Integral-Gradient Formulae for Structured Deformations

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1. Introduction

The name *integral-gradient theorem* has been introduced by Noll and Virga to denote a special version of the Gauss-Green formula stated without proof in their paper [3]. Their formula involves a class of domains which they call *fit regions* and which they show to be appropriate for the description of regions in space occupied when continuous bodies undergo *classical deformations* [1, Sect. 2].

In [1] we considered a larger class of deformations which includes such non-classical deformations as the formation of macroscopic fractures, as well as the occurrence of microscopic changes in structure that we call microfractures. Here we prove for such deformations appropriate versions of the integral-gradient formula. Technically, this requires us to relax the assumptions on the regularity of the domain made in [3] in order to allow for the presence of finite unions of fit regions, which we call *piecewise fit regions.* By contrast, we are obliged to strengthen the regularity assumed in [3] for the integrands by requiring that their restrictions to each of the fit regions forming their domains have C^1 extensions to the whole space.

Our proof is based on the fact that piecewise fit regions are sets with finite perimeter and that, as proved in Section 3, the integrands considered are functions of bounded variation. These facts allow us to make extensive use of the definitions and results developed in [5], [6], starting from concepts in geometric measure theory. Moreover, we take advantage of the additional regularity enjoyed by our functions, and not by arbitrary functions of bounded variation, to prove a number of useful results. Namely, we are able to prove that not only an inward trace is defined area-almost everywhere on the essential boundary of the domain, but also that the inward trace is area-summable there. Moreover, when dealing with the generalization from scalar-valued to vector-valued functions, we prove that the set of all jump points of each element of our class of vector-valued functions not only is included in the union of the set of jump points of the components, but also is area-equivalent to it.

These developments are carried out in Section 3, whose final result is an integral-gradient formula for piecewise fit regions and for vector-valued functions with the regularity mentioned above. In Section 4 we first show that this formula applies directly to the class of *simple deformations* defined in [1]. We then consider *limits of simple deformations* and prove a regularity property of the trace beyond the properties of limits of simple deformations established in [1]. Among other things, we proved there that, if (κ, g, G) is the limit, in the sense of [1, Def. 4.1], of a sequence $n \longmapsto (\kappa_n, f_n)$ of simple deformations from the piecewise fit region $\mathcal{A}$, then g, the L^∞-limit of the sequence $n \longmapsto f_n$, has a representative g_0 which is continuous on $\mathcal{A}$. Here, we prove that g_0 has an inward trace g_0^+ on the essential boundary of $\mathcal{A}$ which is summable there, and that g_0^+ is the L^∞-limit of the sequence $n \longmapsto f_n^+$ of the inward traces of the functions f_n.

This result enables us to establish an integral-gradient formula, equation (4.19), for limits of simple deformations. We then turn to the class of *structured deformations*, whose study was the main object of the paper [1]. Each structured deformation is a triple (κ, g, G), in which (κ, g) is a simple deformation and G is a tensor field whose properties, as proved in the Approximation Theorem in [1, Sect. 5], are sufficient to ensure that (κ, g, G) can be identified with a limit of simple deformations. By repeating the procedure used in [1, Sect. 6] for the fundamental formula of calculus, in Section 4 of the present paper we compare the integral-gradient formula for (κ, g, G) as a limit of simple deformations with that for (κ, g) as a simple deformation and obtain formula (4.20) for structured deformations. In Section 5, we give interpretations for the integrals appearing in the formulae (4.19) and (4.20), and we find that the total deformation due to microfracture admits the tensor field $\nabla g - G$ as a volume density. This extends our earlier result [1, Sect. 6] which established $\nabla g - G$ as a density of deformation due to microfracture along straight lines.

2. Notation and preliminaries

We denote by $\mathcal{E}$ a finite-dimensional Euclidean point space. The associated inner product space is denoted by $\mathcal{V}$, and $Lin\mathcal{V}$ denotes the set of all linear mappings of $\mathcal{V}$ into itself. Both $\mathcal{V}$ and $Lin\mathcal{V}$ are made into normed spaces with the norms

$$|v| := (v \cdot v)^{\frac{1}{2}}, \quad |H| := \sup_{v \in \mathcal{V}\setminus\{0\}} \frac{|Hv|}{|v|}, \quad v \in \mathcal{V},\ H \in Lin\mathcal{V}. \tag{2.1}$$

If N is the dimension of $\mathcal{E}$, we denote by V and A the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure on $\mathcal{E}$, and we call them the volume measure and the area measure, respectively. If $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{E}$, the notations

$$\mathcal{A} \approx \mathcal{B}, \quad \mathcal{A} \stackrel{A}{\approx} \mathcal{B}, \tag{2.2}$$

mean that $\mathcal{A}$ differs from $\mathcal{B}$ by a set of volume zero and by a set of area zero, respectively. By $int\mathcal{A}$, $clo\mathcal{A}$, $bdy\mathcal{A}$ we denote the interior, the closure, and the boundary of $\mathcal{A}$, respectively, and by $\mathcal{B}(x, \delta)$ we denote the open ball of $\mathcal{E}$, centered at x and of radius δ.

We now recall some measure-theoretic concepts, for which we refer to [6]. Let x be a point of $\mathcal{E}$, and let $\mathcal{A}$ be a subset of $\mathcal{E}$. Consider the limit

$$\lim_{\delta \to 0} \frac{V(\mathcal{B}(x,\delta) \cap \mathcal{A})}{V(\mathcal{B}(x,\delta))}. \tag{2.3}$$

We say that x is a *point of density* for $\mathcal{A}$ if the above limit exists and is 1, a *point of rarefaction* if the above limit exists and is zero, and that x belongs to the *essential boundary* of $\mathcal{A}$ in any other case, [6, Sect. 4.4.1]. It is clear that the three sets just defined, which we denote by *dens*$\mathcal{A}$, *rar*$\mathcal{A}$, *eby* $\mathcal{A}$, respectively, are pairwise disjoint and form a partition of $\mathcal{E}$.

Let ν be a unit vector in $\mathcal{V}$. Consider the closed hemiball

$$\mathcal{H}(x,\delta,\nu) := \{y \in \mathcal{E} \mid y \in clo\, \mathcal{B}(x,\delta),\ (y-x)\cdot\nu \leq 0\}. \tag{2.4}$$

We say that ν is an *outward normal* to $\mathcal{A}$ at x if

$$\lim_{\delta\to 0} \frac{V(\mathcal{H}(x,\delta,\nu)\cap\mathcal{A})}{V(\mathcal{H}(x,\delta,\nu))} = 1, \quad \lim_{\delta\to 0} \frac{V(\mathcal{H}(x,\delta,-\nu)\cap\mathcal{A})}{V(\mathcal{H}(x,\delta,-\nu))} = 0. \tag{2.5}$$

An outward normal, if it exists, is unique [6, Sect. 4.5.2]. The set of all points of $\mathcal{E}$ for which an outward normal to $\mathcal{A}$ exists is called the *reduced boundary* of $\mathcal{A}$ and will be denoted here by *rby* $\mathcal{A}$. It follows from equations (2.5) that, if $x \in$ *rby* $\mathcal{A}$, then the limit (2.3) exists and equals 1/2, and therefore x belongs to *eby* $\mathcal{A}$. Thus,

$$rby\,\mathcal{A} \subset eby\,\mathcal{A}. \tag{2.6}$$

We now consider properties of real-valued functions defined on $\mathcal{E}$. We say that the real number d is the *approximate limit* of $u : \mathcal{E} \to \mathbb{R}$ at the point x of $\mathcal{E}$, and write

$$d = \lim_{y\to x} u(y) \tag{2.7}$$

if, for each $\epsilon > 0$,

$$\lim_{\delta\to 0} \frac{V(\mathcal{B}(x,\delta)\cap\{y\in\mathcal{E} \mid |u(y)-d|<\epsilon\})}{V(\mathcal{B}(x,\delta))} = 1, \tag{2.8}$$

[2, Sect. 1.7.2], i.e., if x is a density point for the set $\{y \in \mathcal{E} \mid |u(y)-d| < \epsilon\}$

[6, Sect. 4.4.2]. If $\mathcal{A}$ is a subset of $\mathcal{E}$, the definition of the *approximate limit of u at x relative to $\mathcal{A}$*

$$\lim_{y\to x, y\in\mathcal{A}} u(y) \tag{2.9}$$

is obtained from the condition (2.8) with $\mathcal{B}(x,\delta)$ replaced by $\mathcal{A}\cap\mathcal{B}(x,\delta)$ [6, Sect. 4.4.2]. Let α be a unit vector in $\mathcal{V}$, and denote by $\Pi_\alpha(x)$ the half-space $\{y\in\mathcal{E} \mid (y-x)\cdot\alpha>0\}$, and by $u_\alpha(x)$ the approximate limit

$$u_\alpha(x) := \lim_{y\to x, y\in\Pi_\alpha(x)} u(y). \tag{2.10}$$

We say that x is a *regular point* for u if there exists a unit vector α such that the approximate limits $u_\alpha(x)$ and $u_{-\alpha}(x)$ exist. If the two limits are equal, their common value is the approximate limit of u at x, and x is said to be a *point of approximate continuity* for u. If the two limits are not equal, then α and $-\alpha$ are the only unit vectors for which the approximate limits in (2.10) exist, x is said to be a *jump point* for u, and α and $-\alpha$ are called the *determining vectors* at x. For each jump point x with determining vectors α and $-\alpha$, the vector

$$Ju(x) := (u_\alpha(x) - u_{-\alpha}(x))\alpha \tag{2.11}$$

is called the *directed jump* of u at x [6, Sect. 4.5.4]. It is clear that $Ju(x)$ does not change if α is replaced by $-\alpha$; thus, the vector $Ju(x)$ is defined unambiguously. We extend the definition of directed jump of u to points x of approximate continuity of u by setting $Ju(x)=0$ at those points.

Let $\mathcal{A}$ be a measurable subset of $\mathcal{E}$, and let x be a point in the essential boundary of $\mathcal{A}$. If the approximate limits

$$u^+(x) := \lim_{y \to x, y \in \mathcal{A}} u(y), \quad u^-(x) := \lim_{y \to x, y \in \mathcal{E} \setminus \mathcal{A}} u(y) \tag{2.12}$$

exist, they are called the *inward trace* and the *outward trace* of u at x, respectively. It can be proved that, if the outward normal ν to $\mathcal{A}$ exists at x, then the inward trace exists if and only if the approximate limit $u_{-\nu}(x)$ defined by (2.10) exists and that, in this case, $u^+(x) = u_{-\nu}(x)$. The same property and the equality $u^-(x) = u_\nu(x)$, hold for the outward trace [6, Sect. 5.1.2].

Let $\mathcal{A}$ be an open subset of $\mathcal{E}$, and let $u : \mathcal{A} \to \mathbf{R}$ be given. We say that u is a *function of bounded variation*, if u is summable on $\mathcal{A}$ and if the distributional derivative Du of u is a measure [6, Sect. 9.3.1]. The last requirement is met if and only if

$$sup \left\{ \int_{\mathcal{A}} u(x) div\varphi(x) dV_x \mid \varphi \in C_0^\infty(\mathcal{A}, \mathcal{V}),\ |\varphi(x)| \leq 1 \ \ \forall x \in \mathcal{A} \right\} < +\infty \tag{2.13}$$

[2, Sect. 5.1]. A bounded measurable set $\mathcal{A}$ is called a *set with finite perimeter* if the distributional derivative of its characteristic function $\chi_{\mathcal{A}} : \mathcal{E} \to \mathbf{R}$ is a measure [6, Sect. 4.2.1]. It can be proved that, if $\mathcal{A}$ is a set with finite perimeter, then the measure $D\chi_{\mathcal{A}}$ is concentrated on *eby* $\mathcal{A}$ [6, Sect. 5.1.1] and *eby* $\mathcal{A}$ is area-equivalent to the reduced boundary of $\mathcal{A}$ [5, Sect. 4]:

$$eby\ \mathcal{A} \overset{A}{\approx} rby\ \mathcal{A}. \tag{2.14}$$

We will use the notation $BV(\mathcal{A})$ to denote the set of all functions $u : \mathcal{A} \rightarrow \mathbf{R}$ which are functions of bounded variation, and we will use the symbol BV for $BV(\mathcal{E})$. We list below some properties of the functions belonging to the space BV which are relevant to our purposes.

(BV1) If $u \in BV$ (indeed, if u is summable [6, Sect. 4.4.4]), the points which are not points of approximate continuity for u form a set of volume zero.

(BV2) If $u \in BV$, the points which are not regular points for u form a set of area zero [6, Sect. 4.5.5].

(BV3) If $u \in BV$, then for every set with finite perimeter $\mathcal{A}$ the inward and the outward trace of u exist A-almost everywhere on $eby\,\mathcal{A}$ [6, Sect. 5.1.2].

(BV4) If $u \in BV$, if $\mathcal{A}$ is bounded and with finite perimeter, and if the inward trace u^+ is summable on $eby\,\mathcal{A}$, then

$$\int_{dens\,\mathcal{A}} Du = \int_{eby\,\mathcal{A}} u^+(x)\nu(x)\,dA_x\,, \tag{2.15}$$

where Du is the distributional derivative of u and $\nu(x)$ is the outward normal to $eby\,\mathcal{A}$ at x [6, Sect. 5.1.4].

(BV5) Let $u \in BV$ and let $\mathcal{A}$ be a bounded set with finite perimeter. Then for every A-measurable subset $\mathcal{B}$ of $eby\,\mathcal{A}$,

$$\int_{\mathcal{B}} Du = \int_{\mathcal{B}} (u_{\nu(x)}(x) - u_{-\nu(x)}(x))\nu(x)dA_x = \int_{\mathcal{B}} Ju(x)\,dA_x \tag{2.16}$$

[6, Sect. 5.1.4].

(BV6) For every $u \in BV$ and for every set $\mathcal{B}$ of area zero,

$$\int_{\mathcal{B}} Du = 0 \tag{2.17}$$

[6, Sect. 4.5.5].

(BV7) If u is a C^1 function on $\mathcal{E}$ with bounded support, then $u \in BV$ [3, Sect. 2].

(BV8) If h and u are in BV, are bounded and have bounded support, then the product hu is in BV, is bounded and has bounded support [3, Sect. 3].

We conclude our preliminaries by recalling that a *fit region* is a subset $\mathcal{A}$ of $\mathcal{E}$ satisfying the following properties [3]:

(i) $\mathcal{A}$ is bounded,

(ii) $intclo\mathcal{A} = \mathcal{A}$,

(iii) $\mathcal{A}$ is a set with finite perimeter,

(iv) $V(bdy\mathcal{A}) = 0$,

and that a *piecewise fit region* is a finite union of fit regions [1].

3. An integral-gradient theorem for a class of piecewise C^1 functions

In this section we prove a result that is a starting point for establishing an integral-gradient theorem for structured deformations. We begin by defining a class of piecewise C^1 functions which satisfies the assumptions of the statement $(BV4)$ in the preceding section.

Lemma 3.1. *Let $\mathcal{A}$ be a piecewise fit region, and let $v : \mathcal{A} \to \mathbf{R}$ be a C^1 function satisfying the following requirement: there is a finite cover $\{\mathcal{A}_j \mid j \in \{1, ..J\}\}$ of $\mathcal{A}$ consisting of fit regions, such that the restriction $v|_{\mathcal{A}_j}$ of v to each $\mathcal{A}_j$ has a C^1 extension to $\mathcal{E}$. Then the function $u : \mathcal{E} \to \mathbf{R}$ defined by*

$$u(x) = \begin{cases} v(x) & \text{if } x \in \mathcal{A} \\ 0 & \text{if } x \in \mathcal{E}\backslash\mathcal{A} \end{cases} \tag{3.1}$$

is a function of bounded variation. Moreover, u has an inward trace on the essential boundary of $\mathcal{A}$ which is A-summable there.

Proof. First of all, we observe that there is no loss in generality in assuming that the C^1 extension of each $v|_{\mathcal{A}_j}$ to $\mathcal{E}$ has compact support. Indeed, if it is not so, it is sufficient to multiply the given extension by a real-valued C^1 function with compact support which takes the value 1 in *clo* A to get a C^1 extension with compact support. Denote this extension by u_j. Consider next the sets $\mathcal{C}_j$, defined recursively by

$$\mathcal{C}_1 := \mathcal{A}_1, \ \ \mathcal{C}_j := \mathcal{A}_j \backslash \bigcup_{p=1}^{j-1} \mathcal{C}_p, \ \ j \in \{2, ...J\}. \tag{3.2}$$

They form a partition of $\mathcal{A}$, and each $\mathcal{C}_j$ is included in the corresponding region $\mathcal{A}_j$. Moreover, each $\mathcal{C}_j$ is a set with finite perimeter and the interior of each $\mathcal{C}_j$ is a fit region. This can be proved by observing that (3.2) implies $\bigcup_{p=1}^{j} \mathcal{C}_p = \bigcup_{p=1}^{j} \mathcal{A}_p$

for all j, and, therefore,

$$\mathcal{C}_j = \mathcal{A}_j \setminus \bigcup_{p=1}^{j-1} \mathcal{A}_p = \bigcap_{p=1}^{j-1} (\mathcal{A}_j \setminus \mathcal{A}_p), \tag{3.3}$$

and

$$int\mathcal{C}_j = \bigcap_{p=1}^{j-1} int(\mathcal{A}_j \setminus \mathcal{A}_p), \tag{3.4}$$

and by recalling that the interior of the difference of two fit regions is a fit region and that the intersection of fit regions is a fit region [3, Sect. 5]. $\mathcal{C}_j$ is a set with finite perimeter because it differs by a set of volume zero from $int\mathcal{C}_j$, which is a fit region and, therefore, a set with finite perimeter.

If we denote by χ_j the characteristic function of $\mathcal{C}_j$, we see that the function $u : \mathcal{E} \to \mathbf{R}$ defined by

$$u(x) := \sum_{j=1}^{J} u_j(x)\chi_j(x) \tag{3.5}$$

coincides with the function u defined in (3.1). Each function u_j is of class C^1 with bounded support, and therefore, by ($BV7$), belongs to the space BV. The same holds for the function χ_j, because the characteristic function of a set with finite perimeter belongs to BV. In view of ($BV8$), we deduce that each product $\chi_j u_j$ belongs to BV, and, therefore, the sum of these products, i.e., the function u, belongs to BV as well.

We now use ($BV3$) and the fact that a piecewise fit region is a set with finite perimeter to deduce that u has an inward trace u^+ A-almost everywhere on $eby\,\mathcal{A}$. To prove that u^+ is summable on $eby\,\mathcal{A}$, we will show that each term $u_j\chi_j$ of the sum in (3.5) has an inward trace with respect to $\mathcal{A}$ that is summable on $eby\,\mathcal{A}$.

Because $u_j : \mathcal{E} \to \mathbf{R}$ is of class C^1, it has a limit at every point of $eby\,\mathcal{A}$ equal to the value of u at that point, and we conclude that the inward trace u_j^+ with respect to $\mathcal{A}$ equals $u_j|_{eby\,\mathcal{A}}$. Moreover, u_j^+ is bounded and measurable, because $u_j\,|_{clo\,\mathcal{A}}$ is bounded and u_j is continuous. We next wish to verify that, for A-almost every x in $eby\,\mathcal{A}$,

$$\chi_j^+(x) = \begin{cases} 1 & \text{if} \quad x \in eby\,\mathcal{A} \cap eby\,\mathcal{C}_j \\ 0 & \text{if} \quad x \in eby\,\mathcal{A} \backslash eby\,\mathcal{C}_j. \end{cases} \tag{3.6}$$

Equivalently, we wish to show that for A-almost every x in $eby\,\mathcal{A}$.

$$\chi_j^+(x) = \chi_{eby\,\mathcal{C}_j \cap eby\,\mathcal{A}}(x). \tag{3.7}$$

To this end, we first note that a lemma of Volpert [5, Sect. 2.5] implies that, for $i \neq j$, the set $eby\,\mathcal{C}_i \cap eby\,\mathcal{C}_j \cap eby\,\mathcal{A}$ has area zero. Therefore, for A-almost every $x \in eby\,\mathcal{A}$, there exists only one $j(x) \in \{1, \cdots, J\}$ such that $x \in eby\,\mathcal{C}_{j(x)} \cap eby\,\mathcal{A}$. The same lemma also implies that the outward normal $\nu_{j(x)}(x)$ to $\mathcal{C}_{j(x)}$ at x and the outward normal $\nu(x)$ to $\mathcal{A}$ at x agree for almost every x in $eby\,\mathcal{A}$. Therefore, for A-almost every x in $eby\,\mathcal{A}$,

$$\lim_{y \to x, y \in \mathcal{A}} \chi_j(y) = \lim_{y \to x, y \in \mathcal{C}_{j(x)}} \chi_j(y) = \begin{cases} 1 & \text{if} \quad j = j(x) \\ 0 & \text{if} \quad j \neq j(x). \end{cases} \tag{3.8}$$

This relation implies the desired relations (3.6) and (3.7). Relation (3.7) tells us immediately that χ_j^+ is measurable, because $eby\,\mathcal{C}_j \cap eby\,\mathcal{A}$ is an A-measurable subset of $eby\,\mathcal{A}$. Therefore, we have

$$(u_j\chi_j)^+(x) = u_j^+(x)\chi_j^+(x) \tag{3.9}$$

for A-almost every x in $eby\,\mathcal{A}$, which shows that $(u_j\chi_j)^+$ is bounded and measurable. Because $A(eby\,\mathcal{A}) < +\infty$, it follows that $(u_j\chi_j)^+$ is summable. ∎

We are now ready to state the integral-gradient theorem for the piecewise C^1 functions v which form the object of the preceding lemma.

Theorem 3.2. *Let $\mathcal{A}$ be a piecewise fit region, and let $v : \mathcal{A} \to \mathbf{R}$ be a function satisfying the assumptions of Lemma 3.1. Denote by $\Gamma(v)$ the set of all jump points of v and by ∇v the gradient of v. Then ∇v is V-summable on $\mathcal{A}$, and*

$$\int_{\mathcal{A}} \nabla v(x) dV_x = - \int_{\Gamma(v)} Jv(x) dA_x + \int_{rby\,\mathcal{A}} v^+(x)\nu(x) dA_x . \tag{3.10}$$

Proof. We have just proved that the function u defined by (3.1) satisfies the requirements made in $(BV4)$, so that equation (2.15) holds for u. We now claim that

$$\int_{eby\,\mathcal{A}} u^+(x)\nu(x) dA_x = \int_{rby\mathcal{A}} v^+(x)\nu(x) dA_x . \tag{3.11}$$

Indeed, because u and v agree in $\mathcal{A}$, the inward trace of v on $eby\,\mathcal{A}$ exists and coincides with that of u. The equality (3.11) then follows from the fact that, by (2.14), $rby\,\mathcal{A}$ differs from $eby\,\mathcal{A}$ by a set of area-measure zero. Comparison between (2.15) and (3.10) then shows that what remains to be proved is

$$\int_{dens\mathcal{A}} Du = \int_{\mathcal{A}} \nabla v(x) dV_x + \int_{\Gamma(v)} Jv(x) dA_x . \tag{3.12}$$

To prove this, we invoke again the lemma from [5, Sect. 2.5] which tells us that, for the finite partition $\{C_j \mid j \in \{1, ..., J\}\}$ of $\mathcal{A}$ defined by (3.2),

$$dens\,\mathcal{A} \overset{A}{\approx} (\bigcup_{j=1}^{J} dens\,C_j) \cup (\bigcup_{\substack{k,\ell=1 \\ k\neq\ell}}^{J} \Gamma_{k\ell}), \tag{3.13}$$

where

$$\Gamma_{k\ell} := rby\, C_k \cap rby\, C_\ell\,; \tag{3.14}$$

moreover, all the sets appearing in the right member of Eq. (3.13) are disjoint. Thus, by $(BV6)$ and by the additivity of the measure Du,

$$\int_{dens A} Du = \sum_{j=1}^{J} \int_{dens\, C_j} Du + \sum_{\substack{k,\ell=1 \\ k\neq\ell}}^{J} \int_{\Gamma_{k\ell}} Du. \tag{3.15}$$

We claim that

$$\sum_{j=1}^{J} \int_{dens\, C_j} Du = \int_{A} \nabla v\,(x)\, dV_x\,, \tag{3.16}$$

and that

$$\sum_{\substack{k,\ell=1 \\ k\neq\ell}}^{J} \int_{\Gamma_{k\ell}} Du = \int_{\Gamma(v)} Jv(x) dA_x. \tag{3.17}$$

To prove our first claim, we use formula (3.2) in [6, Sect. 5.1.3] to write, for each $h \in \{1, ..., J\}$,

$$D(u_h \chi_h) = \bar{u}_h D\chi_h + \bar{\chi}_h Du_h, \tag{3.18}$$

where a superimposed bar denotes the average value as defined in [6, Sect. 4.5.6]. Let us evaluate the measure $D(u_h\chi_h)$ at the set $dens\, C_j$, for a fixed $j \in \{1, ..., J\}$. The fact that the distributional derivative of a characteristic function is concentrated at the essential boundary, together with the fact that $eby\, C_h \cap dens\, C_j = \emptyset$ for all sets C_j, C_h in the partition (3.2), tells us that $D\chi_h$ evaluated at $dens\, C_j$ is zero for all h. It is also true that, for all $x \in dens\, C_j$,

$$\bar{\chi}_h(x) = \begin{cases} 1 & \text{if } h = j \\ 0 & \text{if } h \neq j, \end{cases} \tag{3.19}$$

so that, from (3.18),

$$\int_{dens\, C_j} \sum_{h=1}^{J} D(u_h \chi_h) = \int_{dens\, C_j} Du_j\,. \tag{3.20}$$

Consequently, by (3.5),

$$\int_{dens\, C_j} Du = \int_{dens\, C_j} Du_j = \int_{dens\, C_j} \nabla u_j(x) dV_x\,, \tag{3.21}$$

the last step coming from the fact that u_j is of class C^1, and, therefore, its distributional derivative is absolutely continuous with respect to volume. We can now use the fact that $int\, C_j$ is a fit region, and, therefore, that $int\, C_j \approx clo\, C_j$, to assert that $dens\ C_j \approx int\, C_j$. Moreover, by (3.1) and (3.5), the equality $u_j(x) = v(x)$ holds for all $x \in int\, C_j$, and this yields the desired equality (3.16).

To prove the remaining equality (3.17), we observe that each set $\Gamma_{k\ell}$ is A-measurable. Indeed, for each region C_j the essential boundary is A-measurable, because C_j is a set with finite perimeter, and the reduced boundary is A-measurable, because it differs from $eby\, C_j$ by a set of area zero. Therefore, by (3.14), $\Gamma_{k\ell}$ is the intersection of A-measurable sets and therefore it is A-measurable. We can invoke the formula (2.16) to assert that

$$\int_{\Gamma_{k\ell}} Du = \int_{\Gamma_{k\ell}} Ju(x) dA_x = \int_{\Gamma_{k\ell}} Jv(x) dA_x\,, \tag{3.22}$$

where in the last step we take advantage of the fact that the inward traces of u and v relative to each of the sets C_j coincide. It remains to prove that, to within sets of area zero, the set $\Gamma(v)$ of all jump points of v is included in the union of the sets $\Gamma_{k\ell}$ and is A-measurable. The first assertion rests on the observation that, by its very definition, a jump point for v is a density point for its domain $\mathcal{A}$ and that, because v is continuous in each fit region $int\, \mathcal{C}_j$, no density point of $\mathcal{C}_j$ can be a jump point for u. Thus, relation (3.13) tells us that, to within a set of area zero, the set $\Gamma(u)$ is included in the union of the sets Γ_{kl}. The A-measurability of $\Gamma(v)$ follows from the relation $\Gamma(v) = \Gamma(u)\backslash eby\, \mathcal{A}$; indeed, $eby\, \mathcal{A}$ is A-measurable because $\mathcal{A}$ is a set with finite perimeter, and $\Gamma(u)$ is A-measurable because it is the set of jump points of a function in BV [6, Sect. 5.1.6].∎

We now wish to establish a version of the integral-gradient formula (3.10) for vector-valued functions $u : \mathcal{A} \to \mathcal{U}$, where $\mathcal{U}$ is a finite-dimensional inner product space. First of all we note that the definition of approximate limit, given by relations (2.7), (2.8) and (2.10), can be extended to vector-valued functions u, with the only change that the symbol $|\cdot|$ in (2.8) now denotes the norm in $\mathcal{U}$ instead of the absolute value. The same observation applies to the concepts of a point of approximate continuity of u, a jump point for u, and traces of u. By contrast, we modify the definition (2.11) of directed jump at a jump point x as follows:

$$Ju(x) := (u_\alpha(x) - u_{-\alpha}(x)) \otimes \alpha. \tag{3.23}$$

We collect together some properties of vector-valued functions in the following proposition.

Proposition 3.3. *Let $\mathcal{U}$ an n-dimensional inner product space, $\mathcal{A}$ and $\mathcal{C}$ subsets of $\mathcal{E}$, $x \in \mathcal{A}$, and $u : \mathcal{A} \longrightarrow \mathcal{U}$ be given. For each orthonormal basis $\{e^i \mid i \in \{1, ..., n\}\}$ of $\mathcal{U}$, let u_i, $i \in \{1, ..., n\}$, denote the components of u. Then:*

(i) u has an approximate limit relative to C at x if and only if for all $i \in \{1, ..., n\}, u_i$ has an approximate limit relative to C at x.

(ii) u has an inward trace $u^+(x)$ [outward trace $u^-(x)$] at a point $x \in eby\ C$ if and only if, for all $i \in \{1, ..., n\}$, u_i has an inward trace $u_i^+(x)$ [outward trace $u_i^-(x)$] at x; in this case,

$$u^+(x) = \sum_{i=1}^{n} u_i^+(x) e^i, \tag{3.24}$$

$$u^-(x) = \sum_{i=1}^{n} u_i^-(x) e^i, \tag{3.25}$$

(iii) x is a jump point of u with determining vectors $\{\alpha, -\alpha\}$ if and only if x is a regular point for every component of u, x is a jump point for at least one component of u, and all components of u having x as a jump point have the same pair $\{\alpha, -\alpha\}$ as determining vectors at x.

(iv) If x is a jump point of u, then the directed jump Ju defined in (3.23) and the directed jumps Ju_i of the components of u, as defined in (2.11), satisfy

$$Ju(x) = \sum_{i=1}^{n} e^i \otimes Ju_i(x). \tag{3.26}$$

Proof. Item (i) follows immediately from the definitions of approximate limits for u and u_i and from the inequalities $|u_i| \leq |u| \leq \sum_{i=1}^{n} |u_i|$, and item (ii) follows from the definition of approximate limit relative to C. To verify item (iii), we assume first that x is a jump point for u with determining vectors $\{\alpha, -\alpha\}$. Then the approximate limits limits $u_\alpha(x)$ and $u_{-\alpha}(x)$ both exist but are not equal. It follows from item (i) that, for all $i \in \{1, ..., n\}$, the approximate limits $(u_i)_\alpha(x)$ and $(u_i)_{-\alpha}(x)$ both exist, so that x is a regular point for every component of u. Moreover, because $u_\alpha(x) \neq u_{-\alpha}(x)$, we have $(u_i)_\alpha(x) \neq (u_i)_{-\alpha}(x)$ for at least one $i \in \{1, ..., n\}$. For every $i \in \{1, ..., n\}$ at which $(u_i)_\alpha(x) = (u_i)_{-\alpha}(x)$, u_i is approximately continuous at x. This verifies the "only if" part of item (iii). To prove the "if" part it suffices to observe that the three conditions on the components of u specified in item (iii) imply that there is a single pair of unit vectors $\{\alpha, -\alpha\}$ for which the one-sided approximate limits $(u_i)(x)$ and $(u_i)_{-\alpha}(x)$ exist for all $i \in \{1, ..., n\}$. By item (i), $u_\alpha(x)$ and $u_{-\alpha}(x)$ both exist. However, because there is at least one component of u that has a jump point at x, it follows that $\{\alpha, -\alpha\}$ is the only pair of vectors for which both $u_\alpha(x)$ and $u_{-\alpha}(x)$ exist. Thus, u has a jump point at x. This completes the proof of item (iii).

Lastly, to prove item (iv), we assume that x is a jump point for u and use the definition (3.23) to write

$$\begin{aligned} Ju(x) &= (u_\alpha(x) - u_{-\alpha}(x)) \otimes \alpha \\ &= \sum_{i=1}^{n} ((u_\alpha(x) \cdot e^i)e^i - (u_{-\alpha}(x) \cdot e^i)e^i) \otimes \alpha \\ &= \sum_{i=1}^{n} e^i \otimes ((u_\alpha(x) \cdot e^i) - (u_{-\alpha}(x) \cdot e^i))\,\alpha \\ &= \sum_{i=1}^{n} e^i \otimes ((u_i)_\alpha(x)\alpha - (u_i)_{-\alpha}(x))\,\alpha. \end{aligned} \tag{3.27}$$

The last equality above follows from the relation

$$\lim_{y \to x, y \in C} (u \cdot e^i)(y) = \left(\lim_{y \to x, y \in C} u(y) \right) \cdot e^i, \tag{3.28}$$

which is valid whenever u has an approximate limit with respect to C at x. Relations (3.27) and (2.11) immediately yield (3.26).∎

Item (iii) of Proposition 3.3 provides the following relation between $\Gamma(u)$, the set of jump points of u, and $\{\Gamma(u_i) | i \in \{1, ..., n\}\}$, the sets of jump points of the components of u:

$$\Gamma(u) \subset \bigcup_{i=1}^{n} \Gamma(u_i). \tag{3.29}$$

Again, according to item (iii), in order that a point $x \in \mathcal{E}$ satisfy $x \in \bigcup_{i=1}^{n} \Gamma(u_i) \backslash \Gamma(u)$, it must be either that x is not a regular point for at least one component of u or that there are at least two components of u having jump points at x with different pairs of determining vectors. We prove below that $\bigcup_{i=1}^{n} \Gamma(u_i)$ and $\Gamma(u)$ are area-equivalent.

Proposition 3.4. *Let $\mathcal{A}$ be a piecewise fit region, and let $u : \mathcal{A} \longrightarrow \mathcal{U}$ be a vector-valued function satisfying the following requirement: there is a finite cover $\{\mathcal{A}_j \,|\, j \in \{1, ..., J\}\}$ consisting of fit regions such that, for each $j \in \{1, ..., J\}$, the restriction $u|_{\mathcal{A}_j}$ of u to $\mathcal{A}_j$ has a C^1 extension to $\mathcal{E}$. Moreover, let an orthonormal basis $\{e^i \,|\, i \in \{1, ..., n\}\}$ of $\mathcal{U}$ be given. The jump sets $\Gamma(u)$ and $\Gamma(u_i)$, $i \in \{1, ..., n\}$, then satisfy not only (3.29), but also*

$$\Gamma(u) \overset{A}{\approx} \bigcup_{i=1}^{n} \Gamma(u_i) \tag{3.30}$$

Proof. For each $u_i : \mathcal{A} \longrightarrow \mathbf{R}$, we have from Lemma 3.1 that the function

$$u_i^e(x) := \begin{cases} u_i(x) & \text{for } x \in \mathcal{A} \\ 0 & \text{for } x \in \mathcal{E} \backslash \mathcal{A} \end{cases} \tag{3.31}$$

is of bounded variation. By (BV2), the set of points of $\mathcal{E}$ which are not regular points of u_i^e form a set of area zero. Because the points in $dens\,\mathcal{A}$ that are not regular points for u_i also are not regular points for u_i^e, it follows that the set S_i of points of $dens\,\mathcal{A}$ that are not regular points of u_i has area zero. Consequently, $\bigcup_{i=1}^{n} S_i$ has area zero. The observation following the proof of Proposition 3.3 tells us that (3.30) is satisfied if the set

$$\mathcal{D} := \left\{ x \in dens\mathcal{A} \;\middle|\; \begin{array}{l} \text{there are at least two components} \\ \text{of } u \text{ having jump points at } x \text{ with} \\ \text{different pairs of determining vectors} \end{array} \right\} \tag{3.32}$$

has area zero. In order to verify that $\mathcal{D}$ has zero area, we consider the partition $\{C_j \mid j \in \{1, ..., J\}\}$ of $\mathcal{A}$ obtained as in (3.2) with the scalar-valued function v there replaced by the given vector-valued function u, and we again use the lemma [5, Sect. 2.5] cited prior to (3.13) to write

$$dens\,\mathcal{A} \overset{\mathcal{A}}{\approx} \bigcup_{j=1}^{J} dens\,C_j \cup \bigcup_{\substack{k,\ell=1 \\ k\neq\ell}}^{J} \Gamma_{k\ell}, \tag{3.33}$$

with $\Gamma_{k\ell}$ given by (3.14). Moreover, according to the same lemma, at all points x_0 in $\Gamma_{k\ell}$ the outward normals $\nu_k(x_0)$ and $\nu_\ell(x_0)$ to C_k and C_ℓ satisfy

$$\nu_k(x_0) + \nu_\ell(x_0) = 0. \tag{3.34}$$

We observe that, at each density point of C_j, u is approximately continuous, because $u|_{intC_j}$ extends to $\mathcal{E}$ as a C^1 function. Therefore, no component of u can have a jump point in $\bigcup_{j=1}^{J} densC_j$. Consequently, the intersection of $\mathcal{D}$ and $\bigcup_{j=1}^{J} densC_j$ is empty. To complete the proof, we only need to prove that the intersection of $\mathcal{D}$ and $\bigcup_{\substack{k,\ell=1\\k\neq\ell}}^{J} \Gamma_{k\ell}$ is also empty, i.e., that

$$\bigcup_{\substack{k,\ell=1\\k\neq\ell}}^{J} (\Gamma_{k\ell} \cap \mathcal{D}) = \emptyset. \tag{3.35}$$

Moreover, the fact that each $\Gamma_{k\ell}$ is a subset of the reduced boundaries of both C_k and C_ℓ and the fact that $u|_{C_k}$ and $u|_{C_\ell}$ each extend to $\mathcal{E}$ as a C^1 function imply that each component u_i of u has approximate limits with respect to both C_k and C_ℓ at every point in $\Gamma_{k\ell}$. Therefore, (3.34) permits us to conclude that every point x_0 in $\Gamma_{k\ell}$ is a regular point for each component u_i, and $\{\nu_k(x_0), \nu_\ell(x_0)\} = \{\nu_k(x_0), -\nu_k(x_0)\}$ is a pair of determining vectors for each u_i. Consequently, all components of u share a pair of determining vectors at x_0, and this implies that $\mathcal{D}$ and $\Gamma_{k\ell}$ are disjoint. ∎

Remark 3.5. *By using the fact that, if $u_i \in BV$, then the set $\Gamma(u_i)$ is a set of the class Γ as defined in [6, Sect. 5.1.5], one can prove that the relation (3.30) holds for all mappings $u : \mathcal{E} \to \mathcal{U}$ whose components are in BV* [4].

We are now in a position to extend Theorem 3.2 to vector-valued functions.

Theorem 3.6. *Let $\mathcal{A}$ be a piecewise fit region, and let $u : \mathcal{A} \to \mathcal{U}$ be a vector-valued function satisfying the smoothness condition in Proposition 3.4. Then*

$$\int_{\mathcal{A}} \nabla u(x) dV_x = -\int_{\Gamma(u)} Ju(x) dA_x + \int_{\mathrm{rb}_{\mathrm{r}}\mathcal{A}} u^+(x) \otimes \nu(x) dA_x. \tag{3.36}$$

Proof. We choose an orthonormal basis $\{e^i \mid i \in \{1, ..., n\}\}$ of $\mathcal{U}$ and note that each component $u_i = u \cdot e^i$ of u satisfies the hypothesis of Theorem 3.2. Writing (3.10) for each u_i, taking the tensor product with e^i, and summing the resulting equations, we obtain:

$$\begin{aligned}\sum_{i=1}^{n} \int_{\mathcal{A}} e^i \otimes \nabla u_i(x) dV_x &= -\sum_{i=1}^{n} \int_{\Gamma(u_i)} e^i \otimes J u_i(x) dA_x \\ &\quad + \sum_{i=1}^{n} \int_{rby\mathcal{A}} e^i \otimes (u_i^+(x)\nu(x)) dA_x.\end{aligned} \tag{3.37}$$

By (3.30), we have

$$\Gamma(u_i) \subset \bigcup_{j=1}^{n} \Gamma(u_j) \overset{A}{\approx} \Gamma(u), \tag{3.38}$$

and we note that A-almost every point of $\Gamma(u) \backslash \Gamma(u_i)$ is a point of approximate continuity for u_i. Therefore, $J u_i(x) = 0$ for A-almost every x in $\Gamma(u) \backslash \Gamma(u_i)$. Thus, $\Gamma(u_i)$ can be replaced by $\Gamma(u)$ in (3.37), and (3.36) follows from (3.26) and the relations

$$\nabla u(x) = \sum_{i=1}^{n} e^i \otimes \nabla u_i(x), \tag{3.39}$$

$$u(x) \otimes \nu(x) = \sum_{i=1}^{n} e^i \otimes u_i(x) \nu(x). \blacksquare \tag{3.40}$$

4. Integral-gradient formulae for structured deformations

Integral-gradient formulae appropriate for the classes of deformations introduced in [1] can be deduced easily from Theorem 3.6. Here we consider the classes of deformations $Sid, LimSid, Std$ defined in [1], and for each of them we write the appropriate integral-gradient formula.

According to Definition 3.2 in [1], a *simple deformation* from a piecewise fit region $\mathcal{A}$ is a pair (κ, f), where κ is a subset of $\mathcal{A}$ of volume zero such that $\mathcal{A}\backslash\kappa$ is a piecewise fit region, and f, the *transplacement* associated with the given simple deformation, is a C^1 mapping of $\mathcal{A}\backslash\kappa$ into $\mathcal{E}$ which, among others, has the following property: there is at least one finite cover of $\mathcal{A}\backslash\kappa$ by fit regions $\mathcal{A}_j$ such that the restriction of f to each $\mathcal{A}_j$ has a C^1 extension to $\mathcal{E}$.

For the point-valued mapping $f : \mathcal{A}\backslash\kappa \rightarrow \mathcal{E}$, one can define approximate limits, traces and the directed jump as done for vector-valued functions. If we choose a fixed arbitrary point o of $\mathcal{E}$ and define $u : \mathcal{A}\backslash\kappa \rightarrow \mathcal{V}$ by

$$u(x) := f(x) - o, \tag{4.1}$$

we can relate the gradient, the inward trace and the directed jump of f with those of u by

$$\nabla f(x) = \nabla u(x), \quad f^+(x) = o + u^+(x), \quad Jf(x) = Ju(x). \tag{4.2}$$

For a simple deformation (κ, f), u satisfies the assumptions of Theorem 3.6, and therefore Eq. (3.36) holds. Note that the fact that κ has volume zero implies $\mathcal{A} \approx \mathcal{A}\backslash\kappa$ and $rby\mathcal{A} = rby(\mathcal{A}\backslash\kappa)$, so that Eq. (3.36) can be written with $\mathcal{A}$ instead of $\mathcal{A}\backslash\kappa$. This equation, together with relations (4.2), yields

$$\int_{\mathcal{A}} \nabla f(x) dV_x = -\int_{\Gamma(f)} Jf(x) dA_x + \int_{rby\mathcal{A}} (f^+(x) - o) \otimes \nu(x) dA_x. \tag{4.3}$$

This is the integral-gradient formula for simple deformations. Note that the last integral in the formula is independent of the choice of the point o.

Let $\mathcal{A}$ be a piecewise fit region of $\mathcal{E}$. According to [1, Def. 4.1], a *limit of simple deformations from* $\mathcal{A}$ is a triple (κ, g, G) with $\kappa \subset \mathcal{A}$, $g \in L^\infty(\mathcal{A}, \mathcal{E})$, $G \in L^\infty(\mathcal{A}, Lin\mathcal{V})$, for which there is a sequence $n \mapsto (\kappa_n, f_n)$ of simple deformations from $\mathcal{A}$, called a *determining sequence* for (κ, g, G), such that:

$$\kappa = \liminf_{n\to\infty} \kappa_n = \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} \kappa_n, \tag{4.4}$$

$$\lim_{n\to\infty} \|g - f_n\|_{L^\infty(\mathcal{A},\mathcal{V})} = 0, \tag{4.5}$$

$$\lim_{n\to\infty} \|G - \nabla f_n\|_{L^\infty(\mathcal{A},Lin\mathcal{V})} = 0. \tag{4.6}$$

It can be proved [1, Theorem 4.10] that κ has volume zero and that g and G have representatives g_0, G_0 which are continuous in $\mathcal{A}\backslash\kappa$. Moreover ([1, Lemma 4.11]), $n \mapsto f_n$ and $n \mapsto \nabla f_n$ converge to g_0 and G_0 uniformly, in the sense that for every $\epsilon > 0$ there is an $n_\epsilon \in \mathrm{N}$ such that, for all $n > n_\epsilon$,

$$\sup_{x\in\mathcal{A}\backslash(\kappa\cup\kappa_n)} |f_n(x) - g_0(x)| < \epsilon, \tag{4.7}$$

$$\sup_{x\in\mathcal{A}\backslash(\kappa\cup\kappa_n)} |\nabla f_n(x) - G_0(x)| < \epsilon. \tag{4.8}$$

We now prove an analogous property of uniform convergence for the inward traces of the functions f_n on the essential boundary of $\mathcal{A}$.

The proof of Theorem 3.6 shows that each of the functions $u_n(x) := f_n(x) - o$ has an inward trace u_n^+ defined A-almost everywhere and summable on $eby\,(\mathcal{A}\backslash\kappa_n) = eby\ \mathcal{A}$. By (4.2), the same holds for f_n^+, the inward trace of f_n. We denote by $\mathcal{F}_n$ the domain of f_n^+, i.e. the set of all points $x \in eby\,\mathcal{A}$ at which the trace $f_n^+(x)$ is defined, and we observe that $\mathcal{F}_n \overset{A}{\approx} eby\,\mathcal{A}$ by (BV3). We also denote by $\mathcal{F}_0$ the set

$$\mathcal{F}_0 := \bigcap_{n=1}^{\infty} \mathcal{F}_n, \tag{4.9}$$

which also is a subset of $eby\,\mathcal{A}$ with full area measure.

Theorem 4.1. *Let $\mathcal{A}$ be a piecewise fit region and let $n \mapsto (\kappa_n, f_n)$ be a sequence of simple deformations from $\mathcal{A}$. Assume that the sequence $n \mapsto f_n$ has a uniform limit $g_0 : \mathcal{A}\backslash\kappa \to \mathcal{E}$ in the sense of relation (4.7), with κ given by (4.4). Then the sequence $n \mapsto f_n^+$ of the inward traces of the functions f_n on $eby\,\mathcal{A}$ has a uniform limit defined over the set $\mathcal{F}_0$ defined in (4.9). Moreover, this limit is summable in $eby\,\mathcal{A}$ and is the inward trace of g_0 on $eby\,\mathcal{A}$.*

Proof. Let $\epsilon > 0, n \in \mathbf{N}, x \in \mathcal{F}_n$ be given. For every $\delta > 0$, define

$$\mathcal{D}_\epsilon(n, x, \delta) := \left\{ y \in \mathcal{B}(x, \delta) \cap \mathcal{A}\backslash(\kappa_n \cup \kappa) \,\middle|\, |f_n^+(x) - f_n(y)| < \epsilon \right\}. \tag{4.10}$$

It is clear from the definitions of approximate limit and of trace given in Section 2 that

$$\lim_{\delta \to 0} \frac{V(\mathcal{D}_\epsilon(n, x, \delta))}{V(\mathcal{B}(x, \delta) \cap \mathcal{A})} = 1. \tag{4.11}$$

Now let $x \in \mathcal{F}_0$ and $m, n \in \mathbf{N}$ be such that $m, n > n_\epsilon$, where $n_\epsilon \in \mathbf{N}$ is such that the inequality (4.7) holds for all $n > n_\epsilon$. We also choose δ such that the set $\mathcal{D}_\epsilon(m, x, \delta) \cap \mathcal{D}_\epsilon(n, x, \delta)$ has positive volume; in view of (4.11), this can be done by choosing δ such that

$$\min\{V(\mathcal{D}_\epsilon(m,x,\delta)), V(\mathcal{D}_\epsilon(n,x,\delta))\} > \frac{1}{2}V(\mathcal{A}\cap\mathcal{B}(x,\delta)). \tag{4.12}$$

We then choose $y \in \mathcal{D}_\epsilon(m,x,\delta)\cap\mathcal{D}_\epsilon(n,x,\delta)$ and consider the inequality

$$\begin{aligned}|f_m^+(x)-f_n^+(x)| \le |f_m^+(x)-f_m(y)| \;+\; |f_m(y)-g_0(y)|\\ +|g_0(y)-f_n(y)| \;+\; |f_n(y)-f_n^+(x)|\end{aligned} \tag{4.13}$$

which, by (4.10) and the property (4.7) of uniform convergence of $n \mapsto f_n(y)$, allows us to write

$$|f_m^+(x)-f_n^+(x)| < 4\epsilon, \quad \forall\, x\in\mathcal{F}_0. \tag{4.14}$$

This implies that for every $x \in \mathcal{F}_0$ the sequence $n \mapsto f_n^+(x)$ is a Cauchy sequence and, therefore, converges to a point which we call $f_0(x)$. In this manner, we have constructed a mapping $f_0 : \mathcal{F}_0 \to \mathcal{E}$ such that the sequence $n \mapsto f_n^+$ of the inward traces converges pointwise to f_0. Moreover, the inequality (4.14) tells us that the convergence is uniform on $\mathcal{F}_o$, because n_ϵ does not depend upon the point x in $\mathcal{F}_o$. Finally, f_0 is summable because it is the L^∞-limit of a sequence of summable functions. It remains to prove that f_0 is the trace on *eby* $\mathcal{A}$ of the limit g_0 of the sequence $n \mapsto f_n$. First of all we note that the uniform convergence of $n \mapsto f_n^+$ to f_0 on $\mathcal{F}_o$ implies that for every $\epsilon > 0$ there is an $n'_\epsilon \in \mathrm{N}$ such that $n > n'_\epsilon$ implies

$$\left|f_n^+(x)-f_0(x)\right| < \epsilon \quad \forall x\in\mathcal{F}_0. \tag{4.15}$$

For a fixed $\epsilon > 0$, we then choose $x \in \mathcal{F}_0$ and $n > \max\{n_\epsilon, n'_\epsilon\}$. We also let $\delta > 0$ be given and choose $y \in \mathcal{D}_\epsilon(n, x, \delta)$, to get the inequality

$$|f_0(x) - g_0(y)| \leq \left|f_0(x) - f_n^+(x)\right| + \left|f_n^+(x) - f_n(y)\right| + |f_n(y) - g_0(y)|. \tag{4.16}$$

In the right-hand side, we have $|f_0(x) - f_n^+(x)| < \epsilon$ by (4.15), $|f_n(y) - g_0(y)| < \epsilon$ by (4.7), and $|f_n^+(x) - f_n(y)| < \epsilon$ by (4.10). We then conclude that

$$|f_0(x) - g_0(y)| < 3\epsilon \qquad \forall\, x \in \mathcal{F}_0,\ \forall\, y \in \mathcal{D}_\epsilon(n, x, \delta), \tag{4.17}$$

and we deduce from (4.11) that the set $\mathcal{D}_\epsilon(n, x, \delta)$ is sufficiently large to ensure that $f_0(x)$ is the inward trace of g_0 at x.∎

It is now easy to obtain an integral-gradient formula for limits of simple deformations. Indeed, if $n \mapsto (\kappa_n, f_n)$ is a determining sequence for the limit of simple deformations (κ, g, G), writing the integral-gradient formula (4.3) for each (κ_n, f_n)

$$\int_{\mathcal{A}} \nabla f_n(x) dV_x = -\int_{\Gamma(f_n)} J f_n(x) dA_x + \int_{rby\mathcal{A}} (f_n^+(x) - o) \otimes \nu(x) dA_x\,, \tag{4.18}$$

in the limit as $n \to \infty$ we obtain from Eqs. (4.6) and (4.15)

$$\int_{\mathcal{A}} G(x) dV_x = -\lim_{n\to\infty} \int_{\Gamma(f_n)} J f_n(x) dA_x + \int_{rby\mathcal{A}} (g_0^+(x) - o) \otimes \nu(x) dA_x\,. \tag{4.19}$$

Note that, whereas the limits of the first and third integral in (4.18) take an explicit expression in terms of the limiting fields G and g_0^+, the same does not occur for the second integral. Nevertheless, Eq. (4.19) tells us that the limit of the second integral exists and is determined by the two remaining integrals in (4.19).

In [1, Sect. 5], a ***structured deformation from a piecewise fit region*** $\mathcal{A}$ has been defined to be a triple (κ, g, G), in which (κ, g) is a simple deformation from $\mathcal{A}$ and G is a tensor field defined on $\mathcal{A}\backslash\kappa$ and subject to appropriate regularity assumptions. The Approximation Theorem, also proved in [1, Sect. 5], shows that every structured deformation is a limit of simple deformations. Consequently, for a structured deformation both formula (4.3) for simple deformations and formula (4.19) for limits of simple deformations hold. Subtracting (4.19) from (4.3), with f replaced there by g, leads to the equation

$$\int_{\mathcal{A}} (\nabla g(x) - G(x))dV_x = -\int_{\Gamma(g)} Jg(x)dA_x + \lim_{n\to\infty} \int_{\Gamma(f_n)} Jf_n(x)dA_x. \qquad (4.20)$$

5. Applications to continua undergoing fracture

In this section we review our earlier interpretation [1] of simple deformations, limits of simple deformations, and structured deformations as mathematical objects that describe geometrical changes in a continuous body undergoing macroscopic fracture (macrofracture) and microscopic fracture (microfracture). The integral-gradient formulae of Section 4 then permit us to identify measures of total deformation due to fracture, total deformation due to microfracture, and total deformation due to macrofracture, as well as a volumetric density of deformation due to microfracture.

We consider in a three-dimensional Euclidean space a continuous body that occupies a given piecewise fit region $\mathcal{A}$. The points of $(int\ clo\mathcal{A})\backslash\mathcal{A}$ are viewed as pre-existing crack sites or unopened cracks. Each simple deformation (κ, f)

from $\mathcal{A}$ is viewed as introducing new cracks in the body at the points of κ and then moving each material point x in $\mathcal{A}\backslash\kappa$ to the point $f(x)$ in $f(\mathcal{A}\backslash\kappa)$. The set of jump points $\Gamma(f)$ is included in $\kappa \cup ((int\ clo\mathcal{A})\backslash\mathcal{A})$, the points on the new and on the pre-existing crack sites. At a point x in $\Gamma(f)$, the determining vectors $\{\nu(x), -\nu(x)\}$ distinguish the two sides of the crack, and $f_{\nu(x)}(x) - f_{-\nu(x)}(x)$ gives the displacement of points near x on the $+\nu(x)$-side of the crack, relative to points near x on the $-\nu(x)$-side of the crack. Of course, there is no reason to choose as the reference for measuring displacements one side (here $-\nu(x)$) over the other. The tensor $Jf(x) = (f_{\nu(x)}(x) - f_{-\nu(x)}(x)) \otimes \nu(x)$ keeps track of the relative displacement without the necessity of making a choice of one side of the crack site over the other, and we call $Jf(x)$ the *tensor of deformation due to macrofracture at the point x in $\Gamma(f)$*. The area integral $\int\limits_{\Gamma(f)} Jf(x)dA_x$ then represents a net or *total deformation in $\mathcal{A}$ due to macrofracture* for the simple deformation (κ, f) from $\mathcal{A}$. At a point x in $\mathcal{A}\backslash\kappa$, f is differentiable, no fracture occurs, and we call $\nabla f(x)$ the *macroscopic deformation at x*. Similarly, we call $\int\limits_{\mathcal{A}} \nabla f(x)dV_x$ the *total macroscopic deformation in $\mathcal{A}$*. The integral-gradient formula for simple deformations (4.3) can be interpreted as follows: for a simple deformation, the total macroscopic deformation of $\mathcal{A}$ plus the total deformation in $\mathcal{A}$ due to macrofracture is determined by the displacements of the boundary of $\mathcal{A}$.

Next we consider a limit of simple deformations (κ, g, G) along with a determining sequence of $n \mapsto (\kappa_n, f_n)$ of simple deformations. Because $G = \lim\limits_{n\to\infty} \nabla f_n$ and ∇f_n measures deformation away from sites of fracture, we have called G the *tensor* *of deformation without fracture*, [1, Sect. 6], and we here call $\int\limits_{\mathcal{A}} G(x)dV_x$ the *total deformation in $\mathcal{A}$ without fracture*. Similarly, we call the limit $\lim\limits_{n\to\infty} \int\limits_{\Gamma(f_n)} Jf_n(x)dA_x$ the *total deformation in $\mathcal{A}$ due to fracture*. The integral-gradient formula (4.19) for limits of simple deformations then admits the interpretation: for a limit of simple deformations the total deformation in $\mathcal{A}$ without fracture plus the total deformation in $\mathcal{A}$ due to fracture is determined by the displacements of the boundary of $\mathcal{A}$.

For a structured deformation (κ, g, G), not only is (κ, g) a simple deformation but also (κ, g, G) can be regarded as a limit of simple deformations. The integral-gradient formula (4.20) results from subtracting the two corresponding versions, one for simple deformations and one for limits of simple deformations. In particular, the displacements on the boundary of $\mathcal{A}$ do not appear in (4.20). The right-hand side of (4.20) is the total deformation in $\mathcal{A}$ due to fracture minus the total deformation in $\mathcal{A}$ due to macrofracture; therefore, we interpret the difference

$$\lim_{n\to\infty} \int\limits_{\Gamma(f_n)} Jf_n(x)dA_x - \int\limits_{\Gamma(g)} Jg(x)dA_x$$

as the *total deformation in $\mathcal{A}$ due to microfracture*. Relation (4.20) now yields the result: *the total deformation in $\mathcal{A}$ due to microfracture has a volume density which is given by the tensor $M := \nabla g - G$.*

In [1], we have called M the *Burgers microfracture tensor*, and we showed there that M is a density of deformation due to microfracture along lines. The present analysis extends the interpretation of M from a one-dimensional density to a three-dimensional density.

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