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Singular limits of scalar Ginzburg-Landau equations with multiple-well potentials

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Abstract

We characterize the limiting behavior of scalar phase-field equations with infinitely many potential wells as the density of potential wells tends to infinity. An example of such a family of equations is

$$u_t^\epsilon = \Delta u^\epsilon - \frac{1}{\epsilon^{1+\alpha}} W'(\frac{u^\epsilon}{\epsilon^{1-\alpha}}),$$

where W is a periodic function. We prove that solutions of the above equation converge to solutions of the Mean Curvature PDE for a range of positive values of the parameter α, and we also determine the limiting equation when $\alpha = 0$. We show that our techniques can be modified to apply to fully nonlinear equations and to other classes of infinite-well equations. We discuss some applications to questions of interaction between wave fronts in dynamic phase transitions.

1 Introduction

In this paper we examine singular limits of phase field equations corresponding to potentials with multiple wells. In the limits we consider, we let the number of potential wells (more precisely, the density of wells) tend to infinity. This is a departure from previous work on the asymptotic behavior of scaled phase field equations, most of which has considered equations corresponding to double-well potentials.

Consider the following equation:

$$u_t^\epsilon = \Delta u^\epsilon - \frac{1}{\epsilon^{1+\alpha}} W'(\frac{u^\epsilon}{\epsilon^{1-\alpha}}) \qquad \text{on } R^n \times [0, \infty), \tag{1.1}$$

$$u^\epsilon(x, 0) = u_0(x). \tag{1.2}$$

This is the equation for L^2-gradient flow with respect to the functional

$$E^\epsilon(u) = \int \frac{1}{2}|Du(x)|^2 + W^\epsilon(u(x))dx,$$

where $W^\epsilon(x) = \epsilon^{-2\alpha}W(x/\epsilon^{1-\alpha})$. If $\alpha = 1$ and W is a symmetric double-well potential, (1.1) becomes the Allen-Cahn equation, solutions of which are known (see, for example, [8], [2]) to converge to (discontinuous) solutions of the Mean Curvature PDE

$$u_t = (\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2})u_{x_ix_j}. \tag{1.3}$$

We will instead assume $\alpha < 1$ and that W is an appropriate periodic function with one or more wells in each period, so that the potential W^ϵ oscillates rapidly as ϵ becomes small.

One expects solutions of (1.1) to have a terraced structure, so that $|Du^\epsilon|$ is very small where $u^\epsilon/\epsilon^{1-\alpha}$ is approximately equal to one of the minima of the potential W, and very large otherwise. The scaling is written in such a way that the transitions between adjacent stable phases have width of order $\sim \epsilon$, whereas adjacent transitions are separated by distances $\sim \epsilon^{1-\alpha}$. Thus adjacent transitions are widely separated when $\alpha > 0$. The fact that this does not hold when $\alpha = 0$ makes this case more difficult.

The bulk of this paper is given over to a detailed analysis of the asymptotic behavior of the equation (1.1) for a range of values of the parameter α. We obtain the following result, under certain loose assumptions on the initial data and the potential W.

Theorem 1.1 *Then there exists $\alpha_0 \in (0,1]$ such that if $\alpha \in [0, \alpha_0)$, the sequence of functions u^ϵ converges locally uniformly to a unique limit u. If $\alpha \in (0, \alpha_0)$ then u solves the mean curvature PDE; if $\alpha = 0$, then u solves the equation*

$$u_t = (\delta_{ij} - \theta(|Du|^2)\frac{u_{x_i}u_{x_j}}{|Du|^2})u_{x_ix_j}, \tag{1.4}$$

where $\theta(0) = 1$, and $0 < \theta(z) < 1$ if $z > 0$.

The result for $0 < \alpha < \alpha_0$ differs from earlier results on singular limits of the Allen-Cahn equation in that, whereas in the two-well case, the limiting function takes on only two values a.e. (corresponding to the minima of the potential wells), in the infinite well limit we can obtain solutions of the mean curvature PDE with Lipschitz initial data. Also, we do not require the initial data to have any special form and can take it independent of ϵ. Typically, $E^\epsilon(u_0)$ blows up as ϵ tends to zero.

An explicit expression for the function θ, which occurs in the limiting equation when $\alpha = 0$, is given in (2.6). In this case, our result does not correspond to anything in two-well theory.

This undertaking is motivated in part by the work of Rubinstein, Sternberg, and Keller [20], which examines interactions between different phase boundaries in the case of three-well potentials modelling materials with both surface tension effects across the interface and differences in bulk energy between phases. They consider the equation

$$u_t^\epsilon = \Delta u^\epsilon - \frac{1}{\epsilon^2}(W_0'(u^\epsilon) + \epsilon W_1'(u^\epsilon)),$$

where W_0 is a three-well potential and W_1 is a smooth function. In particular, they assume W_0 is nonnegative with zeros at exactly three points $a < b < c$. As ϵ shrinks to zero, one expects to see an interface Γ^{ab} separating regions in which $u^\epsilon \sim a$ and $u^\epsilon \sim b$, and likewise an interface Γ^{bc}. In this situation, if Γ^{ab} and Γ^{bc} are sufficiently far apart, each interface should evolve as it would in the two-well problem, so that as ϵ approaches 0, the velocity V^{ab} of Γ^{ab} should be given by

$$V^{ab} = H + [W_1]_a^b/\beta_a^b, \qquad \text{where}$$

$$[W_1]_{z_1}^{z_2} = W_1(z_2) - W_1(z_1), \qquad \beta_{z_1}^{z_2} = \int_{z_1}^{z_2} \sqrt{2W_0(u)}du,$$

Here H denotes the mean curvature of the interface. A similar formula would hold for Γ^{bc}. In the two-well case, formal computations are given in [19] and a full proof of this and more general results in [2].

Rubinstein et al. in [20] address the more subtle question of what happens when the interfaces are close together, so that they interact with each other. They show that under certain circumstances, stable two-stepped fronts can exist, and they present formal computations which characterize the motion of these composite fronts. Their calculations suggest that, as $\epsilon \to 0$, a two-stepped front Γ^{ac} should move according to the rule

$$V^{ac} = H + ([W_1]_a^b + [W_1]_b^c)/(\beta_a^b + \beta_b^c) = H + [W_1]_a^c/\beta_a^c. \tag{1.5}$$

The first equality above characterizes the motion of the two-stepped front in terms of the two single-stepped fronts out of which it is formed.

In a much earlier paper, Fife and MacLeod [10] looked at similar questions in a 1-dimensional setting, and with different scaling. They show the existence of multi-stepped travelling front solutions to certain reaction-diffusion equations from population biology, and they also prove that, under weak assumptions on the initial data, general solutions converge to these travelling fronts solutions. The velocity of the composite fronts is given by essentially the same formula as above (although of course the mean curvature is zero in the 1-dimensional case.) Similar problems were addressed at about the same time by Aronson and Weinberger in [1].

In the context of rapidly oscillating potentials, we can consider the same question from a somewhat different perspective. We can look at the equation

$$u_t^\epsilon = \Delta u^\epsilon - \frac{1}{\epsilon^{1+\alpha}}(W_0'(\frac{u^\epsilon}{\epsilon^{1-\alpha}}) + \epsilon W_1'(\frac{u^\epsilon}{\epsilon^{1-\alpha}})), \tag{1.6}$$

where W_0 and W_1 are appropriate periodic extensions of the functions above. The techniques presented in this paper could be used to show that for appropriate $\alpha > 0$ the solutions u^ϵ converge to solutions of the PDE

$$u_t = (\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|Du|^2})u_{x_i x_j} + \frac{[W_1]_a^c}{\beta_a^c}|Du|.$$

Formally, this equation specifies that every level set moves as in (1.5); this equation can also be used to define a generalized evolution of hypersurfaces via the rule given in (1.5). We interpret the term involving $[W^1]_a^c/\beta_a^c$ as reflecting averaging of the normal velocities associated with the two different pairs of potential wells in each repetition of the period.

In this way we can use phase field equations with rapidly oscillating potentials to study averaging effects that arise from interaction between wavefronts with different kinetic properties.

We also note that a one-dimensional, hyperbolic version of (1.1) has been used by material scientists to model the propagation of dislocations along a row of atoms resting upon a rigid, oscillating substrate. For a discussion of this, see Sections 3.2 and 7.1.1.2 of Nabarro, [16].

In addition to the papers mentioned above, other work related to ours includes [14], which looks at singular limit problems with two different length scales in the context of turbulent diffusion; [3] and [18] examine different aspects of problems with multiple-well potentials.

The proof of Theorem 1.1 uses a moderately elaborate version of Evans' perturbed test function method, cf [7]. It relies on an asymptotic expansion of (1.1), as well as on explicit sub- and supersolutions of (1.1) which provide rough, qualitative bounds on limiting behavior of sequences $\{u^\epsilon\}_{\epsilon>0}$. We start in Section 2 by presenting the asymptotic expansion of (1.1). This expansion contains a number of error terms, which we estimate in Section 3. In Section 4 we construct sub- and supersolutions of (1.1) for $0 < \alpha < \alpha_0$. The construction of a supersolution for the case $\alpha = 0$ is more delicate and is given in Section 5. In Section 6 we finally prove Theorem 1.1.

In Section 7 we examine applications to questions of front interaction, as discussed above. We indicate how one could prove the above assertion on the asymptotic behavior of (1.6), and we examine a related example in greater detail.

Our results are not sharp in that, although Theorem 1.1 is almost certainly true for $\alpha_0 = 1$, for technical reasons our proof is only valid for $\alpha_0 = 1/6$. In the case $\alpha > 0$, one could give an alternate proof of Theorem 1.1 by constructing explicit families of sub- and supersolutions of (1.1) which converge to solutions of the Mean Curvature PDE as $\epsilon \to 0$. This could be done by modifying, scaling, and superposing the supersolutions for the Allen-Cahn equation constructed in [8]. Such a proof could be expected to yield a sharper estimate of α_0, and it would probably also be simpler than the argument presented here. However, our proof has the advantage that it extends easily to the case $\alpha = 0$ as well as to the slightly more exotic phase field equations discussed in Section 7.

Notation and Preliminaries

We assume about the potential $W(\cdot)$ that W is C^2, $W \geq 0$, and that

$$W(x+1) = W(x),$$

It is important for the applications in Section 7 that we allow W to have multiple wells with different structures in each repetition of the period. We therefore assume that $W(\cdot)$ has finitely many zeros in [-1/2,1/2), which we denote $z_1, \ldots, z_J$, and that

$$W''(z_i) > 0$$

for $i = 1, \ldots, J$. It is convenient to assume that $0 = z_i$ for some i, so that $W(0) = 0$.

These assumptions imply that there are numbers $-1/2 = a_0 < \ldots < a_J = 1/2$ and a constant $M > 0$ such that $a_{i-1} < z_i < a_i$ and

$$M^{-1}(x - z_i)^2 \leq W(x) \leq M(x - z_i)^2 \tag{1.7}$$

for all $x \in [a_{i-1}, a_i]$ In fact, for any z_i as above, the upper bound may be assumed to hold for all $x \in R$. In particular, we have

$$W(x) \leq Mx^2 \tag{1.8}$$

for all $x \in R$. Since W is C^2 and nonnegative, elementary calculus gives

$$\frac{W'}{\sqrt{W}} \leq C. \tag{1.9}$$

As a final consequence of the above assumptions, we note that there exists some small number μ such that

$$W''(u) \geq C^{-1} > 0 \qquad \text{whenever } |u - z_i| \leq 2\mu \pmod 1 \quad \text{for some } i = 1, \ldots, J, \tag{1.10}$$

and

$$W(u) \geq C^{-1} > 0 \qquad \text{whenever } |u - z_i| \geq \mu \pmod 1 \quad \text{for all } i = 1, \ldots, J, \tag{1.11}$$

We will frequently appeal to the theory of viscosity solutions of nonlinear PDE. In particular, when we refer to solutions of the Mean Curvature PDE (1.3) or equation (1.4), we always mean solutions in the viscosity sense. Uniqueness and existence theorems for viscosity solutions of wide classes of degenerate, discontinuous parabolic equations, including these examples, have been established in [5] and [9]. We will use the following comparison theorem, which is proven under considerably weaker hypotheses and in much greater generality in [11].

Theorem 1.2 *Suppose that u is an u.s.c subsolution and v a l.s.c supersolution of (1.3) or (1.4) on $R^n \times [0,\infty)$, and that $u(x,0) \le v(x,0)$ for all $x \in R^n$. Suppose also that*

$$\sup_{t\ge 0} ||Du(\cdot,t)||_{L^\infty}, \quad \sup_{t\ge 0} ||Dv(\cdot,t)||_{L^\infty} < +\infty.$$

Then

$$u \le v$$

in $R^n \times [0,\infty)$.

For a comprehensive introduction to the theory of viscosity solutions, see the User's Guide of Crandall, Ishii, and Lions [6].

We employ the usual analysts' convention of letting C denote a large constant which may change from line to line but which could in principle be computed. Similarly, C^{-1} will denote a small constant. When we want to be more careful about constants, we use other letters such as K and M.

We will work throughout this paper with auxilliary functions of two variables, ξ and λ. When $f(\xi,\lambda)$ is such a function, the notation

$$\int_a^b f d\sigma$$

will always mean

$$\int_a^b f(\sigma,\lambda)d\sigma.$$

Thus an integral of the above form is always a function of the variable λ.

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While working on this paper I discussed these and related issues with Craig Evans and with Mete Soner. These discussions were very helpful. The fact that equation (1.1) is related to certain models of dislocations in material science is something I learned from David Kinderlehrer, who also gave me some relevant references.

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2 Asymptotic Expansion

We start by introducing auxilliary functions g and h. Both of these are functions of two dummy variables, which we call $\xi \in R$ and $\lambda \in (0,\infty)$. They are defined by families of ODEs in the variable ξ, in which the variable λ appears as a parameter. Thus, for every $\lambda > 0$ we stipulate that $g(\cdot,\lambda)$ satisfies

$$W'(g) - \lambda g_{\xi\xi} = 0, \tag{2.1}$$

$$g(0,\lambda) = 0, \tag{2.2}$$

$$g(\xi+1,\lambda) = g(\xi,\lambda) + 1. \tag{2.3}$$

The auxilliary function h is given by the family of ODEs

$$W''(g)h - \lambda h_{\xi\xi} = 4g_{\xi\lambda} + \frac{\theta(\lambda)}{\lambda} g_\xi \tag{2.4}$$

$$h(\xi+1,\lambda) = h(\xi,\lambda), \qquad h(0,\lambda) = 0. \tag{2.5}$$

Here $\theta(\lambda)$ is defined by the condition

$$\theta(\lambda) = \frac{4\lambda \int_0^1 g_{\xi\lambda} g_\xi d\sigma}{\int_0^1 g_\xi^2 d\sigma} \tag{2.6}$$

if $\lambda > 0$. This is just a solvability condition for the ODE defining h. We also define $\theta(0) = 1$; we will see in Lemma 2.3 that this choice makes θ continuous at $\lambda = 0$.

We will later need some estimates of the derivatives of these functions, and in the course of obtaining these estimates we integrate the ODEs, obtaining fairly explicit formulas for g and h. Thus existence of solutions of these equations is not a problem.

The reason for defining these functions as above is clear from the proof of the following

Lemma 2.1 *Suppose $\phi : Q \to R$ is smooth. On the set where $D\phi \neq 0$, define*

$$v^\epsilon = \epsilon^{1-\alpha} g(\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2) + \epsilon^{2+2\alpha}\phi_{x_i}\phi_{x_j}\phi_{x_i x_j} h(\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2).$$

Then v^ϵ is smooth where it is defined, and on this set we have

$$v_t^\epsilon - \Delta v^\epsilon + \frac{1}{\epsilon^{1+\alpha}} W(\frac{v^\epsilon}{\epsilon^{1-\alpha}}) =$$
$$g_\xi(\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2)\left[\phi_t - (\delta_{ij} - \theta(\epsilon^{2\alpha}|D\phi|^2)\frac{\phi_{x_i}\phi_{x_j}}{|D\phi^2|})\phi_{x_i x_j}\right] + E$$

where the error term E has the form

$$
\begin{aligned}
E &= O(\epsilon^{1+\alpha}g_\lambda + \epsilon^{1+3\alpha}g_{\lambda\lambda} + \epsilon^{1+3\alpha}h_\xi + \epsilon^{1+5\alpha}h_{\xi\lambda} \\
&\quad + \epsilon^{2+4\alpha}h_\lambda + \epsilon^{2+6\alpha}h_{\lambda\lambda} + \epsilon^{2+2\alpha}h + \epsilon^{1+5\alpha}h^2).
\end{aligned}
$$

Moreover, E is uniform on sets on which ϕ and its derivatives up to fourth order are uniformly bounded.

Remark. The credulous reader can see this as constituting a formal proof of Theorem 1.1. Indeed, if v^ϵ has the given form and is a solution of (1.1), then v^ϵ converges uniformly to ϕ (if h is well-behaved) and, at least formally, ϕ satisfies

$$
\phi_t - (\delta_{ij} - \lim_{\epsilon\to 0}\theta(\epsilon^{2\alpha}|D\phi|^2)\frac{\phi_{x_i}\phi_{x_j}}{|D\phi^2|})\phi_{x_ix_j} = 0.
$$

We will see that the limit in the above equation equals 1 if $\alpha > 0$.

Moreover, one can check that $\lim_{\lambda\to\infty}\theta(\lambda) = 0$. For this reason, we expect that the limiting equation in the case $\alpha < 0$ is the heat equation. It is not clear whether the methods developed in this paper can be modified to establish this result, since we rely in a couple of places on the degeneracy of the limiting equation when $\alpha \geq 0$.

Proof. Let $N(\phi) = \phi_{x_i}\phi_{x_j}\phi_{x_ix_j}$. It is clear from the definitions of g and h that they are smooth as long as $\lambda > 0$. The definition of v^ϵ and the chain rule thus give us

$$
\begin{aligned}
&v_t^\epsilon - \Delta v^\epsilon + \frac{1}{\epsilon^{1+\alpha}}W'(\frac{v^\epsilon}{\epsilon^{1-\alpha}}) = \\
&\frac{1}{\epsilon^{1+\alpha}}\left[W'(g + \epsilon^{1+3\alpha}N(\phi)h) - \epsilon^{2\alpha}|D\phi|^2 g_{\xi\xi}\right] \\
&\quad + \left[g_\xi(\phi_t + \Delta\phi) - 4\epsilon^{2\alpha}N(\phi)g_{\xi\lambda} - \epsilon^{4\alpha}N(\phi)|D\phi|^2 h_{\xi\xi}\right] + E1,
\end{aligned}
$$

where the error term $E1$ has the form

$$
\begin{aligned}
E1 &= O(\epsilon^{1+\alpha}g_\lambda + \epsilon^{1+3\alpha}g_{\lambda\lambda} + \epsilon^{1+3\alpha}h_\xi + \epsilon^{1+5\alpha}h_{\xi\lambda} \\
&\quad + \epsilon^{2+4\alpha}h_\lambda + \epsilon^{2+6\alpha}h_{\lambda\lambda} + \epsilon^{2+2\alpha}h).
\end{aligned}
$$

It is clear that the terms in $E1$ depend on derivatives of ϕ of no higher than fourth order, so that the error is uniform on sets on which these derivatives are uniformly bounded. We now simplify the leading-order term using Taylor's theorem and the ODE (2.1) evaluated at the point $(\xi^\epsilon, \lambda^\epsilon) \equiv (\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2)$.

$$
\begin{aligned}
&W'(g + \epsilon^{1+3\alpha}N(\phi)h) - \epsilon^{2\alpha}|D\phi|^2 g_{\xi\xi} \\
&\quad = W'(g) - \epsilon^{2\alpha}|D\phi|^2 g_{\xi\xi} + \epsilon^{1+3\alpha}W''(g)N(\phi)h + O(\epsilon^{2+6\alpha})h^2 \\
&\quad = \epsilon^{1+3\alpha}W''(g)N(\phi)h + O(\epsilon^{2+6\alpha})h^2.
\end{aligned}
$$

Substituting this into the above expression now yields

$$v_t^\epsilon - \Delta v^\epsilon + \frac{1}{\epsilon^{1+\alpha}} W'(\frac{v^\epsilon}{\epsilon^{1-\alpha}}) =$$
$$g_\xi(\phi_t + \Delta\phi) + \epsilon^{2\alpha} N(\phi)(W''(g)h - \epsilon^{2\alpha}|D\phi|^2 h_{\xi\xi} - 4g_{\xi\lambda}) + E.$$

The lemma now follows from evaluating the ODE (2.4) at the point $(\xi^\epsilon, \lambda^\epsilon)$ and substituting into the above identity. □

In order to make use of Lemma 2.1 we need to know how the function $\theta(\lambda)$ behaves as $\lambda \to 0$ for the case $\alpha > 0$. We will study the asymptotic behavior of θ in the remainder of this section.

We also need to estimate the error term E. We defer this task to the next section.

We start our computations by noting that (2.1) can be integrated to give

$$g_\xi = \lambda^{-1/2}(2W(g) + c(\lambda))^{1/2}, \tag{2.7}$$

where the constant $c(\lambda)$ is chosen to satisfy condition (2.3). Integrating once more, we see that

$$\xi_2 - \xi_1 = \int_{g(\xi_1,\lambda)}^{g(\xi_2,\lambda)} \frac{\lambda^{1/2}}{(2W(u) + c(\lambda))^{1/2}} du \tag{2.8}$$

In particular, (2.3) is equivalent to

$$1 = \int_0^1 \frac{\lambda^{1/2}}{(2W(u) + c(\lambda))^{1/2}} du. \tag{2.9}$$

Differentiating (2.7) with respect to λ yields

$$g_{\lambda\xi} = -\frac{g_\xi}{2\lambda} + \frac{1}{2\lambda g_\xi} c'(\lambda) + \frac{g_{\xi\xi} g_\lambda}{g_\xi}. \tag{2.10}$$

We rewrite (2.10) in the form

$$\left(\frac{g_\lambda}{g_\xi}\right)_\xi = -\frac{1}{2\lambda} + \frac{1}{2\lambda g_\xi^2} c'(\lambda)$$

It follows from (2.2) that $g_\lambda(0, \lambda) = 0$, so the previous equation can be integrated to find

$$\frac{g_\lambda}{g_\xi}(\xi, \lambda) = -\frac{\xi}{2\lambda} + \frac{1}{2\lambda} c'(\lambda) \int_0^\xi \frac{d\sigma}{g_\xi(\sigma, \lambda)^2} \tag{2.11}$$

Since $g(1, \lambda) \equiv 1$, we may evaluate the above equation at $\xi = 1$ to derive a formula for $c'(\lambda)$:

$$c'(\lambda) = \left(\int_0^1 \frac{d\sigma}{g_\xi(\sigma, \lambda)^2} \right)^{-1} \tag{2.12}$$

Our next lemma will be used in our analysis of the function $\theta(\cdot)$.

Lemma 2.2 *There exist positive numbers C, λ_0 such that for all $0 < \lambda < \lambda_0$ we have*

$$c(\lambda) \le C\exp(-1/C\sqrt{\lambda}),$$
$$c'(\lambda) \le C\lambda^{-3/2}c(\lambda).$$

Proof. Using (1.10) and (2.9), we have

$$\begin{aligned}\lambda^{-1/2} &= \int_0^1 \frac{du}{(W(u)+c(\lambda))^{1/2}} \\ &\le \Sigma_{i=i}^{J} \int_{a_{i-1}}^{a_i} \frac{du}{(M^{-1}(u-z_i)^2+c(\lambda))^{1/2}}.\end{aligned}$$

The above integrals can be evaluated explicitly; in fact, the ith integral equals equals

$$\sqrt{M}\sinh^{-1}(\sqrt{\frac{M}{c(\lambda)}}(a_i-z_i)) + \sqrt{M}\sinh^{-1}(\sqrt{\frac{M}{c(\lambda)}}(z_i-a_{i-1})).$$

Since $-1/2 \le a_{i-1} < z_i < a_i \le 1/2$, we have

$$\lambda^{-1/2} \le 2J\sqrt{M}\sinh^{-1}(\sqrt{\frac{M}{c(\lambda)}})$$

Thus $C(c(\lambda))^{-1/2} \ge \sinh((C\lambda)^{-1/2})$. Observe that $\sinh(x) \ge e^x/4$ if x is sufficiently large. Thus if λ is sufficiently small,

$$c(\lambda)^{-1/2} \ge C^{-1}\exp(1/C\sqrt{\lambda}).$$

This implies the first conclusion of the lemma.

To estimate $c'(\lambda)$ we start by rewriting (2.12) using (2.7) and the change of variables $u = g(\sigma, \lambda)$. (We will make this change of variables a number of times.) This gives

$$\begin{aligned}c'(\lambda)^{-1} &= \lambda^{3/2}\int_0^1 \frac{du}{(2W(u)+c(\lambda))^{3/2}} \\ &\ge \lambda^{3/2}(3c(\lambda))^{-3/2}\text{meas}\{u \in [0,1] | W(u) \le c(\lambda)\} \\ &\ge \lambda^{3/2}(3c(\lambda))^{-3/2}\text{meas}\{u \in [0,1] | Mu^2 \le c(\lambda)\} \\ &\ge C\lambda^{3/2}c(\lambda)^{-1},\end{aligned}$$

which is the stated estimate of $c'(\lambda)$. The constant M is the one from (1.8). The final inequality holds only if λ is small enough that, say, $c(\lambda) \le M/4$. □

We now prove a lemma which characterizes the function θ.

Lemma 2.3 *The definition of θ is equivalent to*

$$\theta(\lambda) = 1 - \left(\int_0^1 g_\xi^{-2}(\sigma, \lambda) d\sigma \int_0^1 g_\xi^2(\sigma, \lambda) d\sigma \right)^{-1}. \tag{2.13}$$

Moreover, $\theta(\lambda) \in (0,1)$ for all $\lambda > 0$. Finally, for every positive integer n there exists a constant $C = C(n)$ such that

$$1 - \theta(\lambda) \leq C\lambda^n.$$

In particular, θ is continuous on $[0, \infty)$.

Proof. From (2.6) we have

$$\begin{aligned}
(1-\theta(\lambda)) \int_0^1 g_\xi^2(\sigma,\lambda) d\sigma &= 4\lambda \int_0^1 g_{\xi\lambda}(\sigma\lambda) g_\xi(\sigma\lambda) d\sigma + \int_0^1 g_\xi^2(\sigma\lambda) d\sigma \\
&= 2\lambda^{1/2} \frac{\delta}{\delta\lambda} \left(\lambda^{1/2} \int_0^1 g_\xi^2(\sigma\lambda) d\sigma \right) \\
&= 2\lambda^{1/2} \frac{\delta}{\delta\lambda} \left(\int_0^1 (2W(u) + c(\lambda))^{1/2} du \right) \\
&= c'(\lambda) \int_0^1 \frac{\lambda^{1/2}}{(2W(u) + c(\lambda))^{1/2}} du \\
&= c'(\lambda),
\end{aligned}$$

where we have used (2.9). Now (2.13) follows from the above identity and (2.12)

From (2.13) we immediately deduce that $\theta < 1$ and (using Holder's inequality) that $\theta(\lambda) \geq 0$, with equality iff $g_\xi(\cdot, \lambda)$ is constant. From (2.7) it is evident that $g_\xi(\cdot, \lambda)$ cannot be constant, so $\theta > 0$.

To study the behavior of $\theta(\cdot)$ as $\lambda \to 0$, first note that

$$\int_0^1 g_\xi^2 d\sigma \geq \left(\int_0^1 g_\xi d\sigma \right)^2 = 1$$

by Jensen's inequality and (2.3). Thus, from (2.13), to finish the proof it suffices to show that given any n we can find C such that

$$c'(\lambda) = \left(\int_0^1 g_\xi^{-2} d\sigma \right)^{-1} \leq C\lambda^n$$

as $\lambda \to 0$. This follows directly from the estimate of $c'(\lambda)$ in Lemma 2.2. □

3 Controlling the error

In this section we estimate the error terms which arises in the asymptotic expansion carried out in Lemma 2.1. In our estimates we will seek to bound various derivatives of g and h in terms of g_ξ, since we ultimately need to show that $E = o(g_\xi)$.

The restrictions on the range of values of the parameter α for which Theorem 1.1 holds stem from the fact that our estimates, although rather lengthy, are quite far from sharp. Refining them would apparently require delicate arguments involving cancellation between different terms in the ODEs below. Thus the main point here may be that, since the moderately elaborate estimates we present in this section end up giving a result which is very far from optimal, we are justified in relying on quick and rough estimates. We will do this in Section 7, when further estimates are required.

We start our estimates with the following

Lemma 3.1 *For all $\beta_1, \beta_2 > 0$ such that $1 \le \beta_1 + \beta_2 \le 3$, we have*

$$\partial_\xi^{\beta_1} \partial_\lambda^{\beta_2} g = g_\xi O(\lambda^{\frac{1}{2} - \frac{\beta_1}{2} - \frac{3\beta_2}{2}})$$

as $\lambda \longrightarrow 0$, uniformly for $\xi \in R$,

Remark. 1. In fact, if $W \in C^k$ then the above formula remains valid for all $\beta_1 + \beta_2 \le k+1$. We need third derivatives of g to estimate the second derivatives of h which appear in the error term in Lemma 2.1; hence our assumption that $W \in C^2$.

Proof. 1. It is clear from its definition that $c(\lambda) > 0$, so equations (2.1), (2.7), and (1.9) imply that

$$\begin{aligned} \frac{g_{\xi\xi}}{g_\xi} &= \lambda^{-1/2} \frac{W'(g)}{(2W(g) + c(\lambda))^{1/2}} \\ &= O(\lambda^{-1/2}). \end{aligned}$$

Next, differentiating (2.1) with respect to ξ, we find that $g_{\xi\xi\xi} = \lambda^{-1} W''(g) g_\xi = O(\lambda^{-1}) g_\xi$.

2. It is obvious from (2.11) and (2.12) that $g_\lambda = O(\lambda^{-1}) g_\xi$.

From equation (2.10) and the estimates of $g_{\xi\xi}$ and g_λ above we see that

$$\begin{aligned} g_{\xi\lambda} &= g_\xi [O(\lambda^{-3/2}) + \frac{1}{2\lambda g_\xi^2} c'(\lambda)] \\ &= g_\xi O(\lambda^{-3/2})(1 + \frac{c(\lambda)}{2\lambda g_\xi^2}) && \text{by Lemma 2.2} \\ &= g_\xi O(\lambda^{-3/2})(1 + \frac{c(\lambda)}{4W(g) + 2c(\lambda)}) && \text{by (2.7)} \\ &= g_\xi O(\lambda^{-3/2}). \end{aligned}$$

Next we differentiate (2.1) to find that

$$g_{\xi\xi\lambda} = \frac{1}{\lambda}W''(g)g_\lambda - \frac{1}{\lambda}g_{\xi\xi} = O(\lambda^{-2})g_\xi \tag{3.1}$$

by our earlier estimates of g_λ and $g_{\xi\xi}$.

3. To estimate $g_{\lambda\lambda}$, first note that

$$\frac{g_{\lambda\lambda}}{g_\xi} = \frac{g_{\xi\lambda}}{g_\xi}\frac{g_\lambda}{g_\xi} + \left(\frac{g_\lambda}{g_\xi}\right)_\lambda .$$

The first term on the right-hand side is $O(\lambda^{-5/2})$ by our estimates of g_λ and $g_{\xi\lambda}$ above. Thus we only need to show that

$$(\frac{g_\xi}{g_\lambda})_\lambda = O(\lambda^{-5/2}) = (\frac{g_\lambda}{g_\xi})O(\lambda^{-3/2}).$$

as $\lambda \to 0$. From (2.11) and (2.12) we see that $\frac{g_\lambda}{g_\xi}$ has the general form

$$\frac{g_\lambda}{g_\xi} = \frac{1}{2\lambda}(\xi + \Theta_1 \times \Theta_2) \tag{3.2}$$

where for $i = 1,2$, Θ_i is of the form

$$\Theta \equiv \left(\int_a^b g_\xi^{-2} d\sigma\right)^k , \tag{3.3}$$

where k is an integer, and a and b are may depend on ξ but not on λ. Thus it suffices to show that $\Theta_\lambda = O(\lambda^{-3/2})\Theta$. Using our earlier estimate of $g_{\xi\lambda}$,

$$\begin{aligned}
|\Theta_\lambda| &= k\left(\int_a^b g_\xi^{-2} d\sigma\right)^{k-1} \left|\int_a^b -2g_\xi^{-2}\frac{g_{\xi\lambda}}{g_\xi}d\sigma\right| \\
&\leq k\left(\int_a^b g_\xi^{-2} d\sigma\right)^{k-1} \sup_\xi \left|\frac{g_{\xi\lambda}}{g_\xi}\right| \left(\int_a^b 2g_\xi^{-2} d\sigma\right) \\
&= O(\lambda^{-3/2})\Theta.
\end{aligned}$$

Therefore $g_{\lambda\lambda}$ satisfies the stated estimate.

4. We next use (2.10) and (2.11) to express $g_{\xi\lambda}$ entirely in terms of derivatives of g_ξ, g_λ, and $g_{\xi\xi}$. Our earlier estimates and a little bookkeeping then quickly allow us to deduce that $g_{\xi\lambda\lambda}/g_{\xi\lambda} = O(\lambda^{-3})$.

The estimate of $g_{\lambda\lambda\lambda}$ follows in a similar way from our earlier estimates. An easy way to see this is to start by observing that

$$\frac{g_{\lambda\lambda\lambda}}{g_\xi} = \frac{g_{\xi\lambda\lambda}}{g_\xi}\frac{g_\lambda}{g_\xi} + 2\frac{g_{\xi\lambda}}{g_\xi}\left(\frac{g_\lambda}{g_\xi}\right)_\lambda + \left(\frac{g_\lambda}{g_\xi}\right)_{\lambda\lambda} .$$

We know that the first two terms on the right-hand side are $O(\lambda^{-4})$, so it suffices to show that $\left(\frac{g_\lambda}{g_\xi}\right)_{\lambda\lambda} = O(\lambda^{-4}) = (\frac{g_\lambda}{g_\xi})O(\lambda^{-3})$. Returning to our representation of $\frac{g_\lambda}{g_x}$ in (3.2) and (3.3), we see that the desired estimate is implied by the estimates $\Theta_\lambda = O(\lambda^{-3/2})\Theta$, which we have already established, and $\Theta_{\lambda\lambda} = O(\lambda^{-3})\Theta$. This follows by differentiating Θ and using our estimates of $g_{\xi\lambda}$ and $g_{\xi\lambda\lambda}$. We omit the computations. □

We still need to estimate the auxilliary function h and its derivatives. Before we undertake this, we first seek to represent h in a form from which the estimates we want can be deduced without too much difficulty.

Lemma 3.2 *Let $\rho = h/g_\xi$. Then ρ solves the equation*

$$\rho_\xi = -\frac{2g_\lambda}{\lambda g_\xi} - \frac{c'(\lambda)}{\lambda^2 g_\xi^2}\left(\xi - \frac{\int_0^\xi g_\xi^2 d\sigma}{\int_0^1 g_\xi^2 d\sigma}\right). \tag{3.4}$$

Remark. We could integrate the above equation and multiply by g_ξ to obtain an explicit formula for h. By differentiating that formula, we could then find representations of all the derivatives we want to estimate. The procedure we actually follow in Lemma 3.3 below is equivalent to this but slightly more efficient.

Proof. First observe that

$$\begin{aligned} -\lambda(g_\xi^2\rho_\xi)_\xi &= \lambda(hg_{\xi\xi\xi} - h_{\xi\xi}g_\xi) \\ &= hW''(g)g_\xi - \lambda h_{\xi\xi}g_\xi && \text{by (2.1)} \\ &= 4g_\xi g_{\xi\lambda} + (\theta(\lambda)/\lambda)g_\xi^2 && \text{by (2.4).} \end{aligned}$$

Note that θ is defined precisely so that the integral of the right-hand side vanishes. Since we want ρ_ξ to be periodic, this means that we do not need to add a constant to the right-hand when integrating to find ρ_ξ. (More precisely, the appropriate constant of integration is 0.)

Next, we rewrite the above expression and substitute from (2.10), (2.12), and (2.13) as follows:

$$\begin{aligned} 4g_{\xi\lambda}g_\xi + (\theta(\lambda)/\lambda)g_\xi^2 &= 2g_{\xi\lambda}g_\xi + 2g_\xi(g_{\xi\lambda} + \frac{1}{2\lambda}g_\xi) + \frac{\theta - 1}{\lambda}g_\xi^2 \\ &= 2g_{\xi\lambda}g_\xi + 2(\frac{c'(\lambda)}{2\lambda} + g_{\xi\xi}g_\lambda) - \frac{c'(\lambda)}{\lambda}\left(\int_0^1 g_\xi^2 d\sigma\right)^{-1} g_\xi^2 \\ &= 2(g_\xi g_\lambda)_\xi + \frac{c'(\lambda)}{\lambda}\left(1 - \frac{g_\xi^2}{\int_0^1 g_\xi^2 d\sigma}\right). \end{aligned}$$

Putting together the last two computations we deduce that the lemma holds. □

Lemma 3.3 *For all β_1, β_2 such that $0 \le \beta_1 \le 1, 0 \le \beta_2 \le 2$, we have*

$$\partial_\xi^{\beta_1}\partial_\lambda^{\beta_2} h = g_\xi O(\lambda^{\frac{-5}{2}-\frac{\beta_1}{2}-\frac{3\beta_2}{2}})$$

as $\lambda \to 0$, uniformly for $\xi \in R$. Finally, we also have $h = O(\lambda^{-3})$ as $\lambda \to 0$.

Proof. 1. We first claim that

$$\rho_\xi = O(\lambda^{-5/2}) \qquad \text{as } \lambda \to 0. \tag{3.5}$$

Indeed, it is clear that

$$\xi - \frac{\int_0^\xi g_\xi^2 d\sigma}{\int_0^1 g_\xi^2 d\sigma} \tag{3.6}$$

is bounded uniformly in ξ and λ. Also, we have seen in Step 3 of the proof of Lemma 3.1 that $c'(\lambda)/g_\xi^2 = O(\lambda^{-1/2})$ as $\lambda \to 0$. These facts, with (3.4) and the estimate of g_λ in Lemma 3.1, imply that (3.5) holds.

2. We next claim that

$$\rho_{\xi\lambda} = O(\lambda^{-4}), \qquad \rho_{\xi\lambda\lambda} = O(\lambda^{-11/2}) \quad \text{as } \lambda \to 0. \tag{3.7}$$

These follow from differentiating (3.4) and using the estimates in Lemmas 3.1. As in Steps 3 and 4 of the proof of Lemma 3.1, it suffices to check that the right-hand side of equation (3.4) can be written as sums and products of factors F^i, each of which satisfies

$$F_\lambda^i = O(\lambda^{-3/2})F^i, \quad F_{\lambda\lambda}^i = O(\lambda^{-3})F^i, \qquad \text{as } \lambda \to 0.$$

This follows immediately from earlier estimates for $F^1 \equiv \frac{g_\lambda}{g_\xi}$ and $F^2 \equiv g_\xi^2$, and it is obvious for $F^3 \equiv \lambda^{-2}$. We have already verified that $F^4 \equiv c'(\lambda) = (\int_0^1 g_\xi^{-2} d\sigma)^{-1}$ has exactly these properties. The remaining nontrivial factors have the form

$$\tilde{\Theta} \equiv \left(\int_a^b g_\xi^2 d\sigma\right)^k$$

for some integer k. Estimating these in exactly the same way that we estimated quantities of the form Θ in Lemmas 3.1, we find factors of the form $\tilde{\Theta}$ have the required asymptotic behavior as $\lambda \to 0$. Thus (3.7) holds.

3. Since ρ is periodic, Step 1 implies that

$$\sup_\xi \rho(\xi, \lambda) = \sup_{\xi\in[0,1]} \int_0^\xi \rho_\xi d\sigma.$$

Thus

$$\rho = O(\lambda^{-5/2}) \quad \text{as } \lambda \to 0.$$

This is the estimate which is claimed for h/g_ξ in the statement of the lemma. Since $g_\xi = O(\lambda^{-1/2})$ as $\lambda \to 0$, this implies the final estimate of h. Similarly, by integrating our estimates for $\rho_{\xi\lambda}$ and $\rho_{\xi\lambda\lambda}$ we find that

$$\rho_\lambda = O(\lambda^{-4}) \quad \text{as } \lambda \to 0.$$

$$\rho_{\lambda\lambda} = O(\lambda^{-11/2}) \quad \text{as } \lambda \to 0.$$

4. In all of the following calculations, we differentiate both sides of the identity $h = \rho g_\xi$ and divide the result by g_ξ, then apply estimates from Lemma 3.1 and Steps 1–3 above.

$$\frac{h_\xi}{g_\xi} = \frac{g_{\xi\xi}}{g_\xi}\rho + \rho_\xi = O(\lambda^{-3}) \qquad \text{as } \lambda \to 0,$$

$$\frac{h_\lambda}{g_\xi} = \rho_\lambda + \frac{g_{\xi\lambda}}{g_\xi}\rho = O(\lambda^{-4}) \qquad \text{as } \lambda \to 0,$$

$$\frac{h_{\xi\lambda}}{g_\xi} = \rho_{\xi\lambda} + \frac{g_{\xi\lambda}}{g_\xi}\rho_\xi + \frac{g_{\xi\xi}}{g_\xi}\rho_\lambda + \frac{g_{\xi\xi\lambda}}{g_\xi}\rho = O(\lambda^{-9/2}) \qquad \text{as } \lambda \to 0,$$

$$\frac{h_{\lambda\lambda}}{g_\xi} = \rho_{\lambda\lambda} + 2\frac{g_{\xi\lambda}}{g_\xi}\rho_\lambda + \frac{g_{\xi\lambda\lambda}}{g_\xi}\rho = O(\lambda^{-11/2}) \qquad \text{as } \lambda \to 0.$$

Thus we conclude the proof of this lemma. □

Now we are finally able to deduce the following

Proposition 3.1 *Let ϕ be as in Lemma 2.1, and suppose that all derivatives of ϕ of fourth order or less are uniformly bounded. If $0 < \alpha < 1/6$ then the error term E in Lemma 2.1 satisfies*

$$E = o(g_\xi)$$

as $\epsilon \to 0$, uniformly on any set of the form

$$S_\delta \equiv \{(x,t) | \delta < |D\phi(x,t)|\}$$

for any $\delta > 0$.

Proof. From Lemma 2.1 we have

$$\begin{aligned}\frac{E}{g_\xi} &= O(\epsilon^{1+\alpha}\frac{g_\lambda}{g_\xi} + \epsilon^{1+3\alpha}\frac{g_{\lambda\lambda}}{g_\xi} + \epsilon^{1+3\alpha}\frac{h_\xi}{g_\xi} + \epsilon^{1+5\alpha}\frac{h_{\xi\lambda}}{g_\xi} \\ &\quad + \epsilon^{2+4\alpha}\frac{h_\lambda}{g_\xi} + \epsilon^{2+6\alpha}\frac{h_{\lambda\lambda}}{g_\xi} + \epsilon^{2+2\alpha}\frac{h}{g_\xi} + \epsilon^{1+5\alpha}\frac{h^2}{g_\xi}),\end{aligned}$$

First suppose that $\alpha > 0$. The functions $g_\lambda, g_{\lambda\lambda}$, etc. are evaluated at the point $(\xi^\epsilon, \lambda^\epsilon) \equiv (\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi^2|)$, so λ^ϵ tends to zero with ϵ and the asypmtotic behavior of the functions appearing above is given by Lemmas 3.1 and 3.3. Also, on the set S_δ it is clear that λ^ϵ is bounded above and below by $\epsilon^{2\alpha}$. This fact and the estimates yield

$$\frac{E}{g_\xi} = O(\epsilon^{1-\alpha} + \epsilon^{1-2\alpha} + \epsilon^{1-3\alpha} + \epsilon^{1-4\alpha} + \epsilon^{2-4\alpha} + \epsilon^{2-5\alpha} + \epsilon^{2-3\alpha} + \epsilon^{1-6\alpha}).$$

Thus the conclusion of the lemma holds for $0 < \alpha < 1/6$. If $\alpha = 0$ then all the terms involving the auxilliary functions are $O(1)$ as $\epsilon \longrightarrow 0$ and the conclusion of the lemma is obvious. □

Finally, note that

Lemma 3.4 *Suppose ϕ is smooth and define v^ϵ as in the statement of Lemma 2.1 and the set S_δ as above. Assume also that $\alpha < 1/6$. Then $v^\epsilon \longrightarrow \phi$ uniformly on S_δ for every $\delta > 0$.*

Proof. Since $g_\xi > 0$ and $g(k, \lambda) = g(k, \lambda) + 1$ for every integer k and every $\lambda > 0$, it is clear that $|g(\xi, \lambda) - \xi| < 1$. Thus the conclusion of the lemma follows immediately from the definition of v^ϵ and the fact that $h = o(\epsilon^{2+2\alpha})$ uniformly on S_δ as $\epsilon \longrightarrow 0$. □

4 Supersolutions: $0 < \alpha < \alpha_0$

In this section we construct explicit supersolutions of (1.1) in the case $\alpha > 0$. We will not discuss subsolutions other than to note that they could be constructed in exactly the same manner.

The construction will yield a family of supersolutions converging as $\epsilon \longrightarrow 0$ to a function $\phi \vee B$, where ϕ is a supersolution of the mean curvature PDE and B is an appropriately chosen constant. In building these functions, we start with the function ϕ and make extensive modifications, guided by the asymptotic expansion in Lemma 2.1. Because the construction is quite complicated, we will present it first for a specific choice of ϕ and B. We indicate afterwards how the construction can be modified to achieve slightly greater generality.

Let

$$\phi(x,t) = \sqrt{|x|^2 + 2(n-1)t} - 1 + t.$$

Fix $T > 0$ so that $\phi(0,t) < 0$ for $t \in [0,T]$. For the remainder of this construction we will confine our attention to the set $Q_T \equiv R^n \times [0,T]$. (Thus the supersolutions we construct will be local in time.) Having restricted our attention to this time interval, we may define $\rho(t) > 0$ so that $\rho^2(t) + 2(n-1)t = (1-t)^2$ for $t \in [0,T]$. We have chosen ρ so that

$$\phi(x,t) \geq 0 \qquad \text{iff } |x| \geq \rho(t).$$

Note that ρ is smooth and decreasing.

We then have the following

Lemma 4.1 *ϕ satisfies the following on Q_T:*

$$\phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} = 1,$$

$$C^{-1} \leq |D\phi| \leq C \qquad \textit{on } \{(x,t) \in Q_T | \phi(x,t) \geq 0\}.$$

Proof. These follow directly from the definition of ϕ. In fact, one easily verifies that $\sqrt{|x|^2 + 2(n-1)t}$ is an exact solution of the Mean Curvature PDE, so

$$\phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} = 1.$$

Similarly, we have

$$|D\phi(x,t)| = \frac{|x|}{\sqrt{|x|^2 + 2(n-1)t}}.$$

This is clearly bounded. Moreover, if $\phi(x,t) \geq 0$ then $|x| \geq \rho(t) \geq \rho(T)$, from which we easily deduce that $|D\phi| \geq C^{-1}$. □

Following Lemma 2.1, we next define

$$v^\epsilon = \epsilon^{1-\alpha} g(\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2) + \epsilon^{2+2\alpha}\phi_{x_i}\phi_{x_j}\phi_{x_ix_j} h(\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2).$$

We would like to use Lemma 2.1 and the error estimates to conclude that v^ϵ is a supersolution of the PDE. However, this is not yet true, so we have to modify v^ϵ is several ways.

First, the asymptotic expansion in Lemma 2.1 breaks down at points where $D\phi = 0$, so we need to modify v^ϵ near the origin. We do this by using a cutoff function to replace v^ϵ by a constant in a neighborhood of the origin, so that we only need to rely on the expansion on a set in which $D\phi$ is bounded away from zero.

We now define this cutoff function: Let $\zeta^\epsilon(x,t) = \eta^\epsilon(|x| - \rho(t))$, where η^ϵ is a smooth function on $[0,\infty)$ such that

$$\eta^\epsilon(z) = 0 \qquad \text{if } z \le 0,$$

$$\eta^\epsilon(z) = 1 \qquad \text{if } z \ge \epsilon,$$

$$\eta^{\epsilon\prime} \le C/\epsilon, \qquad \eta^{\epsilon\prime\prime} \le C/\epsilon^2.$$

Finally, we define

$$V^\epsilon = \zeta^\epsilon v^\epsilon + \epsilon^2.$$

The role of the ϵ^2 is to absorb error that is introduced by our use of the cutoff function ζ^ϵ.

We will need several estimates of v^ϵ on the support of $D\zeta^\epsilon$. The crucial fact is that Dv^ϵ is extremely small, so that v^ϵ can be melded smoothly to a constant function.

Lemma 4.2 *On* $suppD\zeta^\epsilon$ *we have*

$$g_\xi(\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2) = O(\epsilon^{-\alpha}\exp(-\frac{1}{C\epsilon^\alpha})),$$

$$v^\epsilon = o(\exp(-\frac{1}{C\epsilon^\alpha})),$$

$$|Dv^\epsilon| = O(\epsilon^{-\alpha}\exp(-\frac{1}{C\epsilon^\alpha})).$$

Proof. 1. Let $(\xi^\epsilon, \lambda^\epsilon) = (\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2)$ as usual. We first claim that

$$\xi^\epsilon \in [0, C\sqrt{\lambda^\epsilon}] \tag{4.1}$$

on $\text{supp}D\zeta^\epsilon$. Indeed, it is clear from Lemma 4.1 that on this set we have $C^{-1}\epsilon^{2\alpha} \le \lambda^\epsilon \le C\epsilon^{2\alpha}$, so we only need to show that $0 \le \xi^\epsilon \le C\epsilon^\alpha$. We know from the definition of ζ^ϵ that $\rho(t) \leq |x| \leq \rho(t) + \epsilon$ whenever $(x,t) \in \mathrm{supp} D\zeta^\epsilon$. Writing $\phi(x,t) = \overline{\phi}(|x|,t)$, it is clear that $\overline{\phi}(\cdot,t)$ is an increasing function, so

$$\begin{aligned} 0 &= \overline{\phi}(\rho(t),t) \\ &\leq \phi(x,t) \\ &\leq \overline{\phi}(\rho(t)+\epsilon,t) \\ &\leq C\epsilon \end{aligned}$$

since ϕ is smooth. This implies that $0 \leq \xi^\epsilon \leq \epsilon^\alpha$ as claimed.

2. We next claim that (4.1) implies that $0 \leq g(\xi^\epsilon,\lambda^\epsilon) \leq C\sqrt{c(\lambda^\epsilon)}$ on $\mathrm{supp} D\zeta^\epsilon$. We know that $g(0,\lambda^\epsilon) = 0$ and that $g(\cdot,\lambda^\epsilon)$ is increasing, so we only need to estimate $g(C\sqrt{\lambda^\epsilon},\lambda^\epsilon)$. For this we have, by (2.8) and (1.8),

$$\begin{aligned} \xi^\epsilon &= \int_0^{g(\xi^\epsilon,\lambda^\epsilon)} \left(\frac{\lambda^\epsilon}{2W(u)+c(\lambda^\epsilon)}\right)^{1/2} du \\ &\geq \int_0^{g(\xi^\epsilon,\lambda^\epsilon)} \left(\frac{\lambda^\epsilon}{2Mu^2+c(\lambda^\epsilon)}\right)^{1/2} du \\ &= \sqrt{\frac{\lambda^\epsilon}{2M}} \sinh^{-1}\left(\sqrt{\frac{2M}{c(\lambda^\epsilon)}} g(\xi^\epsilon,\lambda^\epsilon)\right). \end{aligned}$$

Thus

$$\sinh^{-1}\left(\sqrt{\frac{C(g(\xi^\epsilon,\lambda^\epsilon)}{c(\lambda^\epsilon)}}\right) \leq \frac{C\xi^\epsilon}{\sqrt{\lambda^\epsilon}} \leq C$$

on $\mathrm{supp} D\zeta^\epsilon$, by (4.1). Hence the argument of $\sinh^{-1}$ is bounded, which was the claim we set out to prove.

3. The ODE (2.7) now implies that for $(\xi^\epsilon,\lambda^\epsilon)$ corresponding to points $(x,t) \in \mathrm{supp} D\zeta^\epsilon$, we have

$$\begin{aligned} g_\xi(\xi^\epsilon,\lambda^\epsilon) &= \left(\frac{2W(g(\xi^\epsilon,\lambda^\epsilon))+c(\lambda^\epsilon)}{\lambda^\epsilon}\right)^{1/2} \\ &\leq \left(\frac{2Mg(\xi^\epsilon,\lambda^\epsilon)^2+c(\lambda^\epsilon)}{\lambda^\epsilon}\right)^{1/2} \\ &\leq C\left(\frac{c(\lambda^\epsilon)}{\lambda^\epsilon}\right)^{1/2}, \end{aligned}$$

by Step 2 and (1.8). This inequality, together with Lemma 2.2 and the fact that λ^ϵ is bounded above and below by $\epsilon^{2\alpha}$, yields the stated estimate of g_ξ.

4. It follows from the above and the estimate of h in Lemma 3.3 that

$$\begin{aligned} \epsilon^{2+2\alpha}\phi_{x_i}\phi_{x_j}\phi_{x_ix_j}h(\xi^\epsilon,\lambda^\epsilon) &\leq C\epsilon^{2+2\alpha}(\lambda^\epsilon)^{-5/2}g_\xi \\ &\leq C\epsilon^{2-4\alpha}\sqrt{c(\lambda^\epsilon)} \end{aligned}$$

on supp$D\zeta^\epsilon$. Since $v^\epsilon = \epsilon^{1-\alpha} g(\xi^\epsilon, \lambda^\epsilon) + \epsilon^{2+2\alpha}\phi_{x_i}\phi_{x_j}\phi_{x_i x_j} h$, this fact and Step 2 imply that $v^\epsilon = o(\sqrt{c(\lambda^\epsilon)})$. Now the stated estimate of v^ϵ follows from Lemma 2.2.

Finally, we have

$$Dv^\epsilon = g_\xi D\phi + O(g_\lambda \epsilon^{1+\alpha} + h\epsilon^{2+2\alpha} + h_\xi \epsilon^{1+3\alpha} + h_\lambda \epsilon^{2+4\alpha}).$$

Using Lemmas 3.1 and 3.3 and, again, the fact that λ^ϵ is bounded above and below by $\epsilon^{2\alpha}$ on supp$D\zeta^\epsilon$, this becomes

$$Dv^\epsilon = g_\xi[D\phi + O(\epsilon^{1+\alpha} + \epsilon^{2-3\alpha} + \epsilon^{1-3\alpha} + \epsilon^{2-3\alpha})] = O(g_\xi)$$

as $\epsilon \to 0$. Using Step 3, we find that the proof of the lemma is finished. □

The proof of the following proposition is now quite straightforward.

Proposition 4.1 *V^ϵ is a supersolution of (1.1) on Q_T for all sufficiently small ϵ.*

Proof. 1. We begin by looking at the case where $\zeta^\epsilon = 1$, so that $V^\epsilon = v^\epsilon + \epsilon^2$. By Lemma 4.1, $|D\phi| > C^{-1}$ on this region, so we may use the asymptotic expansion in Lemma 2.1 and the error will be negligible, by Proposition 3.1. We thus have

$$\begin{aligned} V_t^\epsilon - \Delta V^\epsilon + \frac{1}{\epsilon^{1+\alpha}} W'(\frac{V^\epsilon}{\epsilon^{1-\alpha}}) & \\ = \; & g_\xi(C^{-1} + o(1)) + \frac{1}{\epsilon^{1+\alpha}}\left(W'(\frac{V^\epsilon}{\epsilon^{1-\alpha}}) - W'(\frac{v^\epsilon}{\epsilon^{1-\alpha}})\right) \\ = \; & g_\xi(C^{-1} + o(1)) + W''(\frac{v^\epsilon}{\epsilon^{1-\alpha}}) + O(\epsilon^{1+\alpha}). \end{aligned} \tag{4.2}$$

We consider two subcases. The notation comes from (1.10) and (1.11).

Subcase 1: $|g - z_i| \le \mu \pmod 1$ for some $i = 1, \ldots, J$.

Estimates of the auxilliary function h in Lemma 3.3 imply that $v^\epsilon/\epsilon^{1-\alpha} = g + o(1)$ uniformly, since $D\phi$ is bounded away from zero. In particular, if ϵ is sufficiently small we have $|v^\epsilon/\epsilon^{1-\alpha} - z_i| \le 2\mu$, in which case (1.10) assures us that $W''(v^\epsilon/\epsilon^{1-\alpha}) \ge C^{-1} > 0$. Together with (4.2), this implies the conclusion of the lemma.

Subcase 2: $|g - z_i| \ge \mu \pmod 1$ for all $i = 1, \ldots, J$.

In this case (1.11) tells us that $W(g) \ge C^{-1} > 0$, and so

$$g_\xi = \left(\frac{2W(g) + c(\lambda^\epsilon)}{\lambda^\epsilon}\right)^{1/2} \ge \frac{1}{C\sqrt{\lambda^\epsilon}} = C^{-1}\epsilon^{-\alpha}.$$

This estimate and (4.2) imply that V^ϵ is a supersolution in this case.

2. On the region where $\zeta^\epsilon = 0$, it is evident that $V^\epsilon = \epsilon^2$, and so

$$V_t^\epsilon - \Delta V^\epsilon + \frac{1}{\epsilon^{1+\alpha}} W'(\frac{V^\epsilon}{\epsilon^{1-\alpha}}) = \frac{1}{\epsilon^{1+\alpha}} W'(\epsilon^{1+\alpha}).$$

Because $W''(0) > 0$, this number is positive if ϵ is small.

3. Lemma 4.2 implies that on $\text{supp} D\zeta^\epsilon$, $V^\epsilon = \epsilon^2 + o(\epsilon^2)$, so we see as in Step 2 that

$$\frac{1}{\epsilon^{1+\alpha}} W'(\frac{V^\epsilon}{\epsilon^{1-\alpha}}) > C^{-1}$$

for small ϵ. Thus we only need to show that $V_t^\epsilon - \Delta V^\epsilon = o(1)$ as $\epsilon \to 0$. We have

$$V_t^\epsilon - \Delta V^\epsilon = (\zeta_t^\epsilon - \Delta\zeta^\epsilon) v^\epsilon - 2D\zeta^\epsilon \cdot Dv^\epsilon + \zeta^\epsilon(v_t^\epsilon - \Delta v^\epsilon).$$

It is obvious from the estimates in Lemma 4.2 that the first two terms on the right-hand side vanish as $\epsilon \to 0$. As for the final term, using Lemma 2.1 and the error estimates as in Step 1, we compute that

$$v_t^\epsilon - \Delta v^\epsilon = g_\xi(C^{-1} + o(1)) - \frac{1}{\epsilon^{1+\alpha}} W'(\frac{v^\epsilon}{\epsilon^{1-\alpha}}).$$

By Lemma 4.2,

$$\frac{1}{\epsilon^{1+\alpha}} W'(\frac{v^\epsilon}{\epsilon^{1-\alpha}}) = \frac{1}{\epsilon^{1+\alpha}} W'(o(\epsilon^{1+\alpha})) = o(1)$$

on $\text{supp} D\zeta^\epsilon$ as $\epsilon \to 0$. Also, $g_\xi \to 0$ on $\text{supp} D\zeta^\epsilon$ by Lemma 4.2, so we are finished with the proof. □

Remark. Given $A, B \in R$ with $A < B$, $x_0 \in R^n$, $M > 0$ and $\alpha \in (0, 1/6)$ we can construct a family of supersolutions V^ϵ as above such that

$$V^\epsilon(x,t) \to (M\sqrt{|x - x_0|^2 + 2(n-1)t} + A + t) \vee B$$

uniformly in Q_T as $\epsilon \to 0$. We simply modify the above construction in a number of ways. We translate the origin in the obvious way, then define $\phi(x,t) = M\sqrt{|x|^2 + 2(n-1)t} + A + t$. The cutoff function ζ^ϵ is defined as above, where $\rho^\epsilon(t)$ now solves

$$M^2(\rho^\epsilon(t)^2 + 2(n-1)t) = (B^\epsilon - A - t)^2$$

where B^ϵ is a number such that $B^\epsilon \to B$ as $\epsilon \to 0$ and $B^\epsilon/\epsilon^{1-\alpha}$ is an integer. The functions ρ^ϵ may be taken to be defined on some fixed interval $[0,T]$, independent of ϵ as $\epsilon \to 0$. In fact, it is clear that T depends only on $B - A$ and M and that, if M is fixed, T may be made arbitrarily large by increasing $B - A$.

Finally, define

$$V^\epsilon = \zeta^\epsilon v^\epsilon + (1 - \zeta^\epsilon) B^\epsilon + \epsilon^2.$$

These are defined on Q_T, where T is as discussed above. All the proofs then proceed exactly as before, except that the estimate of v^ϵ in Lemma 4.2 is now replaced by an estimate of $v^\epsilon - B^\epsilon$.

5 Supersolutions: $\alpha = 0$

The construction of supersolutions in this case follows very closely the construction presented above. However, several modifications are necessary. First, in the earlier construction we took advantage of the fact that it is extremely easy to construct explicit (radially symmetric) supersolutions of the mean curvature PDE. Doing the same for solutions of the limiting PDE in the $\alpha = 0$ case is a little more difficult.

Second, the proof of Lemma 4.2 relies on the fact that $\lambda^\epsilon = \epsilon^{2\alpha}|D\phi|^2$ tends to zero as $\epsilon \longrightarrow 0$. To establish a version of the same fact for the case $\alpha = 0$, we need to use a sequence of supersolutions ϕ^ϵ of the limiting equation (1.4) such that $D\phi^\epsilon$ tends to zero on a certain set as $\epsilon \longrightarrow 0$.

We start by defining this sequence. We take ϕ^ϵ of the form

$$\phi^\epsilon(r,t) = q^\epsilon(\sqrt{r^2 + 2Lt}) + \epsilon^\gamma t,$$

where $r \equiv |x|$, and L and γ are constants which will be fixed below. For each ϵ, $q^\epsilon(\cdot)$ is chosen to be a smooth function on $[0,\infty)$ such that

$$q^\epsilon(t) = \epsilon^\gamma(t - 1/4) \qquad \text{if } 0 \le t \le 1/2,$$

$$q^{\epsilon\prime\prime}(t) = 0 \qquad \text{if } t \ge 1$$

$$q^\epsilon(t) \ge t - 1,$$

$$q^{\epsilon\prime} \le 1, \quad q^{\epsilon\prime\prime} \ge 0, \quad q^{\epsilon\prime} q^{\epsilon\prime\prime} \le C.$$

As in our earlier construction, we fix $T > 0$ such that $\phi^\epsilon(0,t) < 0$ for $0 \le t \le T$. This can be done independent of ϵ; for example, we may take $T = 1/KL$, for some large number K. For the remainder of this section we restrict our attention to $Q_T \equiv R^n \times [0,T]$.

We also may define a smooth, positive, decreasing function ρ^ϵ on $[0,T]$ such that

$$\phi^\epsilon(x,t) \ge 0 \qquad \text{iff } |x| \ge \rho^\epsilon(t).$$

We now have

Lemma 5.1 *If L is sufficiently large, then for all $\epsilon \le 1$ we have*

$$\phi^\epsilon_t - (\delta^{ij} - \theta(|D\phi^\epsilon|^2)\frac{\phi^\epsilon_{x_i}\phi^\epsilon_{x_j}}{|D\phi^\epsilon|^2})\phi^\epsilon_{x_i x_j} \ge \epsilon^\gamma.$$

Also, $C^{-1}\epsilon^\gamma \le |D\phi^\epsilon(x,t)| \le C$ on the set $\{(x,t) \in Q_T | \phi^\epsilon(x,t) \ge 0\}$.

Proof. First, note that Lemma 2.3 immediately implies that $1 - \theta(\lambda) \le C\lambda^2$ for some appropriate constant C. Using this fact and the radial symmetry of

ϕ^ϵ, we have

$$\begin{aligned}\phi^\epsilon_t - (\delta^{ij} - \theta(|D\phi^\epsilon|^2)\frac{\phi^\epsilon_{x_i}\phi^\epsilon_{x_j}}{|D\phi^\epsilon|^2})\phi^\epsilon_{x_ix_j} &= \phi^\epsilon_t - \frac{n-1}{r}\phi^\epsilon_r - \phi^\epsilon_{rr}(1-\theta({\phi^\epsilon_r}^2)) \\ &\geq \phi^\epsilon_t - \frac{n-1}{r}\phi^\epsilon_r - C{\phi^\epsilon_r}^2\phi^\epsilon_{rr}.\end{aligned}$$

Next we use the definition of ϕ^ϵ to write out the above expression in terms of q^ϵ. The left-hand side above then becomes

$$\frac{q^{\epsilon\prime}}{\sqrt{r^2+2Lt}}\left[L-(n-1)-(q^{\epsilon\prime})^2\frac{r^2 2Lt}{(r^2+2Lt)^2} - q^{\epsilon\prime}q^{\epsilon\prime\prime}\frac{r^4}{(r^2+2Lt)^{3/2}}\right] + \epsilon^\gamma.$$

We need to show that the quantity in brackets is positive if L is sufficiently large. For this, it clearly suffices to show that the negative terms are bounded independent of ϵ. This fact follows from our choice of q^ϵ. Indeed,

$$(q^{\epsilon\prime})^2\frac{r^2 2Lt}{(r^2+2Lt)^2} \leq (q^{\epsilon\prime})^2 \leq 1,$$

and

$$q^{\epsilon\prime}q^{\epsilon\prime\prime}\frac{r^4}{(r^2+2Lt)^{3/2}} \leq q^{\epsilon\prime}q^{\epsilon\prime\prime} \leq C.$$

We have used the fact that $q^{\epsilon\prime\prime}(\sqrt{r^2+2Lt}) = 0$ if $\sqrt{r^2+2Lt} \geq 1$.

The estimate of $D\phi^\epsilon$ is proved in exactly the same way as the corresponding estimate in Lemma 4.1. □

Next we define

$$v^\epsilon = \epsilon g(\frac{\phi^\epsilon}{\epsilon}, |D\phi^\epsilon|^2) + \epsilon^2 {\phi^\epsilon}_{x_i}{\phi^\epsilon}_{x_j}{\phi^\epsilon}_{x_ix_j} h(\frac{\phi^\epsilon}{\epsilon}, |D\phi^\epsilon|^2).$$

As before, we want to use a cutoff function to wed v^ϵ to a constant function, which we will again take to be 0. We accordingly define a cutoff function which will truncate v^ϵ on a set where ϕ^ϵ (and hence v^ϵ) are very close to 0. Thus we define $\zeta^\epsilon(x,t) = \eta^\epsilon(|x| - \rho^\epsilon(t))$, where η^ϵ is exactly as in Section 4, i.e.

$$\eta^\epsilon(z) = 0 \qquad \text{if } z \leq 0,$$

$$\eta^\epsilon(z) = 1 \qquad \text{if } z \geq \epsilon,$$

$$\eta^{\epsilon\prime} \leq C/\epsilon, \qquad \eta^{\epsilon\prime\prime} \leq C/\epsilon^2.$$

Finally, we define

$$V^\epsilon = \zeta^\epsilon v^\epsilon + \epsilon^{2+2\gamma}.$$

First we need to establish some estimates like those found in Lemma 4.2.

Lemma 5.2 *If $\gamma < 1/6$, then on supp$D\zeta^\epsilon$ we have*

$$g_\xi(\frac{\phi^\epsilon}{\epsilon}, |D\phi^\epsilon|^2) = O(\epsilon^{-\gamma}\exp(-\frac{1}{C\epsilon^\gamma})),$$

$$v^\epsilon = o(\exp(-\frac{1}{C\epsilon^\gamma})),$$

$$|Dv^\epsilon| = O(\epsilon^{-\gamma}\exp(-\frac{1}{C\epsilon^\gamma})).$$

Proof. 1. Let $(\xi^\epsilon, \lambda^\epsilon) = (\frac{\phi^\epsilon}{\epsilon}, |D\phi^\epsilon|^2)$. From the definitions of ζ^ϵ and ϕ^ϵ it is clear that

$$\lambda^\epsilon = O(\epsilon^{2\gamma}) \qquad \text{in supp} D\zeta^\epsilon. \tag{5.1}$$

We may also easily check, as in the proof of Lemma 4.2, that $0 \leq \xi^\epsilon \leq C\epsilon^\gamma$. Thus

$$\xi^\epsilon \in [0, C\sqrt{\lambda^\epsilon}] \qquad \text{in supp} D\zeta^\epsilon. \tag{5.2}$$

We have already proven in Step 2 of the proof of Lemma 4.2 that (5.2) implies that

$$0 \leq g(\xi^\epsilon, \lambda^\epsilon) \leq C\sqrt{c(\lambda^\epsilon)}. \tag{5.3}$$

We then deduce, exactly as before, that

$$g_\xi(\xi^\epsilon, \lambda^\epsilon) \leq C\left(\frac{c(\lambda^\epsilon)}{\lambda^\epsilon}\right)^{1/2} \tag{5.4}$$

on supp$D\zeta^\epsilon$. The first conclusion of the lemma now follows from (5.1), (5.4), and Lemma 2.2.

2. It follows from the Step 1 above, Lemma 5.1, and the estimate of h in Lemma 3.3 that

$$\begin{aligned} \epsilon^2 {\phi^\epsilon}_{x_i} {\phi^\epsilon}_{x_j} {\phi^\epsilon}_{x_i x_j} h(\xi^\epsilon, \lambda^\epsilon) &\leq C\epsilon^2(\lambda^\epsilon)^{-5/2} g_\xi \\ &\leq C\epsilon^{2-5\gamma}\sqrt{c(\lambda^\epsilon)} \end{aligned}$$

on supp$D\zeta^\epsilon$. Since $v^\epsilon = \epsilon g(\xi^\epsilon, \lambda^\epsilon) + \epsilon^2 {\phi^\epsilon}_{x_i} {\phi^\epsilon}_{x_j} {\phi^\epsilon}_{x_i x_j} h(\xi^\epsilon, \lambda^\epsilon)$, the above inequality and (5.3) imply that $v^\epsilon = o(\sqrt{c(\lambda^\epsilon)})$ if $\gamma < 1/6$. Now the desired estimate of v^ϵ follows from Lemma 2.2.

3. Finally, we have

$$\begin{aligned} Dv^\epsilon &= g_\xi D\phi^\epsilon + O(\epsilon g_\lambda + \epsilon^2 h + \epsilon h_\xi + \epsilon^2 h_\lambda) \\ &= g_\xi(O(\epsilon^\gamma + \epsilon(\lambda^\epsilon)^{-1} + \epsilon(\lambda^\epsilon)^{-3} + \epsilon^2(\lambda^\epsilon)^{-3} + \epsilon^2(\lambda^\epsilon)^{-4}) \\ &= g_\xi O(\epsilon^\gamma + \epsilon^{1-6\gamma} + \epsilon^{2-8\gamma}), \end{aligned}$$

by (5.1) and Lemmas 3.1 and 3.3. It is therefore evident that $Dv^\epsilon = O(g_\xi)$ as $\epsilon \longrightarrow 0$ if $\gamma < 1/6$. Since we have already estimated g_ξ, we are finished. □

The proof of the next proposition is very similar to that of Proposition 4.1

Proposition 5.1 *Let γ be some fixed number such that $\gamma < 1/12$. Then V^ϵ is a supersolution of (1.1) on Q_T for all sufficiently small ϵ.*

Proof. 1. We consider separately the cases $\zeta^\epsilon = 1$, $D\zeta^\epsilon \neq 0$, and $\zeta^\epsilon = 0$. The final case follows exactly as in Proposition 4.1. For the first two cases we need to use the asymptotic expansion in Lemma 2.1, and we need to be able to assert that the error term E is small. Using the expression for E in Lemma 2.1, the error estimates from Section 3, and Lemma 5.1, we find that

$$E = O(g_\xi \epsilon (\lambda^\epsilon)^{-11/2}) = O(g_\xi \epsilon^{1-11\gamma}) \qquad \text{as } \epsilon \to 0.$$

(We exhibit only the largest term.) Since $\gamma < 1/12$, this gives

$$E = o(g_\xi \epsilon^\gamma). \tag{5.5}$$

2. On the set where $\zeta^\epsilon = 1$, the asymptotic expansion from Lemma 2.1, together with (5.5) and Lemma 5.1, yields

$$\begin{aligned} V_t^\epsilon - \Delta V^\epsilon + \frac{1}{\epsilon} W'(\frac{V^\epsilon}{\epsilon}) & \\ &= g_\xi(\epsilon^\gamma + o(\epsilon^\gamma)) + \frac{1}{\epsilon}\left(W'(\frac{V^\epsilon}{\epsilon}) - W'(\frac{v^\epsilon}{\epsilon^{1-\alpha}})\right) \\ &= g_\xi(\epsilon^\gamma + o(\epsilon^\gamma)) + \epsilon^{2\gamma} W''(\frac{v^\epsilon}{\epsilon}) + O(\epsilon^{1+4\gamma}) \end{aligned}$$

Recall that the constant μ is defined in (1.10) and (1.11). We consider two subcases.

Subcase 1: $|g - z_i| \leq \mu \pmod 1$ for some $i = 1 \ldots, J$. Then $|V^\epsilon/\epsilon - z_i| \leq 2\mu \pmod 1$ for small ϵ, and we deduce from (1.10) that

$$V_t^\epsilon - \Delta V^\epsilon + \frac{1}{\epsilon} W'(\frac{V^\epsilon}{\epsilon}) = g_\xi(\epsilon^\gamma + o(\epsilon^\gamma)) + C^{-1}\epsilon^{2\gamma} + O(\epsilon^{1+4\gamma}).$$

This quantity is positive if ϵ is sufficiently small.

Subcase 2: $|g - z_i| \geq \mu \pmod 1$ for all $i = 1, \ldots, J$. Then $W(u) \geq C^{-1} > 0$, by (1.11) and

$$g_\xi = \left(\frac{2W(g) + c(\lambda^\epsilon)}{\lambda^\epsilon}\right)^{1/2} \geq \frac{1}{C\sqrt{\lambda^\epsilon}} = C^{-1}.$$

It follows that

$$V_t^\epsilon - \Delta V^\epsilon + \frac{1}{\epsilon} W'(\frac{V^\epsilon}{\epsilon})) \geq \epsilon^\gamma + o(\epsilon^\gamma) + \epsilon^{2\gamma} W''(\frac{v^\epsilon}{\epsilon}) + O(\epsilon^{1+4\gamma}).$$

In this case too we find that V^ϵ is a supersolution as claimed.

3. The final case to consider, $D\zeta^\epsilon \neq 0$, is handled very much as in the proof of Proposition 4.1, the only difference being that, as in Step 2 above, one has

to keep track of factors of ϵ^γ which multiply various quantities. We omit the details. □

Remark. Given $A, B \in R$ with $A < B$, $x_0 \in R^n$, and $M > 0$, we can construct a family of supersolutions V^ϵ as above such that

$$\liminf_{\epsilon \to 0} V^\epsilon(x,t) \geq (M\sqrt{|x - x_0|^2 + 2Lt} + A + \epsilon^\gamma t) \vee B$$

and

$$\lim_{\epsilon \to 0} V^\epsilon(x,t) = B$$

for all $(x,t) \in \mathcal{N} \times [0,T]$, where $\mathcal{N}$ is some neighborhood of x_0. This is done by modifying the above construction in various ways similar to those described in the remark following the proof of Proposition 4.1. As in that case, the time interval T for which the supersolutions are defined depends only on A, B, and M, and it can be made arbitrarily large by taking $B - A$ large compared to M.

6 Proof of Theorem 1.1

Although the limiting behavior of solutions of (1.1) is very different according to whether $\alpha = 0$ or $0 < \alpha < \alpha_0$, the proofs are nearly identical in the two cases. In order to discuss the two cases simultaneously, it is convenient to introduce the function

$$\theta(\lambda;\alpha) = \begin{cases} 1 & \text{if } \alpha > 0 \\ \theta(\lambda) & \text{if } \alpha = 0 \end{cases}$$

In the proof which follows, we employ the method of Barles and Perthame for weak passage to limits, together with a version of Evans' perturbed test function argument. The chief technical difficulty rests in the fact that the asymptotic expansion of Lemma 2.1 breaks down at points at which the test function ϕ has zero gradient. The proof of Lemma 6.5 is chiefly devoted to circumventing this problem.

We first assume several lemmas and use them to complete the proof of the theorem, then afterwards present the proofs of the lemmas.

Theorem 1.1 *Suppose that u^ϵ solves (1.1) with initial data $u^\epsilon(x,0) = h(x)$, where h is uniformly Lipschitz on R^n. If $0 \leq \alpha < \alpha_0 = 1/6$, then u^ϵ converges uniformly to a function u, and u solves the equation*

$$u_t - \left(\delta_{ij} - \theta(|Du|^2;\alpha)\frac{u_{x_i}u_{x_j}}{|Du|^2}\right)u_{x_ix_j} = 0 \qquad \text{on } Q = R^n \times (0,\infty) \quad (6.1)$$

with initial data $u(x,0) = h(x)$.

Proof. We start by defining

$$u^*(x,t) = \limsup_{\epsilon\to 0,(y,s)\to(x,t)} u^\epsilon(y,s).$$

$$u_*(x,t) = \liminf_{\epsilon\to 0,(y,s)\to(x,t)} u^\epsilon(y,s).$$

By Lemma 6.4, both u^* and u_* are finite.

We prove in Lemma 6.2 that $u^*(x,0) = u_*(x,0) = h(x)$, and in Lemma 6.5 we show that u^* (resp., u_*) is a subsolution (resp., supersolution) of (6.1). It is easy to check that u^* is upper semicontinuous and that u_* is lower semicontinuous. Finally, Lemma 6.1 shows that both u^* and u_* are uniformly Lipschitz. Thus the Comparison Principle, Theorem 1.2, immediately implies that

$$u^*(x,t) \leq u_*(x,t)$$

in Q. On the other hand, the opposite inequality is evident from the definitions of u^* and u_*. It follows that $u^* = u_* \equiv u$, and that u is both a supersolution and a subsolution. □

Let L denote the Lipschitz constant of h. We start by showing that u^* inherits the same Lipschitz constant.

Lemma 6.1 *For every $t > 0$ and every $x, y \in R^n$,*

$$|u^*(x,t) - u^*(y,t)| \le L|x-y|.$$

Proof. Fix $z \in R^n$. It suffices to show that

$$u^*(x+z,t) \ge u^*(x,t) - L|z| \tag{6.2}$$

for all $(x,t) \in Q$.

Let u^ϵ as above denote the solution of (1.1) with the given initial data h, and $\tilde{u}^\epsilon(x,t) = u^\epsilon(x+z,t) + K^\epsilon$, where $K^\epsilon = \epsilon^{1-\alpha}([L|z|/\epsilon^{1-\alpha}]+1)$, and $[s]$ here denotes the integer part of s. Then $K^\epsilon > L|z|$, so

$$\tilde{u}^\epsilon(x,0) = h(x+z) + K^\epsilon \ge h(x) - L|z| + K^\epsilon > h(x) = u^\epsilon(x,0).$$

Also, since K^ϵ is a multiple of $\epsilon^{1-\alpha}$, $\tilde{u}^\epsilon$ is again a solution of (1.1). The comparison principle thus implies that

$$u^\epsilon(x+z,t) + K^\epsilon = \tilde{u}^\epsilon(x,t) \ge u^\epsilon(x,t).$$

Because $K^\epsilon \to L|z|$ as $\epsilon \to 0$, this readily implies that (6.2) holds. □

Our next lemma shows that the initial data is assumed.

Lemma 6.2 $u^*(x,0) = u_*(x,0) = h(x)$.

Proof. First suppose that $\alpha > 0$. Fix any point $x_0 \in R^n$. We will show that $u^*(x_0,0) \le h(x_0)$. The inequality $u_*(x_0,0) \ge h(x_0)$ is established in exactly the same way, and together these imply the conclusion of the lemma.

Let $M = L+1$, so that $h(x) < h(x_0) + M|x_0 - x|$ for all $x \ne x_0$, and fix some $\delta > 0$. As noted following the proof of Propositions 4.1, if $0 < \alpha < 1/6$, we can construct supersolutions V^ϵ on $Q_\tau \equiv R^n \times [0,\tau]$, for some $\tau > 0$ independent of ϵ, such that

$$V^\epsilon(x,t) \to (M\sqrt{|x-x_0|^2 + 2(n-1)t} + h(x_0) + t) \vee h(x_0) + \delta.$$

In particular, for ϵ sufficiently small, $\overline{V}^\epsilon(x,0) \ge h(x) = u^\epsilon(x,0)$. The comparison principle then implies that

$$u^\epsilon \le V^\epsilon \qquad \text{on } Q_\tau.$$

Thus $u^\epsilon(x,t) \le h(x_0) + \delta$ for all x in some small neighborhood of x_0 and all $t \in [0,\tau]$. This clearly implies that $u^*(x_0,0) \le h(x_0)$.

If $\alpha = 0$ we argue in a similar way, using the remark which follows the proof of Proposition 5.1. □

Lemma 6.3 *Suppose $(x_0, t_0) \in Q$ and $r > 0$ are such that*

$$u^*(y, t_0) \le u^*(x_0, t_0) \qquad \textit{for all } y \in B_r(x_0).$$

Then there exists $\tau > 0$, $r' > 0$, depending only on r and the Lipschitz constant L of u^*, such that

$$u^*(y, t) \le u^*(x_0, t_0) \qquad \textit{for all } y \in B_r(x_0).$$

for all $(y, t) \in B_{r'}(x_0) \times [t_0, t_0 + \tau]$.

Proof. The hypotheses and Lemma 6.1 imply that for any $\delta > 0$,

$$u^*(y, t_0) < u^*(x_0, t_0 + \delta) + [L(|x - x_0| - r)]^+,$$

where $[f]^+ = f \vee 0$. By a translation we may set $t_0 = 0$, to simplify notation. Now we repeat the argument from Lemma 6.2, constructing supersolutions V^ϵ on $R^n \times [0, \tau]$ for some $\tau > 0$ such that

$$V^\epsilon(x, t) \longrightarrow u^*(x_0, t_0) + \delta + [L\sqrt{|x - x_0|^2 + 2(n-1)t} - Lr + t]^+.$$

Recall from the from the construction of V^ϵ that the only condition on our choice of the time τ is that we need $V^\epsilon(x, t) \equiv 0$ on a neighborhood of x_0, i.e., we must have

$$L\sqrt{2(n-1)t} - Lr + t < 0$$

for all $t \in [0, \tau]$. The parameter δ does not enter into this condition, so we may construct the above supersolutions on a time interval $[0, \tau]$ which is uniform for $\delta > 0$ (This is not the case in the proof of Lemma 6.2 above.)

We deduce as above that $u^\epsilon \le V^\epsilon$ on Q_τ for all sufficiently small ϵ, and in particular that, for all x in some small ball $B_{r'}(x_0) \times [0, \tau]$, we have $u^\epsilon(x, t) \le u^*(x_0, t_0) + \delta$. Passing to the upper-$*$ limit and then letting $\delta \longrightarrow 0$, we find that $u^*(x, t) \le u^*(x_0)$ as claimed.

If $\alpha = 0$ we reach the same conclusion using Proposition 5.1 and the remark which follows it. □

We also need the following

Lemma 6.4 *u^* and u_* are finite.*

Proof. Given any $T > 0$, we can construct a family of supersolutions $\{V^\epsilon\}_{\epsilon > 0}$ which are defined on $R^n \times [0, T]$ and such that $V^\epsilon(x, 0) \ge h(x)$. This is evident from the remarks following the proof of Proposition 4.1 (if $0 < \alpha < 1/6$) and Proposition 5.1 (if $\alpha = 0$). The comparison principle thus implies that u^ϵ is locally uniformly bounded above on Q_T for small ϵ. A similar argument shows that u^ϵ is bounded below. The conclusion of the lemma is now immediate. □

A final, long lemma will complete the proof of the theorem.

Lemma 6.5 u^* *(respectively u_*) is a subsolution (resp. supersolution) of (6.1).*

We show that u^* is a subsolution of the appropriate limiting equation. The proof that u_* is a supersolution follows exactly the same pattern, so we will omit it.

Fix some smooth function ϕ such that

$$u^* - \phi \leq 0$$

with equality at (x_0, t_0) and strict inequality elsewhere. We will show that

$$\phi_t - \left(\delta_{ij} - \theta(|D\phi^2|;\alpha)\frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} \leq 0 \tag{6.3}$$

at (x_0, t_0), if $D\phi(x_0, t_0) \neq 0$, and that if $D\phi(x_0, t_0) = 0$, then

$$\phi_t - (\delta_{ij} - \nu^i \nu^j \phi_{x_i x_j}) \leq 0 \tag{6.4}$$

at (x_0, t_0) for some $\nu \in R^n$ such that $|\nu| = 1$.

1. First, suppose that $D\phi(x_0, t_0) \neq 0$. We define v^ϵ by

$$v^\epsilon = \epsilon^{1-\alpha} g(\xi^\epsilon, \lambda^\epsilon) + \epsilon^{2+2\alpha}\phi_{x_i}\phi_{x_j}\phi_{x_i x_j} h(\xi^\epsilon, \lambda^\epsilon), \tag{6.5}$$

where $\lambda^\epsilon = \epsilon^{2\alpha}(|D\phi|^2 \vee \frac{1}{2}|D\phi(x_0, t_0)|^2)$ and $\xi^\epsilon = \frac{\phi}{\epsilon^{1-\alpha}} + d^\epsilon$. Here d^ϵ is some constant, which we *claim* can be selected so that, at least along a subsequence, we have

$$u^\epsilon - v^\epsilon \leq 0$$

with equality at points (x_ϵ, t_ϵ) such that $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$ as $\epsilon \to 0$.

2. We now verify the above claim. First define $v^\epsilon(\cdot, \cdot; s)$ for $s \in R$ by taking v^ϵ as in (6.5), λ^ϵ as above, and $\xi^\epsilon = \frac{\phi}{\epsilon^{1-\alpha}} + s$. From the definitions of g and h we have

$$v^\epsilon(x, t; s+1) = v^\epsilon(x, t; s) + \epsilon^{1-\alpha} \tag{6.6}$$

for all (x, t, s). By Lemma 3.4, $v^\epsilon(\cdot, \cdot, 0)$ converges uniformly to $\phi(\cdot, \cdot)$, so the same holds for $v(\cdot, \cdot, s)$ uniformly for $s \in [0, 1]$. The definition of u^* now implies that there is some subsequence of the ϵ's tending to zero along which, for every $s \in [0, 1]$, $u^\epsilon(\cdot, \cdot) - v^\epsilon(\cdot, \cdot; s)$ attains a max at some point $(x^{\epsilon,s}, t^{\epsilon,s})$, and that $(x^{\epsilon,s}, t^{\epsilon,s}) \to (x_0, t_0)$ uniformly for $s \in [0, 1]$ as $\epsilon \to 0$. This follows from a straightforward modification of a standard, elementary argument, which I omit here.

Now select some ϵ in the subsequence identified above, and define

$$f(s) = \max_{(x,t)\in Q} u^\epsilon(x, t) - v^\epsilon(x, t; s).$$

Clearly $f(\cdot)$ is continuous, and (6.6) implies that $f(s+1) = f(s)+\epsilon^{1-\alpha}$. Thus the intermediate value theorem implies that there is some number, which we can call d^ϵ, such that $f(d^\epsilon) = 0$. Let (x_ϵ, t_ϵ) be a point at which $\max(u^\epsilon(\cdot,\cdot) - v^\epsilon(\cdot,\cdot,d^\epsilon))$ is attained. We see from (6.6) that for any integer k, $\max(u^\epsilon(\cdot,\cdot)-v^\epsilon(\cdot,\cdot,d^\epsilon+k))$ is attained at the same point (x_ϵ, t_ϵ). In particular, there is some $s \in [0,1]$ such that (x_ϵ, t_ϵ) can be identified with $(x^{\epsilon,s}, t^{\epsilon,s})$. It follows that $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$ as claimed.

3. The comparison principle now implies that

$$v_t^\epsilon - \Delta v^\epsilon + \frac{1}{\epsilon^{1+\alpha}} W'(\frac{1}{\epsilon^{1-\alpha}}) \leq 0$$

at (x_ϵ, t_ϵ). We have assumed that $D\phi(x_0,t_0) \neq 0$ and we know that $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$, so $|D\phi(x_\epsilon, t_\epsilon)| > \frac{1}{2}|D\phi(x_0,t_0)|$ for small ϵ. Thus $\lambda^\epsilon = \epsilon^{2\alpha}|D\phi|^2$ in a neighborhood of (x_ϵ, t_ϵ), for small ϵ. Also, $|D\phi(x_\epsilon, t_\epsilon)|^2$ is bounded away from zero as $\epsilon \to 0$. We may therefore use the asymptotic expansion in Lemma 2.1 and Proposition 3.1 to deduce that

$$g_\xi[\phi_t - \Delta\phi + \theta(\lambda^\epsilon)\frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\phi_{x_i x_j} + o(1)] \leq 0$$

at (x_ϵ, t_ϵ). Letting ϵ tend to zero we find that (6.3) holds.

4. Now suppose $D\phi(x_0, t_0) = 0$. We consider several cases.

Case 1: There is some vector $\eta \in R^n$, $|\eta| = 1$, such that $D_{\eta\eta}\phi < 0$.

Let $\mathcal{N}$ be a neighborhood of (x_0,t_0) in which $D_{\eta\eta}\phi \leq -\delta < 0$ for some number δ. For each $h > 0$, we define $\phi^h(x,t) = \phi(x - h\eta, t)$. Clearly $\phi^h \to \phi$ uniformly as $h \to 0$, so for h sufficiently small, $u^* - \phi^h$ attains a maximum at some point, say (x_h, t_h), and $(x_h, t_h) \to (x_0, t_0)$ as $h \to 0$. We moreover claim that for h sufficiently small $D\phi^h(x_h, t_h) \neq 0$.

Let (y,t) be any point in $\mathcal{N}$ such that $D\phi(y,t) = 0$, so that $D\phi^h(y+h\eta, t) = 0$. Any maximum of $u^* - \phi^h$ must eventually be contained in $\mathcal{N}$, so it suffices to show that $u^* - \phi^h$ cannot attain its maximum at $(y+h\eta, t)$ if h is very small. First note that, since $D\phi(x_0,t_0) = 0$ and $D_{\eta\eta}\phi(x_0,t_0) < -\delta$, we have

$$\begin{aligned}(u^* - \phi^h)(x_0,t_0) &= \phi(x_0,t_0) - \phi(x_0 - h\eta, t_0) \\ &\geq \delta h^2/2 + o(h^2)\end{aligned}$$

as $h \to 0$. Thus if h is sufficiently small, $u^* - \phi^h$ is positive at the point at which its maximum is attained. On the other hand, similar calculations yield

$$\begin{aligned}(u^* - \phi^h)(y + h\eta, t) &\leq \phi(y + h\eta, t) - \phi(y,t) \\ &\leq -\delta h^2/2 + o(h^2).\end{aligned}$$

as $h \to 0$. This is negative if h is small, so the maximum cannot be attained at $(y + h\eta, t)$, as claimed.

Since $D\phi^h(x_h, t_h) \neq 0$, we may use the results of Steps 2 and 3 above to conclude that

$$\phi_t^h - \left(\delta_{ij} - \theta(|D\phi^h|^2;\alpha)\frac{\phi_{x_i}^h\phi_{x_j}^h}{|D\phi^h|^2}\right)\phi_{x_ix_j}^h \leq 0$$

at (x_h, t_h). By the definition of ϕ^h, this says that

$$\phi_t - \left(\delta_{ij} - \theta(|D\phi|^2;\alpha)\frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_ix_j} \leq 0$$

at $(y_h, t_h) \equiv (x_h - h\eta, t_h)$. Because the unit sphere is compact in Euclidean space, we may extract a subsequence along which $D\phi(y_h, t_h)/|D\phi(y_h, t_h)|$ converges to a limit $\nu \in R^n$ with $|\nu| = 1$. Also, $D\phi(y_h, t_h) \longrightarrow D\phi(x_0, t_0) = 0$, so $\theta(|D\phi(y_h, t_h)|^2;\alpha) \longrightarrow \theta(0;\alpha) = 1$. Letting h tend to zero along this subsequence, we pass to limits in the above expression to deduce (6.4)

5. We may now safely assume that $D^2\phi(x_0, t_0) \geq 0$. Without any loss we may in fact go farther and assume that

$$D^2\phi(x_0, t_0) > 0 \qquad \text{and} \tag{6.7}$$

$$\phi(x,t) \geq u^*(x,t) + s(|x - x_0|^2 + (t - t_0)^2) \qquad \text{for some } s > 0. \tag{6.8}$$

Indeed, if (6.7) and (6.8) do not hold, select some $s > 0$ and define $\tilde{\phi}(x,t) = \phi(x,t) + s(|x - x_0|^2 + (t - t_0)^2)$. The above conditions are clearly satisfied by $\tilde{\phi}$. If we can show that $\tilde{\phi}$ satisfies (6.4) for arbirtary $s > 0$, we can then let $s \longrightarrow 0$ to find that ϕ satisfies (6.4).

Conditions (6.7) and (6.8) imply that ϕ attains a minimum near x_0 for all t in a neighborhood $\mathcal{N}_t$ of t_0. In other words, there is a neighborhood $\mathcal{N}_x$ of x_0 in R^n such that, for any $t \in \mathcal{N}_t$, there is a unique point $x(t) \in \mathcal{N}_x$ at which $D\phi(\cdot, t) = 0$.

We remark for future use that clearly

$$x(\cdot) \text{ is continuous, and } x(t_0) = x_0. \tag{6.9}$$

We can now state the remaining 2 cases.

Case 2: For every t sufficiently close to t_0, $u^*(\cdot, t)$ attains a local maximum at $x(t)$.

Case 3: Case 2 does not hold, so that in every neighborhood of t_0 there are points t such that $u^*(\cdot, t)$ does not have a local maximum at $x(t)$.

6. We consider first Case 3. As in Step 4, we will modify ϕ slightly so that a maximum is attained at a point near (x_0, t_0) at which the gradient is nonzero.

Fix $h > 0$ and let $f_h : R \longrightarrow R$ be a smooth function such that

$$f_h(s) = \gamma s^2 \quad \text{if } s \leq h^2, \qquad f_h(s) \geq s - h;$$

$$0 \leq f_h{}' \leq 1, \qquad 0 \leq f_h{}'' \leq C/h.$$

The number γ will be chosen below; here we only specify that $\gamma \in (0,1]$. We now define

$$\phi^h(x,t) = \phi(x(t),t) + f_h(\phi(x,t) - \phi(x(t),t)).$$

Since f_h converges uniformly to the identity function, ϕ^h converges uniformly to ϕ, so $u^* - \phi^h$ must attain a maximum at a point (x_h, t_h), with (x_h, t_h) tending to (x_0, t_0) as $h \to 0$. (So it does not matter that ϕ^h is only defined on the neighborhood of (x_0, t_0) for which the function $x(t)$ is defined.)

7. We now claim that if γ is sufficiently small, then $D\phi^h(x_h, t_h) \neq 0$. Note first that $D\phi^h = f_h{}'(\cdots)D\phi$, so $D\phi^h(x,t) = 0$ if and only if $x = x(t)$. Thus we need to show that the maximum is not attained at one of the points $(x(t),t)$, if we select γ correctly. Regardless of our choice of γ, we have

$$(u^* - \phi^h)(x(t),t) = (u^* - \phi)(x(t),t) \leq 0.$$

On the other hand, if we let γ approach 0, $\phi^h(\cdot,t)$ converges uniformly to $\phi(x(t),t)$ on some neighborhood of $x(t)$, for every $t \in \mathcal{N}_t$. The condition that defines Case 3 may be restated by saying that the function $(y,t) \mapsto u^*(y,t) - \phi(x(t),t)$ takes on positive values in every neighborhood of (x_0,t_0). Thus if $\gamma > 0$ is small enough, $u^* - \phi^h$ is positive somewhere in $\mathcal{N}_x \times \mathcal{N}_t$. This verifies our claim.

8. Since $D\phi(x_h, t_h) \neq 0$, we may now appeal to the results of Steps 2 and 3 to assert that

$$\phi_t^h - \left(\delta_{ij} - \frac{\phi^h_{x_i}\phi^h_{x_j}}{|D\phi^h|^2}\right)\phi^h_{x_i x_j} \leq (1 - \theta(|D\phi^h|^2;\alpha))\frac{\phi^h_{x_i}\phi^h_{x_j}}{|D\phi^h|^2}\phi_{x_i x_j} \tag{6.10}$$

at (x_h, t_h).

From the definition of ϕ^h we compute, recalling that $D\phi(x(t),t) = 0$,

$$\phi_t^h(x,t) = \phi_t(x(t),t) + f_h{}'(\phi(x,t) - \phi(x(t),t))(\phi_t(x,t) - \phi_t(x(t),t).$$

Both (x_h, t_h) and $(x(t_h), t_h)$ converge to (x_0, t_0) as h tends to zero, so $|(x_h, t_h) - (x(t_h), t_h)|$ is arbitrarily small as $h \to 0$. Thus the above identity and (6.9) imply that

$$\phi_t^h(x_h, t_h) = \phi_t(x(t^h), t^h) + o(1) = \phi_t(x_h, t_h) + o(1) \tag{6.11}$$

as $h \to 0$. Also, for all (x,t) for which ϕ^h is defined and $D\phi^h \neq 0$ we have

$$\left(\delta_{ij} - \frac{\phi^h_{x_i}\phi^h_{x_j}}{|D\phi^h|^2}\right)\phi^h_{x_i x_j} = f_h'\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} \leq \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j}. \tag{6.12}$$

The inequality holds because $D^2\phi \geq 0$ and $f_h' \leq 1$.

Now (6.10), (6.11), and (6.12) imply that

$$\phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_ix_j} \leq (1-\theta(|D\phi^h|^2;\alpha))\frac{\phi^h_{x_i}\phi^h_{x_j}}{|D\phi^h|^2}\phi_{x_ix_j} + o(1) \qquad (6.13)$$

at (x_h, t_h) as $h \to 0$.

9. The right-hand side of (6.13) is zero if $\alpha > 0$. If $\alpha = 0$, it equals

$$\text{r.h.s.} = (1-\theta(|D\phi^h|^2))\frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}(f_h{}'\phi_{x_ix_j} + f_h{}''\phi_{x_i}\phi_{x_j})$$

evaluated at (x_h, t_h). We now claim that this quantity vanishes as $h \to 0$. Using the properties of f_h, we have

$$\text{r.h.s.} \leq (1-\theta(|D\phi|^2))(C + C|D\phi^2|/h).$$

To show that this vanishes as $h \to 0$, we must estimate $|D\phi(x_h, t_h)|$. We have

$$\begin{aligned} |D\phi(x_h,t_h)| &= |D\phi(x_0,t_0)| + O(|x_h - x_0| + |t_h - t_0|)| \\ &= O(|x_h - x_0| + |t_h - t_0|). \end{aligned}$$

Next, from the construction of ϕ^h, the fact that $u^* - \phi^h$ attains its maximum at (x_h, t_h), and (6.8), we have

$$\begin{aligned} 0 &< u^*(x_h,t_h) - \phi^h(x_h,t_h) \\ &\leq u^*(x_h,t_h) - (\phi(x_h,t_h) - h) \\ &\leq h - s(|x_h - x_0|^2 + (t_0 - t)^2). \end{aligned}$$

The above two estimates imply that $|D\phi(x_h, t_h)|^2 = O(h)$. Thus

$$\text{r.h.s} \leq C(1 - \theta(Ch)) = o(1)$$

at (x_h, t_h) as $h \to 0$, using Lemma 2.3. We may finally let $h \to 0$ and pass to limits in (6.13) as we did in Step 4 above to deduce that (6.4) holds.

10. Finally, we consider Case 2. Fix some $t \in \mathcal{N}_t$ such that $t < t_0$ and such that $u^*(x(t),t) \geq u^*(y,t)$ for all $y \in B_r(x(t))$, for some small number r. Taking $\mathcal{N}_t$ smaller as necessary, we may assume that $r \geq C^{-1}$ uniformly for $t \in \mathcal{N}_t$. According to Lemma 6.3,

$$u^*(y,s) \leq u^*(x(t),t) \qquad \text{for all } (y,s) \in B_{r'}(x(t)) \times [t, t+\tau) \qquad (6.14)$$

for some numbers r' and τ which depend only on $r > C^{-1}$ and on the Lipschitz constant L of $u^*(\cdot, t)$. Since we have uniform control over these quantities in $\mathcal{N}_t$, we may assume that $r' \geq C^{-1}$ and $\tau \geq C^{-1}$. In particular, if we take $t < t_o$ sufficiently close to t_0, then (6.14) implies that $u^*(x_0, t_0) \leq u^*(x(t), t)$. Since

$u^* \leq \phi$ with equality at (x_0, t_0), we see that $\phi(x_0, t_0) \leq \phi(x(t), t)$ for t such that $0 < t_0 - t$ is small. Therefore

$$0 \leq \frac{d}{dt}\phi(x(t), t)|_{t=t_0} = D\phi(x_0, t_0) \cdot \dot{x}(t) + \phi_t(x_0, t_0) = \phi_t(x_0, t_0).$$

Since $D^2\phi \geq 0$ this implies that (6.4) holds for any choice of $\nu \in S^{n-1}$. □

7 Application: Averaged Kinetic Constants

In the problem considered above, we find the same limiting equation for $\alpha > 0$ for an extremely wide range of potentials $W(\cdot)$. In some sense, this says that information about the structure of the potential is lost in the passage to the limit. We now consider situations in which the limiting PDE depends more delicately on the structure of the approximating PDEs. The two examples we will consider are fully nonlinear versions of (1.1) and semilinear equations in which the potential varies as $\epsilon \longrightarrow 0$. In both cases, if the potential W has a number of potential wells in each periodic repetition, a constant appears in the limiting equation which we interpret as reflecting a sort of average of the kinetic properties of fronts associated with the different potential wells. This procedure thus predicts properties of "composite fronts" formed when various fronts impinge on and interact with one another, from a knowledge of the properties of the constituent fronts.

We first consider a fully nonlinear version of (1.1). Assume that $W(\cdot)$ is as described in the introduction, and in particular that $W(\cdot)$ satisfies (1.7)-(1.11). We further assume that $|W''| < 1$ and that $W \in C^4(R/Z)$. Then the function $x \longmapsto x + W'(x)$ is increasing, so we may define

$$T(\cdot) \quad \text{is the inverse of the function } x \longmapsto x + W'(x). \tag{7.1}$$

We will look at the asymptotic behavior of solutions of the equation

$$u_t^\epsilon = \frac{1}{\epsilon^{1+\alpha}} \left(\frac{-u^\epsilon}{\epsilon^{1-\alpha}} + T\Big(\frac{u^\epsilon}{\epsilon^{1-\alpha}} + \epsilon^{1+\alpha} \Delta u^\epsilon\Big) \right). \tag{7.2}$$

If $\alpha = 1$, this is essentially the phase-field equation proposed by O. Penrose [17] and analyzed by the author in [12]. We will instead look at values $0 \leq \alpha < \alpha_0 \leq 1$.

Existence and uniqueness of solutions of (7.2) with $\alpha = 1$, under some restrictions on the initial data, is proven in [12]. Exactly the same arguments can be used to establish existence and uniqueness for arbitrary α for continuous initial data which is periodic or can be approximated monotonically by periodic functions. The latter holds, for example, if $u(x, 0)$ equals a constant, say u_∞, outside a compact set and if u_∞ is an extremal value of $u(x, 0)$. No doubt one could establish existence under more general hypotheses; we will not pursue that here.

Recall that (1.1) is the equation for L^2-gradient flow with respect to the functional E^ϵ, defined in the introduction. It is easy to verify, at least formally, that the same energy E^ϵ is a Lyapunov functional for equation (7.2). Indeed, if u^ϵ is a smooth solution of (7.2), then we have

$$\frac{d}{dt} E^\epsilon(u^\epsilon) \;=\; \int Du^\epsilon \cdot Du_t^\epsilon + \frac{1}{\epsilon^{1+\alpha}} W'\Big(\frac{u^\epsilon}{\epsilon^{1-\alpha}}\Big) u_t^\epsilon \, dx$$

$$= \frac{1}{\epsilon^{1+\alpha}} \int [(W'(\frac{u^\epsilon}{\epsilon^{1-\alpha}}) + \frac{u^\epsilon}{\epsilon^{1-\alpha}}) - (\epsilon^{1+\alpha}\Delta u^\epsilon + \frac{u^\epsilon}{\epsilon^{1-\alpha}})] u_t^\epsilon dx$$

$$= \frac{-1}{\epsilon^{2+2\alpha}} \int (A_1 - A_2)(T(A_1) - T(A_2)) dx, \qquad \text{where}$$

$$A_1 = W'(\frac{u^\epsilon}{\epsilon^{1-\alpha}}) + \frac{u^\epsilon}{\epsilon^{1-\alpha}}, \qquad A_2 = \epsilon^{1+\alpha}\Delta u^\epsilon + \frac{u^\epsilon}{\epsilon^{1-\alpha}}.$$

We have used (7.1) and (7.2). The integrand in the last integral can be seen to be nonnegative, since $T(\cdot)$ is an increasing function.

This suggests that we may think of (7.2) as a fully nonlinear phase field equation corresponding to the infinite-well potential W^ϵ.

We now translate our earlier results to this context. We start by proving a lemma corresponding to Lemma 2.1, which contains the asymptotic expansion in the semilinear case.

First we rewrite the ODEs defining g and h in terms of T, defined in (7.1). It is easy to see that the ODE (2.1) is equivalent to

$$g - T(g + \lambda g_{\xi\xi}) = 0. \tag{7.3}$$

Also, (7.1) and (7.3) imply that

$$W''(g) - 1 = \frac{1}{T'(g + \lambda g_{\xi\xi})}. \tag{7.4}$$

We may use this to rewrite (2.4) in the form

$$\frac{h}{T'(g + \lambda g_{\xi\xi})} - h - \lambda h_{\xi\xi} = 4 g_{\xi\lambda} + \frac{\theta(\lambda)}{\lambda} g_\xi. \tag{7.5}$$

We also define another auxilliary function $\tilde{h}$ by the ODE

$$\tilde{h}_{\xi\xi} + 2\frac{g_{\xi\xi}}{g_\xi}\tilde{h}_\xi = \frac{1}{\lambda}\left(\frac{\kappa(\lambda)}{T'(g + \lambda g_{\xi\xi})} - 1\right) \tag{7.6}$$

$$\tilde{h}(0, \lambda) = 0, \qquad \tilde{h}(\xi + 1, \lambda) = \tilde{h}(\xi, \lambda).$$

Here $\kappa(\lambda)$ is defined by the condition

$$\int_0^1 \frac{\kappa(\lambda) g_\xi^2}{T'(g + \lambda g_{\xi\xi})} - g_\xi^2 d\sigma = 0.$$

As usual, this is a solvability condition for (7.6).

It will be convenient to use the notation

$$N(\phi) \equiv \phi_{x_i}\phi_{x_j}\phi_{x_i x_j}, \qquad M^\epsilon(\phi) \equiv \Delta\phi - \theta(\epsilon^{2\alpha}|D\phi|^2)\frac{N(\phi)}{|D\phi|^2}.$$

We now carry out an asymptotic expansion.

Lemma 7.1 *Suppose $\phi : Q \to R$ is smooth, and on the set where $D\phi \neq 0$ define*

$$\phi^\epsilon = \phi + \epsilon^2 M^\epsilon(\phi)\tilde{h}(\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2), \qquad and$$

$$v^\epsilon = \epsilon^{1-\alpha} g(\frac{\phi^\epsilon}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2) + \epsilon^{2+2\alpha} N(\phi) h(\frac{\phi^\epsilon}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2).$$

Then v^ϵ is smooth where it is defined, and on this set we have

$$\begin{aligned} & v_t^\epsilon + \frac{1}{\epsilon^{1+\alpha}}\left(\frac{v^\epsilon}{\epsilon^{1-\alpha}} - T(\frac{v^\epsilon}{\epsilon^{1-\alpha}} + \epsilon^{1+\alpha}\Delta v^\epsilon)\right) \\ = \;& g_\xi(\frac{\phi^\epsilon}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2)\,[\phi_t - \kappa(\epsilon^{2\alpha}|D\phi|^2)M^\epsilon(\phi)] + O(\epsilon) \end{aligned}$$

where the $O(\epsilon)$ error term contains derivatives of ϕ up to fourth order, and derivatives of the auxilliary functions g, h, and $\tilde{h}$ up to second order.

Proof. Let $\xi^\epsilon = \frac{\phi}{\epsilon^{1-\alpha}}$, $\tilde{\xi}^\epsilon = \frac{\phi^\epsilon}{\epsilon^{1-\alpha}}$, $\lambda^\epsilon = \epsilon^{2\alpha}|D\phi|^2$, and $\tilde{\lambda}^\epsilon = \epsilon^{2\alpha}|D\phi^\epsilon|^2$.

We start by computing

$$\begin{aligned} \frac{v^\epsilon}{\epsilon^{1-\alpha}} + \epsilon^{1+\alpha}\Delta v^\epsilon &= g + \tilde{\lambda}^\epsilon g_{\xi\xi} \\ &+ \epsilon^{1+\alpha}[g_\xi \Delta\phi^\epsilon + 4\epsilon^{2\alpha}\phi^\epsilon_{x_i}\phi_{x_j}\phi_{x_i x_j} g_{\xi\lambda} + \epsilon^{2\alpha}N(\phi)(h + \tilde{\lambda}^\epsilon h_{\xi\xi})] \\ &+ O(\epsilon^2). \end{aligned}$$

Here and in what follows, we combine derivatives of ϕ up to fourth order and derivatives of the auxilliary functions up to second order in the error terms indiscriminately. All the auxilliary functions explicitly appearing in the above expression are evaluated at $(\tilde{\xi}^\epsilon, \lambda^\epsilon)$. Recall, however, that $\tilde{h}$ and its derivatives, which we do not yet exhibit explicitly, are evaluated at $(\xi^\epsilon, \lambda^\epsilon)$.)

From the definition of ϕ^ϵ, we easily compute that

$$\tilde{\lambda}^\epsilon = \lambda^\epsilon(1 + 2\epsilon^{1+\alpha}M^\epsilon(\phi)\tilde{h}_\xi) + O(\epsilon^2)$$

We use the above substitution for $\tilde{\lambda}^\epsilon$ where it appears in the leading order term. In the lower-order term, we use the cruder estimates $\tilde{\lambda}^\epsilon = \lambda^\epsilon + O(\epsilon)$, $\phi^\epsilon_{x_i} = \phi_{x_i} + O(\epsilon)$. This gives

$$\begin{aligned} \frac{v^\epsilon}{\epsilon^{1-\alpha}} + \epsilon^{1+\alpha}\Delta v^\epsilon &= g + \lambda^\epsilon g_{\xi\xi} \\ &+ \epsilon^{1+\alpha}(g_\xi\Delta\phi^\epsilon + 2M^\epsilon(\phi)\lambda^\epsilon g_{\xi\xi}\tilde{h}_\xi + \epsilon^{2\alpha}N(\phi)(h + \lambda^\epsilon h_{\xi\xi} + 4g_{\xi\lambda})) \\ &+ O(\epsilon^2). \end{aligned}$$

Now equation (7.3), Taylor's theorem, and the above equation give

$$\begin{aligned} & \frac{1}{\epsilon^{1+\alpha}}[g - T(\frac{v^\epsilon}{\epsilon^{1-\alpha}} + \epsilon^{1+\alpha}\Delta v^\epsilon)] \\ = \;& -T'(g + \lambda^\epsilon g_{\xi\xi})[g_\xi\Delta\phi^\epsilon + 2M^\epsilon(\phi)\lambda^\epsilon g_{\xi\xi}\tilde{h}_\xi + \epsilon^{2\alpha}N(\phi)(h + \lambda^\epsilon h_{\xi\xi} + 4g_{\xi\lambda})] \\ & + O(\epsilon) \end{aligned}$$

We next substitute for the terms $T'(\cdots)(h + \lambda^\epsilon h_{\xi\xi} + 4g_{\xi\lambda})$ from (7.5) to find that

$$\begin{aligned}
&\epsilon^{2\alpha} N(\phi)h + \frac{1}{\epsilon^{1+\alpha}}[g - T(\frac{v^\epsilon}{\epsilon^{1-\alpha}} + \epsilon^{1+\alpha}\Delta v^\epsilon)] \\
&= -T'(g + \lambda^\epsilon g_{\xi\xi})[g_\xi \Delta\phi^\epsilon + 2M^\epsilon(\phi)\lambda^\epsilon g_{\xi\xi}\tilde{h}_\xi - \epsilon^{2\alpha}\frac{N(\phi)}{\lambda^\epsilon}\theta(\lambda^\epsilon)g_\xi] \\
&\quad + O(\epsilon) \qquad (7.7)
\end{aligned}$$

Using the definition of ϕ^ϵ, we now compute that

$$\Delta\phi^\epsilon = \Delta\phi + \lambda^\epsilon M^\epsilon(\phi)\tilde{h}_{\xi\xi} + O(\epsilon).$$

It follows that

$$g_\xi \Delta\phi^\epsilon - \epsilon^{2\alpha}\frac{N(\phi)}{\lambda^\epsilon}\theta(\lambda^\epsilon)g_\xi = g_\xi M^\epsilon(\phi)(1 + \lambda^\epsilon \tilde{h}_{\xi\xi}) + O(\epsilon).$$

We substitute the above into the right-hand side of (7.7), and we rewrite the left-hand side using the definition of v^ϵ. This yields

$$\begin{aligned}
&\frac{1}{\epsilon^{1+\alpha}}[\frac{v^\epsilon}{\epsilon^{1-\alpha}} - T(\frac{v^\epsilon}{\epsilon^{1-\alpha}} + \epsilon^{1+\alpha}\Delta v^\epsilon)] \\
&= -T'(g + \lambda^\epsilon g_{\xi\xi})M^\epsilon(\phi)g_\xi[1 + 2\lambda^\epsilon \frac{g_{\xi\xi}}{g_\xi}\tilde{h}_\xi + \lambda^\epsilon \tilde{h}_{\xi\xi}] + O(\epsilon).
\end{aligned}$$

Recall that g and its derivatives are evaluated at the point $(\tilde{\xi}^\epsilon, \lambda^\epsilon)$, whereas $\tilde{h}$ and its derivatives are evaluated at $(\xi^\epsilon, \lambda^\epsilon)$. We therefore replace $\tilde{h}_\xi(\xi^\epsilon, \lambda^\epsilon)$ by $\tilde{h}_\xi(\tilde{\xi}^\epsilon, \lambda^\epsilon)$, and similarly for $\tilde{h}_{\xi\xi}$. Of course this adds another $O(\epsilon)$ term to the error, since $\xi^\epsilon - \tilde{\xi}^\epsilon = O(\epsilon)$. Having done this, we can use the ODE (7.6) to conclude from the above that

$$\frac{1}{\epsilon^{1+\alpha}}[\frac{v^\epsilon}{\epsilon^{1-\alpha}} - T(\frac{v^\epsilon}{\epsilon^{1-\alpha}} + \epsilon^{1+\alpha}\Delta v^\epsilon)] = -M^\epsilon(\phi)g_\xi \ + O(\epsilon).$$

The statement of the lemma now follows once we observe that

$$v_t^\epsilon = g_\xi \phi_t^\epsilon + O(\epsilon) = g_\xi \phi_t + O(\epsilon).$$

□

We now estimate the error terms. All our earlier estimates still hold for g and h and their derivatives, so we only need to estimate $\tilde{h}$.

Lemma 7.2 *There exists some number $k > 0$ such that*

$$\partial_\xi^{\beta_1}\partial_\lambda^{\beta_2}\tilde{h} = O(\lambda^{-k}g_\xi)$$

for all $\beta_1, \beta_2 \geq 0$ such that $\beta_1 + \beta_2 \leq 2$.

Proof. We may integrate (7.6) to find that

$$\tilde{h}_\xi(\xi,\lambda) = \frac{1}{\lambda g_\xi^2}\int_0^\xi g_\xi^2[\kappa(\lambda)W''(g) + (\kappa(\lambda)-1)]d\sigma.$$

This can be verified by differentiating the above expression and using (7.1) and (7.4). Our definition of κ guarantees that the right-hand side is periodic in ξ, so we do not need to add a constant of integration.

In principle, we could now use the definition of $\kappa(\lambda)$ to express $\tilde{h}_\xi$ entirely in terms of g and its derivatives.

Now we can derive explicit expressions for $\tilde{h}$ and its derivatives by integrating and differentiating the above expression. (It is here that we use our assumption that W is C^4.) The only terms occurring in these expressions will be g and its derivatives up to third order, all of which we have already estimated, and it is clear from our earlier estimates that the conclusion of the lemma is satisfied. □

With the estimates from Section 3, Lemma 7.2 implies that if $\alpha > 0$ is sufficiently small, then the error term in Lemma 7.1 is $O(\epsilon^{C^{-1}})g_\xi$, for some $C^{-1} > 0$. For such α and for $\alpha = 0$, we may now proceed exactly as above to construct supersolutions of the PDE (7.2). The proof of Theorem 1.1 uses only the asymptotic expansion, the error estimates, the constructed supersolutions, and certain facts about the structure of the limiting PDEs which still hold in this case. Thus the machinery assmebled above now allows us immediately to deduce the following theorem. We are implicitly assuming that the initial data h belongs to a class of functions for which the PDE has a solution.

Theorem 7.1 *Suppose that u^ϵ solves (7.2) with initial data $u^\epsilon(x,0) = h(x)$, where h is Lipschitz. Then there exists some $\alpha_0 > 0$ such that if $0 \le \alpha < \alpha_0$, then u^ϵ converges uniformly to a function u, and u solves the equation*

$$u_t - \kappa(|Du^2|;\alpha)\left(\delta_{ij} - \theta(|Du|^2;\alpha)\frac{u_{x_i}u_{u_j}}{|Du|^2}\right)u_{x_ix_j} = 0$$

with initial data $u(x,0) = h(x)$. Here

$$\kappa(\lambda;\alpha) = \begin{cases} \kappa(\lambda) & \text{if } \alpha = 0 \\ \bar{\kappa} = \lim_{\lambda\to 0}\kappa(\lambda) & \text{if } \alpha > 0 \end{cases}$$

□

We briefly discuss the possible significance of this result. We will focus on the case $0 < \alpha < \alpha_0$.

Suppose that $W(\cdot)$ has exactly J zeroes in each period, so that in the reference period $[-1/2, 1/2)$, we have $-1/2 \le z_1 < z_2 < \ldots < z_J < 1/2 \le z_{J+1} =$

z_1+1. We may then let $\alpha = 1$ in (7.2) and solve the equation with appropriate initial data having roughly the form

$$h^\epsilon(x) \sim z^i \chi(x) + z_{i+1}(1-\chi(x)),$$

where χ is the characteristic function of some open, bounded, reasonably smooth set. Results in [12] show that, as $\epsilon \to 0$, solutions u^ϵ exhibit an interface evolving via generalized mean curvature, rescaled by a factor $\kappa_i = \sigma_i \mu_i$, where

$$\sigma_i = \int_{z_i}^{z_i+1} \sqrt{2W(u)}du, \qquad \mu_i = \left(\int_{z_i}^{z_i+1} (W''(u)+1)\sqrt{2W(u)}du.\right)^{-1}.$$

We refer to κ_i as the kinetic constant, for want of a better name. Very similar results are obtained for a nonlocal, double-well version of (7.2) in [13], see also [15]. Formal computations carried out by Buttá in [4] show that we may interpret σ_i and μ_i as the surface tension and the mobility, respectively, of the wave front associated with the pair of adjacent zeroes $\{z_i, z_{i+1}\}$ of $W(\cdot)$.

In the infinite-well case, with $0 < \alpha < \alpha_0$, the limiting equation specifies that every level set moves via generalized mean curvature motion, with kinetic constant $\overline{\kappa}$. Using (2.6) and (2.7), we may rewrite the definition of $\kappa(\lambda)$ in the form

$$\kappa(\lambda) = \frac{\int_0^1 \sqrt{2W(u)+c(\lambda)}du}{\int_0^1 (W''(u)+1)\sqrt{2W(u)+c(\lambda)}du}.$$

In particular, we have $\overline{\kappa} = \overline{\sigma}\,\overline{\mu}$, where

$$\overline{\sigma} = \int_0^1 \sqrt{2W(u)}du, \qquad \overline{\mu} = \left(\int_0^1 (W''(u)+1)\sqrt{2W(u)}du\right)^{-1}.$$

Thus the limiting kinetic constant in the infinite-well case has the same form as in the two-well case. Note also that

$$\overline{\sigma} = \sigma_1 + \cdots + \sigma_J, \qquad \overline{\mu}^{-1} = \mu_1^{-1} + \cdots + \mu_J^{-1}.$$

We interpret $\overline{\sigma}$, $\overline{\mu}$, and $\overline{\kappa}$ as reflecting the average kinetic properties of a wavefront formed from the interactions of the J fronts associated with the distinct pairs of potential wells. Thus our results suggest that the surface tension and the mobility of a front composed of a number of smaller fronts can be expressed as above in terms of the surface tension and mobility of the constituent wavefronts.

Finally we examine the family of equations

$$u_t^\epsilon - \Delta u^\epsilon + \frac{1}{\epsilon^{1+\alpha}}(W_0'(\frac{u^\epsilon}{\epsilon^{1-\alpha}}) + \epsilon W_1'(\frac{u^\epsilon}{\epsilon^{1-\alpha}})) = 0 \tag{7.8}$$

discussed in the introduction. Here W_1 may be any smooth function the derivative of which is periodic with period 1. As above, the asymptotic behavior of this equation for a range of values $0 < \alpha < \alpha_0$ can be established using the machinery developed for (1.1). In this case we modify the asymptotic expansion by letting the auxilliary functions depend on ϵ as follows. Let $W^\epsilon = W_0 + \epsilon W_1$. For every $\lambda > 0$, we require that

$$W^{\epsilon\prime}(g^\epsilon) - \lambda g^\epsilon_{\xi\xi} - \epsilon\omega^\epsilon(\lambda)\sqrt{\lambda}g^\epsilon_\xi = 0.$$

$$g^\epsilon(0,\lambda) = 0, \qquad g^\epsilon(\xi+1,\lambda) = g^\epsilon(\xi,\lambda) + 1.$$

Multiplying the ODE by g^ϵ_ξ and integrating gives a compatibility condition which we use to define ω^ϵ:

$$\omega^\epsilon(\lambda) = \frac{W_1(1) - W_1(0)}{\sqrt{\lambda}\int_0^1 g_\xi^{\epsilon 2} d\sigma}.$$

One can verify that

$$\lim_{\epsilon\to 0}\omega^\epsilon(\lambda) \equiv \omega(\lambda) = \frac{W_1(1) - W_1(0)}{\sqrt{\lambda}\int_0^1 g_\xi^2 d\sigma} = \frac{[W_1]_0^1}{\int_0^1 \sqrt{2W(u) + c(\lambda)}du},$$

where g is the solution of the above ODE with $\epsilon = 0$.

We also define h^ϵ by

$$W^{\epsilon\prime\prime}(g^\epsilon)h^\epsilon - \lambda h^\epsilon_{\xi\xi} - \epsilon\omega^\epsilon(\lambda)\sqrt{\lambda}h^\epsilon_\xi = 4g^\epsilon_{\xi\lambda} + \frac{\theta^\epsilon(\lambda)}{\lambda}g^\epsilon_\xi$$

with periodicity conditions as before and θ^ϵ defined by an appropriate modification of (2.6).

We now have

Lemma 7.3 *If ϕ is a smooth function, let $(\xi^\epsilon,\lambda^\epsilon) = (\frac{\phi}{\epsilon^{1-\alpha}}, \epsilon^{2\alpha}|D\phi|^2)$, and on the set where $\lambda^\epsilon \neq 0$ define*

$$v^\epsilon = \epsilon^{1-\alpha}g^\epsilon(\xi^\epsilon,\lambda^\epsilon) + \epsilon^{2+2\alpha}\phi_{x_i}\phi_{x_j}\phi_{x_ix_j}h^\epsilon(\xi^\epsilon,\lambda^\epsilon).$$

Then v^ϵ is smooth where defined, and on this set we have

$$\begin{aligned} & v^\epsilon_t - \Delta v^\epsilon + \frac{1}{\epsilon^{1+\alpha}}W^{\epsilon\prime}(\frac{v^\epsilon}{\epsilon^{1-\alpha}}) \\ &= g^\epsilon_\xi(\xi^\epsilon,\lambda^\epsilon)\left[\phi_t - [\delta_{ij} - \theta^\epsilon(\lambda^\epsilon)\frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}]\phi_{x_ix_j} + \omega^\epsilon(\lambda^\epsilon)|D\phi|\right] + O(\epsilon), \end{aligned}$$

where the $O(\epsilon)$ error term contains derivative of ϕ of up to fourth order and derivatives of g^ϵ and h^ϵ of up to second order.

The proof of this lemma is a calculation very similar to that given in the proof of Lemma 2.1. To complete the proof of the assertion contained in the introduction, we would estimate the error terms and then argue exactly as the proof of Theorem 1.1. Because the ODEs depend on ϵ, the required estimates now involve some work beyond what we have already done in Section 3. We therefore do not complete the proof, because very little insight would be gained by going through these estimates. Formally, though, the above lemma shows that solutions of (7.8) are governed in small-ϵ limit by the equation

$$u_t - (\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}) u_{x_i x_j} + \overline{\omega}|Du| = 0$$

if $0 < \alpha < \alpha_0$, where $\overline{\omega} = \lim_{\lambda \to 0} \omega(\lambda)$. This is essentially the result stated in the introduction. If $\alpha = 0$, the (formal) limiting equation is

$$u_t - (\delta_{ij} - \theta(|Du|^2) \frac{u_{x_i} u_{x_j}}{|Du|^2}) u_{x_i x_j} + \omega(|Du|^2)|Du| = 0.$$

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