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Yury Grabovsky
Carnegie Mellon University

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Bounds and extremal microstructures for two-component composites: A unified treatment based on the translation method.

*Yury Grabovsky**

Department of Mathematics
Carnegie-Mellon University
Pittsburgh, PA 15213

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Abstract

The overall energy or stiffness of an elastic composite depends on the microgeometry. Recently there has been a lot of work on "extremal microstructures" for elastic composites, for example microstructures which minimize the elastic energy at a given macroscopic strain. However, most attention has been focused on composites made of the elastically isotropic component materials. Breaking with this tradition we consider composites made of two *fully anisotropic* phases. Our approach, based on the well-known translation method, provides not only the energy bound but also necessary and sufficient conditions for optimality in terms of the local strain field. These optimality conditions enable us to find optimal microstructures in a more systematic way than before. They also provide clarification of the relations between different problems, for example bounding effective conductivity of a conducting composite versus minimizing strain energy of an elastic composite.

Our analysis shows that anisotropy of the constituent materials is very important in determining optimal microgeometries. Some constructions of extremal matrix-inclusion composites made from isotropic components cease to be available when the matrix material is anisotropic, even when the degree of anisotropy is small.

1 Introduction.

A composite material is by definition a mixture of homogeneous continua on a length scale small compared to loads and boundary conditions, but large enough for continuum theory to apply. On a macroscopic level it is characterized by its own elastic moduli, which are

*This work was done while Y. G. was a student at the Courant Institute.

called the *effective moduli* of the composite. They depend on both the composition and the *microstructure* — the geometric arrangement of the component materials in the composite.

There is by now a large literature on bounds for the effective moduli of composites and on microstructures that are in some sense extremal. Some of this work is in the mechanics literature, e.g. [17], [18], [30]. But recently such problems have received considerable attention from mathematicians, e.g. [21], [28], [34], [41]. Close links have also emerged with other areas, including nonconvex variational problems (e.g. [5]) and Young measure limits of gradients (e.g. [19]).

The main focus of this article is a direct link between extremal microstructures and the translation method—one of the recently developed tools for obtaining geometry-independent lower bounds. To prove the optimality of a bound it is necessary to show that it can be attained by a microstructure. The translation method, the way it was frequently used, provided few clues about what such a microstructure should be. Therefore, in many instances one had to resort to some *ad hoc* means to find an extremal microgeometry. In this article we present a systematic way of obtaining optimal microstructures directly from the derivation of the bounds (see sections 3.3 and 3.4). A similar approach has previously been used by Avellaneda, Cherkaev, Lurie and Milton [4] and also by Milton and Nesi [32] for a different problem involving conductivity in polycrystals. For a general treatment of the translation method we refer to the forthcoming book of Milton [27].

The microstructures we find to be optimal have already made their appearance in prior work, sometimes under different guises. Indeed, sequentially layered composites have been known since at least the work of Schulgasser [33]. The confocal ellipsoid construction has been used to saturate various conductivity bounds [7], [22], [23], [24], [27], [34]. The Vigdergauz construction was first discovered by S. Vigdergauz for the problem of minimizing stress concentrations in a plate with a regular system of holes [35], [36]; he later observed that this microstructure also minimizes elastic energy [39]. Still, we provide some new insight as to why the above geometries are optimal, while others are not.

To avoid repeating much of the known results, we test the power of the translation method on applications to composites made from two *fully anisotropic* component materials—a setting that has been avoided so far in the mathematics literature on explicit optimal bounds. Our analysis includes a detailed discussion of the minimization of elastic and complementary energies of periodic composites in two space dimensions (see Chapters 3 and 4). We achieve a level of understanding for this problem in the anisotropic setting which is comparable to that already available in the isotropic one [1], [2], [11]. In addition to these two dimensional results this paper includes several new results in higher dimensions. We will discuss them a little later in this introduction.

An interesting and somewhat unexpected fact emerges from our analysis. We find that some of the microstructures known to be optimal in the case of composites made from two isotropic materials, namely the confocal ellipse construction [15] and the Vigdergauz construction [16], [37], [39], cease to be available when the matrix material is anisotropic, even when the degree of anisotropy is small.

Another achievement of our approach is a link between elastic energy minimization and bounding the effective conductivity of a composite conductor. This link explains why both problems have similar extremal geometries, and it allows us to transfer optimal microstructures from one setting to the other. In particular, the Vigdergauz construction that mini-

mizes the strain energy of an elastic composite also saturates the conductivity trace bounds. This particular fact has already been observed by Vigdergauz in [38]. (Unfortunately there is a significant misprint in [38]: in the formulas (14) and (15) all occurrences of ψ should be read as ϕ.) Conversely, the confocal ellipsoid construction, which is known to attain the conductivity trace bounds [27], [34], can also be regarded as an extremal geometry for one of the elastic energy lower bounds. This fact was established in [15] for the two dimensional case with isotropic components; the treatment here applies in any dimension and even to some anisotropic component materials.

It should be emphasized, however, that we are not the first to link these problems. One early connection is provided by Milton's work [25], which bounds the effective conductivity and bulk modulus of an isotropic composite in terms of a certain "geometrical parameter" ζ_1. This parameter ranges over $[0,1]$, and the bounds reduce to those of Hashin and Shtrikman when $\zeta_1 = 0$ and $\zeta_1 = 1$. It follows that a microstructure achieving extremal effective conductivity and producing an isotropic composite must also have extremal bulk modulus. The same conclusion also follows from more recent work on bounds coupling the effective conductivity and bulk modulus [9], [12], [13]. A quite different link between elasticity and conductivity has recently been achieved by Milton and Movchan [29]. They find that for two-component 2-D composites made from *special* (anisotropic) materials, the equilibrium equations of linear elasticity can be reduced to a problem of electrostatics. To see that this unification is different from the one achieved here, we observe that their special Hooke's laws are never isotropic.

2 Problem Formulation and Notation.

We focus for the moment on elastic composites in two space dimensions. The elastic properties at any point x of a periodic composite are described by the fourth order tensor (Hooke's law) $C(x/\varepsilon)$, where $C(x)$ is a periodic function on R^2 with the period cell $[0,1]^2$, and ε is small. $C(x)$ takes just two values C_1 and C_2; the tensor C_i is the elasticity tensor of the i^{th} component material. Let $\chi_i(x/\varepsilon)$ be the characteristic function of a set occupied by the i^{th} component material. Then

$$C(x) = C_1\chi_1(x) + C_2\chi_2(x),$$
$$\chi_1(x) + \chi_2(x) = 1$$

In the limit as $\varepsilon \to 0$ the composite represents a homogeneous elastic body described by the effective Hooke's law C^*. According to the periodic homogenization theory [6], C^* is defined in terms of certain "cell problems", as follows. Let us assume that the composite is subjected to constant average strain ξ. Then the corresponding vector of displacements $v(x)$ solves

$$\nabla \cdot C(x)e(v) = 0 \tag{2.1}$$

$$e(v) = \frac{1}{2}(\nabla v + (\nabla v)^t) \tag{2.2}$$

in the period cell $Q = [0,1]^2$, with Q-periodic strain $e(v)$ constrained to have average value ξ:

$$\fint_Q e(v)dx = \xi. \tag{2.3}$$

The average stress associated with ξ is

$$C^*\xi = \fint_Q C(x)e(v)dx. \tag{2.4}$$

As ξ varies, (2.4) defines the effective tensor C^*. The equation (2.1) gives the local fields in the composite, and the overall elastic energy is

$$U = (C^*\xi, \xi). \tag{2.5}$$

Obviously the energy U depends on the microstructure. The problem we address is that of minimizing U under fixed volume fractions of the component materials constraint, i.e. with

$$\fint_Q \chi_1(x)dx = \theta_1 \tag{2.6}$$

held fixed.

3 Optimal bounds and optimal microstructures.

In this Chapter we obtain the optimal lower bounds on the elastic strain energy U of a two phase periodic composite in two space dimensions. We then "derive" the corresponding optimal microgeometries from the optimality conditions which can easily be read off from the translation method—our main tool for obtaining the bounds. What is new in our treatment is the full anisotropy of both component materials.

The microgeometries attaining the bounds will be found in a very systematic way due to the direct link between the translation method and the optimal microstructures. The optimality conditions play a crucial role in establishing this link. Their easy availability and utility has been observed by Milton [27] but he did not apply them to our problem.

Some readers might be more interested in examples of extremal microstructures than in the role of anisotropy or the "technology" of the translation method. Such readers may wish to skip from the end of section 3.2 directly to Chapters 5 and 6, where we discuss the confocal ellipsoid construction as an extremal microstructure for higher dimensional elasticity and the Vigdergauz construction as an extremal microgeometry for conductivity.

3.1 The translation method.

Our present task is to minimize the energy U defined by (2.5) over all possible characteristic functions $\chi_1(x)$ subject to the constraint (2.6):

$$W_{\min} = \inf_{<\chi_1>=\theta} \inf_{e(v)\in\mathcal{E}(\xi)} \fint_Q (C(x)e(v), e(v))dx, \tag{3.1}$$

where $\mathcal{E}(\xi)$ is a subset of $L^2(Q)$ of symmetrized gradients (2.2) with average value ξ, (2.3).

We begin our analysis with the simple yet very important ***harmonic mean bound***, which is obtained by ignoring the differential information (2.2) on the field $e(v)$ but keeping the algebraic one (2.3). This gives a lower bound for (3.1):

$$W_{\min} \geq (H\xi, \xi), \tag{3.2}$$

where

$$H = \left(\fint_Q C^{-1}(x)dx \right)^{-1} \tag{3.3}$$

is the harmonic mean of the tensor $C(x)$. For this reason (3.2) is sometimes called the harmonic mean bound.

We obtained the bound (3.2) using very simple *algebraic* inequalities. It is not difficult to determine when these inequalities become equality. The answer is that (3.2) becomes equality if and only if the local strain field $e(v)$ satisfies

$$e(v) = C^{-1}(x)H\xi. \tag{3.4}$$

Surprisingly, this simple bound is already optimal for certain values of ξ. The translation method elaborates on this property.

The general idea is to use the differential information about the field $e(v)$ being a strain. Some of this information is contained in the following identities:

$$\det e(v) = \det(\nabla v) - \frac{1}{8}|\nabla v - (\nabla v)^t|^2 \tag{3.5}$$

$$\fint_Q \det \nabla v = \det \xi. \tag{3.6}$$

Here we assume without loss of generality that the average infinitesimal rotation is zero. Of course, this idea is not new. It was used by Murat and Tartar [34] for bounding effective conductivity of a two phase composite and also by Gibiansky and Cherkaev [11] for elastic composites made from two isotropic materials. The same method will work for us as well. Notice that in two space dimensions $\det e(v)$ is a quadratic function of $e(v)$. Therefore there exists a fourth order translation tensor T with the symmetry of a Hooke's law (but which is not positive definite) such that for every 2×2 symmetric matrix η

$$(T\eta, \eta) = \det \eta.$$

Let

$$W(v) = \fint_Q (C(x)e(v), e(v))dx.$$

Then we can represent $W(v)$ as follows

$$W(v) = \fint_Q \Big((C(x) + kT)e(v), e(v) \Big) dx - k \fint_Q \det e(v) dx \tag{3.7}$$

for any k. Let us choose $k > 0$ such that $C(x) + kT$ is positive semidefinite. Then we can apply the harmonic mean bound to the first term of (3.7) and identities (3.5) and (3.6) to the second term, thus obtaining a new energy bound. The microstructures attaining the bound can be identified using the optimality conditions, which are essentially an analog of (3.4) with the $C(x)$ replaced by the translated tensor $C(x) + kT$. Let us focus our attention on the bounds first.

3.2 Bounds.

Let

$$\gamma = \sup\{\rho : \ C(x) + \rho T \geq 0\} > 0.$$

Let us assume that $C_2 + \gamma T$ is singular, while $C_1 + \gamma T$ is not. (There is a special case when both $C_1 + \gamma T$ and $C_2 + \gamma T$ are singular, but we will not consider it here.) Then $1/\gamma$ can be characterized as the largest eigenvalue of $-C_2^{-1/2} T C_2^{-1/2}$. In the isotropic case $\gamma = 4\mu_2$, where μ_2 is the shear modulus of the second material.

According to (3.5) and (3.6), for any $\lambda \in [0,1]$

$$W(v) = \fint_Q \Big((C(x) + \lambda\gamma T) e(v), e(v) \Big) dx - \lambda\gamma \det \xi + \frac{1}{8} \lambda\gamma \fint_Q |\nabla v - (\nabla v)^t|^2 dx. \tag{3.8}$$

Applying the harmonic mean bound to the first term and discarding the last one we obtain

$$W(v) \geq \Big(H(C(x) + \lambda\gamma T)\xi, \xi \Big) - \lambda\gamma \det \xi =: B(\lambda), \tag{3.9}$$

where $H(A(x))$ denotes the harmonic mean of the tensor $A(x)$ (cf. (3.2), (3.3)). Equality in (3.9) holds, by analogy with (3.4), if and only if

$$e(v) = (C(x) + \lambda\gamma T)^{-1} H(C(x) + \lambda\gamma T)\xi \tag{3.10}$$

and

$$\lambda(\nabla v - (\nabla v)^t) = 0, \tag{3.11}$$

for $\lambda \neq 1$.

The case $\lambda = 1$ is special in that $(C(x) + \gamma T)^{-1}$ does not exist. Nevertheless the minimization of the first term in (3.8) over the fields $e \in L^2(Q)$ satisfying only the average value restriction (2.3) can easily be carried out. Then it is a matter of simple linear algebra to write out the necessary and sufficient conditions for optimality on the local strain field $e(v)$:

$$e(v) = P_{L(x)} \lim_{\lambda \to 1^-} (C(x) + \lambda\gamma T)^{-1} H(C(x) + \lambda\gamma T)\xi, \tag{3.12}$$

where $P_{L(x)}$ is the orthogonal projection onto a subspace $L(x)$ =Range$(C(x) + \gamma T)$. We would like to note the discontinuity of the optimality conditions at $\lambda = 1$. Even though the right hand side in (3.10) has a well-defined limit as $\lambda \to 1^-$, this limit is not equal to the true optimality condition (3.12) for $\lambda = 1$. Nevertheless, as one can easily verify, the other optimality condition (3.11) and the bound (3.9) are continuous at $\lambda = 1$.

To conclude the derivation of the bound we have to determine the value of λ that gives the best bound (3.9). In other words we need to maximize $B(\lambda)$ over $\lambda \in [0,1]$:

$$B^*(\xi) = \max_{\lambda \in [0,1]} B(\lambda). \tag{3.13}$$

Notice that $B(\lambda)$ is a concave function of λ. This is easy to see from the variational principle defining $B(\lambda)$:

$$B(\lambda) = \inf_{e \in \mathcal{A}(\xi)} \Big\{ \fint_Q ((C(x) + \lambda\gamma T)e, e) dx - \lambda\gamma \det \xi \Big\}, \tag{3.14}$$

where $\mathcal{A}(\xi)$ is the set of second order 2×2 symmetric tensors in $L^2(Q)$ with average value ξ. One can also verify that $B(\lambda)$ is nowhere constant on $[0,1]$. Thus (3.13) has unique maximizer λ^* on $[0,1]$.

In general three different cases are possible:
(i) $\lambda^* = 0$ and $(dB/d\lambda)(0) < 0$;
(ii) λ^* is a critical point of $B(\lambda)$;
(iii) $\lambda^* = 1$ and $(dB/d\lambda)(1) > 0$.
In order to study each of the three possibilities we need to evaluate the derivative $dB/d\lambda$ explicitly. The most enlightening way to do this is to differentiate (3.14) with respect to λ. This can be accomplished by means of a technique from the nonsmooth analysis [10] (Theorem 2.8.2, Corollary 2). For $\lambda \neq 1$

$$\frac{dB}{d\lambda} = \gamma \int_Q \Big(Te(x), e(x) \Big) dx - \gamma(T\xi, \xi),$$

where $e(x)$ is the unique minimizer for (3.14) and it is given by (3.10) above. We remark that since $e(x)$ depends smoothly on λ the same result can also be obtained by a simple application of the chain rule. From (3.10) it follows that $e(x)$ takes just two values which we denote by ε_1 and ε_2 and has average value ξ. Then a simple calculation shows that

$$\frac{dB}{d\lambda} = \gamma \theta_1 \theta_2 \det(\varepsilon_1 - \varepsilon_2),$$

where according to (3.10)

$$\left.\begin{aligned} \varepsilon_1 &= (C_1 + \lambda\gamma T)^{-1} H(C(x) + \lambda\gamma T)\xi, \\ \varepsilon_2 &= (C_2 + \lambda\gamma T)^{-1} H(C(x) + \lambda\gamma T)\xi. \end{aligned}\right\} \tag{3.15}$$

In terms of the physical parameters

$$\frac{dB}{d\lambda} = \gamma \theta_1 \theta_2 \det\Big\{ (\theta_1 C_2 + \theta_2 C_1 + \lambda\gamma T)^{-1} (C_1 - C_2)\xi \Big\}. \tag{3.16}$$

Then it is easy to describe each of the three regimes explicitly.

The Harmonic Mean Bound regime:

$\lambda^* = 0$ and is not a critical point of B if and only if

$$\det\{(\theta_1 C_2 + \theta_2 C_1)^{-1}(C_1 - C_2)\xi\} < 0. \tag{3.17}$$

Then

$$B^*(\xi) = (H\xi, \xi).$$

The Rank-1 Intermediate regime:

λ^* is a critical point of B if and only if

$$\left.\begin{aligned} &\det\{(\theta_1 C_2 + \theta_2 C_1)^{-1}(C_1 - C_2)\xi\} \geq 0 \\ &\det\{(\theta_1 C_2 + \theta_2 C_1 + \gamma T)^{-1}(C_1 - C_2)\xi\} \leq 0. \end{aligned}\right\} \tag{3.18}$$

Then

$$B^*(\xi) = (H(C(x) + \lambda^*\gamma T)\xi, \xi) - \lambda^*\gamma \det \xi,$$

where λ^* is the unique solution of $(dB/d\lambda)(\lambda^*) = 0$.

The Degenerate regime:

$\lambda^* = 1$ and is not a critical point of B if and only if

$$\det\{(\theta_1 C_2 + \theta_2 C_1 + \gamma T)^{-1}(C_1 - C_2)\xi\} > 0. \tag{3.19}$$

Then

$$B^*(\xi) = \Big((C_2 + \gamma T)(\theta_1 C_2 + \theta_2 C_1 + \gamma T)^{-1}(C_1 + \gamma T)\xi, \xi\Big) - \gamma \det \xi.$$

The last two formulas can also be obtained from their respective counterparts for $\lambda \neq 1$ by passing to the limit as $\lambda \to 1^-$.

It turns out that the above bound is optimal. In the case when C_1 and C_2 are isotropic these bounds were obtained by Allaire and Kohn [1], [2], [3] and by Gibiansky and Cherkaev [11]. Our formulas reduce to theirs for isotropic C_1 and C_2.

We knew at the start that our method produces an energy bound, but we did not know that the bound we obtain would be optimal. In the next two sections we will use the optimality conditions in order to find extremal microgeometries, thus proving the sharpness of the bounds.

3.3 The non-degenerate regimes.

Let us concentrate on the first two regimes defined above. The key to our analysis is the "small miracle" that we have already observed deriving the explicit formula for $dB/d\lambda$: for any $\lambda \neq 1$

$$\frac{dB}{d\lambda} = \gamma\theta_1\theta_2 \det(\varepsilon_1 - \varepsilon_2), \tag{3.20}$$

where ε_1 and ε_2 are given by (3.15) above.

Consider the **Harmonic Mean Bound regime** (λ^*=0). In this regime one of the optimality conditions (3.11) is trivially satisfied. As for the other one (3.10), let us recall that if a linear strain takes only two values ε_1 and ε_2 in two space dimensions then these values have to be compatible:

$$\det(\varepsilon_1 - \varepsilon_2) \leq 0. \tag{3.21}$$

Due to (3.20) and according to the definition of the Harmonic Mean Bound regime the compatibility condition (3.21) is satisfied. Then it is easy to check that

$$\varepsilon_1 - \varepsilon_2 = \frac{\alpha}{2}(n_1 \otimes n_2 + n_2 \otimes n_1)$$

for some scalar α and a pair of non-parallel unit vectors n_1 and n_2. It is well-known that a rank-1 layered composite with layers orthogonal to either n_1 or n_2 generates a strain field $e(v)$ given by (3.4). Thus we have found two optimal rank-1 laminates. Conversely, we conjecture that if a microstructure attains the harmonic mean bound then it has to be one

of the two layered composites described above. Perhaps this could be given by an argument analogous to the one of Ball and James [5].

Now let us turn to the **Rank-1 Intermediate regime** (λ^* is a critical point of $B(\lambda)$). Here one of the optimality conditions, (3.11), implies that $e(v) = \nabla v$. Then the inequality (3.21) is not enough for the two values of the strain given by (3.15) for $\lambda = \lambda^*$ to be compatible. Instead we need a stronger relation:

$$\det(\varepsilon_1 - \varepsilon_2) = 0. \tag{3.22}$$

But this conditions is indeed satisfied by (3.20) and the definition of λ^*:

$$\frac{dB}{d\lambda}(\lambda^*) = 0.$$

The equation (3.22) in two space dimensions says that ε_1 and ε_2 are rank-1 related:

$$\varepsilon_1 - \varepsilon_2 = \alpha(n \otimes n)$$

for some scalar α and some unit vector n. It is well-known that the rank-1 layered composite with layers orthogonal to n generates a strain field $e(v)$ given by (3.10) with $\lambda = \lambda^*$. Conversely, if a microstructure attains the bound then it has to be a rank-1 laminate [5] (hence the name of the regime).

3.4 The Degenerate regime.

The third, degenerate, regime is the most interesting one. We recall that it is defined by $\lambda^* = 1$ and $(dB/d\lambda)(1) > 0$. The two strains ε_1 and ε_2 are no longer compatible. But now the optimality conditions are given by (3.11) and (3.12) and not by (3.10). Let us restate them in a longer but more enlightening form:

In phase **1**

$$e(v) = (\theta_1 C_2 + \theta_2 C_1 + \gamma T)^{-1}(C_2 + \gamma T)\xi; \tag{3.23}$$

in phase **2**

$$P_L e(v) = P_L(\theta_1 C_2 + \theta_2 C_1 + \gamma T)^{-1}(C_1 + \gamma T)\xi, \tag{3.24}$$

and in the whole period cell Q

$$\mathbf{curl}(v) = 0. \tag{3.25}$$

Here P_L denotes the orthogonal projection onto a proper subspace L =Range$(C_2 + \gamma T)$ of a three dimensional space of 2×2 symmetric matrices.

In particular, if C_1 and C_2 are isotropic, i.e. given by

$$C_i\eta = 2\mu_i(\eta - \frac{1}{2}(\mathbf{Tr}\eta)I) + k_i(\mathbf{Tr}\eta)I$$

for any symmetric matrix η, then in place of (3.23), (3.24) we obtain:

In phase **1**

$$e(v) = \varepsilon_0 I = \text{constant}, \tag{3.26}$$

where

$$\varepsilon_0 = \frac{(\mu_2 + k_2)\mathrm{Tr}\xi}{2(\mu_2 + \theta_1 k_2 + \theta_2 k_1)};$$

in phase **2**

$$\mathbf{div}\, v = d, \tag{3.27}$$

where

$$d = \frac{(k_1 + \mu_2)\mathrm{Tr}\xi}{(\mu_2 + \theta_1 k_2 + \theta_2 k_1)};$$

It is not easy to find the optimal microstructures just by looking at the optimality conditions. Yet they will enable us to determine at least some of the extremal geometries.

Let

$$\begin{cases} e_1 = (\theta_1 C_2 + \theta_2 C_1 + \gamma T)^{-1}(C_2 + \gamma T)\xi; \\ e_2 = (\theta_1 C_2 + \theta_2 C_1 + \gamma T)^{-1}(C_1 + \gamma T)\xi. \end{cases} \tag{3.28}$$

These quantities are important in studying the optimal geometries, as is already seen from the optimality conditions (3.23), (3.24). Notice that e_2 is not the value of the local strain in phase **2**. Instead we have $P_L e(v)\chi_2(x) = P_L e_2 \chi_2(x)$. There are two major cases distinguished by the dimension of the projection P_L.

THE GENERIC CASE $\dim L = 2$.

In this case, as in the Rank-1 Intermediate regime, condition (3.25) is equivalent to $e(v) = \nabla v$. It is not hard to see that in two space dimensions if a gradient satisfies (3.24) then it has to be locally constant (i.e. constant in any connected component of the phase **2**). In addition any value that $e(v)$ takes in phase **2** must satisfy (3.24). Therefore

$$e(v) = e_2 + ca$$

in phase **2**, where a spans the one-dimensional subspace $L^\perp$ and c is a locally constant scalar.

The local constancy of the strain field in each phase is the characteristic feature of the layered geometries and thus suggests that we have to look for optimal microstructures among laminates. A rank-1 laminate is not enough because e_1 and $e_2 + ca$ cannot produce average ξ and be rank-1 related at the same time. We are going to show that a rank-2 laminate will suffice. In the innermost laminate the phases **1** and **2** are layered together. This is achieved by the choice of the free parameter c:

$$\det(\Delta + ca) = 0, \tag{3.29}$$

where

$$\Delta = e_2 - e_1. \tag{3.30}$$

Notice that (3.29) is a quadratic equation in c:

$$c^2 + 2c\frac{(Ta, \Delta)}{\det a} + \frac{\det \Delta}{\det a} = 0. \tag{3.31}$$

The last term in the above equation is negative because from (3.20)

$$\det \Delta = \frac{1}{\gamma\theta_1\theta_2}\frac{dB}{d\lambda}(1) > 0,$$

while $a \in \mathrm{Ker}(C_2 + \gamma T)$, which implies that $\det a = -(1/\gamma)(C_2 a, a) < 0$. Thus the equation has two distinct real roots, which we denote by c_1 and c_2. Choosing any one of them will provide compatible gradients with optimal values. Suppose we have chosen c_1 and suppose that in the innermost laminate the phases are mixed in the volume fractions ρ_1 and $1 - \rho_1$. So the average field in that laminate will be

$$e_3 = \rho_1 e_1 + (1 - \rho_1)(e_2 + c_1 a)$$

and the layering normal will be n_1 defined by

$$\Delta + c_1 a = \alpha(n_1 \otimes n_1),$$

due to (3.29).

Next we have to "layer" e_3 with either e_1 or $e_2 + ca$. Choosing e_1 will produce the layering normal n_1 again. Effectively we will obtain the same rank-1 laminate with an increased volume fraction ρ_1. To layer e_3 with $e_2 + ca$ we have to choose the value of c such that

$$\det(e_2 + ca - e_3) = 0,$$

or equivalently

$$\det(\Delta + \frac{c - (1 - \rho_1)c_1}{\rho_1} a) = 0.$$

Comparing the last equation with (3.29) we obtain:

$$\frac{c - (1 - \rho_1)c_1}{\rho_1} = c_i.$$

If $i = 1$ then $c = c_1$ and the layering normal is n_1 again, so we obtain the same rank-1 laminate but with increased volume fraction of the second material. Thus we need to use the remaining possibility $i = 2$. Then

$$c = \rho_1 c_2 + (1 - \rho_1)c_1.$$

If we mix e_3 and $e_2 + ca$ in the volume fractions ρ_2 and $1 - \rho_2$ then the average strain is

$$< e >= \rho_2 e_3 + (1 - \rho_2)(e_2 + ca)$$

and the layering direction n_2 is defined by

$$\Delta + c_2 a = \alpha(n_2 \otimes n_2),$$

where α, by convenient abuse of notation, denotes any scalar.

Let us show that choosing ρ_1 and ρ_2 appropriately we can achieve the average strain ξ and the volume fraction θ_1 of the material **1**. The total volume fraction of the material **1** in our construction is obviously $\rho_1\rho_2$. Then we need to solve the system

$$\rho_1\rho_2 = \theta_1,$$

$$< e >= \xi.$$

We can easily find ρ_i, $i = 1, 2$:

$$1 - \rho_1 = \theta_2 \frac{c_2}{c_2 - c_1},$$

$$1 - \rho_2 = -\theta_2 \frac{c_1}{\theta_1 c_2 - c_1}.$$

Since $c_1 c_2 < 0$ it follows that $\rho_i \in (0, 1)$. By interchanging c_1 and c_2 we obtain another rank-2 laminate that is also optimal. It seems unlikely that anything radically different from the two rank-2 laminates and their mixtures can be optimal, but at present we don't know how to formulate such an assertion rigorously.

We note in passing that a rank-2 laminate is not strictly speaking a geometry. In fact it is a sequence of geometries, which attains the energy minimum only in the limit, while each member of the sequence falls slightly short of the mark. The optimality conditions (3.11), (3.12) are also satisfied only asymptotically in $L^2(Q)$.

THE DEGENERATE CASE $\dim L = 1$.

This case is non-generic among all anisotropic Hooke's laws. Yet it is very important, as composites made of isotropic component materials fall in this category. We start with an obvious remark that all of the above arguments extend to the case $\dim(L) = 1$, since we used only the existence of a matrix $a \in L^\perp$ and not its uniqueness. However, in this case the microstructure ***does not have to*** be a laminate at all. For example, in the isotropic case there are essentially different microstructures that attain the energy minimum, namely, the "confocal ellipse construction" [15] and the "Vigdergauz construction" [16], [39]. We will devote the remainder of this section to extending these geometries to the case of the anisotropic matrix material C_2 that falls in the class $\dim L = 1$.

The rest of this section is essentially an addendum to our articles [15], [16]. It is not necessary for understanding Chapters 4–6.

Let the 2×2 symmetric matrix b span the one-dimensional subspace L. Then

$$C_2 + \gamma T = \alpha(b \otimes b)$$

for some constant positive real number α. (The isotropic case corresponds to the matrix b being isotropic.) From the fact that C_2 is positive definite it follows that $\det(z) < 0$ whenever z is orthogonal to b. It is not hard to show that this implies $\det(b) > 0$. Thus without loss of generality we can suppose that the matrix b is positive definite. We also normalize b so that $\det b = 1$, for later convenience.

Now let us rewrite the optimality conditions (3.24), (3.25) as

$$v = \nabla\phi \tag{3.32}$$

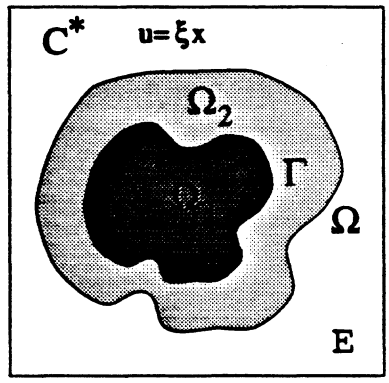

Figure 1: The concentric "something" construction.

for some scalar potential ϕ, and

$$(e(v), b) = (e_2, b) \tag{3.33}$$

in phase **2**. Substituting (3.32) into (3.33) we obtain that in phase **2**

$$(\nabla\nabla\phi, b) = (e_2, b) \tag{3.34}$$

The formula (3.34) is equivalent to Laplace's equation since b is positive definite. Now we are ready to calculate the optimal microstructures explicitly.

The stretched confocal ellipse construction.

In this section we look for an optimal microgeometry which is an analogue of the confocal ellipse construction described in detail in [15] (see Figure 1). We will reduce our problem to a problem in complex variables that has already been solved in [15]. In order to do so we rewrite the optimality conditions in terms of ϕ:
in phase **1**

$$\nabla\phi = e_1 x; \tag{3.35}$$

in phase **2**

$$(\nabla\nabla\phi, b) = (e_2, b). \tag{3.36}$$

We also have the continuity of displacements across the interfaces Γ and $\partial\Omega$:

$$\begin{aligned} \nabla\phi &= e_1 x \quad \text{on } \Gamma, \\ \nabla\phi &= \xi x \quad \text{on } \partial\Omega. \end{aligned} \tag{3.37}$$

We do not have to use the continuity of tractions by the same argument as in [15]; alternatively, one can check directly that the continuity of tractions follow from the conditions listed above. We are now in a position to reformulate our problem in terms of complex variables. Let $x = b^{1/2}y$ and

$$\tilde{\phi}(y) = \phi(b^{1/2}y) - \frac{1}{4}(e_2, b)|y|^2.$$

The function $\tilde{\phi}(y)$ is constructed in such a way as to be harmonic. Now let $z = y_1 + iy_2$ and

$$\Phi(z) = \frac{\partial \tilde{\phi}}{\partial y_1} - i\frac{\partial \tilde{\phi}}{\partial y_2}.$$

One can easily check that $\Phi(z)$ is an analytic function. The major difference from the isotropic case is that the problem in complex variables is formulated not in physical space but in the linearly transformed variables (by the operator $b^{-1/2}$). The transformed phases will be denoted $1'$ and $2'$ respectively. We remark that the volume fractions of the phases and the total area of Ω do not change under the linear transformation due to the normalization of b, namely $\det b = 1$.

Let us reformulate conditions (3.36)–(3.37) in terms of the analytic potential Φ:

$$\left.\begin{aligned} \Phi(z) &= -\frac{1}{2}(\Delta, b)\bar{z} - \frac{1}{2}a(b^{1/2}e_1 b^{1/2})z, \text{ on } \Gamma' \\ \Phi(z) &= -\frac{1}{2}\theta_1(\Delta, b)\bar{z} - \frac{1}{2}a(b^{1/2}\xi b^{1/2})z, \text{ on } \partial\Omega'. \end{aligned}\right\}$$

Here $a(A)$ denotes the "complex deviatoric part" of the matrix A:

$$a(A) = A_{22} - A_{11} + 2iA_{12},$$

and the quantity Δ is defined in (3.30). If we consider the function

$$\Psi(z) = -2\Phi(z) - a(b^{1/2}e_1 b^{1/2})z,$$

we obtain a problem analogous to the one considered in [15]:

$$\left.\begin{aligned} \Psi(z) &= (\Delta, b)\bar{z}, \text{ on } \Gamma' \\ \Psi(z) &= \theta_1(\Delta, b)\bar{z} + \theta_2 a(b^{1/2}\Delta b^{1/2})z, \text{ on } \partial\Omega'. \end{aligned}\right\} \tag{3.38}$$

Now, we can continue exactly as in [15]. We discovered there that the problem (3.38) has a solution if and only if $|q| < \theta_2$ ([15], formula (4.4)), where

$$q = \frac{\theta_2 a(b^{1/2}\Delta b^{1/2})}{(\Delta, b)}. \tag{3.39}$$

When this condition holds, the interfaces Γ' and $\partial\Omega'$ are the confocal ellipses. Taking into account that $\det b = 1$, the existence condition (3.39) reduces to $\det \Delta > 0$, which coincides exactly with the definition of the Degenerate regime (3.19).

In summary, the stretched confocal ellipse geometry is obtained by starting with the confocal ellipse construction (as presented in [15]) corresponding to volume fraction θ_1 and parameter q given by (3.39), then transforming the geometry by $b^{1/2}$. The ellipses in the construction will no longer be confocal on account of the linear transformation. That is why this geometry has been called the "Stretched confocal ellipse construction" by G. Milton [27].

The Vigdergauz construction.

This construction can be obtained as a generalization of the Vigdergauz construction, described in detail in [16]. The argument is entirely parallel to the one given above, so we need not give further details. We have to comment on one thing, though. In [16] we showed how to solve the periodic problem only with a *rectangular* period cell Q. The analogous problem with general parallelogram of fundamental periods seems to be much more difficult technically. (The only exception is the rhombic period cell, see [40].) Therefore, here we have to make a rather artificial technical assumption that $b^{-1/2}Q$ is a rectangle. Thus the Vigdergauz geometry in the period cell Q is obtained from another Vigdergauz geometry in the rectangular period cell Q' by transforming it back by the linear operator $b^{1/2}$. The Vigdergauz construction in Q' has the volume fraction θ_1 and the parameter q given by (3.39). In physical space the requirement that Q' should be rectangular is equivalent to having the sides of the period cell Q parallel to the two eigenvectors of the nonsymmetric matrix $b\Delta$.

In the next chapter we consider another important minimization problem for which the translation method is applicable. It is the problem of minimizing the complementary energy. This problem is related to *maximizing* of the elastic energy via convex duality [1]. But as we will see, it is also related to the problem of elastic energy *minimization* considered above.

4 Minimization of the complementary energy.

In many problems of optimal design it is important to find the "stiffest" material obtained as a mixture of two given components taken in fixed volume fractions. One such problem, the minimization of the complementary elastic energy for a given average stress, is considered here:

$$V_{\min} = \inf_{<x>=\theta} \inf_{\substack{\nabla\cdot\sigma=0 \\ <\sigma>=\sigma_0}} \int_Q (C^{-1}(x)\sigma,\sigma)dx. \tag{4.1}$$

The key to the solution in two space dimensions is the reduction of the problem (4.1) to another one which is very much like the minimization of the strain energy considered in Chapter 3. The reduction is done via an Airy stress potential.

Let us recall that any 2×2 symmetric tensor σ with $\operatorname{div}\sigma = 0$ can be represented in terms of a scalar Airy stress potential ϕ by

$$\sigma = \mathcal{R}\nabla\nabla\phi,$$

where

$$\mathcal{R}\eta = \begin{pmatrix} \eta_{22} & -\eta_{12} \\ -\eta_{12} & \eta_{11} \end{pmatrix}$$

for any 2×2 symmetric matrix η. Denoting $\mathcal{R}\sigma_0$ by ξ we obtain

$$V_{\min} = \inf_{<x>=\theta} \inf_{<\nabla\nabla\phi>=\xi} \int_Q (S(x)\nabla\nabla\phi, \nabla\nabla\phi)dx, \tag{4.2}$$

where

$$S(x) = \mathcal{R}^t C^{-1}(x)\mathcal{R}.$$

Comparing (4.2) with (3.1) we conclude that in general

$$V_{\min} \leq W_{\min}, \tag{4.3}$$

where $W_{\min}$ is computed using the Hooke's law $S(x)$. The equality in (4.3) is obviously achieved if and only if the optimal v for $W_{\min}$ is curl-free. We notice that this is exactly the case in the Rank-1 Intermediate regime and in the Degenerate regime. Thus, if ξ lies in one of the above regimes (calculated using $S(x)$ instead of $C(x)$), the problem of finding $V_{\min}$ is equivalent to the problem of finding $W_{\min}$ but with a different Hooke's law. See [16] for an application of this idea to the problem of structural optimization involving isotropic component materials.

If ξ lies in the Harmonic Mean Bound regime then the two problems are different and the inequality in (4.3) becomes strict. In order to analyze the lower bound on complementary energy in this regime we need to apply the translation method once again. The calculations and formulas are very similar to those done in sections 3.2 and 3.4.

Let

$$V(\phi) = \int_Q (S(x)\nabla\nabla\phi, \nabla\nabla\phi)dx.$$

Then making use of the usual translation, the determinant, we obtain:

$$V(\phi) = \int_Q ((S(x) + \lambda T)\nabla\nabla\phi, \nabla\nabla\phi)dx - \lambda \det\xi; \tag{4.4}$$

but now λ ranges on the interval $[\gamma_-, \gamma_+]$, where $\gamma_- < 0$ and $\gamma_+ > 0$. The interval (γ_-, γ_+) is the maximal interval of the values of λ on which the tensor $S(x) + \lambda T$ remains positive definite.

Let us assume that $S_1 + \gamma_+ T$ is singular, while $S_2 + \gamma_+ T$ is not. We also assume that $S_- + \gamma_- T$ is singular, while $S_+ + \gamma_- T$ is not, where $\{S_+, S_-\} = \{S_1, S_2\}$. Since we do not assume the well-orderedness of the original Hooke's laws, S_- (or S_+) can be either S_1 or S_2.

Now we return to the formula (4.4) and apply the harmonic mean bound:

$$V(\phi) \geq (H(S(x) + \lambda T)\xi, \xi) - \lambda \det\xi = B(\lambda).$$

To obtain the best bound we need to maximize $B(\lambda)$:

$$B^* = \max_{\lambda \in [\gamma_-, \gamma_+]} B(\lambda) \tag{4.5}$$

The study of (4.5) parallels the one for (3.13). As before, λ^* denotes the unique maximizer of (4.5) and we distinguish the three cases:

(i) $\lambda^* = \gamma_-$ and is not a critical point of $B(\lambda)$. In this case $(dB/d\lambda)(\gamma_-) < 0$.

(ii)$\lambda^* \in [\gamma_-, \gamma_+]$ and is a critical point of $B(\lambda)$. In this case $(dB/d\lambda)(\gamma_-) \leq 0$ and $(dB/d\lambda)(\gamma_+) \geq 0$.

(iii) $\lambda^* = \gamma_+$ and is not a critical point of $B(\lambda)$. In this case $(dB/d\lambda)(\gamma_+) > 0$.

If $\lambda^* \geq 0$ then, as we remarked above, $V_{\min} = W_{\min}$ and similar microgeometries attain the bounds for the two minimization problems.

If $\lambda^* < 0$ and belongs to the second regime, then it is clear that the situation here is analogous to the Rank-1 Intermediate regime of the strain energy bounds. We leave the details to the reader.

If $\lambda^* = \gamma_-$ then we obtain another degenerate regime. Here, one might think, we have to distinguish between the two cases $\dim L = 2$ and $\dim L = 1$, where $L = \text{Range}(S_- + \gamma_- T)$. However the following lemma rules out the second possibility.

Lemma 1 *For any positive definite Hooke's law S_-*

$$\text{Rank}(S_- + \gamma_- T) = 2.$$

Proof. Suppose that $\text{Rank}(S_- + \gamma_- T) = 1$. Then there exists a symmetric 2×2 matrix b and a positive number α such that

$$S_- + \gamma_- T = \alpha(b \otimes b).$$

Let us show that we can choose a matrix ξ such that $(\xi, b) = 0$ and $(T\xi, \xi) < 0$. Let ξ be trace free, for example. Then

$$(T\xi, \xi) = -\frac{1}{2}|\xi|^2 < 0,$$

and we need only $(b, \xi) = 0$. We have

$$(b, \xi) = (b_{11} - b_{22})\xi_{11} + 2b_{12}\xi_{12}.$$

If $b_{11} = b_{22}$ and $b_{12} = 0$, then $(b, \xi) = b_{11}\text{Tr}\xi = 0$. Otherwise, let $\xi_{11} = 2b_{12}$ and $\xi_{12} = b_{22} - b_{11}$. Then $(b, \xi) = 0$.

For such a ξ

$$(S_-\xi, \xi) = -\gamma_-(T\xi, \xi) + \alpha(b, \xi)^2 = \frac{1}{2}\gamma_-|\xi|^2 < 0,$$

which contradicts the positivity of the tensor S_-. The Lemma is proved. □

Thus we conclude that the regime $\lambda^* = \gamma_-$ is analogous to the case $\dim L = 2$ of the Degenerate regime of the strain energy bound. The actual calculations are so parallel to the ones in section 3.4 that we do not do them here. Below we give the summary of our results on the complementary energy bounds.

Summary.

1. **The rank-2 regime.**

$$\det\{(\theta_1 S_2 + \theta_2 S_1 + \gamma_- T)^{-1}(S_2 - S_1)\xi\} < 0;$$

$$V_{\text{min}} = ((S_2 + \gamma_- T)(\theta_1 S_2 + \theta_2 S_1 + \gamma_- T)^{-1}(S_1 + \gamma_- T)\xi, \xi) - \gamma_- \det\xi.$$

The optimality conditions on $\nabla\nabla\phi$ are

In phase "+"

$$\nabla\nabla\phi = (\theta_1 S_2 + \theta_2 S_1 + \gamma_- T)^{-1}(S_- + \gamma_- T)\xi;$$

in phase "−"

$$P_{L_-}\nabla\nabla\phi = P_{L_-}(\theta_1 S_2 + \theta_2 S_1 + \gamma_- T)^{-1}(S_+ + \gamma_- T)\xi,$$

where P_{L_-} is the orthogonal projection onto a two-dimensional subspace

$$L_- = \text{Range}(S_- + \gamma_- T)$$

in the three-dimensional space of 2×2 symmetric matrices. The last condition implies that $\nabla\nabla\phi$ is locally constant throughout phase "$-$". The known optimal microstructures are the two variants of a rank-2 laminate. (Hence the name of the regime.)

2. **The Rank-1 Intermediate regime**

$$\begin{cases} \det\{(\theta_1 S_2 + \theta_2 S_1 + \gamma_- T)^{-1}(S_2 - S_1)\xi\} \geq 0, \\ \det\{(\theta_1 S_2 + \theta_2 S_1 + \gamma_+ T)^{-1}(S_2 - S_1)\xi\} \leq 0; \end{cases}$$

$$V_{\min} = (H(S(x) + \lambda^* T)\xi, \xi) - \lambda^* \det \xi.$$

The optimality condition on $\nabla\nabla\phi$ is

$$\nabla\nabla\phi = (S(x) + \lambda^* T)^{-1} H(S(x) + \lambda^* T)\xi,$$

where λ^* is the unique solution of

$$\det\{(\theta_1 S_2 + \theta_2 S_1 + \lambda^* T)^{-1}(S_2 - S_1)\xi\} = 0.$$

The optimal microstructure must be a particular rank-1 laminate.

3. **The Degenerate regime**

$$\det\{(\theta_1 S_2 + \theta_2 S_1 + \gamma_+ T)^{-1}(S_2 - S_1)\xi\} > 0;$$

$$V_{\min} = \Big((S_2 + \gamma_+ T)(\theta_1 S_2 + \theta_2 S_1 + \gamma_+ T)^{-1}(S_1 + \gamma_+ T)\xi, \xi\Big) - \gamma_+ \det \xi.$$

The optimality conditions on $\nabla\nabla\phi$ are
In phase **1**

$$P_{L_+}\nabla\nabla\phi = P_{L_+}(\theta_1 S_2 + \theta_2 S_1 + \gamma_+ T)^{-1}(S_2 + \gamma_+ T)\xi,$$

in phase **2**

$$\nabla\nabla\phi = (\theta_1 S_2 + \theta_2 S_1 + \gamma_+ T)^{-1}(S_1 + \gamma_+ T)\xi,$$

where P_{L_+} is the orthogonal projection onto the subspace $L_+ = \text{Range}(S_1 + \gamma_+ T)$. The set of optimal microstructures depends on $r = \dim L_+$. If $r = 2$ then there are two variants of the second rank laminate which are known to be optimal. If $r = 1$ then the optimal microgeometries that we know of include the rank-2 laminates just mentioned, the stretched confocal ellipse construction and the Vigdergauz construction.

5 Anisotropic conductivity revisited.

Continuing our systematic application of the translation method we now use it to make precise the relation between the conductivity and elasticity problems. We shall obtain by this method a new result for anisotropic conductivity, as well as some new results in elasticity. More specifically, the translation method establishes a link that allows us to regard any microstructure optimal for n-dimensional conductivity as an extremal geometry for the elasticity problem and, under some conditions, vice versa. In particular we are going to show that the Vigdergauz microstructure saturates the conductivity trace bounds while the confocal ellipsoid construction attains the minimum energy of an elastic composite. The optimality conditions will play crucial part in establishing these results.

Let us consider a periodic conductivity problem in R^n with period cell Q:

$$\begin{cases} \nabla \cdot \sigma(x)(\nabla\phi + e_0) = 0 \\ \nabla\phi \in \mathcal{E}, \end{cases} \tag{5.1}$$

where $\sigma(x) = A\chi_1(x) + B\chi_2(x)$, and $\mathcal{E}$ is the subspace of $L^2(Q)$ consisting of Q-periodic gradients with average zero. This problem has a unique (up to an additive constant) solution for every fixed average field e_0 and microgeometry $\chi_1(x)$ ($\chi_2(x) = 1 - \chi_1(x)$). The conductivities A and B are assumed to be well-ordered, $A > B$. This is a technical assumption which ensures that the translation we propose works. It also ensures the familiar form of the trace bounds, as found in [28]. (For the discussion of the non-well-ordered case see [31].) We note that in the case of isotropic conductivity, well-orderedness is not a restriction.

In this Chapter, as in the previous one, we fix the volume fractions of the components:

$$⨍_Q \chi_1(x)dx = \theta_1. \tag{5.2}$$

Let us consider a problem of minimizing the *sum* of the n energies for n linearly independent average electric fields $e_1, \ldots, e_n$:

$$\Sigma_{\min} = \inf_{<\chi_1>=\theta_1} \inf_{\{\nabla\phi_1,\ldots,\nabla\phi_n\}\subset\mathcal{E}} \sum_{i=1}^{n} ⨍_Q (\sigma(x)(\nabla\phi_i + e_i), \nabla\phi_i + e_i)dx$$

We can significantly simplify the above formula and consequent calculations if instead of handling n vectors of length n each we work with two-dimensional arrays whose n columns correspond to our n vectors. Let $\xi_{ij} = e_j^{(i)}$ and let ϕ denote the vector-function $(\phi_1, \ldots, \phi_n)$. Then our problem simplifies to

$$\Sigma_{\min} = \inf_{<\chi_1>=\theta_1} \inf_{\nabla\phi\in\mathcal{E}} ⨍_Q (\sigma(x)(\nabla\phi + \xi), \nabla\phi + \xi)dx. \tag{5.3}$$

We remark that in our new notation $(\nabla\phi)_{ij} = \partial\phi_j/\partial x_i$. (This is a transpose of the usual convention, using which would make all the formulas much less readable due to the profusion of transpose signs.)

The problem (5.3) has been completely solved [26], [28], [34]. We will solve it again and link it to the elasticity problem via the optimality conditions. Our main tool is once again the translation method.

To apply the method successfully we need a good choice of the translation tensor. The article of Murat and Tartar [34], where (5.3) was solved for isotropic component conductors, provides us with a clue. They employed the translation $R(\xi) = |\xi|^2 - (\mathrm{Tr}\xi)^2$. In the anisotropic case we use the modified translation $T(\xi) = R(B^{1/2}\xi)$, which reduces to R if the tensor B is isotropic. At this point the well-orderedness of the conductivities ($A > B$) becomes important. If the two conductivities are not well ordered then in the formula for the translation T we should use a tensor which is neither A nor B. The choice of that new tensor is not obvious. The appropriate analysis was done by V. Nesi in [31] when A and B are coaxial. What happens in the more general case is not clear. We avoid this problem by assuming well-orderedness.

The translation T has the following useful property: For any $\nabla\phi \in \mathcal{E}$ we have

$$\fint_Q T(\nabla\phi + \xi)dx = T(\xi) + \frac{1}{2}\fint_Q |B^{1/2}\nabla\phi - (B^{1/2}\nabla\phi)^t|^2 dx. \tag{5.4}$$

Now let

$$\Sigma(\phi) = \fint_Q (\sigma(x)(\nabla\phi + \xi), \nabla\phi + \xi)dx.$$

Then using (5.4) we obtain

$$\Sigma(\phi) = \fint_Q ((\sigma(x) - T)(\nabla\phi + \xi), \nabla\phi + \xi)dx + T(\xi) + \frac{1}{2}\fint_Q |B^{1/2}\nabla\phi - (B^{1/2}\nabla\phi)^t|^2 dx.$$

We get the bound by minimizing the first term in the above identity forgetting that $\nabla\phi$ is a gradient and using only the fact that $\nabla\phi$ has average value zero. By removing the differential constraints we can only make the infimum in (5.3) smaller. After some straightforward algebraic calculations that are analogous to the derivation of the harmonic mean bound in section 3.1 we obtain

$$\Sigma_{\min} \geq T(\xi) + \frac{1 + \mathrm{Tr}(B(A-B)^{-1})}{\theta_2 + \mathrm{Tr}(B(A-B)^{-1})}(B^{1/2}, \xi)^2. \tag{5.5}$$

Examining all the inequalities involved in obtaining the bound (5.5) we can easily find the necessary and sufficient conditions on the electric field $\nabla\phi$ for equality in (5.5). They are as follows:

In phase **1** *(conductivity A)*

$$\nabla\phi + \xi = \frac{(B^{1/2}, \xi)(A-B)^{-1}B^{1/2}}{\theta_2 + \mathrm{Tr}(B(A-B)^{-1})}$$

in phase **2** *(conductivity B)*

$$(B^{1/2}, \nabla\phi + \xi) = \frac{1 + \mathrm{Tr}(B(A-B)^{-1})}{\theta_2 + \mathrm{Tr}(B(A-B)^{-1})}(B^{1/2}, \xi)$$

and in the whole period cell Q

$$B^{1/2}\nabla\phi = (B^{1/2}\nabla\phi)^t.$$

Instead of proving the attainability of the bound (5.5) we will reduce it to the well-known trace bound, which is known to be attainable [34]. The reduction is achieved by taking

$$\xi = (\sigma^* - B)^{-1}B^{1/2}, \tag{5.6}$$

where σ^* denotes the effective conductivity tensor, and using the relation $\Sigma_{\min} = (\sigma^*\xi, \xi)$. Then (5.5) becomes

$$\mathbf{Tr}(B(\sigma^* - B)^{-1}) \leq \frac{1}{\theta_1}\mathbf{Tr}(B(A - B)^{-1}) + \frac{\theta_2}{\theta_1}. \tag{5.7}$$

Notice that the choice (5.6) assigns a ξ to every σ^*, but not every ξ arises this way. In fact every ξ has an associated optimal σ^* but the ones given in (5.6) are sufficient to obtain the trace bounds.

Now let us change our point of view. Suppose that we are looking for a microstructure with effective conductivity σ^*, where σ^* achieves equality in (5.7). The microstructure that attains it must have the property that the local electric fields (in matrix form) $\nabla\phi + \xi$ satisfy the optimality conditions presented above when the average fields ξ are given by (5.6). Since ξ is non-singular, the matrix $\nabla\phi + \xi$ contains the local fields associated to n linearly independent vectors in R^n; these determine the local fields corresponding to any average field by taking linear combinations. To proceed further it is more convenient to use the vector-potential Φ corresponding to the standard basis of average fields (represented by the identity in matrix notation) rather than $\nabla\phi + \xi$, which corresponds to the basis of average fields given by ξ. There is a simple relation between the two potentials:

$$\nabla\phi + \xi = \nabla\Phi\xi.$$

We remark that both ϕ and Φ correspond to the same optimal microstructure with effective conductivity σ^*. Rewriting the optimality conditions in terms of σ^* and Φ, we obtain
In phase **1** *(conductivity A)*

$$\nabla\Phi = \frac{1}{\theta_1}(A - B)^{-1}(\sigma^* - B) \tag{5.8}$$

in phase **2** *(conductivity B)*

$$(\nabla\Phi, B(\sigma^* - B)^{-1}) = \frac{1}{\theta_1}(1 + \mathbf{Tr}(B(A - B)^{-1})) \tag{5.9}$$

and in the whole period cell Q

$$\nabla\Phi(\sigma^* - B)^{-1} = (\sigma^* - B)^{-1}\nabla\Phi^t \tag{5.10}$$

$$\fint_Q \nabla\Phi dx = I \tag{5.11}$$

To make a connection with the elasticity optimality conditions (3.23)–(3.25) we let

$$v = (\sigma^* - B)^{-1}\Phi$$

then $\nabla v = \nabla\Phi(\sigma^* - B)^{-1}$ (using the convention that $(\nabla\Phi)_{ij} = \partial\Phi_j/\partial x_i$). In terms of v the optimality conditions read
In phase **1**

$$\nabla v = \frac{1}{\theta_1}(A - B)^{-1} \tag{5.12}$$

in phase **2**

$$(\nabla v, B) = \frac{1}{\theta_1}(1 + \mathbf{Tr}(B(A - B)^{-1})) \tag{5.13}$$

and in the whole period cell Q

$$\mathbf{curl}(v) = 0; \tag{5.14}$$

$$\fint_Q \nabla v dx = (\sigma^* - B)^{-1} \tag{5.15}$$

Now it is easy to relate the two problems in two space dimensions. The existence condition (3.19) for elasticity corresponds to the condition $\det \Delta^c > 0$ for conductivity, where

$$\Delta^c = \frac{1}{\theta_1\theta_2}(A - B)^{-1}(\theta_1 A + \theta_2 B - \sigma^*)(\sigma^* - B)^{-1}.$$

Thus we recovered a theorem proved in [34] (see also [14]) that any tensor σ^* achieving equality in the lower trace bound (5.7) and satisfying

$$B < \sigma^* < \theta_1 A + \theta_2 B$$

is attained by some composite. In 2-D we see that the trace bound is saturated not only by laminates, but also by stretched confocal ellipse construction or Vigdergauz microstructure with parameter $q = q^c$ (see (3.39), also [15], [16]), where

$$q^c = \frac{\theta_2 a(B^{1/2}\Delta^c B^{1/2})}{(\Delta^c, B)}.$$

In three space dimensions we also would like to have a Vigdergauz construction to saturate the trace bounds. Unfortunately there is no rigorous proof that one exists, though the numerical computations in [37] are suggestive.

6 The confocal ellipsoid construction in elasticity.

The confocal ellipse (and more generally, confocal ellipsoid) construction is known to be optimal for the conductivity problem from Chapter 5 in any number of space dimensions [7], [8], [22], [23], [24], [27], [34]. (The results from the literature are easily extendible to the case of anisotropic component materials by a remark of Milton [27] that problems involving general anisotropic conductors can be reduced to similar problems with isotropic

components by a suitable linear transformation.) In this Chapter we show that the same confocal (or stretched confocal) ellipsoid construction minimizes the *elastic* energy of a two phase composite in any space dimensions (the 2-D case was solved directly in [15]).

We begin by deriving a new optimal bound on the elastic strain energy in R^n. Then using the optimality conditions we will establish the equivalence between the new elasticity bound and the conductivity trace bound discussed in the previous chapter. By this equivalence, the confocal ellipsoid construction which is known to saturate the trace bounds (see [34]) will also be seen to saturate the new elastic energy bound in R^n.

Consider the case of two isotropic elastic materials in R^n. Here we assume that the materials are well ordered, i.e. $k_1 > k_2$ and $\mu_1 > \mu_2$. We need this assumption in order to establish the equivalence to conductivity. The well-orderedness will not be required for the derivation of the energy bound, though. Let

$$W(e) = \fint_Q (C(x)e, e)dx.$$

Since $C(x)$ is locally isotropic we can represent the energy $W(e)$ as

$$W(e) = \fint_Q \Big[(k(x) + 2\frac{n-1}{n}\mu(x))(\mathrm{Tr}e)^2 - 4\mu(x)J_2(e)\Big]dx,$$

where $J_2(A)$ is the quadratic rotational invariant of the tensor A:

$$J_2(A) = \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}.$$

The representation above suggests the choice $J_2(e)$ as the translation. It satisfies

$$J_2(e(v)) = J_2(\nabla v) - \frac{1}{8}|\nabla v - (\nabla v)^t|^2$$

and

$$\fint_Q J_2(\nabla v)dx = J_2(\xi).$$

Here as in (3.6) we are fixing the infinitesimal rotations to be zero. Then

$$\begin{aligned} W(e) \;=\; & \fint_Q \Big[(k(x) + 2\frac{n-1}{n}\mu(x))(\mathrm{Tr}e)^2 - 4(\mu(x) - \mu_2)J_2(e)\Big]dx \\ & -4\mu_2 J_2(\xi) + \frac{1}{2}\mu_2 \fint_Q |\nabla v - (\nabla v)^t|^2 dx. \end{aligned}$$

Applying the harmonic mean bound and discarding the last term we obtain

$$W_{\min} \geq H(k(x) + 2\frac{n-1}{n}\mu_2)(\mathrm{Tr}\xi)^2 - 4\mu_2 J_2(\xi). \tag{6.1}$$

The inequality becomes equality if and only if the following optimality conditions are satisfied:

***In phase* 1**

$$e(v) = \varepsilon_0 I = \text{constant}, \tag{6.2}$$

where

$$\varepsilon_0 = \frac{H(k(x) + 2\frac{n-1}{n}\mu_2)}{nk_1 + 2(n-1)\mu_2}\mathrm{Tr}\xi;$$

in phase **2**

$$\mathbf{div}\, v = d, \tag{6.3}$$

where

$$d = \frac{H(k(x) + 2\frac{n-1}{n}\mu_2)}{k_2 + 2\frac{n-1}{n}\mu_2}\mathrm{Tr}\xi;$$

and in the whole period cell Q

$$\mathbf{curl}(v) = 0. \tag{6.4}$$

Let us compare the formulas (5.12)–(5.15) to the formulas (6.2)–(6.4). There are two obvious possibilities to make the identification. One is $v^e = v^c$ and the other one is $v^e = -v^c$, where the superscript c refers to conductivity and e to elasticity. Since the optimality conditions for elasticity are linear in ξ, we can assume without loss of generality that $\mathrm{Tr}\xi > 0$. Then everything we prove for such ξ will also be true for $-\xi$. Comparing the two sets of the optimality conditions we conclude that the conductivities A and B must be isotropic and

$$(\sigma^* - BI)^{-1} = \xi; \tag{6.5}$$

$$\frac{1}{\theta_1(A-B)} = \varepsilon_0; \tag{6.6}$$

$$\frac{A + (n-1)B}{\theta_1 B(A-B)} = d. \tag{6.7}$$

Solving this system for A, B and σ^* we obtain

$$\sigma^* = BI + \xi^{-1}; \tag{6.8}$$

$$A = \frac{1}{\theta_1(d - n\varepsilon_0)} + \frac{1}{\theta_1\varepsilon_0}; \tag{6.9}$$

$$B = \frac{1}{\theta_1(d - n\varepsilon_0)}. \tag{6.10}$$

In order to have $A > B > 0$ we need $d > n\varepsilon_0 > 0$. From (6.5) we see that ξ has to be positive definite. Therefore $d > n\varepsilon_0 > 0$ is equivalent to $k_1 > k_2$, which means that the two elastic materials have to be well-ordered. Finally, we use the fact [14] that σ^* achieving equality in the lower trace bound is attained if and only if

$$B < \sigma^* < \theta_1 A + \theta_2 B.$$

Substituting the values of A, B and σ^* in the above inequality we obtain that the positive definite matrix ξ must satisfy

$$\xi > K\mathrm{Tr}\xi I,$$

where

$$K = \frac{H(k(x) + 2\frac{n-1}{n}\mu_2)}{nk_1 + 2(n-1)\mu_2}.$$

This result is also true for the tensor $-\xi$. Thus we obtain the definition of the regime of values of ξ for which the bound (6.1) is optimal. We write it in terms of the eigenvalues of ξ: All eigenvalues ξ_i, $i = 1, \ldots, n$ have to be of the same sign and for each i

$$|\xi_i| \geq \frac{H(k(x) + 2\frac{n-1}{n}\mu_2)}{nk_1 + 2(n-1)\mu_2} \sum_{j=1}^{n} |\xi_j|. \tag{6.11}$$

In [34] L. Tartar described a class of optimal microstructures, the confocal ellipsoid construction, that saturates the trace bounds for conductivity. By the equivalence considered above this same construction will saturate the elasticity bound (6.1) for all values of the average strain ξ satisfying (6.11).

We remark that by analogy with the two dimensional elasticity (see Chapter 3), a generic anisotropic perturbation of the isotropic Hooke's law of the matrix material should break the degeneracy of this problem by eliminating such geometries as the confocal ellipsoid construction. But there are some special *anisotropic* Hooke's laws for which the confocal ellipsoid construction extends, namely laws like

$$(C\xi, \xi) = (b, \xi)^2 - 4\mu J_2(\xi),$$

where $b > 2\mu(1 - 1/n)I$. This case is equivalent to *anisotropic* conductivity.

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