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FOURTH ORDER MOMENTS OF NONNEGATIVE MEASURES ON S² AND APPLICATIONS

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Abstract: We characterize the set of fourth order moments of a nonnegative measure on S^2. This question is motivated by problems of homogenization in linearized elasticity. As a consequence the minimum number of layering directions for generating general anisotropic, transversely isotropic or isotropic multilayered media is found. A key ingredient is a result about the decomposition of some nonnegative polynomials into sums of squares of polynomials; this particular result is due to HILBERT (1888).

0. Introduction

This paper is mainly devoted to a study of the fifteen dimensional set of all fourth-order moments of a nonnegative RADON (BOREL regular) measure on S^2. A complete characterization of the set, a closed cone, is being sought with the help of the HILBERT decomposition theorem for nonnegative polynomials of degree four in two variables. Special emphasis is placed on the boundary of this set which is shown to be generated by atomic measures made of five DIRAC masses or less, those being located on the intersection of S^2 with the zero set of a quadratic form. As a consequence it is shown that every point of the moment set is generated by atomic measures made of six DIRAC masses or less.

As such, this study may be viewed as a contribution to the moment problem; remark that the analogous two-dimensional case has been analyzed in AVELLANEDA & MILTON [AM]. Potential applications for this result are however manifold, especially in a field familiar to the authors, that of homogenization. In the setting of linearized elasticity, effective properties of fine mixtures of two phases are investigated. Specifically, the goal is to analyze the coefficients of the linear second order elliptic system satisfied by the weak limit u of the solution fields u^ϵ to a sequence of elastic problems of the form

$$-\text{div}\Big(A^\epsilon e(u^\epsilon)\Big) = f \text{ in } \Omega, \quad u^\epsilon = 0 \text{ on } \partial\Omega, \tag{0.1}$$

with

$$e_{ij}(u^\epsilon) = \frac{1}{2}\Big(\frac{\partial u_i^\epsilon}{\partial x_j} + \frac{\partial u_j^\epsilon}{\partial x_i}\Big) \text{ in } \Omega,$$

and

$$A^\epsilon = \chi^\epsilon A + (1-\chi^\epsilon)B. \tag{0.2}$$

In (0.2), A and B are two elastic tensors and χ^ϵ a given arbitrary sequence of characteristic functions on Ω, while in (0.1) f is a given arbitrary element of $H^{-1}(\Omega)$.

It is a well known result in the theory of homogenization (cf. TARTAR [T]), that u satisfies

$$-\text{div}\Big(A^0 e(u)\Big) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with

$$e_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \text{ in } \Omega,$$

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and where A^0 is an elasticity tensor called the effective (or homogenized) tensor.

The only knowledge of the weak $\star$ limit θ (in $L^\infty(\Omega;[0,1])$) of χ^ϵ does not uniquely determine A^0: a whole set G_θ of possible effective tensors is generated. As of yet, no complete characterization of this set is available in spite of the abundant literature on the topic (see e.g. HASHIN [H]).

A special kind of mixtures – laminates – which correspond to a specific kind of characteristic functions χ^ϵ, those that oscillate only in one direction, proves fruitful. On the one hand explicit formulae for the effective tensor are at hand in such a case (FRANCFORT & MURAT [FM']). On the other hand the resulting effective tensors are extreme among all microstructures (AVELLANEDA [A']), i.e. for any A^0 in G_θ there exist two tensors $\underline{A}$ and $\overline{A}$ associated to laminates such that

$$\underline{A} \leq A^0 \leq \overline{A}, \tag{0.3}$$

in the sense of quadratic forms. Hence a thorough knowledge of such tensors will provide useful information on the whole set G_θ.

The connection with the study of fourth order moments stems from the actual expression for the effective tensor of a laminate. Assume that both phases are isotropic, i.e. that

$$A = \lambda_1 i \otimes i + 2\mu_1 I, \text{ with } \mu_1 > 0,\ NK_1 = N\lambda_1 + 2\mu_1 > 0$$

$$B = \lambda_2 i \otimes i + 2\mu_2 I, \text{ with } \mu_2 > 0,\ NK_2 = N\lambda_2 + 2\mu_2 > 0$$

where i is the identity matrix of R^N and I that of $R_s^{N^2}$ (the space of symmetric matrices on R^N with its Euclidean structure). Then the effective tensors associated to (finite-rank) laminates (see Subsection 3.1 for further details) are given by

$$(1-\theta)(A^0-A)^{-1}h = (B-A)^{-1}h + \frac{\theta}{\mu_1}\int_{S^{N-1}}\Big(\frac{ha\otimes a + a\otimes ha}{2} - \frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}(ha.a)a\otimes a\Big)d\nu(a),\ h \in R_s^{N^2}, \tag{0.4}$$

with

$$d\nu = \sum_{i=1}^{p} \eta_i \delta_{a_i}, \quad a_i \in S^{N-1}, \quad 0 \leq \eta_i \leq 1, \quad \sum_{i=1}^{p} \eta_i = 1,$$

or the analogous formula obtained upon permuting A and B, and θ and $(1-\theta)$. Inspection of formula (0.4) immediately evidences the role played by the fourth order moments of atomic probability measures on S^{N-1}. It will be established (see Section 2) that such measures are generic from the standpoint of the fourth order moments among all probability measures on S^{N-1}; this, in conjunction with (0.3), provides the desired link between the moment problem and homogenization and permits to gain further insight into the structure of G_θ. In particular it will be shown (cf. Section 3.2) that rank-6 laminates suffice to generate $\underline{A}$ and $\overline{A}$ in (0.3).

The paper is organized as follows: Section1 is entirely devoted to notation and to a review of the few algebraic results needed for the subsequent study. Section 2 is the main part of the paper and it is devoted to the study of the set of fourth order moments of nonnegative RADON measures on S^{N-1}. In a first subsection a few preliminary results that remain true in any dimension are derived. Firstly the generic character of atomic measures is established (Lemma 2.1). Then a result specific to the two and three dimensional cases provides a complete but unpractical characterization of the studied set (Theorem 2.1). Subsection 2.2 addresses the two-dimensional case; previously known results are rederived [AM]. Subsection 2.3 addresses the three dimensional case; the main results are Theorem 2.3 and Remark 2.7 which pertain to the intimate structure of the boundary of that set. In Subsection 2.4 measures that remain invariant under the action of either one of two symmetry groups are considered and the resulting fourth order moments are characterized. Section 3 is devoted to applications of the results of Section 2 to homogenization in linearized elasticity. After a brief positioning of the concept of homogenization in Subsection 3.1, Subsection 3.2 illustrates on three examples the additional information made available through an adequate use of the results of Section 2.

1. Notation and algebraic preliminaries

The following notation is adopted throughout:

$\mathcal{P}_{p,N}$ is the space of polynomials of degree less than or equal to p in N variables

$\mathcal{P}^h_{p,N}$ is the subset of $\mathcal{P}_{p,N}$ of all homogeneous polynomials of degree p

$\mathcal{P}^+_{p,N}$ is the subset of $\mathcal{P}_{p,N}$ of all nonnegative polynomials

$M(S^{N-1})$ is the space of all RADON (BOREL regular) measures on S^{N-1}

$M^+(S^{N-1})$ is the subset of $M(S^{N-1})$ of all nonnegative measures

$\Pi(S^{N-1})$ that of all probability measures

$M_{p,N}$ is the set of p^{th} order moments of all elements of $M^+(S^{N-1})$

$O(N)$ is the set of all orthogonal matrices on R^N

$SO(N)$ is the subset of $O(N)$ of all rotations

$R^{N^2}_s$ is the space of all symmetric matrices on R^N

$i(\in R^{N^2}_s)$ is the identity matrix on R^N

I is the identity matrix on $R^{N^2}_s$

$M(\alpha,\beta) = \{A(x) \in L^\infty\left(R^N; \mathcal{L}(R^{N^2}_s; R^{N^2}_s)\right), \alpha I \le A(x) \le \beta I$ in the sense of quadratic forms$\}$

$\otimes$ denotes the tensor product of two vectors or two matrices.

Our analysis of the moment problem relies on two classical results. The first one is concerned with quadrature formulae for polynomials while the second one pertains to an old question of HILBERT on whether a nonnegative real polynomial is a sum of squares of real polynomials.

The quadrature result is merely stated and the interested reader is referred to [CM], Theorem 2.2.

<u>Lemma 1.1</u>: Let π be a nonnegative measure on $[\alpha,\beta]$, $-\infty < \alpha,\beta < \infty$, whose support contains strictly more than 4 points in (α,β) and such that $\int_{[\alpha,\beta]}(1+|\tau|^9)d\pi < \infty$. Then there exists exactly one quintuplet of distinct points $(a_1,\dots,a_5)$ in (α,β) and one quintuplet of positive real numbers $(\rho_1,\dots,\rho_5)$ such that

$$\int_\alpha^\beta p\,d\pi = \sum_{i=1}^{5} \rho_i p(a_i) \tag{1.1}$$

for all $p \in \mathcal{P}_{9,1}$.

The support condition on π easily implies that the quantity $\int_\alpha^\beta pq\,d\pi$, with p, $q \in \mathcal{P}_{4,1}$, defines an inner product on $\mathcal{P}_{4,1}$ and the proof of Lemma 1.1 then reduces to that of Theorem 2.2 in [CM]. The positive character of the weights $\rho_i, 1 \le i \le 5$, is readily verified upon choosing $\prod_{1\le j\le 5, j\ne i} (x-a_j)^2$ as test function p in (1.1).

In 1888, HILBERT produced a complete characterization of the real polynomials for which the anounced property holds true [H']. The result is strikingly simple.

<u>Theorem 1.1</u>: (HILBERT) $\mathcal{P}^+_{n,1}$, $\mathcal{P}^+_{2,n}$, $n \in N$, and $\mathcal{P}^+_{4,2}$ are the only sets of nonnegative real polynomials in which every element can be written as a sum of squares of real polynomials.

<u>Remark 1.1</u> In the context of Theorem 1.1, the number of elements in the sum may be taken to be 2 for $\mathcal{P}^+_{4,1}$ and 3 for $\mathcal{P}^+_{4,2}$ (cf. [CL], (1.1), (1.2)). It is worth mentioning that HILBERT's proof of Theorem 1.1 in the cases $\mathcal{P}^+_{4,3}$ and $\mathcal{P}^+_{6,2}$ – from which all other cases $\mathcal{P}^+_{n,m}$ with $n \ge 4$, $m \ge 2$, $(n,m) \ne (4,2)$, are easily deduced (see e.g. [CL], (1.4)) – is nonconstructive. The first simple explicit counterexamples seem to have been constructed in the late 1960's (see e.g. [M'], or [CL]).

In any case we are only interested here in the sets $\mathcal{P}^+_{4,1}$ and $\mathcal{P}^+_{4,2}$ for which the property holds true. A simple proof of the precise result used for $\mathcal{P}^+_{4,2}$ is given in the Appendix.

2. Fourth order moments of a nonnegative RADON measure on S^1 or S^2

This section is essentially devoted to the study of the set $M_{4,N}$ of fourth order moments of all nonnegative RADON measures $M^+(S^{N-1})$ on S^{N-1}. A first subsection investigates a general property of all points of $M_{2p,N}, p \geq 1$, namely that they are obtained by measures supported on a finite number of points. A first characterization of the boundary $\partial M_{2p,N}$ is also proposed, and in the case where $N = 2$ or 3, $p = 2$, a complete (albeit unpractical) characterization of $M_{4,N}$ is given in Theorem 2.1.

The second subsection proposes various characterizations of $M_{4,2}$ (cf. Theorem 2.2) while the third subsection, short of providing a handy characterization of $M_{4,3}$, presents a useful analysis of $\partial M_{4,3}$ (cf. Theorem 2.3). In particular it is shown in Theorem 2.3 that all points on $\partial M_{4,3}$ are obtained by measures supported at five points or less of S^2.

Finally the last subsection investigates the subsets of $M_{4,2}$ or $M_{4,3}$ corresponding to various symmetry restrictions.

2.1 The generic character of atomic measures

In any dimension N, let $m_{p,N}$ be the dimension of the space $\mathcal{P}^h_{p,N}$ of homogeneous polynomials of degree p in N variables,

$$m_{p,N} = \binom{N+p-1}{N-1} = \frac{(N+p-1)!}{p!(N-1)!}.$$

From now onward we will write $m_{p,N}$ as m when used as an index or an exponent. Let $p_1, \ldots, p_m$ be a basis of $\mathcal{P}^h_{p,N}$ and define the linear mapping F from $M(S^{N-1})$ into R^m as

$$F(\mu) = \langle \mu, p_i \rangle, \; i = 1, \ldots, m_{p,N}.$$

The mapping F maps $M^+(S^{N-1})$ into a closed positive cone $M_{p,N}$ of R^m while it maps the set $\Pi(S^{N-1})$ of all probability measures on S^{N-1} (a convex set which is compact in the weak $\star$ topology of $M(S^{N-1})$) into a compact convex subset of $M_{p,N}$ which also lies in the hyperplane

$$\langle \mu, 1 \rangle = 1$$

if p is even since $\left(\sum_{i=1}^{N} x_i^2\right)^{p/2}$ is a homogeneous polynomial of degree p with value 1 on S^{N-1}. The following lemma, based on an argument of ARTSTEIN [A], holds true for any N and p.

<u>Lemma 2.1</u>: Any point in $M_{2p,N}$ yields a set of $2p^{\text{th}}$ order moments of a nonnegative measure on S^{N-1} whose support is made of at most $m_{2p,N}$ points.

Proof. Let Q be a point in $M_{2p,N}$. Then $M^+_Q := F^{-1}(Q) \cap M^+(S^{N-1})$ is a convex subset of $M(S^{N-1})$ which is also weak $\star$ compact, because $\langle \mu, 1 \rangle$ is fixed whenever $\mu \in M^+_Q$. According to KREIN-MILMAN's theorem M^+_Q is the closed convex hull of its extreme points. Let ν be such an extreme point. If B_W is any ball in a finite dimensional subspace W of $M(S^{N-1})$ of dimension greater than $m_{2p,N}$, then

$$\nu + B_W \text{ is not included into } M^+(S^{N-1}). \tag{2.1}$$

Indeed since $dim W > m_{2p,N}$,

$$Ker(F|_W) \neq 0,$$

and there would exist a segment $[-\tau, \tau]$, $\tau \neq 0$ with

$$[-\tau, \tau] \subset B_W,$$

and, since $F(\tau) = 0$,

$$[\nu - \tau, \nu + \tau] \subset M^+(S^{N-1}) \cap F^{-1}(Q) = M^+_Q,$$

which contradicts the extremality of ν.

Thus ν does not belong to the interior of a face of dimension greater than $m_{2p,N}$ of $M^+(S^{N-1})$. Assume that the support of ν is not purely atomic. Then there exists a ν-measurable subset E of S^{N-1} with $\nu(E) > 0$ and no ν-atoms in E. Since ν is a BOREL measure, E can be partitioned into m' ($m' > m_{2p,N}$) BOREL sets $E_1, \ldots, E_{m'}$, with

$$\nu(E_j) > 0, \ 1 \leq j \leq m'.$$

If χ_j denotes the characteristic function of the set E_j then the measures $\nu_j = \chi_j \nu$, $1 \leq j \leq m'$, are linearly independent elements of $M^+(S^{N-1})$ and

$$\nu = \sum_{j=1}^{m'} \nu_j + \nu|_{S^{N-1}\setminus E} = \Big(\sum_{j=1}^{m'} \chi_j\Big)\nu + \nu|_{S^{N-1}\setminus E}. \tag{2.2}$$

Let W be the m' dimensional subspace generated by $\nu_1, \ldots, \nu_{m'}$. Then, in view of (2.2) there exists a small ball B_W around 0 in W such that

$$\nu + B_W \subset M^+(S^{N-1})$$

by taking $\sum_{j=1}^{m'} c_j \nu_j$ with c_j close to 1. But, in view of (2.1), this cannot be so. Consequently the support of ν is purely atomic and the same argument would demonstrate that at most $m_{2p,N}$ points lie in the support of ν, so Lemma 2.1 is proved.

We now focus our attention on the boundary of $M_{2p,N}$ and derive the following

Lemma 2.2: Any point on $\partial M_{2p,N}$ yields a set of $2p^{\text{th}}$ order moments such that all nonnegative measures with those moments are supported in the set of (double) zeroes of a nonnegative homogeneous polynomial of degree $2p$ on S^{N-1}. Furthermore it may be assumed that those zeroes lie strictly inside a hemisphere of S^{N-1}.

Remark 2.1: The last statement of Lemma 2.2 bears a short comment. A nonnegative measure μ on S^{N-1} has moments of order $2p$ that are indistinguishable from those of the nonnegative measure obtained by symmetrizing μ about the origin. Thus it may be assumed that the support of μ is symmetric about 0. Further those moments are also undistinguishable from those of the measure

$$\hat{\mu} = \mu|_{x_N=0} + 2\mu|_{x_N>0},$$

the support of which is contained in the northern hemisphere. The last statement of Lemma 2.2 pertains to measures of the latter form.

Proof of Lemma 2.2: Let Q be an arbitrary point in $\partial M_{2p,N}$ distinct from the origin and let $\mu \in M^+(S^{N-1})$ be such that $F(\mu) = Q$. Because $M_{2p,N}$ is convex there exists at least one tangent hyperplane to $M_{2p,N}$ at Q. It can be viewed as a homogeneous polynomial P_Q of degree $2p$, together with a constant γ, such that

$$\langle \mu, P_Q \rangle = \gamma, \tag{2.3}$$

$$\langle \nu, P_Q \rangle \geq \gamma, \ \nu \in M^+(S^{N-1}). \tag{2.4}$$

Define $|\mu| := \langle \mu, 1 \rangle$, and note that $|\mu| > 0$. Setting

$$P'_Q(x) = P_Q(x) - \frac{\gamma}{|\mu|}|x|^{2p}, \ x \in S^{N-1},$$

transforms (2.3)-(2.4) into

$$\langle \mu, P'_Q \rangle = 0, \tag{2.5}$$

$$\langle \nu, P'_Q \rangle \geq \gamma\Big(1 - \frac{|\nu|}{|\mu|}\Big), \ \nu \in M^+(S^{N-1}). \tag{2.6}$$

The choice of an arbitrary DIRAC mass weighted by $|\mu|$, i.e. $|\mu|\delta_a$, $(a \in S^{N-1})$ as test measure in (2.6) implies that

$$|\mu| P'_Q(a) \geq 0. \tag{2.7}$$

Since a is an arbitrary point on S^{N-1}, (2.7) is equivalent to

$$P'_Q \geq 0 \text{ on } S^{N-1}. \tag{2.8}$$

Denote by Z_Q the set of all zeroes of P'_Q. By virtue of (2.5), (2.8) we conclude that

$$\text{supp}(\mu) \subset Z_Q.$$

The homogeneous character of P'_Q together with (2.8) implies that all elements of Z_Q are (double) zeroes of P'_Q (conversely any nonnegative homogeneous polynomial of degree $2p$ defines a tangent hyperplane at all moments of measure μ whose support is contained in its zero set).

Further P'_Q is even; thus the set Z_Q of all its zeroes may be assumed to lie in the northern hemisphere $x_N \geq 0$. If, for a point a in Z_Q, one has $a_N = 0$ then we are at liberty to assume that $a_{N-1} \geq 0$. Repeating this argument until a positive component of a is found we have identified a set Z'_Q of zeroes of P'_Q which is such that

$$\begin{cases} Z'_Q \subset \{a \in S^{N-1} : a_N \geq 0\}; \\ \text{if } a \in Z'_Q \text{ with } a_N = \ldots = a_k = 0, \text{ then } a_{k-1} \geq 0,\ 2 \leq k \leq N; \\ \text{if } a \in Z'_Q, \text{ with } a_N = \ldots = a_2 = 0, \text{ then } a_1 > 0. \end{cases} \tag{2.9}$$

In view of (2.9) the origin is immediately seen not to belong to the closed convex hull of Z'_Q, which permits to conclude to the existence of an equatorial hyperplane $L(x) = 0$ such that $L > 0$ on $(Z'_Q) > 0$, and completes the proof of Lemma 2.2.

Remark 2.2: In the context of Lemma 2.2 and upon relabeling the N^{th} direction to be that which is normal to the hyperplane $L(x) = 0$ we may take Z'_Q to be such that

$$a_N > 0,\ a \in Z'_Q. \tag{2.10}$$

Under the transformation Γ_N from R^N into $R^N \cup +\infty$ defined as

$$\Gamma_N(x_1, \ldots, x_N) := \left(\frac{x_1}{x_N}, \ldots, \frac{x_{N-1}}{x_N}\right), \tag{2.11}$$

P'_Q is transformed into a nonnegative polynomial P''_Q of degree less than or equal to $2p$ in $(N-1)$ variables (an element of $\mathcal{P}^+_{2p,N-1}$) and, in view of (2.10), the points $\frac{a}{a_N}$, with $a \in Z'_Q$, are (double) zeroes of P''_Q.

Remark 2.3: In the two dimensional case ($N = 2$) any element of $\mathcal{P}^+_{2p,1}$ has at most p double zeroes. Thus in the context of Remark 2.2,

$$\text{card}(Z'_Q) \leq p.$$

By virtue of Remark 2.2, the degree of the nonnegative homogeneous polynomial whose zero set contains the support of the inverse image under F of any point of $\partial M_{2p,N}$ may be drastically reduced whenever HILBERT's theorem is applicable. In the case of interest to us in the remainder of this study P is taken to be equal to 2 and we obtain the following

Theorem 2.1: If $N = 2$ or 3 the set $M_{4,N}$ is the set of all matrices $F_{ijkm}, 1 \leq i,j,k,m \leq N$, invariant under any permutation of the indices and such that

$$\sum_{i,j,k,m=1}^{N} F_{ijkm} A_{ij} A_{km} \geq 0, \tag{2.12}$$

for all symmetric matrices A on R^N. Furthermore the nonnegative measures on S^{N-1} whose moments of order 4 lie on $\partial M_{4,N}$ are those that are supported on the zero set of a homogeneous polynomial of degree 2 $R(x) = \sum_{i,j=1}^{N} B_{ij} x_i x_j$, and inequality (2.12) becomes an equality at such points of $\partial M_{4,N}$ for the symmetric matrix B.

Proof: We recall Remark 2.2 and conclude to the existence, for any point Q of $\partial M_{4,N}$, of a nonnegative polynomial P''_Q of degree less than or equal to 4 in $N-1$ variables. If $N = 2$ or 3, HILBERT's theorem (cf. Theorem 1.1) implies that P''_Q is a sum of squares of real polynomials. The same holds true for P'_Q which thus reads as

$$P'_Q = \sum_{i=1}^{k_Q} R_i^2,$$

where R_i is a homogeneous polynomial of degree 2 and k_Q is an integer (which, in view of Remark 1.1, may be taken to be 2 if $N = 2$ or 3 if $N = 3$). Recalling (2.5) we obtain

$$\langle \mu, R_i^2 \rangle = 0, \ i = 1, \ldots, k_Q. \tag{2.13}$$

Furthermore note that if ν is an arbitrary element of $M^+(S^{N-1})$ and R an arbitrary homogeneous polynomial of degree 2 (an element of $\mathcal{P}^h_{2,N}$), then

$$\langle \nu, R^2 \rangle \geq 0. \tag{2.14}$$

Any element R of $\mathcal{P}^h_{2,N}$ reads as

$$R(x) = \sum_{i,j=1}^{N} A_{ij} x_i x_j, \tag{2.15}$$

with A a symmetric matrix on R^N. Thus, upon setting

$$F(\nu)_{ijkm} = \int_{S^{N-1}} x_i x_j x_k x_m d\nu, \ \nu \in M^+(S^{N-1}), \tag{2.16}$$

relation (2.14) states that, for any ν in $M^+(S^{N-1})$ and any symmetric matrix A,

$$\sum_{i,j,k,m=1}^{N} F(\nu)_{ijkm} A_{ij} A_{km} \geq 0, \tag{2.17}$$

while (2.13) is equivalent to the existence of a symmetric matrix B such that

$$\sum_{i,j,k,m=1}^{N} F(\mu)_{ijkm} B_{ij} B_{km} = 0 \tag{2.18}$$

(Q is the point with components $F(\mu)_{ijkm}$ for the choice of $x_i x_j x_k x_m$ as generating set for $\mathcal{P}^h_{4,N}$). Conversely let A be any symmetric matrix; any nonnegative measure μ with its support on the zero set of R defined through (2.15) will satisfy

$$\sum_{i,j,k,m=1}^{N} F(\mu)_{ijkm} A_{ij} A_{km} = 0,$$

while (2.18) is obviously satisfied. Thus R^2 will define a tangent hyperplane to $M_{4,N}$ ($N = 2$ or 3).

We have thus established a one to one correspondence between the tangent hyperplanes to $M_{4,N}$ and the symmetric matrices on R^N. Since a convex closed set ($M_{4,N}$) is the intersection of closed half spaces located above its tangent hyperplanes (2.17), (2.18) provide the desired characterization (2.12) of $M_{4,N}$ ($N = 2$ or 3), while consideration of (2.15) and (2.18) completes the proof of Theorem 2.1.

2.2 The case of $M_{4,2}$.

In the two dimensional case ($N = 2$) a characterization of $M_{4,2}$ can be proposed with the help of Theorem 2.1 or Remark 2.2 specialized to $p = 2$. This is the object of the Theorem 2.2 which also describes the support of the measure whose moments lie on the boundary of $M_{4,2}$.

Theorem 2.2: If $N = 2$, the set

$$M_{4,2} = \left\{F_j = \int_{S^1} x_1^{4-j} x_2^j d\nu,\ j = 0, \ldots, 4;\ \nu \in M^+(S^1)\right\} \tag{2.19}$$

is the subset of all $F_0, \ldots, F_4$ in R^5 such that

$$F_0, F_2, F_4 \geq 0;\ F_1^2 \leq F_0F_2;\ F_2^2 \leq F_0F_4;\ F_3^2 \leq F_2F_4;\ F_0F_2F_4 + 2F_1F_2F_3 - F_0F_3^2 - F_2^3 - F_4F_1^2 \geq 0. \tag{2.20}$$

The elements of $\partial M_{4,2}$ are the fourth order moments of the measures supported at two points of S^1.

Proof: We firstly establish (2.20). To this effect we recall Theorem 2.1 and remark that a 2×2 symmetric matrix A is characterized by three coefficients

$$A_{11} = \alpha;\ A_{12} = A_{21} = \beta;\ A_{22} = \gamma.$$

Upon recalling (2.12) and the definition of F_j in (2.19) (together with (2.16)) we obtain

$$\alpha^2 F_0 + 4\beta^2 F_2 + \gamma^2 F_4 + 4\alpha\beta F_1 + 2\alpha\gamma F_2 + 4\beta\gamma F_3 \geq 0,$$

for every triplet (α, β, γ) in R^3. Thus the matrix

$$\begin{pmatrix} F_0 & 2F_1 & F_2 \\ 2F_1 & 4F_2 & 2F_3 \\ F_2 & 2F_3 & F_4 \end{pmatrix}$$

must be nonnegative. But a 3×3 matrix is nonnegative if and only if its diagonal elements, its diagonal 2×2 minors and its determinant are nonnegative, which yields (2.20).

The remainder of Theorem 2.2 is immediately derived upon recalling Remark 2.3 (specialized to $p = 2$) and Lemma 2.2, and this completes the proof of Theorem 2.2.

Remark 2.4: Another characterization of $M_{4,2}$ may be found in [AM].

Remark 2.5. In the context of Theorem 2.2 assume that the matrix of inertia of a given nonnegative measure μ is known. At the possible expense of a change of basis, we are at liberty to consider a diagonal basis. Its diagonal elements are

$$I_1 = \int_{S^1} x_1^2 d\mu,\ I_2 = \int_{S^1} x_2^2 d\mu, \tag{2.21}$$

and we suppose, with no loss of generality, that $I_1 \leq I_2$. Then the set of the fourth order moments of all nonnegative measures on S^1 with (2.21) as matrix of inertia is given by

$$\begin{cases} F_2 = I_1 - F_0,\ F_3 = -F_1,\ F_4 = I_2 - I_1 + F_0, \\ I_1^2 \leq F_0(I_1 + I_2),\ F_0 \leq I_1, \\ F_1^2(I_1 + I_2) \leq (I_1 - F_0)\Big(F_0(I_1 + I_2) - I_1^2\Big), \end{cases}$$

as immediately checked upon specializing (2.20) to the case under consideration and taking (2.21) into account.

Remark 2.6: Every point in the interior of $M_{4,2}$ can be attained by an element μ of $M^+(S^1)$ which is supported at three points, one of the points being chosen arbitrarily.

Indeed let ν be an arbitrary element of $M^+(S^1)$ such that its moments of order 4, $F(\nu)_j$, lie in the interior of $M_{4,2}$. If a_0 is an arbitrary point of S^1, the measure $\nu - t\delta_{a_0}$ $(t \geq 0)$ – maybe not an element of $M^+(S^1)$ – has moments of order 4 that form a half line $F(\nu) - tF(\delta_{a_0})$ which must intersect $\partial M_{4,2}$ for a value t_0 of t because $M_{4,2}$ is a closed positive cone and thus contains no straight line; but $M_{4,2}$ certainly contains the half line $F(\nu) + tF(\delta_{a_0})$.

Application of Theorem 2.2 yields the existence of a measure

$$\overline{\nu} = \sum_{i=1}^{2} t_j \delta_{a_j},\ t_1, t_2 \geq 0,$$

such that

$$F(\nu) - t_0 F(\delta_{a_0}) = F(\overline{\nu})$$

Hence

$$F(\nu) = \sum_{i=0}^{2} t_j F(\delta_{a_j}).$$

2.3 The case of $M_{4,3}$.

The three-dimensional case is more intricate than its two-dimensional counterpart and an analytical characterization of $M_{4,3}$ seems to be algebraically inextricable. We will however prove the following theorem which may be viewed as the three-dimensional analogue of the part of Theorem 2.2 concerned with $\partial M_{4,2}$:

Theorem 2.3: Assume that $N = 3$. All elements of $\partial M_{4,3}$ are the fourth order moments of a measure supported at five points of S^2.

If $M_{4,3}$ admits a single tangent hyperplane at the point of $\partial M_{4,3}$ under consideration, then the intersection of $\partial M_{4,3}$ with that hyperplane is a 9-dimensional closed positive cone of $M_{4,3}$.

Remark 2.7: Theorem 2.3 can be enriched by adjunction of the result of Theorem 2.1. The support of any element of $M^+(S^2)$, whose fourth order moments lie at a given point Q of $\partial M_{4,3}$ must be contained in the intersection C_Q of the zero set of the homogeneous polynomial R_Q of degree 2 described in Theorem 2.1 (and denoted there by R) with S^2. Furthermore, since, according to Lemma 2.1, those zeroes may be taken to lie strictly inside a hemisphere of S^2, Remark 2.2 permits to view the image of C_Q under the transformation Γ_3, defined in (2.11) by

$$\Gamma_3(x_1, x_2, x_3) := \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right),$$

as living on the image of R_Q under that transformation, i.e. on a conic section.

Further, if the support of an element μ of $M^+(S^2)$ has its image under Γ_3 that lies on a conic section, then μ belongs to $\partial M_{4,3}$. In particular, all measures supported at five points of S^2 or less belong to $\partial M_{4,3}$.

Remark 2.8: If, in the context of Remark 2.7, the symmetric form B associated to R_i has three nonzero eigenvalues, Theorem 2.3 can be strengthened. Specifically Q corresponds to the fourth order moments of a one parameter family of measures supported at five points of S^2.

Proof of Theorem 2.3: According to Theorem 2.1, specialized to the case $N = 3$, any point Q of $\partial M_{4,3}$ is the image under F of a measure μ of $M^+(S^2)$, whose support lies in (the intersection of) the zero sets of nonzero homogeneous quadratic polynomials. Let R_Q be such a polynomial. It is diagonalizable (or in the notation of Theorem 2.1, the associated matrix B_{ij} is diagonalizable) in an orthonormal basis of R^3. In such a basis, R_Q reads as

$$R_Q(x_1, x_2, x_3) = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2.$$

Since the support of μ is not empty, one of the coefficients α, β or γ is nonpositive. There is no loss of generality in assuming that

$$\alpha \geq 0, \beta \geq 0, \gamma \leq 0.$$

Denote by C_Q the intersection of the zero set of R_Q with the sphere S^2. Three cases will be distinguished.

Case 1. $\alpha > 0, \beta > 0, \gamma < 0$.

Then C_Q is the intersection of the surface

$$x_3^2 = \alpha^* x_1^2 + \beta^* x_2^2, \ (\alpha^* = -\frac{\alpha}{\gamma} > 0, \beta^* = -\frac{\beta}{\gamma} > 0) \tag{2.22}$$

with the sphere S^2. Since $C_Q \subset S^2$, one may replace x_3^2 by $1 - x_1^2 - x_2^2$ in the moments of order 4 of μ and consequently 9 moments are to be considered, namely those with the following integrands

$$x_1^4, \ x_1^3x_2, \ x_1^2x_2^2, \ x_1x_2^3, \ x_2^4, \ x_1^3x_3, \ x_1^2x_2x_3, \ x_1x_2^2x_3, \ x_2^3x_3.$$

Because C_Q is invariant under a sign change of any of the $x_i's$ $(1 \le i \le 3)$ the four sets of homogeneous polynomials

$$\begin{aligned} S_1 &= \{x_1^4, \ x_1^2x_2^2, \ x_2^4\}, \\ S_2 &= \{x_1^3x_2, \ x_1x_2^3\}, \\ S_3 &= \{x_1^3x_3, \ x_1x_2^2x_3\}, \\ S_4 &= \{x_2^3x_3, \ x_1^2x_2x_3\}, \end{aligned}$$

are linearly independent from one another on C_Q.

Further assume that either S_2, S_3 or S_4 is not made of linearly independent polynomials on C_Q. Then C_Q is imbedded in a union of equatorial planes of the type

$$x_1 = 0, \ x_2 = 0, \ x_3 = 0, \ \delta x_1 + \eta x_2 = 0.$$

The same holds true for S_1 except maybe if $x_1^2x_2^2$ is a linear combination of x_1^4 and x_2^4, i.e.

$$x_1^2x_2^2 = \zeta x_1^4 + \lambda x_2^4, \ \zeta\lambda \ne 0.$$

In such a case, solving the above second degree equation in $\left(\frac{x_1}{x_2}\right)^2$ (or $\left(\frac{x_2}{x_1}\right)^2$), we end up with an equation of the type $\delta x_1 + \eta x_2 = 0$ (except when $x_1 = 0$ and $x_2 = 0$).

Thus S_i $(1 \le i \le 4)$ is made of linearly independent polynomials on C_Q unless C_Q is included in a finite union of equatorial planes. Such is not the case as easily checked upon recalling (2.22), together with the constraint $x_3^2 = 1 - x_1^2 - x_2^2$.

We conclude that, if $\alpha\beta\gamma < 0$, then

$$x_1^4, \ x_1^3x_2, \ x_1^2x_2^2, \ x_1x_2^3, \ x_2^4, \ x_1^3x_3, \ x_1^2x_2x_3, \ x_1x_2^2x_3, \ x_2^3x_3 \text{ are linearly independent on } C_Q. \tag{2.23}$$

Case 2. $\alpha = 0$ and $\beta \ne 0 \ne \gamma$, or $\alpha \ne 0 \ne \gamma$ and $\beta = 0$.

The zeroes of R_Q lie on the union of two equatorial planes. If, for example, $\alpha = 0$, the two planes are

$$x_2 = \pm\sqrt{-\frac{\gamma}{\beta}}x_3. \tag{2.24}$$

The measure μ introduced at the beginning of the proof is decomposed into two elements of $M^+(S^2)$, μ_1 and μ_2, each being supported on one and only one equatorial plane.

Since μ_i, $i = 1, 2$, is supported on the intersection of a plane with S^2, it is supported on a circle. An appropriate change of coordinates (specific to i) reduces the analysis of μ_i, $i = 1, 2$, to the two-dimensional case. According to Remark 2.6, the moments $F(\mu_i)$ can be realized by an element μ_i^* of $M^+(S^1)$ supported at three points of S^1, one of the points being chosen arbitrarily. But the two circles corresponding to μ_1 and to μ_2 intersect at two (antipodal) points. Choosing one of these points as the third support point for the measures μ_1^* and μ_2^* permits to construct a measure $\mu^* = \mu_1^* + \mu_2^*$ such that

$$F(\mu^*) = F(\mu) = Q,$$

while its support contains at most five points, which proves, in Case 2, the first assertion of Theorem 2.3.

Note that in the case where relation (2.24) holds true, only 9 moments are to be considered, namely

$$1,\ x_1^2,\ x_1^4,\ x_1x_2,\ x_1^3x_2,\ x_1x_3,\ x_1^3x_3,\ x_2x_3,\ x_1^2x_2x_3,$$

and that an argument similar to that used in Case 1 would show that

$$1,\ x_1^2,\ x_1^4,\ x_1x_2,\ x_1^3x_2,\ x_1x_3,\ x_1^3x_3,\ x_2x_3,\ x_1^2x_2x_3,\ \text{are linearly independent on } C_Q. \tag{2.25}$$

A conclusion similar to (2.25) would be reached if $\beta = 0$ instead of $\alpha = 0$.

Case 3. $\alpha = \beta = 0$ or $\gamma = 0$

Then the zeroes of R_Q lie on a single equatorial plane as well as on S^2. An appropriate change of coordinate reduces this case to the two-dimensional one; the zeroes of R_Q may thus be assumed to lie on

$$x_1^2 + x_2^2 = 1,\ x_3 = 0.$$

Remark 2.6 applies and yields a measure μ^* supported at three points of S^1 or less such that

$$F(\mu^*) = F(\mu) = Q,$$

which proves, in Case 3, the first assertion of Theorem 2.3.

Note that there are many quadratic polynomials which are zero on the support of μ; for example all quadratic expressions with a common x_3 term. In other words the set $M_{4,3}$ admits several tangent hyperplanes at the point Q. By virtue of (2.23), (2.25) we infer that, if $M_{4,3}$ admits only one tangent hyperplane at the point Q, the setting is that of Case 1 or Case 2, yielding 9 linearly independent homogeneous integrands on C_Q. Therefore, when the point a of S^2 spans the curve C_Q, the moments $F(\lambda\delta_a)$, $\lambda \geq 0$, span a 9-dimensional closed positive cone of R^{15}. Since $R_Q(a) = 0$, the point $F(\lambda\delta_a)$ belongs to $\partial M_{4,3}$ and the point Q belongs to a 9-dimensional closed positive cone which establishes the second assertion of Theorem 2.3.

It remains to prove the first assertion of Theorem 2.3 in Case 1. To this effect we recall (2.22) and propose a natural parametrization of C_Q, immediately derived from that of the conic section $\gamma_3(C_Q)$ (cf. Remark 2.7). Specifically we set

$$x_1(t) = \frac{S(t)\cos t}{\sqrt{\alpha^*}};\ x_2(t) = \frac{S(t)\sin t}{\sqrt{\beta^*}};\ x_3(t) = S(t),\ -\pi \leq t \leq \pi \tag{2.26}$$

with

$$S(t) = \frac{1}{\sqrt{\frac{\cos^2 t}{\alpha^*} + \frac{\sin^2 t}{\beta^*} + 1}}.$$

Note that (2.22) is then identically satisfied and that

$$x_1(t)^2 + x_2(t)^2 + x_3(t)^2 = 1,$$

as it should be.

Upon setting $\tau = \tan\left(\frac{t}{2}\right)$, (2.26) reads as

$$x_1(\tau) = \frac{(1-\tau^2)f(\tau)}{\sqrt{\alpha^*}},\ x_2(\tau) = \frac{2\tau f(\tau)}{\sqrt{\beta^*}},\ x_3(\tau) = (1+\tau^2)f(\tau),\ -\infty \leq \tau \leq +\infty, \tag{2.27}$$

with

$$f(\tau) := \frac{S(2\arctan\tau)}{(1+\tau^2)} = \frac{1}{\sqrt{\frac{(1-\tau^2)^2}{\alpha^*} + \frac{4\tau^2}{\beta^*} + (1+\tau^2)^2}}. \tag{2.28}$$

Under the parametrization (2.27) a homogeneous polynomial $P(x_1, x_2, x_3)$ of degree 4 in x_1, x_2, x_3 becomes a polynomial $q(\tau)$ of degree comprised between 1 and 8, multiplied by $f^4(\tau)$ with $f(\tau)$ defined in (2.28), i.e.

$$P\Big(x_1(\tau), x_2(\tau), x_3(\tau)\Big) := q(\tau) f^4(\tau),\ 1 \le d^o(q) \le 8. \tag{2.29}$$

The mapping T from S^2 into $\overline{R}_+$ defined as

$$T(x_1, x_2, x_3) := \frac{\sqrt{|x_3 - \sqrt{\alpha^*} x_1|}}{\sqrt{|x_3 + \sqrt{\alpha^*} x_1|}}, \tag{2.30}$$

is introduced. Its restriction to C_Q is the inverse of the mapping $\tau \mapsto \big(x_1(\tau), x_2(\tau), x_3(\tau)\big)$.

If μ_T denotes the image measure of a measure μ supported on C_Q under T defined through (2.30) we obtain, by virtue of (2.29)

$$\int_{S^2} P d\mu = \int_{-\infty}^{+\infty} q(\tau) f^4(\tau) d\mu_T. \tag{2.31}$$

Upon defining the measure $\pi_T := f^4(\tau)\mu_T$, (2.31) reads as

$$\int_{S^2} P d\mu = \int_{-\infty}^{+\infty} q(\tau) d\pi_T. \tag{2.32}$$

We now appeal to a variant of Lemma 1.1 and conclude that, if the support of π_T contains strictly more than 4 points, there exists a one parameter family of measures π_T^λ with

$$\pi_T^\lambda = \sum_{j=1}^{5} \alpha_j^\lambda \delta_{\tau_j}^\lambda,\ -\infty < \tau_j^\lambda < +\infty,\ \alpha_j^\lambda \ge 0,\ j = 1, \dots, 5,$$

such that the moments of order less than or equal to 8 of π_T^λ are, for every value of λ, those of π_T. Notice that although we know that $\int (1+|\tau|^8) d\pi_T < \infty$, we are only in need of a quadrature formula which is exact on $\mathcal{P}_{8,1}$. Let L_4 and L_5 be two polynomials of degree exactly 4 and 5 which are orthogonal to $\mathcal{P}_{3,1}$ (for the scalar product defined by $(f.g) = \int f(\tau) g(\tau) d\pi$, for which one cannot always compute the scalar product of L_4 and L_5). Then for $\lambda \in R$, the points $\tau_j^\lambda, j = 1, \dots, 5$ are the zeroes of $L_5 + \lambda L_4$, with the corresponding weights which give a quadrature formula exact on $\mathcal{P}_{4,1}$, automatically exact on $\mathcal{P}_{8,1}$ by the orthogonality property. To each τ_j^λ there corresponds a unique point a_j^λ on C_Q through (2.27). Consider the measure

$$\mu^\lambda = \sum_{i=1}^{5} \frac{\alpha_j^\lambda}{f^4(\tau_j^\lambda)} \delta_{\alpha_j^\lambda}.$$

According to (2.32), one has, for any element of $\mathcal{P}_{4,3}^h$,

$$\begin{aligned}\int_{S^2} P d\mu &= \int_{-\infty}^{+\infty} q(\tau) d\pi_T = \int_{-\infty}^{+\infty} q(\tau) d\pi_T^\lambda = \sum_{j=1}^{5} \alpha_j^\lambda q(\tau_j^\lambda) = \sum_{j=1}^{5} \frac{\alpha_j^\lambda}{f^4(\tau_j^\lambda)} f^4(\tau_j^\lambda) q(\tau_j^\lambda) \\ &= \sum_{j=1}^{5} \frac{\alpha_j^\lambda}{f^4(\tau_j^\lambda)} P\Big(x_1(\tau_j^\lambda), x_2(\tau_j^\lambda), x_3(\tau_j^\lambda)\Big) = \sum_{j=1}^{5} \frac{\alpha_j^\lambda}{f^4(\tau_j^\lambda)} P(\alpha_j^\lambda) = \int_{S^2} P d\mu^\lambda,\end{aligned}$$

where the third equality holds true because $d^o(q) \le 8$ and the fourth because all τ_j^λ's $(1 \le j \le 5)$ are finite.

We have thus proved that, in Case 1,

$$Q = F(\mu) = F(\mu^\lambda),$$

where μ^λ is a one parameter family of elements of $M^+(S^2)$ supported at 5 points of S^2, which establishes the first assertion in Case 1 and completes the proof of Theorem 2.3 and of Remark 2.8.

Remark 2.9: If, in the context of Theorem 2.3, the point Q of $\partial M_{4,3}$ admits two distinct tangent hyperplanes, it corresponds to the fourth order moments of a measure in $M^+(S^2)$ supported at 4 points at most. Indeed there exist two nonzero homogeneous quadratic polynomials R_{1Q} and R_{2Q} with common zeroes. The number of zeroes is thus limited to 4 unless both polynomials are degenerate and admit a common affine factor L. Then all the zeroes lie on the equatorial plane $L = 0$. Otherwise, setting

$$R_{1Q} = LL_1,\ R_{2Q} = LL_2,$$

with L_1 and L_2 affine, we would conclude that L_1 and L_2 have a common zero; thus either $L_1 = L_2$ and $R_{1Q} = R_{2Q}$ which contradicts the premise, or the support of μ belongs to $L = 0$ or to the intersection of $L_1 = 0$ with $L_2 = 0$, i.e. a line passing through the origin, but then Q admits an infinity of tangent hyperplanes. Consequently the support of μ is contained in the equatorial plane $L = 0$. But this case corresponds to the Case 3 in the proof of Theorem 2.3 and in such a case $M_{4,3}$ admits an infinity of tangent hyperplanes, which once again is a contradiction.

Remark 2.10: The three-dimensional counterpart of Remark 2.6 holds true. Every point in the interior of $M_{4,3}$ can be attained by an element μ of $M^+(S^2)$ supported at six points of S^2, one of the points being chosen arbitrarily. The proof of the above assertion is identical to its two-dimensional analogue upon application of Theorem 2.3 in place of Theorem 2.2.

2.4 Symmetry restrictions

This short subsection serves as an illustration of the previous results. Specifically Theorem 2.3, Remark 2.7 and 2.10 are used towards a characterization of subsets of $M_{4,3}$ that correspond to fourth order moments of nonnegative measures that remain invariant under certain subgroups of $SO(3)$.

As a first example the isotropic measure da, which is invariant under the action of $SO(3)$ itself, is considered. Consider, on S^2, the north pole, together with 5 equidistributed points on the intersection of S^2 with a plane located at $\frac{1}{\sqrt{5}}$ above the equatorial plane. Note that the six directions defined in such a manner are those of the northern vertices of a regular icosahedron. Their coordinates in the canonical basis of R^3 are

$$a_1 = (0,0,1),\ a_{i+1} = (\sin 2\beta \cos 2i\alpha, \sin 2\beta \sin 2i\alpha, \cos 2\beta),\ i = 1,\dots,5$$

with

$$\alpha = \frac{\pi}{5},\ \beta = \frac{1}{2}\arccos\Big(\frac{1}{\sqrt{5}}\Big).$$

Remark 2.11: The six directions $a_1,\dots,a_6$ are precisely those used in FRANCFORT & MURAT ([FM′], Subsection 4.2) to demonstrate that HASHIN & SHTRIKMAN's bounds on the bulk and shear moduli of an isotropic two-phase composite are optimal. The reader is referred to Section 3 for further details.

A mere algebraic computation would demonstrate that the measure

$$\mu = \frac{1}{6}\sum_{i=1}^{6}\delta_{a_i} \tag{2.33}$$

has the same fourth order moments as the normalized LEBESGUE measure $\frac{da}{4\pi}$, i.e. that

$$F\Big(\frac{da}{4\pi}\Big) = F(\mu).$$

If $F(da)$ were to belong to $\partial M_{4,3}$ then, according to Remark 2.7, the image of the set $A := \{a_i,\ i = 1, \ldots, 6\}$ under Γ_3 would lie on a conic section. The coordinates of the points in $\Gamma_3(A)$ are

$$\Gamma_3(a_1) = (0,0),\ \Gamma_3(a_{i+1}) = \tan 2\beta(\cos 2i\alpha, \sin 2i\alpha),\ i = 1, \ldots, 5$$

and these points cannot lie on a conic section as the last five points are distinct and belong to half a circle whose center is the first point.

Thus, by virtue of Remark 2.7,

$$F(da) \notin \partial M_{4,3}$$

and the number of DIRAC masses in (2.33) is minimal. We have then proved the following.

Corollary 2.1 The fourth order moments of the LEBESGUE measure on S^2 belong to the interior of $M_{4,3}$ and cannot be obtained as fourth order moments of any element of $M^+(S^2)$ supported at less than six points of S^2. The construction (2.33) is then optimal.

As a second example we seek a complete characterization of all fourth order moments of nonnegative *transversely isotropic measures*, i.e. of all measures which remain invariant under any rotation about a given axis (which we take to be the x_3 axis).

The assumed symmetry dramatically reduces the number of independent moments of order four. In fact, three independent nonnegative numbers characterize the set of fourth order moments, namely

$$\begin{aligned} a &= \int_{S^2} x_1^4 d\mu = \int_{S^2} x_2^4 d\mu = 3\int_{S^2} x_1^2 x_2^2 d\mu, \\ b &= \int_{S^2} x_1^2 x_3^2 d\mu = \int_{S^2} x_2^2 x_3^2 d\mu, \\ c &= \int_{S^2} x_3^4 d\mu, \end{aligned} \tag{2.34}$$

all other moments being equal to zero. This last statement is immediately established upon rewriting x_1 as $(x.e)$ and x_2 as $(x.e^\perp)$ with $|e| = |e^\perp| = 1$, e orthogonal to $e^\perp$, e and $e^\perp$ orthogonal to e_3 (the unit vector in the x_3 direction). Thus, for example,

$$\begin{aligned} \int_{S^2} x_1^2 x_2^2 d\mu &= \int_{S^2} (x.e)^2 (x.e^\perp)^2 d\mu = \frac{1}{2\pi}\int_{e\perp e_3, |e|=1} \Big(\int_{S^2} (x.e)^2 (x.e^\perp)^2 d\mu\Big) de \\ &= \frac{1}{2\pi}\int_{S^2} \Big(\int_{e\perp e_3, |e|=1} (x.e)^2 (x.e^\perp)^2 de\Big) d\mu = \frac{1}{2\pi}\int_{S^2} (x_1^2 + x_2^2)^2 \Big(\int_0^{2\pi} \cos^2\theta \sin^2\theta d\theta\Big) d\mu \\ &= \frac{1}{8}\int_{S^2} (x_1^2 + x_2^2)^2 d\mu \end{aligned}$$

where the second equality holds because of the transversely isotropic character of μ and the third results from FUBINI's theorem.

Thus

$$\int_{S^2} x_1^2 x_2^2 d\mu = \frac{1}{3}\int_{S^2} x_1^4 d\mu = \frac{1}{3}\int_{S^2} x_2^4 d\mu$$

which establishes (2.34). Since on S^2 one has $x_1^2 + x_2^2 = 1 - x_3^2$, we obtain

$$a = \frac{3}{8}(|\mu| - 2J_2 + J_4),\ b = \frac{1}{2}(J_2 - J_4),\ c = J_4,$$

where

$$J_2 = \int_{S^2} x_3^2 d\mu,\ J_4 = \int_{S^2} x_3^4 d\mu,$$

i.e.

$$|\mu| = \frac{8a}{3} + 4b + c,\ J_2 = 2b + c,\ J_4 = c. \tag{2.35}$$

For a given total mass $|\mu|$, the set of all possible pairs (J_2, J_4) as μ varies over all nonnegative measures on S^2 with fixed total mass is given by

$$0 \le J_2 \le |\mu|,\ J_2^2 \le |\mu| J_4 \le |\mu| J_2. \tag{2.36}$$

In view of (2.34)-(2.36), the set of values for a, b, c is

$$a, b, c \ge 0,\ ac \ge \frac{3b^2}{2}. \tag{2.37}$$

We have thus proved the following.

Corollary 2.2: The set of all fourth order moments of the nonnegative transversely isotropic measures – with fixed transverse axis, namely the x_3 axis – is characterized by three nonnegative real numbers a, b, c satisfying

$$\frac{3b^2}{2} \le ac$$

with

$$a = \int_{S^2} x_1^4 d\mu = \int_{S^2} x_2^4 d\mu = 3\int_{S^2} x_1^2 x_2^2 d\mu$$
$$b = \int_{S^2} x_1^2 x_3^2 d\mu = \int_{S^2} x_2^2 x_3^2 d\mu$$
$$c = \int_{S^2} x_3^4 d\mu,$$

and all other moments equal to zero.

Remark 2.12: Note that this last result should not, in all rigor, be labeled a Corollary to the extent that its derivation, which is elementary, did not appeal to the previously obtained results.

3. Fourth order moments and homogenization in linearized elasticity

The knowledge, acquired in Section 2, of the intimate structure of $M_{4,2}$ and $M_{4,3}$ has immediate consequences in the field of homogenization when applied to linearized elasticity. It is not our purpose to discuss that theory in great details. We will merely recall the few needed definitions and theorems in Subsection 3.1. One of the outstanding problems in "elastic homogenization" is that of finding bounds – and whenever possible, optimal bounds – on various quantities involving the effective tensor (the limit in the sense of homogenization) associated to (sequences of) two-phase mixtures of isotropic elastic materials (cf. e.g. MILTON [M] for a compendium of available results). We will demonstrate in Subsection 3.2 how the results of Section 2 permit to better circumscribe the task at hand.

3.1 H-convergence: a brief recall in the framework of linearized elasticity

Homogenization aims at describing the weak limits of solutions to partial differential equations with oscillating coefficients. Within the framework of linearized elasticity, the following definition and theorem are applied to an arbitrary sequence A^ϵ of elasticity tensors in $M(\alpha, \beta)$.

Definition 3.1: A sequence A^ϵ in $M(\alpha, \beta)$ H-converges to A^0, element of $M(\alpha, \beta)$, if and only if for every bounded domain Ω of R^N and any element f in $(H^{-1}(\Omega))^N$ the solution u^ϵ – unique in $(H_0^1(\Omega))^N$ – of

$$-\mathrm{div}\Big(A^\epsilon e(u^\epsilon)\Big) = f \text{ in } \Omega, \quad u^\epsilon = 0 \text{ on } \partial\Omega,$$

with

$$e_{ij}(u^\varepsilon) = \frac{1}{2}\Big(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\Big) \text{ in } \Omega,$$

is such that

$$u^\varepsilon \rightharpoonup u^0 \text{ weakly in } \Big(H_0^1(\Omega)\Big)^N$$

$$A^\varepsilon e(u^\varepsilon) \rightharpoonup A^0 e(u^0) \text{ weakly in } \Big(L^2(\Omega)\Big)^N$$

where u^0 is the solution – unique in $(H_0^1(\Omega))^N$ – of

$$-\text{div}\Big(A^0 e(u^0)\Big) = f \text{ in } \Omega, \quad u^0 = 0 \text{ on } \partial\Omega,$$

with

$$e_{ij}(u^0) = \frac{1}{2}\Big(\frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_j^0}{\partial x_i}\Big) \text{ in } \Omega.$$

A^0 is called the H-limit of the sequence A^ε.

Theorem 3.1: TARTAR [T]) Consider a family A^ε of elements of $M(\alpha,\beta)$. There exists a subsequence of A^ε which H-converges to an element A^0 of $M(\alpha,\beta)$.

In physical terms, Definition 3.1 proposes a mathematical conceptualization of the intuitive notion of effective behaviour while Theorem 3.1 asserts the existence of the notion of effective properties for any kind of microscopically heterogeneous material.

A specific type of sequence is of particular interest in applications, that corresponding to mixtures of two phases. If A and B denote the elasticity tensors associated to each phase, the sequence A^ε reads as

$$A^\varepsilon(x) = \chi^\varepsilon(x)A + (1-\chi^\varepsilon(x))B, \tag{3.1}$$

where χ^ε is the characteristic function of the A-phase for fixed ε. Assume that it is a priori known that

$$\chi^\varepsilon \rightharpoonup \theta \text{ weak } \star \text{ in } L^\infty(R^N) \tag{3.2}$$

as ε tends to zero. Then the problem of bounds is the following: what is the set G_θ of all possible H-limits of sequences of the form (3.1) – the existence of such H-limits is guaranteed by Theorem 3.1 – for a given weak $\star$ limit θ (a given local volume fraction of the A-phase)? Note that H-limits are actually local so that it may be assumed, without loss of generality, that θ is a constant element of [0,1] (cf. DAL MASO & KOHN [DK]).

This problem, sometimes referred to as the G_θ-closure problem, is the object of a vast literature; the reader is referred to HASHIN [H] for an application oriented overview and to MILTON [M] for a more theoretical standpoint. It is as of yet an unsolved problem, even in the simplest case where A and B are both isotropic, i.e. when

$$\begin{aligned} A &= K_1 i\otimes i + 2\mu_1(I - \frac{i\otimes i}{N}),\ K_1 \equiv \lambda_1 + \frac{2\mu_1}{N}, \\ B &= K_2 i\otimes i + 2\mu_2(I - \frac{i\otimes i}{N}),\ K_2 \equiv \lambda_2 + \frac{2\mu_2}{N}, \end{aligned} \tag{3.3}$$

and when further only isotropic H-limits are considered. If A^0 is such a H-limit it reads as

$$A^0 = K i\otimes i + 2\mu(I - \frac{i\otimes i}{N}),$$

and optimal bounds are known on K and μ separately whenever A and B are well ordered, i.e. whenever

$$K_1 \le K_2 \text{ and } \mu_1 \le \mu_2.$$

These are the celebrated HASHIN-SHTRIKMAN's bounds (cf. e.g. HASHIN-SHTRIKMAN [HS], FRANCFORT & MURAT [FM′], MILTON & KOHN [MK]).

The optimal character of the bounds has been demonstrated through the use of a special kind of composite (a special kind of characteristic functions), multiple rank laminates [FM′]. Specifically the following characteristic function are considered

$$\chi^\epsilon(x) = \tilde{\chi}^\epsilon\Big((x.a)\Big)$$

where $a \in S^{N-1}$, $\tilde{\chi}^\epsilon$ being a sequence of characteristic functions with θ as weak $\star$ limit. Then the associated sequence A^ϵ (cf. (3.1)) is shown to H-converge to A^0 given by

$$(1-\theta)(A^0 - A)^{-1}h = (B-A)^{-1}h + \frac{\theta}{\mu_1}\Big(\frac{ha \otimes a + a \otimes ha}{2} - \frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}(ha.a)a \otimes a\Big),\ h \in R_s^{N^2}. \qquad (3.4)$$

Note that it is implicitly assumed in (3.4) that A is isotropic and given through (3.3) and that $B - A$ is invertible. A more general formula holds true without such restrictions. The resulting A^0 is a rank-1 laminate. Similarly a rank-p laminate is given by

$$(1-\theta)(A^0-A)^{-1}h = (B-A)^{-1}h + \frac{\theta}{\mu_1}\sum_{i=1}^{p}\eta_i\Big(\frac{ha_i \otimes a_i + a_i \otimes ha_i}{2} - \frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}(ha_i.a_i)a_i \otimes a_i\Big),\ h \in R_s^{N^2}, \quad (3.5)$$

with

$$\eta_i \geq 0,\ i = 1,\dots,p; \quad \sum_{i=1}^{p}\eta_i = 1.$$

The vectors a_i, $i = 1,\dots,p$, are the directions of lamination and θ the total volume fraction of the A-phase in the resulting mixture. The phase with elastic tensor A in formula (3.4), (3.5) is the matrix phase, that with B the inclusion phase. Rank-p layers with the phase with elastic tensor B as matrix phase are defined in a similar manner.

It was shown in [FM′] (Proposition 4.3) that, in a three-dimensional setting, six suitable chosen directions of lamination give rise to an isotropic H-limit that saturates both bounds on K and μ simultaneously. Corollary 3.1 will establish that this number is optimal.

As already mentioned the full G$_\theta$-closure, or even the full G-closure (the union of the G$_\theta$-closures as θ varies between 0 and 1) is not known. A simpler problem with many application most notably in the fields of relaxation and structural optimization is the investigation of quantities such as

$$\sup_{A \in G_\theta} (\text{or} \inf_{A \in G_\theta}) \frac{1}{2}\sum_{i=1}^{p}(Ae_i.e_i),\ e_i \in R_s^{N^2}.$$

These are called energy bounds (cf. e.g. ALLAIRE & KOHN [AK]). Whenever $p = 1$ the problem is well understood (cf. [AK] or [FM]). Bounds are known and their optimality is obtained – at least in the well ordered case, i.e. when $A < B$ as quadratic forms – by virtue of the following result:

Theorem 3.2: (AVELLANEDA [A′]) Assume that A and B are isotropic (cf. (3.3)) and that $A < B$. Let Γ be a subgroup of $O(N)$ – the group of orthogonal matrices – and consider the subset $G_\theta(\Gamma)$ of G_θ of all possible H-limits of the form (3.1), (3.2) that remain invariant under the action of Γ. For every element A^0 of $G_\theta(\Gamma)$, there exist two finite-rank laminates with respective elastic tensors $\underline{A}$ and $\overline{A}$ such that $\underline{A}$ and $\overline{A}$ remain invariant under the action of Γ and satisfy

$$\underline{A} \leq A^0 \leq \overline{A}.$$

Then $\underline{A}$ corresponds to a finite-rank laminate with the A-phase as matrix phase, $\overline{A}$ to a finite-rank laminate with the B-phase as matrix phase.

Corollary 3.2 will deliver the rank of such a laminate in the case $N = 3$ and without any symmetry restrictions ($\Gamma = i$); the case $N = 2$ was firstly derived in AVELLANEDA & MILTON [AM]. Corollary 3.3 will investigate the case where $N = 3$ and Γ corresponds to rotations about a given axis, the x_3 axis.

3.2 A few results about bounds

This last subsection refers extensively to the terminology and to the results mentioned in Subsection 3.1.

Firstly it should be noted that the finite-rank lamination formula (3.5) reads as

$$(1-\theta)(A^0-A)^{-1}h=(B-A)^{-1}h+\frac{\theta}{\mu_1}\int_{S^{N-1}}\Big(\frac{(ha\otimes a+a\otimes ha)}{2}-\frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}(ha.a)a\otimes a\Big)d\mu(a), \quad (3.6)$$

where μ is a probability measure ($\mu=\sum_{i=1}^{p}\eta_i\delta_{a_i}$) on S^{N-1}. One of the main (although not explicitly stated) ingredients in the proof of Theorem 3.2 in AVELLANEDA [A′] is the following remark which becomes obvious in view of Subsection 2.1.

Remark 3.1: According to Lemma 2.1, all fourth order moments of a probability measure on S^{N-1} can be achieved by an atomic measure with a finite number of atoms. Thus formulae (3.5) and (3.6) yield the same set of tensors A^0 and can be used interchangeably.

If both phases are isotropic, as well as the resulting effective tensor A^0, then the tensor X defined as

$$Xh=\frac{1}{\mu_1}\int_{S^{N-1}}\Big(\frac{(ha\otimes a+a\otimes ha)}{2}-\frac{\lambda_1+\mu_1}{\lambda_1+2\mu_1}(ha.a)a\otimes a\Big)d\mu(a),\ h\in R_s^{N^2},$$

must be isotropic. Hence, upon setting

$$X=\Lambda i\otimes i+2MI,\ N\Lambda+2M\equiv NK, \quad (3.7)$$

and choosing h to be of the form

$$h=\frac{1}{2}(e_i\otimes e_j+e_j\otimes e_i),\ 1\le i,j\le N$$

the fourth order moment tensor J defined as

$$J_{ijkm}=\int_{S^{N-1}}a_ia_ja_ka_m d\mu(a),\ 1\le i,j,k,m\le N$$

is easily identified as

$$J_{ijkm}=\frac{\lambda_1+2\mu_1}{2(\lambda_1+\mu_1)}\Big(\big(NK_1(\lambda_1+2\mu_1)-2M\mu_1\big)(\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk})-2\mu_1\Lambda\delta_{ij}\delta_{km}\Big). \quad (3.8)$$

We remark that, because of the symmetry properties of J, together with the relation

$$\sum_{i,k=1}^{N}J_{iikk}=1,$$

(3.8) holds true if and only if

$$K=\frac{1}{N^2(\lambda_1+2\mu_1)},\ M=\frac{K_1+2\mu_1}{2(N+2)\mu_1(\lambda_1+2\mu_1)}, \quad (3.9)$$

in which case,

$$J_{ijkm}=\frac{1}{N(N+2)}(\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk}+\delta_{ij}\delta_{km}). \quad (3.10)$$

In the notation of Section 2, (3.10) reads as

$$F(d\mu) = F\Big(\frac{da}{|S^{N-1}|}\Big), \tag{3.11}$$

where $|S^{N-1}|$ denotes the surface of S^{N-1} and da the LEBESGUE measure on S^{N-1}.

Further, according to [FM′] (Subsection 4.2), there exists a unique tensor X of the form (3.7) compatible with the finite rank lamination formula (3.5), in the sense that the resulting tensor A^0 satisfies

$$(1-\theta)(A^0 - A)^{-1} = (B - A)^{-1} + \theta X. \tag{3.12}$$

In (3.12), A and B are isotropic. It is determined through formulae (4.30) and (4.31) of [FM′] and the resulting LAMÉ coefficients K and M are immediately checked to be those determined in (3.9).

Recalling (3.11) we have proved thus that finite rank layering of two isotropic materials with the A-phase as matrix phase may at most produce – at fixed volume fraction – one isotropic tensor A^0; the tensor A^0 is given through (3.6) with

$$d\mu = \frac{da}{|S^{N-1}|}. \tag{3.13}$$

Remark 3.2: Application of the finite rank layering formula with the B-phase as matrix phase would yield an analogous result.

Remark 3.3: The only two possible isotropic tensors A that could be produced through finite rank layering (cf. Remark 3.2) are known to achieve HASHIN-SHTRIKMAN bounds in both bulk and shear (K and μ), see [FM′], subsection 4.2.

We now restrict our attention to the three-dimensional setting and recall Corollary 2.1. The following Corollary is then an immediate consequence of (3.6), (3.13).

Corollary 3.1: The number of directions (6) used in [FM′], Proposition 4.3, so as to generate through finite rank lamination an isotropic effective material (whose bulk and shear moduli saturate HASHIN-SHTRIKMAN bounds) in three dimensions is optimal. Those directions are given by the north pole together with 5 equidistributed points on the intersection of S^2 with a plane located at $\frac{1}{\sqrt{5}}$ above the equatorial plane.

Next, Remark 2.10 immediately yields a corollary to Theorem 3.2 in [A′]. Specifically we obtain the

Corollary 3.2: In the context of Theorem 3.2, the two finite-rank laminates $\underline{A}$ and $\overline{A}$ may be chosen to be at most of rank 6 when $N = 3$.

Remark 3.4: By virtue of Corollary 3.2, the infimum or supremum over $A \in G_\theta$ of $\frac{1}{2}\sum_{i=1}^{p}(Ae_i.e_i)$, with $e_i \in R_s^{N^2}$, is always achieved by rank-6 laminates in the three-dimensional setting (rank-3 in the two-dimensional analogue). Note that, in the case $p = 1$, it is implicit in [AK] that rank-N laminates are optimal, for any N. The best possible lamination rank for $N \neq 1, 2, 3$ and $p > 1$ is not known.

We conclude this study by a rapid incursion into the set of laminates with transverse isotropic symmetry. This problem was examined at length in LIPTON [L]. The second order moments of a nonnegative transversely isotropic measure μ are immediately derived in terms of a, b, c given in Corollary 2.2, We obtain, in the notation of that Corollary,

$$\int_{S^2} x_1^2 d\mu = \int_{S^2} x_2^2 d\mu = \frac{4a}{3} + b,$$

$$\int_{S^2} x_3^2 d\mu = 2b + c,$$

$$\int_{S^2} x_1 x_2 d\mu = \int_{S^2} x_1 x_3 d\mu = \int_{S^2} x_2 x_3 d\mu = 0.$$

Thus, for any such measure μ, (3.6) reads as

$$(1-\theta)\Big((A^0 - A)^{-1}\Big)_{ijkm} = \Big((B-A)^{-1}\Big)_{ijkm} + \frac{\theta}{\mu_1} Y_{ijkm}$$

with

$$Y_{1111} = Y_{2222} = \frac{4a}{3} + b - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} a, \qquad Y_{3333} = 2b + c - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} c, \tag{3.14$_1$}$$

$$Y_{1122} = -\frac{\lambda_1 + \mu_1}{3(\lambda_1 + 2\mu_1)} a, \qquad Y_{1133} = Y_{2233} = -\frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} b, \tag{3.14$_2$}$$

$$Y_{1212} = \frac{2a}{3} + \frac{b}{2} - \frac{\lambda_1 + \mu_1}{3(\lambda_1 + 2\mu_1)} a, \qquad Y_{1313} = Y_{2323} = \frac{a}{3} + \frac{3b}{4} + \frac{c}{4} - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} b, \tag{3.14$_3$}$$

all other components being equal to zero.

The constraint that μ be a probability measure translates into

$$\frac{8a}{3} + 4b + c = 1.$$

We appeal to Remark 3.1, Corollary 2.2, and conclude with the following

Corollary 3.3: The set of all transversely isotropic finite-rank laminates made from two isotropic phases with elastic tensors A and B given through (3.3) and at volume fraction θ of the A-phase is the set of all tensors A^0 satisfying

$$(1-\theta)(A^0 - A)^{-1} = (B-A)^{-1} + \frac{\theta}{\mu_1} Y$$

with Y defined through (3.14), or

$$\theta(A^0 - B)^{-1} = (A-B)^{-1} + \frac{1-\theta}{\mu_2} Z,$$

with Z obtained from Y by replacing λ_1 by λ_2 and μ_1 by μ_2 in (3.14). In (3.14), a, b, c are three nonnegative real numbers satisfying

$$\frac{3b^2}{2} \leq ac,$$

$$\frac{8a}{3} + 4b + c = 1.$$

Remark 3.5: Combining Corollary 3.3 with Theorem 3.2 would permit to obtain energy bounds on the set of transversely isotropic effective elastic tensors. The reader is referred to [L], Section 4, for details.

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Appendix

After treating the 2-dimensional case using the fact that a nonnegative polynomial in one variable is the sum of squares of polynomials, we wondered if this property was also true for nonnegative polynomials of degree at most four in two variables in order to treat the 3-dimensional case. We first learned of the classical result that a nonnegative polynomial is always the sum of squares of rational fractions but not always the sum of squares of polynomials, but the counterexample that we read dealt with a polynomial of degree six in two variables, so that the sought result could still be true, and we indeed derived a proof for it. A few months later we learned that the result had been proved by HILBERT; we did not try to obtain the smallest number of squares, which HILBERT had proved to be three. As HILBERT's proof is not so transparent, we deem it useful to sketch the only part of the proof which is of use to us, namely that if P is a nonnegative polynomial of degree four which has five distinct zeroes $a_j, j = 1, \ldots, 5$, then P is a sum of (three) squares.

Let us consider the case where three of the points a_j are on a line of equation $L(x, y) = 0$. In that case the intersection with $P(x, y) = 0$ with the line has three double zeroes, and therefore the degree of P being at most four, P is divisible by L, so $P = LS$. As P is nonnegative we must have $S = 0$ on the line $L = 0$, so $P = L^2T$ and T has degree ≤ 2 and is nonnegative and so is the sum of (three) squares.

Assuming that we are not in the above mentioned degenerate case, we want to construct a conic section going through all the points $a_j, j = 1, \ldots, 5$. This is certainly possible as a quadratic polynomial is defined by six homogeneous coefficients and one writes one linear relation for expressing the fact that the conic section goes through a point, and five linear relations can be imposed. Let $Q(x, y) = 0$ be the equation of that conic section. In the intersection of the zero set of P and the zero set of Q, each of the points a_j counts for two, and this gives a counting of ten intersection points instead of eight for intersecting two algebraic curves of degree two and four, and so there is a degeneracy and Q and P should have a common factor, i.e. P is a multiple of Q, so there exists a polynomial R of degree at most two such that $P = QR$. Because P is nonnegative, R must change sign when Q does and so $R = 0$ when $Q = 0$ and R is divisible by Q giving $P = cQ^2$ with $c > 0$.

The preceding argument can be made more analytical by parametrizing the conic section $Q(x, y) = 0$ by $x = \frac{a(t)}{c(t)}$, $y = \frac{c(t)}{c(t)}$ with a, b, c being polynomials of degree ≤ 2, as Q is a nondegenerate quadratic polynomial (taking the origin on the conic section, any line of equation $y = tx$ intersects the conic section at two points, one value of x being 0 and the other being expressed as a rational fraction in t). Writing $P = 0$ gives an equation of degree eight in t with 5 double zeroes and one deduces that $Q = 0$ implies $P = 0$. In order to deduce that P is divisible by Q, we first change basis so that Q can be written as $Q(x, y) = ax^2 + q(y)$ with $a \neq 0$, and write P as $P(x, y) = P_1(x, y) + xP_2(x, y)$ where P_1 and P_2 only contain even powers of x. As $Q(x, y) = 0$ also implies $Q(-x, y) = 0$, it implies not only that $P(x, y) = 0$ but $P_1(x, y) = P_2(x, y) = 0$. By replacing then each occurence of x^2 in P_1 or P_2 by $\frac{1}{a}\Big(Q(x, y) - q(y)\Big)$, one obtains a multiple of Q plus a polynomial in y which must be 0 as $Q = 0$ implies $P_1 = P_2 = 0$.

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