Microstructures Minimizing the Energy of a Two Phase Elastic Composite in Two Space Dimensions.

II: The Vigdergauz Microstructure

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Microstructures minimizing the energy of a two phase elastic composite in two space dimensions. II: the Vigdergauz microstructure.

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Abstract

For modeling coherent phase transformations, and for applications to structural optimization, it is of interest to identify microstructures with minimal energy or maximal stiffness. S. Vigdergauz has shown the existence of a particularly simple microstructure with extremal elastic behavior, in the context of two-phase composites made from isotropic components in two space dimensions. This "Vigdergauz microstructure" consists of a periodic array of appropriately shaped inclusions. We provide an alternative discussion of this microstructure and its properties. Our treatment includes an explicit formula for the shape of the inclusion, and an analysis of various limits. We also discuss the significance of this microstructure (i) for minimizing the maximum stress in a composite, and (ii) as a large volume fraction analog of Michell trusses in the theory of structural optimization.

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1 Introduction.

This paper is devoted to the "Vigdergauz microstructure", a special elastic composite in two space dimensions whose microscopic behavior is in a certain sense elastically extremal. This composite is spatially periodic, consisting of properly shaped elastic inclusions embedded in an elastic matrix. We call it extremal because it minimizes the overall energy at a given strain ξ, among all composites made from the same components in the same volume fractions.

To explain why this microstructure is interesting, we note that extremal composites have recently received a lot of attention. One reason lies in applications to structural optimization, see e.g. [2], [8], [12], [19], [26], [32], [39]. Another lies in theoretical developments in the analysis of composites, see e.g. [3], [11], [13], [36], [37], [38], [44]. A third motivation is the idea that the shapes of coherent precipitates may be explained by elastic energy minimization, see e.g. [20], [21], [22], [28], [31], [40], [45].

As a result of recent progress, we now know a rather general algorithm for computing examples of elastically extremal two-component composites [3], [5], [24], [36]. This algorithm works in principle in three space dimensions, and with anisotropic component materials. It produces the extreme value of the elastic energy, and examples of extremal microstructures obtained by a construction known as "sequential lamination".

It is well-known that extremal microgeometries are generally not unique. To take a familiar example, the Hashin-Shtrikman bounds on the effective bulk modulus [17] are realized both by sequential lamination [11] and by the concentric sphere construction [16]. This degeneracy is by no means restricted to the case of the bulk modulus bound, see e.g. [15]. Thus it is natural to consider other examples of extremal microstructures, particularly ones that are in some sense simple.

Hence our interest in the Vigdergauz microstructure: it achieves the same extremal behavior as the other known constructions ("second-rank lamination" and the "confocal ellipse construction"), but it is in some sense simpler and more ordered. This microstructure was first discovered by S. Vigdergauz in a series of papers [46]–[56]. His first paper [46] simply sought a finite number of "equally strong" holes in an elastic plate. He reduced the task of finding such holes to that of solving a particular integral equation, which he then solved numerically. Subsequently Vigdergauz extended his calculations to periodic arrays of elastic inclusions as well as holes [51], [54], [56]. In [56] he also computed the energy of the microstructure explicitly, and observed that the value coincides with the optimal bounds derived by L. Gibiansky and A. Cherkaev, [12].

The purpose of the present article is to give an alternative treatment of the Vigdergauz microstructure, and to explore its properties in more detail. We take a different starting point, namely the "optimality conditions" for elastic energy minimization as derived in [14]. This leads, through the use of Kolosov-Muskhelishvili potentials, to a problem in complex variables for the shape of the inclusions. The problem is very similar to that addressed by Cherepanov in [9]. Using his ideas we derive an explicit representation for the shape of the inclusions in terms of elliptic functions. We recover almost all of the Vigdergauz's results, with arguments that are somewhat simpler, and a representation of the answer that is much more explicit. However we do not have an explicit formula for the full Hooke's law tensor associated to this composite. Such a formula is asserted in [56] but we were not able to

understand that part of the paper.

The Vigdergauz microstructure consists of a spatially periodic array of inclusions of a particular shape. The shape of the inclusions depends on the average strain ξ (or more precisely, on its eigenvalues). When the deviatoric part of ξ becomes much larger than its hydrostatic part, this microstructure ceases to exist. The condition for its existence, equation (3.23) below, is easy to understand from the viewpoint of energy minimization: it defines one of the regimes in the optimal energy bound as derived in [4] (formula (1.4)). When the existence condition (3.23) breaks down one enters another regime, where the optimal geometries must have a different topological structure (see [14] for a more detailed discussion). The condition (3.23) appears in the work of Vigdergauz, too; there it was derived from the positivity of a Green's function for an elliptic operator.

In this paper we examine the behavior of the Vigdergauz microstructure in various limits. We are able to do so because our representation of the microstructure is very explicit. Remarkably, one can obtain such diverse structures as an optimal elliptical inclusion in an infinite plane, a simply layered composite, and a rank-two laminate as limits of the Vigdergauz geometry. More precisely, in the dilute limit the Vigdergauz microstructure consists of a periodic array of non-interacting optimal elliptical inclusions. As one approaches equality in the existence condition (3.23) the inclusions become increasingly elongated, forming in the limit a layered microstructure. And if we use a rectangle with sides L and $1/L$ as the period lattice rather than a square, then the Vigdergauz microstructure becomes a rank-two laminate in the limit $L \to \infty$.

Let us dwell a bit more on the dilute limit. It has been known since the work of Eshelby that the strain in an isolated elliptical inclusion is constant. Several authors have used this fact to predict the shapes and orientation of coherent precipitates in the dilute limit, by assuming an elliptical geometry and optimizing over orientation and eccentricity [7], [20], [21], [40] (see also [43] for a critical review). The Vigdergauz microstructure can be viewed as a large volume fraction analogue. It has constant strain in the inclusion, *provided that* the average strain takes the specified value ξ. Curiously, the strain in the inclusion is *always* isotropic even when ξ is anisotropic. (See [25], [31] for more on elastic energy minimization and the shapes of coherent precipitates.)

Our treatment of the Vigdergauz microstructure is organized around the minimization of elastic energy at fixed average strain ξ. It is equally natural to consider composites which minimize the complementary energy at fixed average stress σ_0. These are the most rigid composites, so they arise quite naturally in problems of structural optimization, see e.g. [2]. The Vigdergauz microstructure can be used to solve this problem too under certain conditions on σ_0 (see equation (5.6) — the analogue of (3.23) for complementary energy). The case when the "inclusions" are holes is particularly interesting. Then the Vigdergauz microstructure provides a systematic and appealing way to pass from a dilute array of elliptical holes (volume fraction≈ 0) to a Michell truss (volume fraction≈ 1), while maintaining exact optimality at any intermediate volume fraction — but only if the two principal stresses have the same sign, or equivalently if $\det \sigma_0 > 0$.

Questions of elastic energy aside, it is natural to think that the Vigdergauz microstructure might minimize the "stress concentration" within some class of composites. Indeed, this is probably what Vigdergauz had in mind when he first sought a periodic array of "equally strong inclusions", c.f. [6], [9], [10], [49], [57]. We explore this issue in section 5.2.

It seems likely that there should be an extension of the Vigdergauz construction to three space dimensions. However the methods of the present paper are entirely two dimensional. This question remains open aside from some suggestive calculations by Vigdergauz [49], [53], [55], where he gives necessary conditions for the existence of "equally-strong" cavities and presents a numerical scheme for calculating their shapes.

Another natural extension would be to let the component Hooke's laws be anisotropic. We address this topic in [14]. The result is rather interesting. For certain non-generic choices of the (now anisotropic) matrix material the Vigdergauz microstructure extends. But for "most" anisotropic choices of the matrix material, there is no elastically optimal analogue of the Vigdergauz construction. Thus an anisotropic perturbation of an isotropic Hooke's law can break the degeneracy (confocal ellipses vs. rank-two laminates vs. Vigdergauz construction) that is present in the isotropic case.

2 The problem and some notation.

Let us consider a unit square Q in R^2. Assume that it is "made" of two elastic isotropic materials with Hooke's laws C_1 and C_2. Then at any point $x \in Q$ the Hooke's law is given by the 4th order tensor

$$C(x) = C_1\chi_1(x) + C_2\chi_2(x).$$

where $\chi_1(x)$ and $\chi_2(x)$ are the indicator functions of the sets occupied by materials 1 and 2 respectively, with

$$\chi_1(x) + \chi_2(x) = 1.$$

Each material is characterized by a bulk modulus k_i and a shear modulus μ_i. The Hooke's law C_i is then defined by

$$C_i\eta = 2\mu_i\Big(\eta - \frac{1}{2}(\mathrm{Tr}\eta)I\Big) + k_i(\mathrm{Tr}\eta)I \tag{2.1}$$

for any symmetric 2×2 matrix η, where I is the identity matrix. Without loss of generality we may assume that $\mu_1 > \mu_2$.

In linear elasticity the stress and strain tensors $\sigma(x)$ and $e(x)$ are related via the Hooke's law:

$$\sigma(x) = C(x)e(x) \tag{2.2}$$

and the strain is

$$e_{ij} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big),$$

where v is the vector of displacements. The equilibrium equation is

$$\nabla \cdot \sigma(x) = 0. \tag{2.3}$$

We are interested in spatially periodic composites; therefore the equations of elasticity (2.2) and (2.3) must be solved on the period cell Q, with periodic boundary conditions. To be

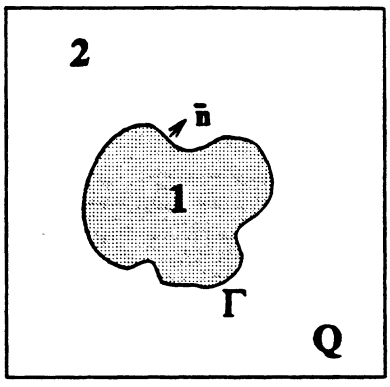

Figure 1: Structure of the period cell.

precise, for any 2×2 symmetric matrix ξ we look for a Q-periodic strain field with average value ξ:

$$\fint_Q e(x)dx = \xi, \tag{2.4}$$

where $\fint_Q$ denotes the average value over Q. Such a field is uniquely determined.

The elastic strain energy is given by

$$W = (\sigma^*, \xi), \tag{2.5}$$

where σ^* is the average stress:

$$\sigma^* = \fint_Q \sigma(x)dx. \tag{2.6}$$

Obviously the energy W depends on the microstructure. The problem we address is that of finding microstructures which minimize the energy W, when the volume fractions of the component materials are fixed, i.e. when the microgeometry is constrained by

$$\fint_Q \chi_1(x)dx = \theta. \tag{2.7}$$

In other words, we look for a characteristic function $\chi_1(x)$ (taking only the values 0 and 1) solving the following minimization problem

$$QW_\theta(\xi) = \inf_{<\chi_1>=\theta} \inf_{<e(v)>=\xi} \fint_Q (C(x)e(v), e(v))dx. \tag{2.8}$$

Specifically, in this paper we look for an **optimal microgeometry which is a Q-periodic array of simply connected inclusions with smooth boundary.** See Figure 1.

From now on we will restrict ourselves to a single period cell, except when we specifically refer to the two dimensional Q-periodic array. Our strategy is exactly as in the

paper [15], henceforth referred to as Part I. We begin with the optimality conditions and apply them to the complex variable formulation of the problem. Here we have a periodic problem, while Part I considers a problem with affine boundary condition, but the optimality conditions for the two cases are the same (see [14]). Therefore the identification of the Kolosov-Muskhelishvili potentials proceeds exactly as in Part I, (equations (3.6), (3.7), (3.8)):

$$\phi_1 = k_1 \varepsilon_0 z, \qquad \psi_1 = 0 \tag{2.9}$$

$$\phi_2(z) = \frac{1}{2} d k_2 z \tag{2.10}$$

$$\psi_2(z) = c\bar{z}, \quad z \in \Gamma \tag{2.11}$$

where the constants ε_0, d and c are given by equations (2.6), (2.7) and (3.9) of Part I. Thus ϕ_1, ψ_1 and ϕ_2 are fully determined in Q. The potential ψ_2 remains unknown, as does the shape of the interface Γ. These will be determined in section 3.

We note in passing that it is not strictly necessary to use the full set of optimality conditions as in Part I. One can actually arrive at (2.9)–(2.11) by starting from the apparently weaker hypothesis that the strain in the inclusion is a constant multiple of the identity. But the argument is long and very tedious, whereas the passage from the full set of optimality conditions to (2.9)–(2.11) is very easy.

Now we use the periodicity to obtain a translation law for the potentials ϕ_2 and ψ_2. From the double periodicity of $e(v)$ and $v(x) - \xi x$ we find, using the method of [34] (p. 251), that the complex potentials ϕ_2 and ψ_2 are single valued, and

$$\left.\begin{aligned} \phi_2 &= \phi_0(z) + \beta z + \gamma \bar{z} \\ \psi_2 &= \psi_0(z) - \bar{z}\Phi_2(z) + \Lambda\bar{\gamma} z + \Lambda\bar{\beta}\bar{z} - 2\mu_2 \overline{\xi z} \end{aligned}\right\} \tag{2.12}$$

where $\Lambda = 1 + 2\mu_2/k_2$; ϕ_0, ψ_0 are Q-periodic functions (they do not have to be analytic); and β, γ are constants to be determined. The expression ξz in (2.12) is understood in the sense of complex variables, i.e.

$$\xi z = \frac{1}{2}(z \mathrm{Tr}\xi - \bar{z}(\xi_{22} - \xi_{11} - 2i\xi_{12})).$$

Now comparing the first line of (2.12) with (2.10) we find that

$$\beta = \frac{1}{2} d k_2, \qquad \gamma = 0.$$

Applying this and (2.10) to the second formula in (2.12) we obtain

$$\psi_2(z) = \psi_0(z) + \theta_1 c\bar{z} + bz, \tag{2.13}$$

where ψ_0 is Q-periodic as before, and b and c are defined in Part I, (equations (3.9), (3.11)).

We note that we haven't used the average strain condition (2.4). However one can check that (2.9), (2.10), (2.11) and (2.13) are consistent with (2.4). The calculations involved are

not completely trivial. The key is to use the formulas

$$\left.\begin{aligned} \Im\mathrm{m}[\frac{1}{k}\oint_\Gamma \overline{\phi}(z)dz] &= \frac{2}{k_2}\beta - \mathrm{Tr}\xi \\ [\frac{1}{\mu}\oint_\Gamma (z\overline{\Phi}(z) + \overline{\psi}(z))dz] &= \frac{2i\Lambda}{\mu_2}\gamma \end{aligned}\right\} \tag{2.14}$$

where β and γ are the same as in (2.12) and square brackets denote the jump across the interface (e.g. $[\phi] = \phi_2 - \phi_1$). Incidentally, the formulas (2.14) do not depend on the optimality conditions: they are valid for *any* smooth, connected inclusion in Q determining a spatially periodic composite.

We also note the following formula for the average stress σ^*:

$$\left.\begin{aligned} \sigma^*_{11} + \sigma^*_{22} &= 2\beta(\Lambda+1) - 2\mu_2\mathrm{Tr}\xi \\ \sigma^*_{22} - \sigma^*_{11} + 2i\sigma^*_{12} &= 2\mu_2(\xi_{22} - \xi_{11} + 2i\xi_{12}) + 2\bar{\gamma}(\Lambda+1) \end{aligned}\right\} \tag{2.15}$$

This is obtained from (2.6) by representing the local stress $\sigma(x)$ in terms of the complex potentials, then evaluating the integral. Therefore the energy W defined in (2.5) is given by

$$W = (\Lambda+1)(\Re\mathrm{e}[\bar{\gamma}(\xi_{22} - \xi_{11} + 2i\xi_{12})] + \beta\mathrm{Tr}\xi) - 4\mu_2 \det\xi. \tag{2.16}$$

Substituting the values of β and γ in (2.16) we obtain an expression that coincides with the minimum value of the energy derived in [4] (formula (1.4)).

3 Solution of the reduced problem.

In this section we find the remaining unknown complex potential ψ_2 and the shape of the inclusion *explicitly*. Here as in Part I we apply the method of Cherepanov, [9]. We recall that the remaining unknown potential ψ_2 satisfies (2.11) and (2.13):

$$\left.\begin{aligned} \psi_2(z) &= c\bar{z}, \qquad z \in \Gamma \\ \psi_2(z) &= \psi_0(z) + \theta_1 c\bar{z} + bz. \end{aligned}\right\} \tag{3.1}$$

Conversely, any analytic function satisfying (3.1) provides, together with (2.9) and (2.10), an optimal field in the matrix. We will show that (3.1) has a solution for a certain regime of values of the average strain ξ.

We begin by mapping the exterior of a periodic array of slits in the ζ plane onto region 2 in the z plane. More precisely, let $z = w(\zeta)$ map the periodic array of slits M of length $2e$ ($e \in (0,1)$) with the period cell $Q' = [0,2) \times [0,2h)$ in the ζ plane to the Q-periodic array of inclusions with smooth boundary in the z plane (see Figure 2). The map w must satisfy the following conditions:

$$w(0) = 0; \qquad w(\zeta+2) = w(\zeta) + 1; \qquad w(\zeta + 2ih) = w(\zeta) + i. \tag{3.2}$$

At the endpoints of the slits $w(\zeta) = O(\sqrt{\zeta - \zeta_M})$ as $\zeta \to \zeta_M$, where ζ_M is an endpoint of a slit, on account of the smoothness of the boundary of the inclusion in the z plane.

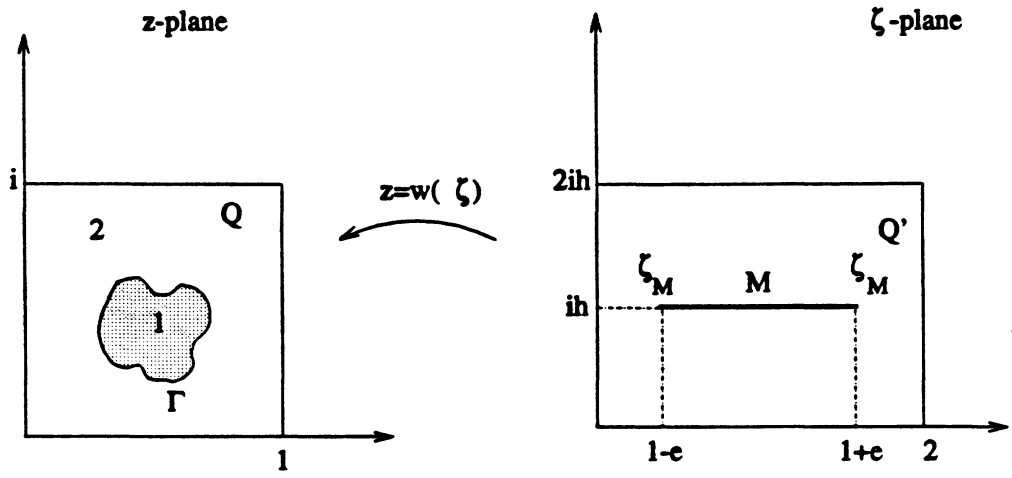

Figure 2: Conformal Mapping

Let us now substitute $z = w(\zeta)$ in the first equation of (3.1) and differentiate along the slit. Using notation $\Psi(\zeta) = \Psi_2(w(\zeta))$ we obtain:

$$\Psi(\zeta)w'(\zeta) = \overline{cw'}(\zeta), \qquad \zeta \in M. \tag{3.3}$$

We can represent (3.3) using the following trick of Cherepanov: Consider two analytic functions F and G chosen such that

$$F'(\zeta) = -\Psi(\zeta)w'(\zeta) + cw'(\zeta), \tag{3.4}$$

$$G'(\zeta) = -\Psi(\zeta)w'(\zeta) - cw'(\zeta). \tag{3.5}$$

Then (3.3) becomes

$$\left.\begin{aligned} \Re\mathrm{e}F'(\zeta) = 0, \quad \zeta \in M, \\ \Im\mathrm{m}G'(\zeta) = 0, \quad \zeta \in M. \end{aligned}\right\} \tag{3.6}$$

Besides (3.6) the analytic functions F and G have the following properties: they are Q' periodic, by (2.13); at the endpoints of the slits

$$F'(\zeta) = O(\frac{1}{\sqrt{\zeta - \zeta_M}}) \text{ and } G'(\zeta) = O(\frac{1}{\sqrt{\zeta - \zeta_M}}) \text{ as } \zeta \to \zeta_M; \tag{3.7}$$

also F' and G' are single valued and have no other singularities. Once such functions are found, using (3.4), (3.5) we can easily reconstruct $w(\zeta)$ and $\Psi(\zeta)$. The result is

$$w(\zeta) = \frac{1}{2c}(F(\zeta) - G(\zeta)) + C_0, \tag{3.8}$$

where C_0 is a constant of integration, determined by (3.2), and

$$\Psi(\zeta) = -c\frac{F'(\zeta)+G'(\zeta)}{F'(\zeta)-G'(\zeta)}. \tag{3.9}$$

Now let's construct the functions F and G. Let $\wp(\zeta)$ be the Weierstrass elliptic function with the period cell Q'. We introduce the notation

$$\wp(1) = e_1, \quad \wp(ih) = e_3, \quad \wp(1+ih) = e_2, \quad \wp(1-e+ih) = \wp(1+e+ih) = \lambda,$$

where $2h$ is the height of a period cell in the ζ-plane and e is the half-length of the slit M (see Figure 2). We remark that e_j and λ are real and $e_1 > e_2 > \lambda > e_3$ (see [33]).

Let us consider the function:

$$v(\zeta) = \sqrt{\frac{\wp(\zeta)-e_2}{\wp(\zeta)-\lambda}}. \tag{3.10}$$

We claim that $v(\zeta)$ has the following properties:

1. $v(\zeta)$ is single valued analytic function in the exterior of the periodic array of slits M;
2. $v(\zeta)$ is Q' periodic;
3. $v(\zeta) = O(\frac{1}{\sqrt{\zeta-\zeta_M}})$ as $\zeta \to \zeta_M$, and v is bounded everywhere else;
4. $\Re e(v(\zeta)) = 0$ on M.

To explain why, let us choose the branch of the square root such that $\sqrt{1} = 1$, with the branch cut along the negative real axis. Then the function $v(\zeta)$ has a branch cut wherever

$$\frac{\wp(\zeta)-e_2}{\wp(\zeta)-\lambda} < 0. \tag{3.11}$$

This is equivalent to the condition that $\wp(\zeta) \in (\lambda, e_2)$, which is satisfied only along the cut M (this is how the function (3.10) was constructed). Thus properties 1 and 4 are proved. Property 2 follows from the Q' periodicity of $\wp(\zeta)$. And property 3 follows from the fact that points ζ_M are simple points for $\wp(\zeta)$ ($\wp'(\zeta_M) \neq 0$).

We look for the functions F' and G' in the form

$$F' = r_1 v(\zeta) + id_1,$$

$$G' = ir_2 v(\zeta) + d_2,$$

where $r_j, d_j \in R$ are constants to be determined. It is easy to see that equations (3.6) are satisfied, as is the condition (3.7). In order to recover F and G from the above formulas we have to use the function

$$V(\zeta) = \int_0^\zeta v(z)dz. \tag{3.12}$$

This function is single valued in the exterior of the periodic system of the slits because $\oint_{\partial Q'} v(\zeta)d\zeta = 0$, since $v(\zeta)$ is doubly periodic. Let $V(2) = 2t_1$, $V(2ih) = 2it_3$, $t_j > 0$. Then according to (3.2) and (3.8) we obtain:

$$r_1 = c\frac{h-1}{t_1 h - t_3}, \qquad r_2 = 0,$$

$$d_1 = 0, \qquad d_2 = c\frac{t_3 - t_1}{t_1 h - t_3},$$

and therefore

$$w(\zeta) = \frac{1}{2}\frac{h-1}{t_1 h - t_3}V(\zeta) + \frac{1}{2}\frac{t_1 - t_3}{t_1 h - t_3}\zeta. \tag{3.13}$$

Thus by (3.9)

$$\psi_2(w(\zeta)) = \frac{1}{2}c\frac{t_1 - t_3}{t_1 h - t_3}\zeta - \frac{1}{2}c\frac{h-1}{t_1 h - t_3}V(\zeta) + \text{const.} \tag{3.14}$$

Now using the translation law for the potential ψ_2 (the second equation in (3.1)) we obtain:

$$\left.\begin{aligned} \theta_1 + q &= \frac{2T - 1 - Th}{Th - 1} \\ \theta_1 - q &= \frac{2h - 1 - Th}{Th - 1}, \end{aligned}\right\} \tag{3.15}$$

where

$$q = \frac{b}{c}, \qquad T = \frac{t_1}{t_3}.$$

Notice that (3.15) implies that q is real, hence so is b. From the formula for b ((3.11) Part I) we see that $\xi_{12} = 0$. This means that the sides of the period cell have to be oriented along the eigendirections of the average strain tensor ξ. Solving the system (3.15) for h and T we obtain:

$$h = \frac{\theta_2 + q}{1 + \theta_1 + q}, \tag{3.16}$$

$$T = \frac{\theta_2 - q}{1 + \theta_1 - q}. \tag{3.17}$$

(The other pair of solutions $h = 1$ and $T = 1$ is not feasible because it leads to a slit of length zero. However, we will see that $h \to 1$ and $T \to 1$ in the limiting case when the volume fraction of the inclusion tends to zero.) Finally substituting (3.16) and (3.17) into (3.13) we obtain:

$$w(\zeta) = \frac{1 + \theta_1 - q}{4t_3}\int_0^\zeta \sqrt{\frac{\wp(z) - e_2}{\wp(z) - \lambda}}dz + \frac{1 + \theta_1 + q}{4}\zeta \tag{3.18}$$

$$\psi_2(w(\zeta)) = \frac{c(1 + \theta_1) + b}{4}\zeta - \frac{c(1 + \theta_1) - b}{4t_3}V(\zeta) + \text{const.} \tag{3.19}$$

Now we have to determine parameter e (the half-length of the slit) or, equivalently, the parameter λ (the value of $\wp(\zeta)$ at the endpoints of the slit):

$$\lambda = \wp(1+e+ih) = \wp(1-e+ih).$$

We notice that by the definition of T

$$T(\lambda) = \frac{t_1(\lambda)}{t_3(\lambda)} = \frac{\int_0^2 \sqrt{\frac{\wp(x)-e_2}{\wp(x)-\lambda}}dx}{\int_0^{2h} \sqrt{\frac{\wp(ix)-e_2}{\wp(ix)-\lambda}}dx}. \tag{3.20}$$

It is easy to check that in the above formula the numerator is an increasing function of λ, and the denominator is a decreasing function function of λ. Thus $T(\lambda)$ is an increasing function of λ. It increases from 0 to $1/h$ as λ increases from e_3 to e_2. Indeed, when $\lambda = e_3$ the numerator in (3.20) is finite, while the integral in the denominator diverges since $\wp'(ih) = 0$, [33]. When $\lambda = e_2$ then $T(\lambda)$ is clearly $1/h$. Therefore in order to be able to solve for λ we need

$$0 < T < \frac{1}{h}. \tag{3.21}$$

By (3.16), (3.17), this condition implies

$$|q| < \theta_2; \tag{3.22}$$

substituting the value of q in the above inequality we find the alternative formulation

$$|\xi_{22} - \xi_{11}| < \theta_2 \frac{|k_1 - k_2| \cdot |\xi_{11} + \xi_{22}|}{\mu_2 + \theta_1 k_2 + \theta_2 k_1}. \tag{3.23}$$

The problem is solved.

Notice that (3.23) coincides with the definition of the third regime in energy bounds in [4]. We have seen this condition before, in Part I, as the condition for the existence of the confocal ellipse construction.

4 Analysis of the solution.

First of all we want to know qualitatively the shape of the optimal inclusion. Of course it depends on some parameters like volume fraction, average strain and material properties. But for all admissible values of the parameters it has rectangular symmetry due to the symmetry of $\wp(\zeta)$ with respect to the center of the period cell. Therefore, it is sufficient to study only one quarter of the inclusion. From the conformal mapping (3.18) we can obtain one quarter of the interface Γ in the parametric form: $x = \Re e[w(t+ih)]$, $y = \Im m[w(t+ih)]$, $t \in [1-e, 1]$. Using (3.11), which holds on the slit, we can write an explicit parametrization of Γ:

$$\begin{cases} x = x_0 + p_1 t \\ y = y_0 + p_2 \int_{1-e}^{t} \sqrt{\frac{\wp(s+ih)-e_2}{\lambda - \wp(s+ih)}}ds \end{cases} \tag{4.1}$$

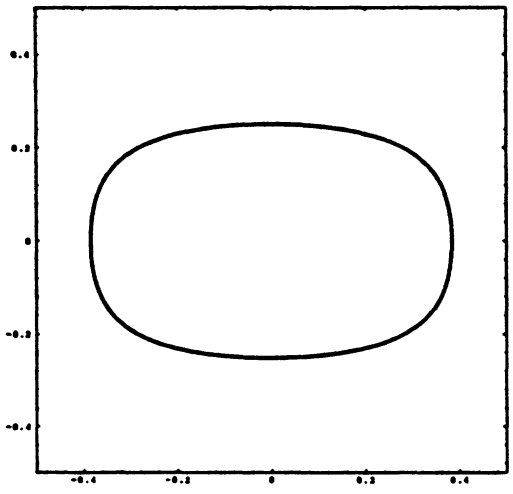

Figure 3: Vigdergauz inclusion in the period cell.

where $t \in [1-e, 1]$ and

$$p_1 = \frac{1+\theta_1+q}{4}$$
$$p_2 = \frac{1+\theta_1-q}{4t_3}.$$

Eliminating t from this expression and ignoring x_0 and y_0 we obtain:

$$y(x) = p_2 \int_{1-e}^{x/p_1} \sqrt{\frac{\wp(s+ih)-e_2}{\lambda-\wp(s+ih)}}\,ds. \tag{4.2}$$

Notice that the coefficients p_1 and p_2 are positive real numbers, so, $x \in [(1-e)p_1, p_1]$. In order to study the shape of the inclusion let us look at the derivative of $y(x)$:

$$y'(x) = \frac{p_2}{p_1}\sqrt{\frac{\wp(ih+x/p_1)-e_2}{\lambda-\wp(ih+x/p_1)}}.$$

It is easy to see that this function is positive, monotone decreasing and becomes 0 only when $x = p_1$, [33]. Therefore $y(x)$ is a monotone increasing and concave function. Thus the whole interface consists of four identical quarters (4.2) glued smoothly together ($\zeta = 1+ih$ is a double point of $\wp(\zeta)$), each of them being a graph of a monotone convex (concave) function. Figure 3 shows a "generic" Vigdergauz inclusion, drawn using the program Mathematica (for more explanation see the Appendix).

4.1 Low volume fraction limit.

Now let's study what happens to the shape of the inclusions as the volume fraction becomes small while all other parameters are fixed. Our goal is to show that the shape becomes

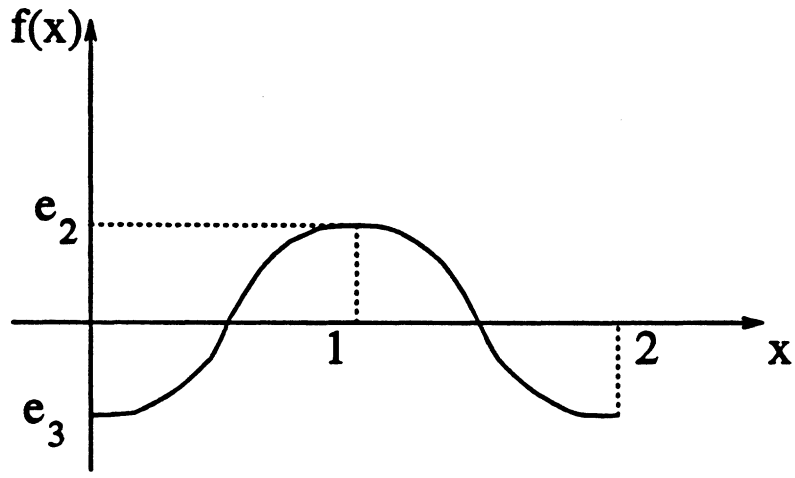

Figure 4: The sketch of the graph of $f(x) = \wp(i + x)$.

asymptotically an ellipse. When $\theta_1 \to 0$ we have $q \to q_0$, where

$$q_0 = \frac{\mu_2 + k_1}{k_1 - k_2} \cdot \frac{\xi_{22} - \xi_{11}}{\mathrm{Tr}\xi}.$$

Also

$$h = 1 - h_0\theta_1 + o(\theta_1),$$
$$T = 1 - T_0\theta_1 + o(\theta_1),$$

where

$$h_0 = \frac{2}{1 + q_0}; \qquad T_0 = \frac{2}{1 - q_0}.$$

Therefore from (3.20) we see that λ approaches e_2 and thus, t_3 approaches 1, while the half length of the slit $\epsilon \to 0^+$. Then we obtain

$$p_1 \to p_1^\circ = \frac{h_0}{2(h_0 + T_0)};$$

$$p_2 \to p_2^\circ = \frac{T_0}{2(h_0 + T_0)}.$$

Let $f(x) = \wp(i + x)$, $x \in [0, 1]$ (here $\wp$ is the Weierstrass' elliptic function with periods 2 and $2i$). The function $f(x)$ is real, with a graph as shown schematically in Figure 4, [33].

In the limit we have for a quarter of the interface boundary:

$$y(x) \approx p_2^\circ \int_{1-\epsilon}^{x/p_1^\circ} \sqrt{\frac{f(1) - f(s)}{f(s) - f(1 - \epsilon)}}\, ds$$

Expanding $f(s)$ in the power series near $s = 1$ and taking into account that $f'(1) = 0$ and $f''(1) \neq 0$ we obtain:

$$y(x) \approx p_2^\circ \int_{1-e}^{s/p_1^\circ} \frac{1-s}{\sqrt{e^2-(1-s)^2}} ds.$$

Evaluating the integral gives

$$y(x) \approx p_2^\circ \sqrt{e^2 - (1 - \frac{x}{p_1^\circ})^2},$$

or, equivalently

$$\frac{y^2}{(ep_2^\circ)^2} + \frac{(x-p_1^\circ)^2}{(ep_1^\circ)^2} = 1.$$

Thus in the small volume fraction limit $y(x)$ represents an ellipse with the eccentricity

$$\frac{p_1^\circ}{p_2^\circ} = \frac{h_0}{T_0}.$$

The fact that the ellipse is an optimal shape in the small volume fraction limit is known as a matter of theory, [30]. The orientation and eccentricity of such an optimal ellipse is uniquely defined by the value of the average strain [7], [20], [21], [23], [29], [40]. The above calculation recovers those parameters precisely, by realizing the optimal ellipse as a limit of microstructures that are optimal for any volume fraction.

4.2 Limiting rank-1 laminate.

Now let's consider a different asymptotic limit, namely what happens as we approach equality in the solvability condition (3.23). Our goal is to show that the inclusions become elongated, approaching in the limit a layered microstructure. Assume, to fix ideas, that q is positive, so the limit of interest is $q \to \theta_2$. In this case $h \to \theta_2$, $T \to 0$. Therefore $\lambda \to e_3$ (see the comment after (3.20)) and thus $t_3 \to \infty$. Therefore $p_1 \to 1/2$, $p_2 \to 0$. Now from (4.2) it easy to see that

$$y(x_1) - y(x_2) = p_2 \int_{x_2/p_1}^{x_1/p_1} \sqrt{\frac{\wp(s+ih)-e_2}{\lambda - \wp(s+ih)}} ds$$

and the integral is bounded as long as the distance between x_i/p_1 and $1-e$ is uniformly positive. Since $p_2 \to 0$, we see that $y(x) \to$ const. Thus we obtain horizontal layers. Figure 5 shows how the interface approaches horizontal layers as ξ changes, while other parameters are held fixed.

If one approaches the other side of the region defined by (3.23), i.e. $q \to -\theta_2$, then one would obtain vertical layers. This statement is obvious in real space because our period cell has square symmetry. On the other hand our formulas do not possess this symmetry since we have chosen a *horizontal* slit to be mapped into the unknown interface. The corresponding calculations require much more effort. It is possible to eliminate this asymmetry by expressing the function (4.2) in parametric form using elliptic integrals of the first kind (see the Appendix, formula (6.4)).

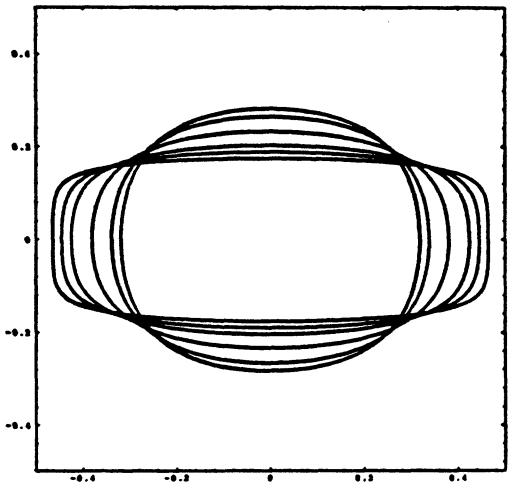

Figure 5: Evolution of Vigdergauz microstructure.

4.3 Limiting rank-2 laminate

So far in this article we have used a square as the period cell. However, the period cell does not have to be a square. It can just as well be a rectangle, with any aspect ratio. Let us summarize briefly how the calculations change. If Q_L is a rectangle with sides L and $1/L$ then equations (3.2), (3.13) and the formulas for r_i and d_i must be adjusted appropriately. It is not hard to see that if h_1 and T_1 are given by (3.16) and (3.17) respectively then the corresponding quantities for a rectangular period cell Q_L are

$$T_L = L^2 T_1,$$

$$h_L = \frac{h_1}{L^2}.$$

Since (3.21) depends only on the product Th we see that the existence condition (3.22) remains the same. Finally the formula for the inclusion changes: the conformal map w is now

$$w_L(\zeta) = \frac{1+\theta_1-q}{4Lt_3^{(L)}} V_L(\zeta) + L\frac{1+\theta_1+q}{4}\zeta \tag{4.3}$$

where V_L is defined as in (3.12) but using the Weierstrass $\wp$ function with periods 2 and $2ih_L$.

As $L \to \infty$, the Vigdergauz microstructure approaches a "sequentially laminated microstructure of rank 2", the microstructure used in [4] and [13] to prove the optimality of the energy bounds. (See e.g. [11] and [35] for further discussion of sequentially laminated microstructures.) To see this we should use a different scaling. We can do so because what is important is the aspect ratios, not the actual sizes. Thus without loss of generality we may assume that the macro-scale corresponds to lengths of order 1. The longer side of the

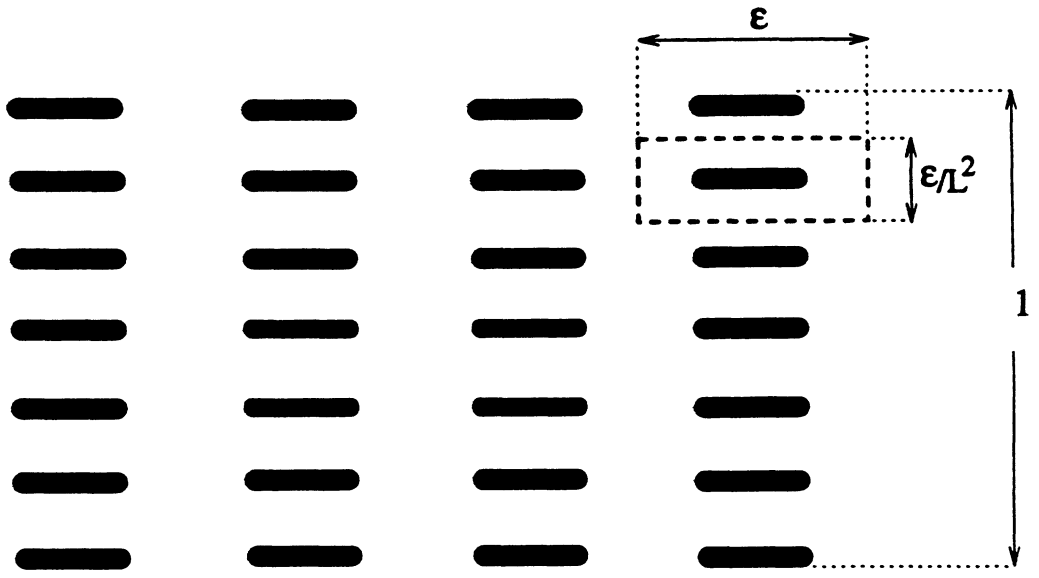

Figure 6: Limiting Second Rank Laminate.

period cell should be of length ε. Then the shorter side must be of length ε/L^2 to make the period cell have aspect ratio L^2. When $L \to \infty$ we have three different length scales 1, ε and ε/L^2. In the limit the Vigdergauz microstructure looks like a doubly periodic array of platelets (see Figure 6). This is exactly what is called a rank two laminate.

It is natural to ask if similar microstructures exist with a non-rectangular period cell. This question remains open.

5 Applications.

5.1 Structural optimization.

In problems of structural optimization one is usually interested in the most rigid possible composite. Therefore, in this framework it is more interesting to minimize the compliance rather than the elastic energy. We will show that the two minimization problems are different but closely related (see [2] or [14] for more detailed treatments).

We start with the dual variational principle for the compliance U:

$$U = \inf_{\substack{\nabla\cdot\sigma=0 \\ <\sigma>=\sigma_0}} \int_Q (C^{-1}(x)\sigma,\sigma)dx. \tag{5.1}$$

The problem we consider in this section is to minimize the compliance U over all microstructures with fixed volume fractions, i.e. to evaluate

$$QU_\theta(\sigma_0) = \inf_{<\chi_1>=\theta} \inf_{\substack{\nabla\cdot\sigma=0 \\ <\sigma>=\sigma_0}} \int_Q (C^{-1}(x)\sigma,\sigma)dx. \tag{5.2}$$

We will show that in two space dimensions the Vigdergauz microstructure achieves the minimum value in (5.2), under certain restrictions on the average stress σ_0. But now, as

one would expect, the more rigid material must be in the matrix and the softer material in the inclusion.

In order to find a relation between (5.2) and our original problem (2.8) we represent the stress field σ via the Airy stress potential ϕ:

$$\sigma = R_\perp^t \nabla\nabla\phi R_\perp,$$

where

$$R_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since σ and $\nabla\nabla\phi$ are linearly related we can express σ as

$$\sigma = \mathcal{R}\nabla\nabla\phi,$$

where $\mathcal{R}$ is a 4^{th} order tensor with the symmetries of a Hooke's law. Then we can rewrite (5.1) as

$$\inf_{<\nabla\nabla\phi>=\xi} \int_Q (\mathcal{R}^t C^{-1}(x)\mathcal{R}\nabla\nabla\phi, \nabla\nabla\phi)dx, \tag{5.3}$$

where

$$\xi = R_\perp^t \sigma_0 R_\perp.$$

We can simplify (5.3) even further if we notice that for any isotropic Hooke's law C and any two symmetric matrices ξ and η

$$(C\mathcal{R}\xi, \mathcal{R}\eta) = (C\xi, \eta).$$

Thus (5.3) is equivalent to

$$\inf_{<e(\nabla\phi)>=\xi} \int_Q (C^{-1}(x)e(\nabla\phi), e(\nabla\phi))dx. \tag{5.4}$$

Here we used the fact that $\nabla\nabla\phi$ is always a symmetric matrix. If instead of $\nabla\phi$ in (5.4) we allow any vector field to be admissible then we will enlarge the space of admissible test fields and therefore

$$\inf_{<e(\nabla\phi)>=\xi} \int_Q (C^{-1}(x)e(\nabla\phi), e(\nabla\phi))dx \geq \inf_{<e(u)>=\xi} \int_Q (C^{-1}(x)e(u), e(u))dx. \tag{5.5}$$

Thus we have found that the minimum of the compliance is bounded below by the minimum of the strain energy with a different Hooke's law, namely the inverse of the actual local Hooke's law tensor. It is interesting to ask when equality holds in (5.5). Obviously this is the case if and only if the optimal field u for the right hand side of (5.5) is curl-free. If the right hand side of (5.5) is optimized by the Vigdergauz microstructure, then the associated field u is indeed curl-free, as can be seen from the optimality conditions presented in Part I (formula (2.8)). Thus, the Vigdergauz microstructure achieves the minimum in (5.2) whenever (3.23) is satisfied with σ_0 in place of ξ, and k_i^{-1} and μ_i^{-1} in place of k_i and μ_i:

$$|\sigma_1 - \sigma_2| < \theta_2 \frac{\mu_2 |k_2 - k_1| \cdot |\sigma_1 + \sigma_2|}{k_1 k_2 + \mu_2(\theta_1 k_1 + \theta_2 k_2)}, \tag{5.6}$$

where σ_1 and σ_2 are the principal stresses of the average stress σ_0.

Now let us consider a particular case of interest: shape optimization. In other words we take one of the materials to be void, so $k_1 = \mu_1 = 0$. In this case the existence condition (5.6) for the Vigdergauz construction becomes

$$\det \sigma_0 > 0. \tag{5.7}$$

If this condition is satisfied then a periodic array of holes with the Vigdergauz shape will minimize the complementary energy at average stress σ_0.

Notice that when $k_1 = \mu_1 = 0$, the volume fraction does not figure in the existence condition (5.7). Therefore, we can study the "large volume fraction limit". When $\theta_1 \approx 1$ we remove almost all of the material, leaving a truss-like structure with optimal properties. It is essentially a Michell truss, see e.g. [18], [27], [41], [42]. At intermediate volume fractions the microstructure can be thought of as a Michell truss with thickened members and rounded corners.

5.2 The problem of minimal stress concentration.

In this section we show that the Vigdergauz construction, besides minimizing the elastic energy, also solves a certain problem of minimizing the stress concentration.

Our objective is to find the shape of a simply connected inclusion of given area (whose characteristic function we denote by χ) that minimizes the stress concentration throughout the periodic composite with period cell Q:

$$\inf_{\chi} \sup_{x \in Q} \|\sigma(x)\|,$$

where $\|\sigma\|$ is the operator norm of matrix σ:

$$\|\sigma\| = \sup_{|v|=1} |\sigma v|.$$

To solve this problem we repeat the argument of Wheeler, [58]. As in [58] we assume that the two materials are well-ordered:

$$k_1 > k_2, \qquad \mu_1 > \mu_2.$$

The idea is to use maximum principle for the trace of the stress [6], which is harmonic in an isotropic material. As usual, quantities with index 1 refer to the inclusion phase while quantities with index 2 refer to the matrix phase.

If t denotes the tangent to the interface and n the normal then due to the continuity of displacements and tractions across the interface we have:

$$(e_1 t, t) = (e_2 t, t)$$

$$\sigma_1 n = \sigma_2 n.$$

Using the constitutive relations

$$\sigma_i = 2\mu_i\left(e_i - \frac{1}{2}(\mathrm{Tr} e_i)I\right) + k_i(\mathrm{Tr} e_i)I$$

we easily see that

$$\mathrm{Tr}\sigma_2 = \frac{k_2\mu_2(k_1+\mu_1)}{k_1\mu_1(k_2+\mu_2)}\mathrm{Tr}\sigma_1 + \frac{2k_2(\mu_1-\mu_2)}{\mu_1(k_2+\mu_2)}(\sigma_i n, n)$$

everywhere on the interface. Taking absolute values and estimating

$$|\mathrm{Tr}\sigma_1| \le 2\|\sigma\|_\infty$$

and

$$|(\sigma_i n, n)| \le \|\sigma\|_\infty,$$

where

$$\|\sigma\|_\infty = \sup_{x\in Q}\|\sigma(x)\|,$$

we obtain

$$|\mathrm{Tr}e_2| \le \frac{k_1+\mu_2}{k_1(k_2+\mu_2)}\|\sigma\|_\infty. \tag{5.1}$$

We also have

$$|\mathrm{Tr}e_1| = \frac{1}{2k_1}|\mathrm{Tr}\sigma_1| \le \frac{\|\sigma\|_\infty}{k_1}. \tag{5.2}$$

To use these estimates we observe that

$$|\mathrm{Tr}\xi| = \left|\fint_Q \mathrm{Tr}e(x)dx\right| \le \theta_1 \sup_{\mathbf{1}}|\mathrm{Tr}e_1| + \theta_2 \sup_{\mathbf{2}}|\mathrm{Tr}e_2| = \theta_1 \sup_{\Gamma}|\mathrm{Tr}e_1| + \theta_2 \sup_{\Gamma}|\mathrm{Tr}e_2|.$$

The last equality is due to the maximum principle and the fact that functions $\mathrm{Tr}e_1(x)$ and $\mathrm{Tr}e_2(x)$ are harmonic on their respective domains. Now we can apply our estimates (5.1) and (5.2):

$$\|\sigma\|_\infty \ge \frac{k_1(k_2+\mu_2)}{\mu_2+\theta_1 k_2+\theta_2 k_1}|\mathrm{Tr}\xi| \tag{5.3}$$

This inequality is valid for any simply connected inclusion in the period cell with volume fraction θ_1.

It remains to show that the bound is attained by the Vigdergauz geometry. It is not difficult to check that the norm of the stress for the Vigdergauz microstructure $\|\sigma_V(x)\|$ is constant on the boundary of the inclusion. We have already seen that the stress in the period cell has trace which is constant in each phase, and it follows (e.g. using the representation in terms of complex potentials) that the stress itself is harmonic. Thus by the maximum principle for harmonic functions, the maximum value of $\|\sigma_V(x)\|$ must occur on the interface boundary. But on the boundary of the Vigdergauz inclusion, as one can easily verify directly, $\|\sigma_V\|_\infty$ attains our lower bound (5.3) if and only if $k_1 > k_2$. Thus our assertion is proved.

We remark that the above argument makes essential use of the hypothesis that the matrix phase be connected. Without this hypothesis the result becomes false. For instance, a simple calculation shows that rank-1 layering often produces a smaller stress concentration than the Vigdergauz construction. This is the case, for example, if the average strain is isotropic and the layers are orthogonal to one side of a square period cell.

6 Appendix: An alternative representation for the inclusion shape.

In section 3 we derived an explicit formula (3.18) for the shape of the inclusions in the Vigdergauz construction. In fact (3.18) is rather inconvenient to use, both with respect to analytic investigation and for obtaining numerical solutions. For this reason we were led to seek an alternative formula.

We start with our initial parametrization (4.1):

$$\begin{cases} x = p_1 t \\ y = p_2 \int_{1-e}^{t} \sqrt{\dfrac{\wp(s+ih)-e_2}{\lambda-\wp(s+ih)}}\, ds, \end{cases}$$

where $t \in [1-e, 1]$. It gives a quarter of Γ. From the theory of the Weierstrass $\wp$-function [33], we know that $\wp(ih+s)$ is strictly monotone increasing from λ to e_2 on $[1-e, 1]$. Therefore we can reparametrize Γ using the new parameter $p = \wp(ih+t)$. After a routine but lengthy calculation we obtain:

$$\begin{cases} x = -p_1 \displaystyle\int_{\wp}^{e_2} \frac{ds}{2\sqrt{(s-e_1)(s-e_2)(s-e_3)}} \\ y = p_2 \displaystyle\int_{\lambda}^{\wp} \frac{ds}{2\sqrt{(s-e_1)(\lambda-s)(s-e_3)}} \end{cases} \tag{6.1}$$

These integrals can be expressed in terms of incomplete elliptic integrals of the first kind, [1]. It is convenient to do so, because many software packages are capable of evaluating such elliptic integrals. The resulting expression is

$$\begin{cases} x = -p_1 F\left(\sqrt{1-\dfrac{e_1-e_2}{e_2-e_3}\cdot\dfrac{\wp-e_3}{e_1-\wp}} \;\middle|\; \dfrac{e_2-e_3}{e_1-e_3}\right) \\ y = p_2 F\left(\sqrt{\dfrac{e_1-e_3}{e_1-\lambda}\cdot\dfrac{\wp-\lambda}{\wp-e_3}} \;\middle|\; \dfrac{e_1-\lambda}{e_1-e_3}\right), \end{cases} \tag{6.2}$$

where

$$F(x \mid m) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}.$$

Let

$$m = \frac{e_2-e_3}{e_1-e_3}, \qquad m_\lambda = \frac{e_1-\lambda}{e_1-e_3},$$

so that

$$\begin{cases} x(\wp) = -p_1 F\left(\sqrt{1-\dfrac{1-m}{m}\cdot\dfrac{\wp-e_3}{e_1-\wp}} \;\middle|\; m\right) \\ y(\wp) = p_2 F\left(\sqrt{\dfrac{1}{m_\lambda}\cdot\dfrac{\wp-\lambda}{\wp-e_3}} \;\middle|\; m_\lambda\right), \end{cases}$$

and let

$$t = \frac{1-m}{m} \cdot \frac{p-e_3}{e_1-p}.$$

Then it is easy to check that

$$\frac{1}{m_\lambda} \cdot \frac{p-\lambda}{p-e_3} = 1 - \frac{M}{t},$$

where

$$M = \frac{1-m}{m} \cdot \frac{1-m_\lambda}{m_\lambda}. \tag{6.3}$$

For one quarter of the interface the parameter t ranges from M to 1.

It remains to recalculate the constants p_1 and p_2 in terms of the physical parameters. Standard formulas (see e.g. [1]) provide the necessary information. The parameters m and m_λ can be found as unique solutions of the equations

$$\frac{K(1-m)}{K(m)} = h, \qquad \frac{K(1-m_\lambda)}{K(m_\lambda)} = T,$$

where h and T are given by (3.16) and (3.17) and $K(m) = F(1 \mid m)$ is the complete elliptic integral of the first kind. Once m and m_λ are known we can write the final parametrization of the interface:

$$\begin{cases} x(t) = -\dfrac{1+\theta_1+q}{4K(m)} F(\sqrt{1-t} \mid m) \\ \\ y(t) = \dfrac{1+\theta_1-q}{4K(m_\lambda)} F(\sqrt{1-\dfrac{M}{t}} \mid m_\lambda), \\ \\ t \in [M, 1], \end{cases} \tag{6.4}$$

where M is given by (6.3) above. This is the parametrization we used to create Figures 3 and 5.

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