NAMT
94-035

Dynamics and Oscillatory Microstructure in a Model of Displacive Phase Transformations

Irene Fonseca
Deborah Brandon
Carnegie Mellon University

Pieter Swart
Courant Institute of Mathematical Sciences

Research Report No. 94-NA-035

November 1994

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

Dynamics and Oscillatory Microstructure in a Model of Displacive Phase Transformations

IRENE FONSECA
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

DEBORAH BRANDON
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

PIETER SWART
Courant Institute of Mathematical Sciences
New York University
New York, NY 10012

1 Introduction

Recently there has been considerable progress in the understanding of material instabilities such as symmetry-breaking solid-solid phase transformations and the associated onset of microstructure. The variational treatment for the corresponding equilibrium problems generally leads to the study of nonconvex energy functionals for which there exist multiple minimizers. This severe nonuniqueness may be due in part to the lack of a surface energy contribution in the total energy, as well as to the fact that in the static framework we are ignoring the dynamical process, responsible in part for selecting physically preferred steady states (see [1], [3], [4], [5], [15]). In this work we present some results on the role of dynamics in the creation and evolution of oscillatory microstructure. A complete description can be found in Brandon, Fonseca & Swart [6].

We study the dynamics of oscillations in a one-dimensional model of a nonlinear visco-elastic solid which can undergo displacive phase transformations, as described by

$$u_{tt} = (\sigma(u_x) + \beta u_{xt})_x - \alpha u, \tag{1.1}$$

where

$$\begin{gathered}\sigma(u_x) := u_x^3 - u_x, \quad \beta > 0,\ \alpha \geq 0, \\ u(0,t) = u(1,t) = 0,\end{gathered} \tag{1.2}$$

and with initial data

$$u(x,0)=a(x), \quad u_t(x,0)=b(x), \quad 0<x<1. \tag{1.3}$$

This problem was introduced by Ball, Holmes, James, Pego & Swart in [3] as a model dissipative evolution equation whose underlying energy functional does not attain a minimum in the space of "classical" functions. Minimizing sequences develop oscillations and their macroscopic weak limits are nonminimizing states.

In order to gain some insight into the dynamical onset and pattern of the microstructure, we analyze the creation and propagation of oscillations through the behavior of the generalized measure-valued solutions (Young measures) (see also [4], [8]).

Using the theory of compensated compactness together with techniques developed by Tartar for the study of the Carleman and Broadwell models [21], we show that, regardless of the presence of oscillations in the initial velocity b, oscillations cannot be created in finite time, but if they exist in the initial strain $\{a'_\epsilon\}$ then they must persist in the strain u_x for all time. Also, in order to follow the propagation of these oscillations we determine the evolution problem solved by the Young's probability measure associated with the weakly converging solution sequence $\{(\frac{\partial u_\epsilon}{\partial t},\frac{\partial u_\epsilon}{\partial x})\}$.

In this setting the evolution is governed by the nonlinear partial differential equation (1.1) which is closely related to that of viscoelasticity with underlying static problem of mixed type. Physically, $u=u(x,t)$ represents the displacement of a one-dimensional viscoelastic continuum bonded, with strength α, to a rigid substrate. The function σ gives the nonconvex stress-strain response corresponding to a strain energy density W, precisely $\sigma(z):=dW(z)/dz$, while βu_{xxt} represents linear Newtonian viscosity. To make things concrete we choose $W(z)=\frac{1}{4}(z^2-1)^2$ as a simple but typical multi-well strain energy density. This choice is not crucial, and any two-well potential W with non-monotone, cubic-like derivative $\sigma(z)$ will produce similar results. We refer to [3] for more details.

The evolution equation (1.1) can be derived as the Euler-Lagrange equation corresponding to the energy functional

$$E[u]:=\tfrac{1}{2}\int_0^1 u_t^2\,dx+I[u], \tag{1.4}$$

with total stored elastic energy

$$\begin{aligned} I[u] &= \textstyle\int_0^1 W(u_x)\,dx+\frac{\alpha}{2}\int_0^1 u^2\,dx \\ &= \tfrac{1}{4}\|u_x\|_{L^4}^4-\tfrac{1}{2}\|u_x\|_{L^2}^2+\tfrac{\alpha}{2}\|u\|_{L^2}^2. \end{aligned}$$

Differentiating this energy along sufficiently smooth solutions gives

$$E\geq 0 \ \text{ and } \ \frac{dE}{dt}=-\beta\|u_{xt}\|_{L^2}^2, \ \text{for all } t\geq 0, \tag{1.5}$$

thus E is a Liapunov function for the dynamics, i.e. E is nonincreasing along solutions (and strictly decreasing provided $\|u_{xt}\|_{L^2} \neq 0$).

The nonuniqueness of solution for the equilibrium problem is readily seen. The equilibrium states of (1.1)–(1.3) are solutions of the boundary value problem

$$\begin{cases} (u_x^3 - u_x)_x - \alpha u = 0 \\ u(0) = u(1) = 0, \end{cases}$$

and are obtained by fitting together segments lying on the level sets of the first integral

$$\frac{3u_x^4}{4} - \frac{u_x^2}{2} - \frac{\alpha u^2}{2} = \text{constant},$$

in the range $1/\sqrt{3} < |u_x| < 2/\sqrt{3}$ with appropriate jump conditions, namely, that u and $\sigma(u_x) = u_x^3 - u_x$ be continuous at discontinuities in u_x (cf. [3]). Such steady states can have arbitrary many discontinuities in u_x, located with complete freedom, the only restriction being the constraint imposed by the boundary conditions, namely $\int_0^1 u_x\,dx = u(1,t) - u(0,t) = 0$. If, in addition, $\sigma'(u_x) = 3u_x^2 - 1 \geq \sigma_0 > 0$ a.e., then such equilibria are exponentially asymptotically stable with respect to perturbations which do not move or introduce strain (u_x) discontinuities, and are therefore weak relative minimizers (in the sense of [11]) of the total energy E. We refer to [3] for more details.

As in [15] and [3], we transform to new variables $p(x,t)$, $q(x,t)$, where

$$p(x,t) := \int_0^x u_t(s,t)\,ds - \int_0^1 \int_0^x u_t(s,t)\,ds\,dx, \quad q := \beta u_x - p,$$

observing that

$$p_x = u_t, \; p + q = \beta u_x$$

and that $\int_0^1 p\,dx = \int_0^1 q\,dx = 0$. Let B denote the solution operator for the Neumann problem $Bw := U$, where

$$U_{xx} = w - \int_0^1 w\,dx \quad \text{for} \quad 0 < x < 1, U_x(0,\cdot) = U_x(1,\cdot) = 0, \quad \int_0^1 U\,dx = 0,$$

so that

$$Bw = \int_0^x \int_0^y w\,dz\,dy - \int_0^1 \int_0^x \int_0^y w\,dz\,dy\,dx.$$

Equations (1.1)–(1.3) then transform into the semilinear degenerate parabolic system

$$\begin{cases} p_t = \beta p_{xx} + F((p+q)/\beta) \\ q_t = \qquad\quad - F((p+q)/\beta), \end{cases} \tag{1.6}$$

where

$$F(w) := \sigma(w) - \int_0^1 \sigma(w)\,dx - \alpha Bw, \tag{1.7}$$

subject to the Neuman boundary conditions

$$p_x(0,t) = p_x(1,t) = 0, \quad t > 0.$$

In both [15] and [3] this representation is exploited extensively. Local existence and uniqueness of solutions follow from standard semigroup results (as presented for example in [10]), and global existence follows from the fact that the equation in (1.6) for the q component can be viewed as an ODE at almost every x. More precisely, if $a \in W^{1,\infty}(0,1)$ and $b \in L^2(0,1)$ then, for any $\tau > 0$, $\nu < \frac{1}{2}$ and some $\gamma > 0$, it is shown in [3] and [15] that

$$u \in C([0,\infty), W^{1,\infty}),\ u_{xt} \in C^{\gamma}([\tau,\infty), L^{\infty}),\ u_{tt} \in C^{\gamma}([\tau,\infty), C^{\nu}).$$

The p component (and therefore also u_t) enjoys a limited amount of smoothing, whereas the q component (and therefore also the strain u_x) is not smoothed at all. In fact, the viscoelastic damping allows only stationary strain discontinuities (i.e. in u_x) and prohibits the creation (in finite time) of any discontinuities not present in the initial data.

2 Creation of oscillations

In this section we are interested in the dynamical behavior of solutions to (1.1) with oscillatory initial data (similar questions were addressed in [13], [16], [17], [19], [20]). For this purpose let $\{u^\epsilon\}$ denote a sequence of solutions to (1.1) arising from increasingly oscillatory initial data as $\epsilon \to 0$, namely

$$\begin{cases} u^\epsilon_{tt} = (\sigma(u^\epsilon_x) + \beta u^\epsilon_{xt})_x - \alpha u^\epsilon & 0 < x < 1 \\ u^\epsilon(0,t) = u^\epsilon(1,t) = 0 \\ u^\epsilon(x,0) = a^\epsilon(x), \quad u^\epsilon_t(x,0) = b^\epsilon(x), & 0 < x < 1 \end{cases}$$

where, as before, $\sigma(u^\epsilon_x) = (u^\epsilon_x)^3 - u^\epsilon_x$. The parameter $\epsilon > 0$ can be thought of as denoting the length-scale of spatial oscillations in the initial data. In the p, q variables we have

$$\begin{cases} p^\epsilon_t = & \beta p^\epsilon_{xx} & + & F((p^\epsilon + q^\epsilon)/\beta) \\ q^\epsilon_t = & & - & F((p^\epsilon + q^\epsilon)/\beta), \end{cases} \tag{2.1}$$

subject to the Neumann boundary conditions

$$p^\epsilon_x(0,t) = p^\epsilon_x(1,t) = 0, \quad t > 0.$$

In view of (1.4) we consider sequences of initial data, $\{a^\epsilon\}$ and $\{b^\epsilon\}$, for which

$$\sup_\epsilon \left(\|a^\epsilon\|_{W_0^{1,4}} + \|b^\epsilon\|_{L^2} \right) \le C. \tag{2.2}$$

For weak solutions of (1.1) - (1.3) we have the weak form of viscoelastic energy dissipation

$$E(t) = E(0) - \beta \int_0^t \|u^\epsilon_{xt}\|^2_{L^2}\, dt$$

and an immediate consequence of the non-increasing nature of $E(t)$ is that

$$u^\epsilon \in Bdd\, L^\infty(0,T; W_0^{1,4}),\ u^\epsilon_t \in Bdd\, L^\infty(0,T; L^2).$$

(We write $u^\epsilon \in Bdd\, L^\infty(X)$ to denote the fact that $\{u^\epsilon\}$ is uniformly bounded in $L^\infty(X)$.) Moreover, from (1.5),

$$\int_0^T \|u^\epsilon_{xt}\|^2_{L^2}\, dt \le \frac{1}{\beta}(E(0) - E(T)),$$

and since $E \geq 0$ it follows that

$$u^\epsilon_x \in Bdd\, H^1(0,T; L^2).$$

The above energy estimates can also be expressed in terms of p and q; from the relations $p^\epsilon_x = u^\epsilon_t$ and $p^\epsilon + q^\epsilon = \beta u^\epsilon_x$ we obtain

$$p^\epsilon \in Bdd\, L^\infty(0,T; H^1) \cap L^2(0,T; H^2),$$
$$q^\epsilon \in Bdd\, L^\infty(0,T; L^4) \cap H^1(0,T; L^2),$$

and therefore, from (1.7) and the choice $\sigma(u_x) = u_x^3 - u_x$, we have that

$$p^\epsilon_t = \beta p^\epsilon_{xx} + F^\epsilon,\ F^\epsilon \in Bdd\, L^\infty(0,T; L^{\frac{4}{3}})$$

where

$$F^\epsilon := F\left(\frac{p^\epsilon + q^\epsilon}{\beta}\right).$$

Note that if we had the slightly sharper bound $F^\epsilon \in Bdd\, L^2(0,T; L^2)$, then there would be a weak solution with $p^\epsilon_t \in BddL^2(0,T; L^2)$ and therefore also $p^\epsilon_{xx} = (p^\epsilon_t - F^\epsilon)/\beta \in Bdd\, L^2(0,T; L^2)$, where we have used (with $V = H^1, H = L^2$)

Theorem 2.1 [Existence of weak solutions, [12]]
If $u_0 \in V$ then there exists a $T > 0$ and a unique solution $u \in C(0,T; V)$ such that

$$u_t, Au \in L^2(0,T; H), \tag{2.3}$$

$$\|u_t\|_{L^2(0,T;H)}, \|Au\|_{L^2(0,T;H)} \le C(T, \|u_0\|_V, \|f\|_{L^2(0,T;H)}), \tag{2.4}$$

and

$$u_t = Au + f \text{ for } t \in (0,T)$$

holds in the distributional sense in $L^2(0,T; H)$.

From compensated compactness (see [7], [14], [18]) it would then follow that

$$p_x^\epsilon \text{ converges strongly in } L^2_{loc}((0,T)\times(0,1))\,. \tag{2.5}$$

Precisely, here we applied the following result

Theorem 2.2 *Let $\{p_\epsilon\} \in BddH^1((0,T)\times(0,1)), p_\epsilon \stackrel{*}{\rightharpoonup} p^0$ in $H^1((0,T)\times(0,1))$, and assume that $\{p^\epsilon_{xx}\} \in BddL^2((0,T)\times(0,1))$. Then p^ϵ_x converges strongly to p^0_x in $L^2_{loc}((0,T)\times(0,1))$.*

Despite having only the weaker bound $F^\epsilon \in Bdd\, L^2(0,T;L^{\frac{4}{3}})$ at our disposal, we now show the existence of weak solutions, and, moreover, that (2.5) remains valid, even in the possible presence of (initial) singularities in F^ϵ. Our proof relies on the following decomposition of the function F^ϵ.

Lemma 2.3 [Decomposition of singularities]
Let $f \in L^p(\Omega)$, with $\Omega \subset \mathbf{R}^n$ open and bounded and $1 < p < 2$. For any $\delta > 0$, there exists a decomposition $f = f_1(\delta) + f_2(\delta)$, such that $f_1(\delta) \in L^1(\Omega)$, $f_2(\delta) \in L^2(\Omega)$, and

$$\|f_1(\delta)\|_{L^1} \le \delta \|f\|_{L^p}, \qquad \|f_2(\delta)\|_{L^2} \le \frac{1}{\delta}\|f\|_{L^p}. \tag{2.6}$$

Using this lemma, we now show that (2.5) holds, by decomposing $F^\epsilon(t) \in L^{\frac{4}{3}}$ as the sum of an L^2 function and an arbitrarily small L^1 function. This representation, together with the lemma below, allows us to extract a strongly convergent subsequence of $\{p^\epsilon_x\}$.

Lemma 2.4 *Let $\{x_n\}$ be a bounded sequence in a complete metric space (X,d) and assume that, for any $\delta > 0$, the sequence $\{x_n\}$ is contained in the open δ-neighborhood of some compact set $K_\delta \subset X$. Then $\{x_n\}$ contains a convergent subsequence.*

Theorem 2.5
For initial data satisfying the minimal regularity requirement (2.2) for finite-energy solutions, there exists a unique sequence of distributional solutions to (2.1). Furthermore, $\{u^\epsilon_t\} = \{p^\epsilon_x\}$ converges strongly in $L^2_{loc}((0,T)\times(0,1))$.

Proof.
For any $\delta > 0$, according to Lemma 2.3 we can decompose F^ϵ as

$$F^\epsilon = F_1^\epsilon(\delta) + F_2^\epsilon(\delta),$$

with $F_1^\epsilon(\delta)(\cdot,t) \in L^1$ and $F_2^\epsilon(\delta)(\cdot,t) \in L^2$ satisfying

$$\|F_1^\epsilon(\delta)\|_{L^\infty(0,T;L^1)} \le \delta\, \|F^\epsilon\|_{L^\infty(0,T;L^{\frac{4}{3}})} \le C\delta,$$

$$\|F_2^\epsilon(\delta)\|_{L^\infty(0,T;L^2)} \leq \frac{1}{\delta}\|F^\epsilon\|_{L^\infty(0,T;L^{\frac{4}{3}})} \leq C/\delta.$$

We now solve for $p^\epsilon = p_1^\epsilon(\delta) + p_2^\epsilon(\delta)$, where

$$\begin{cases} p_1^\epsilon(\delta)_t = \beta p_1^\epsilon(\delta)_{xx} + F_1^\epsilon(\delta), \\ p_1^\epsilon(\delta)(x,0) = 0, \end{cases}$$

and

$$\begin{cases} p_2^\epsilon(\delta)_t = \beta p_2^\epsilon(\delta)_{xx} + F_2^\epsilon(\delta), \\ p_2^\epsilon(\delta)(x,0) = p^\epsilon(x,0) \in Bdd\, H^1. \end{cases}$$

Since $L^1 \hookrightarrow (H^1)'$, there exists a unique weak solution $p_1^\epsilon(\delta) \in Bdd\, L^2(0,T;H^1)$ satisfying

$$\|p_1^\epsilon(\delta)\|_{L^2(0,T;H^1)} \leq C\delta.$$

On the other hand, since $F_2^\epsilon(\delta) \in Bdd\, L^2(0,T;L^2)$, Theorem 2.1 yields a unique weak solution $p_2^\epsilon(\delta) \in BddH^1((0,T)\times(0,1))$ satisfying $\{(p_2^\epsilon(\delta))_{xx}\} \in BddL^2((0,T)\times(0,1))$. Thus, from Theorem 2.2 it follows that $\{p_2^\epsilon(\delta)_x\} \subset K_\delta$, where for each $\delta > 0$, K_δ is a compact subset of $L^2_{loc}((0,T)\times(0,1))$, with diameter proportional to $1/\delta$.

We have therefore established the existence of a weak solution p^ϵ for which

$$\text{dist}(\{p_x^\epsilon\}, K_\delta) \leq \sup_\epsilon \|p_1^\epsilon(\delta)_x\|_{L^2((0,T)\times(0,1))} \leq C\delta,$$

i.e., for each $\delta > 0$ the sequence $\{p_x^\epsilon\}$ lies in a δ-neighborhood of some compact $K_\delta \subset L^2_{loc}((0,T)\times(0,1))$. We are now in a position to apply Lemma 2.4, which guarantees the existence of a subsequence $u_t^{\epsilon'} = p_x^{\epsilon'}$ that converges strongly in $L^2_{loc}((0,T)\times(0,1))$. □

3 Propagation of oscillations

In Theorem 2.5 above, it was shown that any oscillatory microstructure in the transformed system (2.1) must be restricted to the q component, or, equivalently, to u_x in the original evolution equation (1.1). Also, there is a smoothing effect in that oscillations in the initial velocity $u_t(x,0) = p_x(x,0) = b(x)$ immediately vanish due to the parabolic nature of the evolution equation (1.6) for p.

In this section we show that oscillatory microstructure will persist where initially present, by deriving upper and lower limits on its rate of growth (Theorem 3.4). Moreover, for this problem microstructure cannot migrate into regions initially free of oscillations.

We henceforth restrict our attention to solutions for which $u_x \in L^\infty$, and therefore consider the problem (1.1)-(1.3) with initial data a^ϵ, b^ϵ, satisfying

$$\sup_\epsilon \left(\|a^\epsilon\|_{W_0^{1,\infty}} + \|b^\epsilon\|_{L^2}\right) \leq C. \tag{3.1}$$

This is the case studied in [3], where existence, uniqueness and regularity of solution were described as follows.

Theorem 3.1 [Existence and uniqueness of strong solutions, Ball et al. [3]]. *Suppose $a \in W_0^{1,\infty}, b \in L^2$. Then for any $T > 0$, a unique solution of (1.1) - (1.3) exists, satisfying*

$$\{u, u_t\} \in C([0,\infty), W_0^{1,\infty} \times L^2) \cap C^1((0,\infty), W_0^{1,\infty} \times C)$$

and

$$u_{tt} \in C((0,\infty), C), \sigma(u_x) + \beta u_{xt} \in C((0,\infty), C^1).$$

Furthermore, we have

$$\sup_{t \geq 0} \|u(\cdot,t)\|_{W^{1,\infty}} \leq M < \infty, \tag{3.2}$$

$$\sup_{t > \tau} \|u_t(\cdot,t)\|_{W^{1,\infty}} \leq N < \infty, \quad \forall \tau > 0, \tag{3.3}$$

and the energy identity (1.5) holds for all $t > 0$.

From (3.2) we have $\sup_{\epsilon,t} \|q^\epsilon(0,t)\| \leq M(\beta) < \infty$, and we can therefore extract a subsequence of $\{q^\epsilon\}$ that converges weak-* in L^∞. Also, from (3.3) and since $p_x^\epsilon = u_t^\epsilon$, we have that (a subsequence) $\{p_x^\epsilon\}$ converges weak-* in L^∞ and we write:

Definition 3.2

$$\begin{aligned} P_1 &:= \lim p^\epsilon, \\ R_n &:= \text{weak-*} \lim (p_x^\epsilon)^n, \\ Q_m &:= \text{weak-*} \lim (q^\epsilon)^m \\ W_m &:= Q_m - Q_{m-1}Q_1. \end{aligned}$$

Remark 3.3

i) Clearly $R_1 = P_{1x}$ and by Theorem 2.4 $\{p_x^\epsilon\}$ converges strongly in $L^p_{loc}((0,T) \times (0,1)), 1 \leq p < +\infty$. Therefore $R_n = (R_1)^n$.

ii) It is well known that $W_2 \geq 0$ and that $\{q^\epsilon\}$ is a strongly convergent sequence if and only if $W_2 = 0$, *i.e.* $Q_2 = Q_1^2$. Therefore W_2 can be viewed as a measure for the extent to which the sequence $\{q^\epsilon\}$ fails to converge strongly due to the formation of oscillations.

Knowing the dynamical behavior of W_2, we are able to characterize the growth or decay of oscillatory microstructure. For this purpose, we derive an evolution equation for W_2 from which we derive the following constraint on the growth of oscillatory microstructure.

Theorem 3.4

$$W_2(x,0)\, e^{\frac{2}{\beta}(1-3M^2)t} \leq W_2(x,t) \leq W_2(x,0)\, e^{\frac{2}{\beta}t}. \tag{3.4}$$

Remark 3.5

i) If the initial data contains no oscillatory microstructure on a subset (a,b) (i.e. $W(x,0) = 0$ for $x \in (a,b)$), then this result shows that no microstructure can be created there in finite positive time, thereby ruling out the dynamical migration of microstructure into regions initially free of oscilations. On the other hand, in those regions where oscillatory microstructure is present, its "strength" — as measured by $W_2 = Q_2 - Q_1^2$ — cannot increase faster than $e^{\frac{2}{\beta}t}$.

ii) Note that the upper bound (3.4) does not depend on α , and (3.4) is therefore valid even if $\alpha = 0$ (the case studied in [15]). The origin of a possible exponential growth in W_2 then becomes clearer if we observe that (3.4) also holds for the simpler ODE

$$q_t = -\left[\sigma\left(\frac{q}{\beta}\right) - \int_0^1 \sigma\left(\frac{q}{\beta}\right) dx\right], \quad \int_0^1 q\, dx = 0,$$

obtained by setting $p = 0$, $\alpha = 0$. Here a small but highly oscillatory perturbation of the unstable steady state $q = 0$ (corresponding to $u_x = 0$) which separates the two isolated wells at $q = \pm\beta$ (corresponding to $u_x = \pm 1$) will also lead to such a rapid initial increase in $Q_2 - Q_1^2$.

iii) This theorem shows the persistence of oscillatory microstructure, i. e. if initially present, it cannot disappear in finite time.

iv) Note that the exponent in (3.4) can have either sign. In particular, when β becomes very small (the case of small damping), this theorem predicts a strong exponential growth in the existing microstructure (as measured by nonzero W_2).

v) This persistence, or even possible exponential growth, in oscillatory microstructure is in marked contrast with the behavior of entropy solutions of a 2×2 hyperbolic system of conservation laws:

$$u_t + f(u)_x = 0. \tag{3.5}$$

In the genuinely nonlinear case, of which Burger's equation ($f(u) = u^2/2$) is the prototypical example, the pioneering paper of DiPerna [8] showed that, roughly speaking, oscillations of wavelength ϵ have a lifetime of the order of ϵ, i.e. as time evolves, cancellation and acoustic absorption cause the instantaneous destruction of all oscillatory microstructure in the initial data.

Theorem 3.4 above shows that the evolution equation under question has more in common with the linearly degenerate case of (3.5), studied for example in [17], [16] and [22], in which oscillations cannot be created if not initially present, but, if present, can persist and propagate along the linear degenerate field. For our case, however, oscillatory microstructure cannot propagate, due to the ODE-like evolution of the q component.

4 Evolution of Young measures

As in Section 3 we assume that

$$\sup_{\epsilon} (||a^\epsilon||_{W^{1,\infty}} + ||b^\epsilon||_{L^\infty}) < +\infty$$

and suppose further that

$$a^\epsilon \overset{*}{\rightharpoonup} a \text{ in } W^{1,\infty}, \quad b^\epsilon \overset{*}{\rightharpoonup} b \text{ in } L^\infty.$$

If $\{a^\epsilon_x\}$ oscillates, then by Theorem 3.4 so does $\{u^\epsilon_x\}$ and in this case the Young measure $\{\Lambda_{x,t}\}_{(x,y)}$ generated by a subsequence of $\{(u^\epsilon_t, u^\epsilon_x)\}$ helps us identify nonlinear weak limits of $\{(u^\epsilon_t, u^\epsilon_x)\}$. We recall (see [2], [7], [14], [18])

Definition 4.1 *Let $\{u_n\}$ be a sequence in $BddL^\infty(\Omega)$, where Ω is a bounded, open subset of $\mathbf{R}^N$. Then there exists a subsequence $\{u_h\}$ and a family of probability measures $\{\nu_x\}_{x\in\Omega}$ such that if $f : \mathbf{R} \to \mathbf{R}$ is a continuous function then*

$$f(u_h) \overset{*}{\rightharpoonup} \bar{f} \text{ in } L^\infty$$

where

$$\bar{f}(x) := \int_{\mathbf{R}} f(y) d\nu_x(y), \text{ for a. e. } x \in \Omega.$$

The parametrized measure $\{\nu_x\}_{x\in\Omega}$ is said to be the Young measure generated by $\{u_h\}$.

In our context, if $\{\Lambda_{x,t}\}_{(x,t)}$ is the Young measure generated by some subsequence $\{(u^{\epsilon'}_t, u^{\epsilon'}_x)\}$ and if $f : \mathbf{R}^2 \to \mathbf{R}$ is a continuous function, then

$$f(u^{\epsilon'}_t, u^{\epsilon'}_x) \to \left[(x,t) \to \int_{\mathbf{R}^2} f(s,y) d\Lambda_{x,t}(s,y)\right].$$

Note that since the set of polynomials in $\mathbf{R}^2$ is dense in $C_0(\mathbf{R}^2)$, the Young measure $\{\Lambda_{x,t}\}_{(x,t)}$ is completely characterized by its moments

$$\int_{\mathbf{R}^2} s^n y^m d\Lambda_{x,t}(s,y) = \text{weak-}*\text{ limit of } (u^{\epsilon'}_t)^n (u^{\epsilon'}_x)^m.$$

Using the notation introduced in Definition 3.2 and by virtue of the strong convergence of $\{u^\epsilon_t\}$ (see Theorem 2.4) we have

$$\begin{aligned} \int_{\mathbf{R}^2} s^n y^m d\Lambda_{x,t}(s,y) &= \text{weak-}*\text{ limit of } (p^{\epsilon'}_t)^n \left(\frac{p^{\epsilon'} + q^{\epsilon'}}{\beta}\right)^m \\ &= \frac{1}{\beta^m} (P_{1x})^n \sum_{i=0}^m \binom{i}{m} (P_1)^i Q_{m-i}. \end{aligned} \tag{4.1}$$

Theorem 4.2 [**Uniqueness of the Young measure**]
If the subsequences of $(u_t^\epsilon, u_x^\epsilon)$, $\{(u_t^{\epsilon'}, u_x^{\epsilon'})\}$ *and* $\{(u_t^{\epsilon^*}, u_x^{\epsilon^*})\}$, *have corresponding Young measures* $\{\Lambda_{x,t}\}_{(x,t)}$, $\{\widehat{\Lambda}_{x,t}\}_{(x,t)}$, *then* $\widehat{\Lambda}_{x,t} = \Lambda_{x,t}$. *In particular, the sequence* $\{(u_t^\epsilon, u_x^\epsilon)\}$ *converges* weak-$*$ *in* L^∞.

Proof.
The latter part of the statement is a consequence of the uniqueness of the Young measure. Indeed, if $\{\Lambda_{x,t}\}_{(x,t)}$ is the Young measure generated by (a subsequence of) $\{(u_t^\epsilon, u_x^\epsilon)\}$ then any subsequence of $\{(u_t^\epsilon, u_x^\epsilon)\}$ admits a (sub)subsequence $\{(u_t^{\epsilon'}, u_x^{\epsilon'})\}$ generating the same Young measure, and so

$$(u_t^{\epsilon'}, u_x^{\epsilon'}) \stackrel{*}{\rightharpoonup} \left[(x,t) \to \int_{\mathbf{R}^2} (s,y)\, d\Lambda_{x,t}(s,y)\right]$$

and we conclude that there is uniqueness of all weak-* sublimits of $\{(u_t^\epsilon, u_x^\epsilon)\}$.

By (4.1)

$$\Lambda_{x,t} = \widehat{\Lambda}_{x,t} \iff P_1 = \widehat{P}_1 \text{ and } Q_m = \widehat{Q}_m \ \forall m.$$

We now show that $P_1 = \widehat{P}_1$ and $Q_m = \widehat{Q}_m$ at $t = 0$ implies $P_1 = \widehat{P}_1$ and $Q_m = \widehat{Q}_m$ for $t > 0$. Without loss of generality, we assume that $\beta = 1$. From (2.1) we have, for $m = 1, 2, \ldots$

$$\frac{\partial}{\partial t}(q^\epsilon)^m = m(q^\epsilon)^{m-1}\left\{-\sigma\left(\frac{p^\epsilon + q^\epsilon}{\beta}\right) + \int_0^1 \sigma\left(\frac{p^\epsilon + q^\epsilon}{\beta}\right) dx + \frac{\alpha}{\beta}B(p^\epsilon + q^\epsilon)\right\},$$

from which we deduce that

$$\frac{1}{m}\frac{\partial Q_m}{\partial t} = -Q_{m+2} - 3P_1 Q_{m+1} - \sigma'(P_1)Q_m - \sigma(P_1)Q_{m-1} + F_1 Q_{m-1},$$

where

$$F_1 := \int_0^1 \{Q_3 + 3P_1 Q_2 + \sigma'(P_1)Q_1 + \sigma(P_1)\}\, dx + \alpha B(P_1 + Q_1),$$

and similarly for the $\widehat{Q}_m$ (with $\widehat{P}_1$ replacing P_1, etc.). At $t = 0$,

$$Q_m(x,0) = \widehat{Q}_m(x,0) = \widehat{A}_m(x) = \text{weak-}*\lim\left[a^\epsilon - \int_0^1 b^\epsilon\, dx\right]^m.$$

Therefore,

$$\begin{aligned}
\frac{1}{m}\frac{\partial}{\partial t}(Q_m - \widehat{Q}_m) = & - (Q_{m+2} - \widehat{Q}_{m+2}) - 3P_1(Q_{m+1} - \widehat{Q}_{m+1}) - \sigma'(P_1)(Q_m - \widehat{Q}_m) \\
& - \sigma(P_1)(Q_{m-1} - \widehat{Q}_{m-1}) + F_1(Q_{m-1} - \widehat{Q}_{m-1}) \\
& - 3\widehat{Q}_{m+1}(P_1 - \widehat{P}_1) - \widehat{Q}_m(\sigma'(P_1) - \sigma'(\widehat{P}_1)) \\
& - \widehat{Q}_{m-1}(\sigma(P_1) - \sigma(\widehat{P}_1)) + \widehat{Q}_{m-1}(F_1 - \widehat{F}_1).
\end{aligned}$$

Let $\delta_m(t) := \|Q_m(\cdot,t) - \widehat{Q}_m(\cdot,t)\|_{L^\infty(0,1)}$ ($\delta_0 \equiv \delta_1 \equiv 0$). Using the uniform bounds (3.2), (3.3)

$$\|Q_m\|_{L^\infty}, \|\widehat{Q}_m\|_{L^\infty}, \|P_1^m\|_{L^\infty}, \|\widehat{P}_1^m\|_{L^\infty} \leq K^m,$$

for some constant $K > 0$, and it is straightforward to show that, for some constant $C = C(K) > 0$ (which need not necessarily be the same in each step below), we have

$$\begin{aligned}\frac{1}{m}\delta_m(t) &\leq C\int_0^t \{\delta_{m-1} + \delta_m + \delta_{m+1} + \delta_{m+2}\}(s)\,ds \\ &+ CK^{m+1}\int_0^t \left\{\|P_1 - \widehat{P}_1\|_{L^\infty_x} + \|F_1 - \widehat{F}_1\|_{L^\infty_x}\right\}(s)\,ds.\end{aligned}$$

Moreover,

$$\begin{aligned}F_1 - \widehat{F}_1 &:= \int_0^1 \left\{(P_1^3 + 3P_1^2Q_1 + 3P_1Q_2 + Q_3) - (\widehat{P}_1^3 + 3\widehat{P}_1^2\widehat{Q}_1 + 3\widehat{P}_1\widehat{Q}_2 + \widehat{Q}_3)\right\}dx \\ &+ \alpha B(P_1 - \widehat{P}_1) + \alpha B(Q_1 - \widehat{Q}_1),\end{aligned}$$

hence

$$\|F - \widehat{F}\|_{L^\infty_x} \leq C\|P_1 - \widehat{P}_1\|_{L^\infty_x} + C\{\delta_1 + \delta_2 + \delta_3\}.$$

We therefore have that

$$\begin{aligned}\|\delta_m\|_{L^\infty} &\leq (2K)^m \\ \frac{1}{m}\delta_m(t) &\leq CK^{m+1}at + CK^{m+1}\int_0^t \{\delta_1 + \delta_2 + \delta_3\}(s)\,ds \\ &+ C\int_0^t \{\delta_{m-1} + \delta_m + \delta_{m+1} + \delta_{m+2}\}(s)\,ds, \end{aligned} \tag{4.2}$$

where $a := \|P_1 - \widehat{P}_1\|_{L^\infty}$. Without loss of generality, we can take $C, K > 1$ so that

$$\begin{aligned}\delta_m(t) &\leq mC\left\{K^{m+1}at + K^{m+1}\textstyle\int_0^t(\delta_1 + \delta_2 + \delta_3)(s)ds\right. \\ &\left.+ \textstyle\int_0^t(\delta_{m-1} + \delta_m + \cdots + \delta_{m+2})(s)ds\right\}.\end{aligned}$$

This inequality implies that for all k = 0, 1, 2,. . . ,

$$\delta_m(t) \leq a\sum_{i=0}^{k}\alpha_m^{(i)}t^i + C_m^{(k)}(2K)^m t^k \tag{4.3}$$

where

$$\alpha_m^{(i)} := (7C)^i\frac{m(m+2)\ldots(m+2i)}{i!}K^{m+2i+3} \text{ if } i \geq 1$$

and $\alpha_m^{(0)} = 0$, with

$$C_m^{(k)} := [7C(2K)^4]^k\frac{m(m+2)\ldots(m+2(k-1))}{k!} \text{ if } k \geq 1$$

and $C_m^{(0)} = 1$.

Let $t < T^* := \frac{1}{14(2K)^4}$. Since, as $k \to +\infty$,

$$\frac{C_m^{k+1}(2K)^m t^{k+1}}{C_m^k(2K)^m t^k} = 7C(2K)^4 \frac{m+2K}{k+1} t \to 14C(2K)^4 t,$$

$$\frac{\alpha_m^{(k+1)} t^{k+1}}{\alpha_m^{(k)} t^k} = 7C \frac{m+2(k+1)K^2 t}{k+1} \to 14CK^2 T,$$

by the Weierstrass ratio test we have

$$C_m^{(k)}(2K)^m t^k \to 0,\ S_m(t) := \sum_{i=0}^{\infty} \alpha_m^{(i)} t^i \text{ is convergent.}$$

This result, together with (4.2), yields

$$\delta_m(t) \le a S_m(t) \tag{4.4}$$

for all $m, 0 \le t < T^*$. On the other hand, by (1.6) we have

$$\begin{cases} (P_1 - \hat{P}_1)_t = (P_1 - \hat{P}_1)_{xx} + f(P_1 - \hat{P}_1, \delta_1, \delta_2, \delta_3) \\ (P_1 - \hat{P}_1)(x, 0) = 0 \\ (P_1 - \hat{P}_1)_x(0, t) = 0 = (P_1 - \hat{P}_1)_x(1, t),\ x > 0, \end{cases} \tag{4.5}$$

where

$$|f(P_1 - \hat{P}_1, \delta_1, \delta_2, \delta_3)| \le Const.(a + a^2 + a^3 + |\delta_1| + |\delta_2| + |\delta_3|).$$

Multiplying the equation for $(P_1 - \hat{P}_1)$ in (4.8) by $(P_1 - \hat{P}_1)$, integrating by parts, using Gronwall's lemma and (4.7), we conclude that $(P_1 - \hat{P}_1) = 0$ in $[0, T^*)$ and (4.7) yields

$$\delta_m(t) = 0 \text{ in } [0, T^*).$$

Translating the origin to $t = \frac{T^*}{2}$ and repeating the above argument a finite number of times, we conclude that $\delta_m(t) = 0$ for $t \in [0, T)$ and this completes the proof. □

Acknowledgments. This work was supported by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis at Carnegie Mellon University. The authors are profoundly indebted to Luc Tartar for the many interesting mathematical discussions, which motivated this work. Also, IF thanks NSF (NSF/DMS-9201215) for supporting this work, and PJS thanks AFOSR (AFOSR-90-0090) and NSF (NSF/DMS-92-17151).

References

[1] J.M. Ball. Dynamics and minimizing sequences, in *Problems Involving Change of Type*, (ed. K. Kirchgässner) Springer Lecture Notes in Physics No. 359, 3–16, Springer–Verlag, New York, Heidelberg, Berlin, 1990.

[2] J.M. Ball. A version of the fundamental theorem for Young measures, in *PDEs and Continuum Models of Phase Transitions,* (eds. M. Rascle, D. Serre and M. Slemrod) Springer Lecture Notes in Physics No. 344, 207–216, Springer–Verlag, New York, Heidelberg, Berlin, 1989.

[3] J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart. On the dynamics of fine structure. *J. Nonlinear Science* **1**, 1991, 17–70.

[4] J.M. Ball & R.D. James. Fine phase mixtures as minimizers of energy. *Arch. Rat. Mech. Anal.* **100**, 1987, 13–52.

[5] J.M. Ball & R.D. James. Proposed experimental tests of a theory of fine microstructure and the two-well problem. *Phil. Trans. R. Soc. Lond. A* **338**, 1992, 389–450.

[6] D. Brandon, I. Fonseca & P.J. Swart. The creation and propagation of oscillatory microstructure in a dynamical model of displacive phase transformations. *Preprint*, 1994.

[7] B. Dacorogna. *Weak continuity and weak lower semicontinuity of non-linear functionals.* Springer Lecture Notes in Physics No. 922, Springer–Verlag, New York, Berlin, 1982.

[8] R.J. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rat. Mech. Anal.* **82**, 1983, 27–70.

[9] R.J. DiPerna. Measure-valued solutions to conservation laws. *Arch. Rat. Mech. Anal.* **88**, 1985, 223–270.

[10] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Springer Lecture Notes in Mathematics No. 840, Springer–Verlag, New York, 1981.

[11] R.D. James. Finite deformations by mechanical twinning. *Arch. Rat. Mech. Anal.* **77**, 1981, 143–176.

[12] J.L. Lions & E. Magenes *Nonhomogeneous Boundary Value Problems and Applications.* Springer–Verlag, New York, 1972.

[13] D. McLaughlin, G. Papanicolaou & L. Tartar. Weak limits of semilinear hyperbolic systems with oscillatory data, in *Macroscopic Modeling of Turbulent Flows.* (eds. U. Frisch et al.) Springer Lecture Notes in Physics No. 230, 277–298, Springer–Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[14] F. Murat A survey on compensated compactness, in *Contributions to modern calculus of variations.* (ed. L. Cesari), Pitman Research Notes in Math., 145–183. Longman, Harlow, 1987.

[15] R.L. Pego. Phase transitions in one–dimensional nonlinear viscoelasticity: admissibility and stability. *Arch. Rat. Mech. Anal.* **97**, 1987, 353–394.

[16] M. Rascle. On the static and dynamic study of oscillations for some nonlinear hyperbolic systems of conservation laws. *Anal. Nonlin.* **8**, 3-4, 1991, 333–350.

[17] D. Serre. La compacité par compensation pour les systèmes hyperboliques non linéaires de deux équations à une dimension d'espace. *J. Math. Pures et Appl.* **65**, 1986, 423–468.

[18] L. Tartar. Compensated compactness and applications to PDEs, in *Nonlinear Analysis and Mechanics* (ed. R. Knops), Pitman Research Notes in Math., Vol. 4, 136–212. Pitman, London–San Francisco, 1979.

[19] L. Tartar. The compensated compactness method applied to systems of conservation laws, in *Material Instabilities in Continuum Mechanics and Related Mathematical Problems* (ed. J. M. Ball), 263–285. Oxford University Press, 1983.

[20] L. Tartar. Oscillations and Asymptotic Behaviour for Two Semilinear Hyperbolic Systems, in *Dynamics of Infinite Dimensional Systems* (eds. S. N. Chow & J. K. Hale), 341–356. Springer–Verlag, Berlin–Heidelberg, 1987.

[21] L. Tartar. Private communication.

[22] Weinan E & R.V. Kohn. The Initial–Value Problem for Measure–Valued Solutions of a Canonical 2×2 System with Linearly Degenerate Fields. *Comm. Pure Appl. Math.* **44**, 1991, 981–1000.