^0$ and $\phi[w]_t \equiv u_0$. Then by Lemma 5.4 there exists $\eta > 0$ such that

$$F(Y_t, \phi[w]_t, w_t) \geq \frac{\alpha}{2}$$

for all $t \in [0, \eta]$ and $w \in \mathcal{W}_b^0$. Integrating and using Lemma 5.5 to assert that $\eta \wedge \tau = \eta$ for η sufficiently small yields the result. □

Lemma 5.9 *Let $g \in C^1(\overline{G})$ satisfy*

$$0 < \alpha \leq -\theta - H(x_0, g_x(x_0))$$

for some $x_0 \in G$ and $\alpha > 0$. Then there exists $\eta > 0$ and a bounded $w \in \mathcal{W}^0$ such that for all $\phi \in \Phi$

$$\int_0^{\eta\wedge\tau} \left\{ \theta + \frac{1}{2}|w_t|^2 + \langle f(Y_t, \phi[w]_t) + \sigma(Y_t)w_t, g_x(Y_t)\rangle \right\} dt \leq -\frac{\eta\alpha}{2}$$

where Y_t is given by (5.8) with initial condition x_0.

The proof of Lemma 5.9 is very similar to that of Lemma 5.8. In particular, here one lets $w_t \equiv w^*$ where $w^* \in \text{argmin } F(x_0, u, w)$, with F is given by (5.13). (Again note that w^* is independent of u.)

Now the way is clear for the main theorem of this subsection. The proof will be delayed until Section 6.

Theorem 5.10 *$\widetilde{W}$ is the unique continuous viscosity solution of (5.11).*

5.3 Robust Interpretation

First we review the nonlinear H^∞ disturbance attenuation problem, and then the analogous robust interpretation for the limit problem considered here will be presented. For the nonlinear H^∞ disturbance attenuation problem, one typically considers games with payoffs of the form

$$\int_0^T \left[L(X_t, u_t) - \gamma^2 |w_t|^2 \right] dt$$

where X is the state, u is the (true) control for the minimizing player, and w is the disturbance which becomes the control for the maximizing player. L is the running cost, and is often taken to be a quadratic function. Further, one assumes an initial value for the state, $X_0 = x_0$, such that $L(x_0, 0) = 0$. If there exists a strategy for the minimizing player (ideally a feedback control leading to well-defined dynamics) such that the value of the game is zero for all T, then this implies

$$\left[\int_0^T L(X_t, u_t)\, dt \right]^{\frac{1}{2}} \leq \gamma \|w\|_{L^2(0,T)} \qquad \forall T < \infty.$$

That is, there is a bound for the cost in the form of a product of the disturbance attenuation constant, γ, and the L^2 norm of the disturbance for all time horizons and disturbances. See, for instance, [3,30,21,1,23] among others.

For the escape time problem, large time averages do not make any sense, since it is the transient behavior that is of primary importance. Since we cannot eliminate escape entirely, the best one should hope for in the robust problem is a bound on the escape time in terms of the energy of the player representing the disturbance. Clearly, this bound will depend on the initial position of the controlled process. For the case without maximizing control, let

$$\overline{W}(x) \equiv \widetilde{W}(x) = \inf_{w \in \mathcal{W}^0} J(x,w) \qquad \forall x \in G.$$

(This new notation is being introduced so that both cases can be treated together in this section.) For the case with maximizing control, choose some (optimal or suboptimal) strategy ϕ_0 given in a feedback form such that the dynamics are well-defined. Let $\overline{W}$ be the value of the game with this control, i.e.

$$\overline{W}(x) = \inf_{w \in \mathcal{W}^0} J(x, \phi_0[w], w) \qquad \forall x \in G.$$

Then by (5.1) and (5.2) in the case without control, or by (5.8) and (5.11) in the case with maximizing control,

$$\overline{W}(x) \leq \left[\theta + \frac{1}{2\tau}\int_0^\tau |w_t|^2\, dt\right]\tau \qquad \forall w \in L^2, \ \forall x \in G,$$

or

$$\tau \geq \frac{\overline{W}(x)}{\theta + \frac{1}{2\tau}\int_0^\tau |w_t|^2\, dt} \qquad \forall w \in L^2, \ \forall x \in G. \tag{5.14}$$

Let

$$\mathcal{W}^P = \left\{ w \in L^2[0,\infty) : \frac{1}{T}\int_0^T |w_t|^2\, dt \leq P \quad \forall 0 \leq T < \infty \right\}.$$

Then (5.14) has the interpretation

$$\tau \geq \frac{\overline{W}(x)}{\theta + \frac{1}{2}P} \qquad \forall w \in \mathcal{W}^P, \ \forall x \in G.$$

This is a lower bound on the escape time as a function of the energy of the input noise. It is analogous to the attenuation bound of H^∞ control.

6 Proofs

This section contains proofs for results which appeared in Sections 4 and 5.

6.1 Proofs for Section 4

6.1.1 Proof of Theorem 4.2

Let $u \in \mathcal{U}_\nu$ and $x \in G$. By (4.1), for any bounded, F_t–progressively measurable process $w.$, one has

$$\begin{aligned} X_t^\varepsilon &= x + \textstyle\int_0^t f(X_r^\varepsilon, u_r)\,dr + \sqrt{\varepsilon}\int_0^t \sigma(X_r^\varepsilon)\,dB_r \\ &= x + \textstyle\int_0^t [f(X_r^\varepsilon, u_r) + \sigma(X_r^\varepsilon)w_r]\,dr + \sqrt{\varepsilon}\int_0^t \sigma(X_r^\varepsilon)\,dB_r - \int_0^t \sigma(X_r^\varepsilon)w_r\,dr \\ &= x + \textstyle\int_0^t [f(X_r^\varepsilon, u_r) + \sigma(X_r^\varepsilon)w_r]\,dr + \sqrt{\varepsilon}\int_0^t \sigma(X_r^\varepsilon)\,dB_r^0 \end{aligned} \tag{6.1}$$

which defines $B^0_\cdot$.

Fix any $T < \infty$. We would like a probability measure, P^0, under which B^0 is a Brownian motion. By Girsanov's Theorem, [24, p. 191] such a measure exists on (Ω, F_T). For any set A in F_T, this measure satisfies

$$P^0(A) = \int_A \exp\left[\sqrt{\frac{1}{\varepsilon}}\int_0^T \langle w_r, dB_r^0\rangle + \frac{1}{2\varepsilon}\int_0^T |w_r|^2\,dr\right] P(d\omega).$$

In terms of this measure, we can write

$$E_x \exp\{-\theta(\tau^\varepsilon \wedge T)/\varepsilon\} = E_x^0 \exp\frac{1}{\varepsilon}\left[-\int_0^{\tau^\varepsilon\wedge T}\left(\theta + \frac{1}{2}|w_r|^2\right)dr - \varepsilon^{1/2}\int_0^{\tau^\varepsilon\wedge T}\langle w_r, dB_r^0\rangle\right]. \tag{6.2}$$

By applying Ito's rule with the new dynamics (6.1) and using the PDE (4.3), one obtains

$$\begin{aligned} \tilde{V}^\varepsilon(X_t^\varepsilon) - \tilde{V}^\varepsilon(x) &= \int_0^t \left[\frac{\varepsilon}{2}\mathrm{tr}\left[a(X_r^\varepsilon)\tilde{V}_{xx}^\varepsilon(X_r^\varepsilon)\right] + \langle f(X_r^\varepsilon, u_r), \tilde{V}_x^\varepsilon(X_r^\varepsilon)\rangle + \langle \sigma(X_r^\varepsilon)w_r, \tilde{V}_x^\varepsilon(X_r^\varepsilon)\rangle\right] dr \\ &\quad + \varepsilon^{1/2}\int_0^t \langle \tilde{V}_x^\varepsilon(X_r^\varepsilon), \sigma(X_r^\varepsilon)dB_r^0\rangle \\ &\le \int_0^t \left[-\theta + \frac{1}{2}\langle \tilde{V}_x^\varepsilon(X_r^\varepsilon), a(X_r^\varepsilon)\tilde{V}_x^\varepsilon(X_r^\varepsilon)\rangle + \langle \sigma(X_r^\varepsilon)w_r, \tilde{V}_x^\varepsilon(X_r^\varepsilon)\rangle\right] dr \\ &\quad + \varepsilon^{1/2}\int_0^t \langle \tilde{V}_x^\varepsilon(X_r^\varepsilon), \sigma(X_r^\varepsilon)dB_r^0\rangle. \end{aligned} \tag{6.3}$$

If we let the disturbance process be given by $w_r = -\sigma^T(X_r^\varepsilon)\tilde{V}_x^\varepsilon(X_r^\varepsilon)$, then for this choice (6.3) implies

$$\begin{aligned} \tilde{V}^\varepsilon(X_t^\varepsilon) - \tilde{V}^\varepsilon(x) &\le -\textstyle\int_0^t \left(\theta + \frac{1}{2}|w_r|^2\right) dr + \varepsilon^{1/2}\int_0^t \langle \tilde{V}_x^\varepsilon(X_r^\varepsilon), \sigma(X_r^\varepsilon)dB_r^0\rangle \\ &= -\textstyle\int_0^t \left(\theta + \frac{1}{2}|w_r|^2\right) dr - \varepsilon^{1/2}\int_0^t \langle w_r, dB_r^0\rangle. \end{aligned} \tag{6.4}$$

Combining (6.2) and (6.4), one obtains

$$E_x \exp\left[-\frac{\theta(\tau^\varepsilon \wedge T)}{\varepsilon}\right] \ge E_x^0 \exp\frac{1}{\varepsilon}[\tilde{V}^\varepsilon(X_{\tau^\varepsilon\wedge T}^\varepsilon) - \tilde{V}^\varepsilon(x)] = \exp\left[-\frac{1}{\varepsilon}\tilde{V}^\varepsilon(x)\right] E_x^0 \exp\frac{1}{\varepsilon}\tilde{V}^\varepsilon(X_{\tau^\varepsilon\wedge T}^\varepsilon).$$

Sending $T \to \infty$ and applying the dominated convergence theorem gives

$$E_x \exp\left[-\frac{\theta\tau^\varepsilon}{\varepsilon}\right] \geq \exp\left[-\frac{1}{\varepsilon}\tilde{V}^\varepsilon(x)\right],$$

which yields the first assertion of the theorem.

To obtain the second assertion, simply note that such a $\bar{u}$ exists (see, for example, [16]), and employ $u_t = \bar{u}(\bar{X}_t^\varepsilon)$ to obtain equality in the argument above. □

6.1.2 Proof of Lemma 4.3:

This is obtained by standard applications of the Comparison Principle. In particular, for the lower bound, one compares with $Z(x) \equiv 0$. For the upper bound, one compares V^ε with $Z(x) = A + q \cdot x$ where A and q are constants independent of ε. By uniform ellipticity, it is easy to show that for $|q|$ sufficiently large there exists $\delta > 0$ such that

$$0 > -\delta \geq \theta + \frac{\varepsilon}{2}\mathrm{tr}\left[a(x)Z_{xx}(x)\right] + H(x, Z_x(x)) \qquad \forall x \in G,\ \forall \varepsilon > 0.$$

Further, for A sufficiently large, $Z(x) \geq 0 = V^\varepsilon(x)$ for all $x \in \partial G$. From the Comparison Principle and the compactness of $\overline{G}$, one has the result. □

6.1.3 Proof of Lemma 4.4:

The barrier method (see, for instance, [20]) will be used. Fix $x_0 \in \partial G$ and note that by Lemma 4.3

$$V^\varepsilon(x) \geq 0 = V^\varepsilon(x_0) \qquad \forall x \in \overline{G}. \tag{6.5}$$

Recall Condition 4.1, which states that the uniform exterior sphere condition holds. This implies there exists r independent of x_0 and $y \in \overline{G}^c$ such that

$$B_r(y) \cap \overline{G} = \{x_0\}$$

and $|x_0 - y| = r$. Fix this value of y, and define

$$Z(x) \doteq \alpha[|x - y| - r] \qquad \forall x \in \overline{G}. \tag{6.6}$$

where α is a positive constant to be chosen independent of $\varepsilon \in (0, \varepsilon_0)$ and $x_0 \in \partial G$. By (6.5) and (6.6), it is sufficient to prove that

$$V^\varepsilon(x) \leq Z(x) \qquad \forall x \in G,\ \forall \varepsilon \in (0, \varepsilon_0). \tag{6.7}$$

On the boundary of course,

$$V^\varepsilon(x) = 0 \leq Z(x). \tag{6.8}$$

If we set $v_x \doteq \frac{x-y}{|x-y|}$, then

$$\begin{aligned}
&\theta + \frac{\varepsilon}{2}\mathrm{tr}\left[a(x)Z_{xx}(x)\right] + H(x, Z_x(x)) \\
&= \theta + \frac{\varepsilon}{2}\frac{\alpha}{|x-y|}\left[\mathrm{tr}(a(x)) - \langle v_x, a(x)v_x\rangle\right] + \alpha \max_{v \in U}\langle f(x,v), v_x\rangle - \frac{\alpha^2}{2}\langle v_x, a(x)v_x\rangle \\
&\leq \theta + \alpha\left\{\frac{\varepsilon}{2|x-y|}\left[C_a - \langle v_x, a(x)v_x\rangle\right] + C_f - \frac{\alpha}{2}\langle v_x, a(x)v_x\rangle\right\},
\end{aligned}$$

where

$$C_a = \max_{x \in \overline{G}} |\mathrm{tr}(a(x))|$$

and

$$C_f = \max_{(x,v) \in \overline{G} \times U} |f(x,v)|.$$

By the uniform ellipticity of a, this yields

$$\theta + \frac{\varepsilon}{2}\mathrm{tr}\left[a(x)Z_{xx}(x)\right] + H(x, Z_x(x)) \leq \theta + \alpha\left\{\frac{C_a\varepsilon}{2r} + C_f - \frac{\alpha}{2}\mu\right\} < 0 \tag{6.9}$$

for α sufficiently large independent of $\varepsilon \in (0, \varepsilon_0)$ and $x_0 \in \partial G$.

By (6.8), (6.9) and the Comparison Principle, one has (6.7). □

6.1.4 Proof of Theorem 4.5:

We recall the definitions of V^* and V_* given in equation (4.6). As noted in Section 4, it is sufficient to prove that

$$V^*(x) \leq W(x) \leq V_*(x) \qquad \forall x \in \overline{G}. \tag{6.10}$$

By a simple modification of [17], Prop. 7.6.1, one finds that V^* is a subsolution of (4.5) and V_* is a supersolution. Further, by (4.6), the boundary conditions are achieved pointwise. It will be shown that $W \leq V_*$ on G; the proof that $V^* \leq W$ is similar.

Let $\delta > 0$ and suppose

$$\bar{a} \equiv \min_{x \in \overline{G}} [(1+\delta)V_*(x) - W(x)] < 0. \tag{6.11}$$

By (4.7) and the semicontinuity and continuity properties of V_* and $\widetilde{W}$, the minimum occurs at some point $\bar{x} \in G$. Let

$$\phi^\eta(x,y) \doteq (1+\delta)V_*(x) - W(y) + \frac{1}{2\eta}|x-y|^2 \tag{6.12}$$

and

$$(x^\eta, y^\eta) \in \operatorname{argmin} \phi^\eta(x,y)$$

where η will be a small parameter. It is easy to see that

$$|x^\eta - y^\eta| \to 0 \ \text{ as } \ \eta \downarrow 0. \tag{6.13}$$

Further, since (x^η, y^η) minimizes ϕ^η,

$$\phi^\eta(x^\eta, y^\eta) \leq \phi^\eta(x^\eta, x^\eta) \tag{6.14}$$

which by (6.12) implies

$$\frac{|x^\eta - y^\eta|^2}{\eta} \leq 2m_W(|x^\eta - y^\eta|) \tag{6.15}$$

(where $m_W(\cdot)$ is the modulus of continuity of W over $\overline{G}$). Thus by (6.13),

$$\frac{|x^\eta - y^\eta|^2}{\eta} \to 0 \ \text{ as } \ \eta \downarrow 0. \tag{6.16}$$

By the compactness of $\overline{G}$, there exists a sequence $\eta_k \downarrow 0$ and an $x^0 \in \overline{G}$ such that

$$x^{\eta_k} \to x^0 \text{ and } y^{\eta_k} \to x^0 \tag{6.17}$$

as $k \to \infty$. To simplify the notation, we retain η as the index of this convergent sequence. By the choice of (x^η, y^η), (6.12) and (6.11)

$$\phi^\eta(x^\eta, y^\eta) \leq \bar{a} < 0 \qquad \forall \eta > 0. \tag{6.18}$$

Then by (6.16), the lower semicontinuity of V_* and the continuity of W,

$$(1+\delta)V_*(x^0) - W(x^0) < 0. \tag{6.19}$$

If x^0 were in ∂G, then (6.19) would contradict (4.6) and (4.7). Therefore $x^0 \in G$, which implies that for η sufficiently large (in our re-indexed subsequence)

$$x^\eta, y^\eta \in G. \tag{6.20}$$

Now let

$$\psi(x) \doteq \frac{1}{1+\delta}\left[W(y^\eta) - \frac{1}{2\eta}|x - y^\eta|^2\right]$$

and note that $V_* - \psi$ has a minimum at x^η. Therefore, since V_* is a supersolution, one has

$$-\theta - H(x^\eta, \psi_x(x^\eta)) \geq 0$$

which implies

$$-(1+\delta)\theta \geq \frac{1}{\eta}\max_{v\in U}\langle -f(x^\eta, v), (x^\eta - y^\eta)\rangle - \frac{1}{2\eta^2(1+\delta)}\langle (x^\eta - y^\eta), a(x^\eta)(x^\eta - y^\eta)\rangle. \tag{6.21}$$

Also, let

$$\tilde{\psi}(y) \doteq (1+\delta)V_*(x^\eta) + \frac{1}{2\eta}|x^\eta - y|^2$$

which implies that $W - \tilde{\psi}$ has a maximum at y^η.

Therefore, since W is a subsolution (as well as a supersolution), one has

$$-\theta - H(y^\eta, \tilde{\psi}_x(y^\eta)) \leq 0$$

which implies

$$\theta \geq -\frac{1}{\eta}\max_{v\in U}\langle -f(y^\eta, v), (x^\eta - y^\eta)\rangle + \frac{1}{2\eta^2}\langle (x^\eta - y^\eta), a(y^\eta)(x^\eta - y^\eta)\rangle. \tag{6.22}$$

Adding (6.21) and (6.22) yields

$$\begin{aligned} -\delta\theta \geq\ & \tfrac{1}{\eta}\max_{v\in U}\langle -f(x^\eta, v), (x^\eta - y^\eta)\rangle - \tfrac{1}{\eta}\max_{v\in U}\langle -f(y^\eta, v), (x^\eta - y^\eta)\rangle \\ & + \tfrac{1}{2\eta^2}\langle (x^\eta - y^\eta), \left[a(y^\eta) - \tfrac{1}{1+\delta}a(x^\eta)\right](x^\eta - y^\eta)\rangle. \end{aligned} \tag{6.23}$$

Recall that by Conditions 4.2 and 4.3 f and a are Lipschitz continuous with constants K_f and K_a on $\overline{G}$, respectively.

Consequently (6.23) implies

$$-\delta\theta \geq \frac{|x^\eta - y^\eta|^2}{2\eta^2}[-2\eta K_f - K_a|x^\eta - y^\eta|] + \frac{1}{2\eta^2}\langle (x^\eta - y^\eta), \left(1 - \frac{1}{1+\delta}\right) a(x^\eta)(x^\eta - y^\eta)\rangle,$$

and since a is uniformly elliptic with constant μ,

$$-\delta\theta \geq \frac{|x^\eta - y^\eta|^2}{2\eta^2}\left[-2\eta K_f - K_a|x^\eta - y^\eta| + \left(1 - \frac{1}{1+\delta}\right)\mu\right]. \tag{6.24}$$

But, by (6.13), for η sufficiently small

$$-2\eta K_f - K_a|x^\eta - y^\eta| + \left(1 - \frac{1}{1+\delta}\right)\mu > 0.$$

This implies that for η sufficiently small the right hand side of (6.24) is nonnegative, which is a contradiction. Therefore, (6.11) is false, and consequently

$$\min_{x \in \overline{G}} (1+\delta)V_*(x) - W(x) \geq 0.$$

Since this is true for all $\delta > 0$, one has $W \leq V_*$ for all $x \in \overline{G}$. □

6.2 Proofs for Section 5

6.2.1 Proof of Lemma 5.2:

Since the lower bound is obvious, only the upper bound will be considered. It is sufficient to prove that for each $\phi \in \Phi$, there exists a $w \in \mathcal{W}^0$ (depending on ϕ) such that $J(x, \phi[w], w) \leq M_3$. The control $w.$ will be constructed in a feedback fashion; the existence of an open–loop $w.$ with the same values will be clear.

Let $b \in \Re^n$ be any vector with $|b| = 1$. Let $t_n = n\Delta$ for all nonnegative integer n where the value of Δ is yet to be specified. The control $w.$ will be constant over each interval $[t_n, t_{n+1})$. Let C_f be a bound for f over $\overline{G}$, and fix a $\phi \in \Phi$. Define

$$w_t = w^0 \qquad \forall t \in [t_0, t_1)$$

where $w^0 = 2C_f\sigma^{-1}(x)b$ and x is the initial state. Let the dynamics over $[t_0, t_1)$ be given by (5.8) with controls w and $\phi[w]$. Then for $t \in [t_0, t_1]$,

$$\begin{aligned} \langle Y_t - x, b\rangle &\geq -C_f t + 2C_f t + 2C_f \textstyle\int_0^t \langle b, [\sigma(Y_r)\sigma^{-1}(x) - I]\, b\rangle\, dr \\ &\geq C_f t - 2C_f \textstyle\int_0^t \langle b, [\sigma(Y_r) - \sigma(x)]\sigma^{-1}(x) b\rangle\, dr. \end{aligned} \tag{6.25}$$

But by Condition 4.3, there exists $\overline{m}_\sigma < \infty$ such that $\|\sigma^{-1}(x)\| \leq \overline{m}_\sigma$ for all $x \in \overline{G}$. Further, since $\sigma \in C^1(\overline{G})$, it is Lipschitz on $\overline{G}$ with constant K_σ. Employing these in (6.25) yields

$$\langle Y_t - x, b\rangle \geq C_f t - 2C_f \overline{m}_\sigma K_\sigma \int_0^t |Y_r - x|\, dr.$$

Since there exists $\overline{C}$ (depending on the bounds on f, σ and σ^{-1}) such that $|Y_t - x| \leq \overline{C}t$, one has

$$\langle Y_t - x, b\rangle \geq C_f t - C_f \overline{m}_\sigma K_\sigma \overline{C} t^2$$

which implies that there exists $\Delta > 0$ (independent of x) such that

$$\langle Y_t - x, b\rangle \geq \frac{C_f}{2} t \qquad \forall t \in [0, \Delta]. \tag{6.26}$$

This is the desired choice for Δ.

Turning now to the second segment, let

$$w_t = w^1 \equiv 2C_f \sigma^{-1}(Y_{t_1})b \qquad \forall t \in [t_1, t_2).$$

Proceeding as for the first segment, one finds

$$\langle Y_t - Y_{t_1}, b\rangle \geq \frac{C_f}{2}(t - t_1) \qquad \forall t \in [t_1, t_2]. \tag{6.27}$$

Combining (6.26) and (6.27) yields

$$\langle Y_{t_2} - x, b\rangle \geq 2\frac{C_f}{2}\Delta.$$

Continuing this process, one has

$$w_t = 2C_f \sigma^{-1}(Y_{t_n})b \qquad \forall t \in [t_n, t_{n+1}]$$

and

$$\langle Y_{t_{n+1}} - x, b\rangle \geq (n+1)\frac{C_f}{2}\Delta \qquad \forall n.$$

Therefore,

$$\tau \leq \frac{2\text{diam}(\overline{G})}{C_f}, \tag{6.28}$$

and consequently

$$J(x, \phi[w], w) \leq \frac{2\text{diam}(\overline{G})}{C_f}\left[\theta + \frac{1}{2}[2C_f \overline{m}_\sigma]^2\right] \doteq M_3. \square$$

6.2.2 Proof of Theorem 5.6:

The proof follows the standard form. The equality in the dynamic programming principle is obtained by proving inequalities in both directions. Let τ_x indicate the time to escape given the initial state x.

Let

$$R(x) = \sup_{\phi \in \Phi} \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{T \wedge \tau_x} \left(\theta + \frac{1}{2}|w_t|^2\right) dt + \widetilde{W}(Y_{T \wedge \tau_x})\right]. \tag{6.29}$$

Let $\varepsilon \in (0, 1]$. Then there exists $\tilde{\phi} \in \Phi$ such that

$$R(x) \leq \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{T \wedge \tau_x} \left(\theta + \frac{1}{2}|w_t|^2\right) dt + \widetilde{W}(Y_{T \wedge \tau_x})\right] + \varepsilon \tag{6.30}$$

when $\tilde{\phi}[w]$ is used in the dynamics (5.8). Now, for any $y \in G$,

$$\widetilde{W}(y) = \sup_{\phi \in \Phi} \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{\tau_y} \left(\theta + \frac{1}{2}|w_t|^2\right) dt\right] \tag{6.31}$$

which implies there exists $\tilde{\phi}' \in \Phi$ such that

$$\widetilde{W}(y) \leq \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{\tau_y} \left(\theta + \frac{1}{2}|w_t|^2 \right) dt \right] + \varepsilon \tag{6.32}$$

when $\tilde{\phi}'[w]$ is used in the dynamics (5.8).

Define $\hat{\phi}$ as follows. For each $w \in \mathcal{W}_b^0$, let

$$\hat{\phi}[w]_t = \begin{cases} \tilde{\phi}[w]_t & \text{if } t \leq T \\ \tilde{\phi}'[w_{\cdot - T}]_{t-T} & \text{if } t > T \end{cases}$$

where the choice of $\tilde{\phi}'$ depends on Y_T. Note that $\hat{\phi} \in \Phi$. Then (6.30) and (6.32) imply

$$R(x) \leq \inf_{w^1 \in \mathcal{W}_b^0} \inf_{w^2 \in \mathcal{W}_b^0} \left[\int_0^{T \wedge \tau_x} \left(\theta + \frac{1}{2}|w_t^1|^2 \right) dt \ + I(T < \tau_x) \int_T^{\tau_{Y_T}} \left(\theta + \frac{1}{2}|w_{t-T}^2|^2 \right) dt \right] + 2\varepsilon$$

where

$$Y_t = x + \int_0^t \left[f(Y_r, \hat{\phi}[\hat{w}]_r) + \sigma(Y_r)\hat{w}_r \right] dr$$

and

$$\hat{w}_t = \begin{cases} w_t^1 & \text{if } t \leq T \\ w_t^2 & \text{if } t > T. \end{cases}$$

This implies

$$R(x) \leq \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{\tau_x} \left(\theta + \frac{1}{2}|w_t|^2 \right) dt \right] + 2\varepsilon \leq \sup_{\phi \in \Phi} \inf_{w \in \mathcal{W}_b^0} J(x, \phi[w], w) + 2\varepsilon$$

which, since $\varepsilon \in (0, 1]$ was arbitrary, implies

$$R(x) \leq \widetilde{W}(x).$$

Now the reverse inequality is proved. Given $\varepsilon > 0$, there exists $\tilde{\phi} \in \Phi$ such that

$$\widetilde{W}(x) \leq \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{\tau_x} \left(\theta + \frac{1}{2}|w_t|^2 \right) dt \right] + \varepsilon \tag{6.33}$$

when $\tilde{\phi}[w]$ is used in the dynamics. On the other hand, by (6.29)

$$R(x) \geq \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{T \wedge \tau_x} \left(\theta + \frac{1}{2}|w_t|^2 \right) dt + \widetilde{W}(Y_{T \wedge \tau_x}) \right]$$

where again $\tilde{\phi}[w]$ is being used in the dynamics. This implies there exists $\tilde{w} \in \mathcal{W}_b^0$ such that

$$R(x) \geq \int_0^{T \wedge \tau_x} \left(\theta + \frac{1}{2}|\tilde{w}_t|^2 \right) dt + \widetilde{W}(Y_{T \wedge \tau_x}) - \varepsilon. \tag{6.34}$$

From (6.31), if $T < \tau_x$ then

$$\widetilde{W}(Y_{T \wedge \tau_x}) = \widetilde{W}(Y_T) \geq \inf_{w \in \mathcal{W}_b^0} \int_0^{\tau_{Y_T}} \left(\theta + \frac{1}{2}|\hat{w}_t|^2 \right) dt$$

where the dynamics over $[T, \tau_{Y_T}]$ are governed by the controls $\widehat{\phi}$ and $\widehat{w}$,

$$\widehat{w}_t = \begin{cases} \widetilde{w}_t & \text{if } t \le T \\ w_{t-T} & \text{if } t > T \end{cases} \quad \text{and } \widehat{\phi}[w]_t = \begin{cases} \widetilde{\phi}[w]_t & \text{if } t \le T \\ \widetilde{\phi}'[w_{\cdot - T}]_{t-T} & \text{if } t > T. \end{cases}$$

(Note that $\widetilde{\phi}'$ depends on Y_T and that $\widehat{\phi} \in \Phi$.) Therefore, there exists $\widehat{w}' \in \mathcal{W}_b^0$ such that (if $T < \tau_x$)

$$\widetilde{W}(Y_{T \wedge \tau_x}) \ge \int_T^{\tau_{Y_T}} \left(\theta + \frac{1}{2} |\widehat{w}_t'|^2 \right) dt - \varepsilon \tag{6.35}$$

where the other control is given by strategy $\widehat{\phi}$. Now specifically let

$$\widehat{w}_t = \begin{cases} \widetilde{w}_t & \text{if } t \le T \\ \widehat{w}_{t-T}' & \text{if } t > T \end{cases}$$

Then (6.34) and (6.35) imply

$$R(x) \ge \int_0^{\tau_x} \left(\theta + \frac{1}{2} |\widehat{w}_t|^2 \right) dt - 2\varepsilon$$

which, by (6.33), yields $R(x) \ge \widetilde{W}(x) - 3\varepsilon$. Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

6.2.3 Proof of Theorem 5.10:

First note that continuity of $\widetilde{W}$ follows from Lemma 5.4 and constructions similar to those used in the proof of Lemma 5.2. Once it has been proved that $\widetilde{W}$ is a viscosity solution, the uniqueness will then follow from the comparison principle used in the proof of Theorem 4.5. Thus, it is sufficient to prove that $\widetilde{W}$ is a viscosity solution. This will be achieved by proving that it is both a supersolution and a subsolution.

Suppose there exists $g \in C^1(\overline{G})$ such that $\widetilde{W} - g$ has a local minimum at some $x_0 \in G$. To prove that $\widetilde{W}$ is a viscosity supersolution, it must be shown that

$$-\theta - H(x_0, g_x(x_0)) \ge 0.$$

If the last inequality is not valid, then there exists $\alpha > 0$ such that

$$-\alpha \ge -\theta - H(x_0, g_x(x_0)).$$

Then, by Lemmas 5.5 and 5.8, there exist $\phi \in \Phi$ and $\eta > 0$ such that for all $w \in \mathcal{W}_b^0$

$$\int_0^{\eta} \left\{ \theta + \frac{1}{2} |w_t|^2 + \langle [f(Y_t, \phi[w]_t) + \sigma(Y_t) w_t], g_x(Y_t) \rangle \right\} dt \ge \frac{\eta \alpha}{2}$$

where Y_t is given by (5.8) with initial condition x_0 and controls w and $\phi[w]$. This implies

$$\sup_{\phi \in \Phi} \inf_{w \in \mathcal{W}_b^0} \left[\int_0^{\eta} \left\{ \theta + \frac{1}{2} |w_t|^2 + \langle [f(Y_t, \phi[w]_t) + \sigma(Y_t) w_t], g_x(Y_t) \rangle \right\} dt \right] \ge \frac{\eta \alpha}{2}. \tag{6.36}$$

On the other hand, since $\widetilde{W} - g$ has a local minimum at x_0, Lemma 5.4 implies

$$\widetilde{W}(x_0) - g(x_0) \le \widetilde{W}(Y_t) - g(Y_t) \qquad \forall t \le \eta, \ \forall \phi \in \Phi, \ \forall w \in \mathcal{W}_b^0 \tag{6.37}$$

where η may have to be reduced in size. Also, from Theorem 5.6,

$$\widetilde{W}(x_0) = \sup_{\phi\in\Phi} \inf_{w\in\mathcal{W}_b^0} \left[\int_0^\eta \left(\theta + \frac{1}{2}|w_t|^2\right) dt + \widetilde{W}(Y_\eta)\right]. \tag{6.38}$$

Substituting (6.37) into (6.38) yields

$$\begin{aligned} 0 &\geq \sup_{\phi\in\Phi} \inf_{w\in\mathcal{W}_b^0} \left[\int_0^\eta \left(\theta + \tfrac{1}{2}|w_t|^2\right) dt + g(Y_\eta) - g(x_0)\right] \\ &= \sup_{\phi\in\Phi} \inf_{w\in\mathcal{W}_b^0} \left[\int_0^\eta \left\{\left(\theta + \tfrac{1}{2}|w_t|^2\right) + \langle [f(Y_t,\phi[w]_t) + \sigma(Y_t)w_t], g_x(Y_t)\rangle\right\} dt\right]. \end{aligned} \tag{6.39}$$

But (6.36) and (6.39) form a contradiction. Therefore $\widetilde{W}$ is a supersolution.

The analogous proof that $\widetilde{W}$ is a subsolution is as follows. Suppose there exists $g \in C^1(\overline{G})$ such that $\widetilde{W} - g$ has a local maximum at some $x_0 \in G$. To prove that $\widetilde{W}$ is a subsolution, it must be shown that

$$-\theta - H(x_0, g_x(x_0)) \leq 0.$$

If this inequality is not true then there exists $\alpha > 0$ such that

$$\alpha \leq -\theta - H(x_0, g_x(x_0)).$$

Then, by Lemmas 5.5 and 5.9, there exist a bounded $w \in \mathcal{W}^0$ and $\eta > 0$ such that for all $\phi \in \Phi$

$$\int_0^\eta \left\{\theta + \frac{1}{2}|w_t|^2 + \langle [f(Y_t,\phi[w]_t) + \sigma(Y_t)w_t], g_x(Y_t)\rangle\right\} dt \leq -\frac{\eta\alpha}{2}$$

where Y_t is given by (5.8) with initial condition x_0 and controls w and $\phi[w]$. This implies

$$\sup_{\phi\in\Phi} \inf_{w\in\mathcal{W}_*^0} \left[\int_0^\eta \left\{\theta + \frac{1}{2}|w_t|^2 + \langle [f(Y_t,\phi[w]_t) + \sigma(Y_t)w_t], g_x(Y_t)\rangle\right\} dt\right] \leq -\frac{\eta\alpha}{2} \tag{6.40}$$

where $\mathcal{W}_*^0 = \{w \in \mathcal{W}^0 : |w_t| \leq M_* \ \forall t\}$ and M_* is the bound on the function w whose existence is asserted in Lemma 5.9.

On the other hand, since $\widetilde{W} - g$ has a local maximum at x_0, Lemma 5.4 implies

$$\widetilde{W}(x_0) - g(x_0) \geq \widetilde{W}(Y_t) - g(Y_t) \qquad \forall t \leq \eta,\ \forall \phi \in \Phi,\ \forall w \in \mathcal{W}_*^0 \tag{6.41}$$

where η may have to be reduced in size. Also, from Theorem 5.7,

$$\widetilde{W}(x_0) \leq \sup_{\phi\in\Phi} \inf_{w\in\mathcal{W}_*^0} \left[\int_0^\eta \left(\theta + \frac{1}{2}|w_t|^2\right) dt + \widetilde{W}(Y_\eta)\right]. \tag{6.42}$$

Combining (6.41) and (6.42) yields

$$\begin{aligned} 0 &\leq \sup_{\phi\in\Phi} \inf_{w\in\mathcal{W}_*^0} \left[\int_0^\eta \left(\theta + \tfrac{1}{2}|w_t|^2\right) dt + g(Y_\eta) - g(x_0)\right] \\ &= \sup_{\phi\in\Phi} \inf_{w\in\mathcal{W}_*^0} \left[\int_0^\eta \left\{\left(\theta + \tfrac{1}{2}|w_t|^2\right) + \langle [f(Y_t,\phi[w]_t) + \sigma(Y_t)w_t], g_x(Y_t)\rangle\right\} dt\right]. \end{aligned} \tag{6.43}$$

But (6.40) and (6.43) form a contradiction. Therefore $\widetilde{W}$ is a subsolution. $\square$

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