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# Dynamics as a mechanism preventing the formation of finer and finer microstructure

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**Abstract**

We study the dynamics of pattern formation in the one-dimensional partial differential equation

$$u_{tt} - (W'(u_x))_x - u_{xxt} + u = 0 \qquad (u = u(x,t),\ x \in (0,1),\ t > 0)$$

proposed recently by Ball, Holmes, James, Pego & Swart [BHJPS] as a mathematical "cartoon" for the dynamic formation of microstructures observed in various crystalline solids. Here $W$ is a double-well potential like $\frac{1}{4}((u_x)^2 - 1)^2$. What makes this equation interesting and unusual is that it possesses as a Lyapunov function a free energy (consisting of kinetic energy plus a nonconvex "elastic" energy, but no interfacial energy contribution) which does not attain a minimum but favours the formation of finer and finer phase mixtures:

$$E[u, u_t] = \int_0^1 \left( \frac{u_t^2}{2} + W(u_x) + \frac{u^2}{2} \right) dx.$$

Our analysis of the dynamics confirms the following surprising and striking difference between statics and dynamics, conjectured in [BHJPS] on the basis of numerical simulations of Swart & Holmes [SH]:

- While minimizing the above energy predicts infinitely fine patterns (mathematically: *weak but not strong convergence* of all minimizing sequences $(u_n, v_n)$ of $E[u,v]$ in the Sobolev space $W^{1,p}(0,1) \times L^2(0,1)$), solutions to the evolution equation of Ball et al. typically develop patterns of small but finite length scale (mathematically: *strong convergence* in $W^{1,p}(0,1) \times L^2(0,1)$ of all solutions $(u(t), u_t(t))$ with low initial energy as time $t \to \infty$).

Moreover, in order to understand the finer details of why the dynamics fails to mimic the behaviour of minimizing sequences and how solutions select their limiting pattern, we present a detailed analysis of the evolution of a restricted class of initial data – those where the strain field $u_x$ has a transition layer structure; our analysis includes proofs that

- at low energy, the number of phases is in fact exactly preserved, that is, there is no nucleation or coarsening
- transition layers lock in and steepen exponentially fast, converging to discontinuous stationary sharp interfaces as time $t \to \infty$
- the limiting patterns – while not minimizing energy globally – are 'relative minimizers' in the weak sense of the calculus of variations, that is, minimizers among all patterns which share the same strain interface positions.

# 1 Introduction

In recent years an expanding and increasingly sophisticated literature has been devoted to the analysis of minimizing sequences and minimizing Young measures of nonconvex variational integrals and their role in modelling microstructures in crystals (see e.g. [BCJ], [BFJK], [Bh], [BJ1], [BJ2], [CK], [DM], [JK], [KM], [KP], [Ma], [Sv1], [Sv2], [Zh]).

By contrast, the *dynamic processes* by which such microstructural patterns may be created or evolve have received little attention, and many of the most basic associated mathematical issues have not been addressed. The goal of this paper is to study some of these fundamental issues in the perhaps simplest possible mathematical setting, that of a one-dimensional model equation proposed for this purpose by Ball, Holmes, James, Pego & Swart [BHJPS]. The key feature of this equation is that it is an evolution equation with a nonconvex underlying free energy (decreasing with time along solutions) which energetically favours the formation of finer and finer phase mixtures.

Our analysis confirms several remarkable features of the evolution equation of Ball et al., conjectured in [BHJPS] and observed numerically in [SH] but until now largely unexplained, which may be not only of mathematical but also of some physical interest ([BHJPS, Sec. 7]; [BJ2, Sec. 10]). Most prominently, while minimizing the free energy predicts infinitely fine patterns (mathematically: *weak but not strong convergence* of minimizing sequences in the appropriate Sobolev spaces), all solutions of the evolution equation with low initial energy fail to nucleate more and more phases (mathematically: *strong convergence* as time $t \to \infty$, see Thm 3.1), typically tending instead to 'relative minimizers' with finitely many interfaces (see Thm 4.1).

In the remainder of this Introduction, we explore in more detail the motivation and agenda of this work, coming from the area of

(1) nonconvex variational integrals and their role in modelling microstructures in solids,

and comment on some of the mathematical intricacies of associated dynamic models and on our methods of analysing the mechanisms of pattern formation from the viewpoint of

(2) the geometric theory of dissipative dynamical systems as presented e.g. in [CFNT], [Ha], [He1], [Te].

The latter area has provided a successful framework in which to study issues like those relevant here (nucleation, layer dynamics, approach to equilibrium) in PDE's whose underlying equilibrium problems are elliptic (see e.g. the articles [ABF], [BF], [BX1], [BX2], [CP], [FH] on the one-dimensional Allen-Cahn and Cahn-Hilliard equations), but due to its focus on evolution problems exhibiting 'regularity' it has until now remained more or less disjoint from (1).

## 1.1 Connection of our work with the study of nonconvex variational integrals and microstructures in solids.

Phase transformations in elastic crystals, induced by imposed stresses, changes in temperature or applied electric or magnetic fields, often lead to the formation of fine mixtures of spatial domains with different (or differently oriented) atomic lattice structure. In recent years the presence of such 'microstructures' and various of its features have been successfully explained by the minimization of free energy in continuum models (see Ball & James [BJ1] and after them e.g. [BJ2], [BCJ], [JK], [KM], [Bh]). To summarize briefly the main line of thought underlying these studies: Elastic energy functionals for crystals which account for crystallographic symmetry are necessarily nonelliptic[1], and, in certain cases, do not attain a minimum in the appropriate function spaces; the finer and finer oscillations of minimizing sequences – or the associated Young measures – can be used to model (various features of) the experimentally observed microstructures. We record here the by now well-known mathematical structure of the relevant energy functionals:

$$I[y] = \int_\Omega \Phi(Dy(x))\, dx \quad (\Omega \subset \mathbb{R}^n,\ y : \Omega \to \mathbb{R}^n) \tag{1.1}$$

[1] more precisely: neither poly- nor quasi- nor rank-1-convex

where $\Omega$ is an open and bounded reference configuration, the deformations $y$ satisfy appropriate (say linear) boundary conditions

$$y(x)|_{\partial\Omega} = Fx \quad (F \in M^{n\times n}), \tag{1.2}$$

and the integrand $\Phi$ is a nonnegative function on $M^{n\times n}$ which is zero exactly on a set of several potential wells

$$\Phi^{-1}(0) = \bigcup_{i=1}^{N} SO(n)A_i, \tag{1.3}$$

with the $A_i$ playing the role of 'material parameters' reflecting the underlying atomic lattice structure.

In contrast with an expanding knowledge about variational integrals like (1.1) and their minimizing microstructures (for a state-of-the-art and a recent literature survey see [BFJK]), the *dynamic processes* leading to the formation of such fine phase mixtures have as yet largely withstood a successful mathematical description. This paper, which continues the work begun by [BHJPS] and [SH], presents some progress in the direction of such a mathematical description. In contrast with much of the mathematical literature on dynamic models for nonconvex elastic materials (which has focused on studying questions of existence, uniqueness and regularity – of course an interesting area in its own right [HS], [NS], [Ry]) the present paper pursues a different agenda, aimed at understanding the mechanisms of pattern formation in such systems and initiated by John Ball in his survey article [Ba] (and taken up in [BHJPS], [SH], [Fr3]).

For a discussion of why an understanding of these mechanisms (in the form of answers to questions like those below for appropriate model equations) may be of physical interest in the study of phase transformations in crystals, we refer to [Ba], [BJ2, Sec. 10]; we will however discuss some of these issues from a mathematical point of view (see subsection 1.2 of this Introduction). Even though the present paper only studies these issues in a one-dimensional situation, we state them here in the context of any evolution equation with an underlying nonelliptic variational integral (1.1), a prototype example being equation (1.4) below.

(A) For a given initial state, does the dynamics select a unique limit state as time $t \to \infty$?[1]

(B) If $I[y]$ does not attain its infimum, can solutions mimic the behaviour of minimizing sequences and form finer and finer microstructure, or does dynamics act as a mechanism preventing the nucleation of more and more phases?[2]

(C) Can we justify a precise version of a variational principle 'Minimize $I[y]$' from dynamics? This would involve determining (i) the exact function class in which typical limit states lie, and (ii) the exact class of variations with respect to which such limit states are energetically stable.[3]

(D) How do transition layers or interfaces between different preferred phases $Dy \approx A$, $Dy \approx B$ ($A, B \in \Phi^{-1}(0)$) deform and propagate?

(E) At what time scales do solutions relax to equilibrium – is the relaxation process fast and the dynamics essentially 'quasistatic' or do solutions get stuck in metastable states with a large lifetime before settling into their final pattern?

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[1] Energy miniminimization typically predicts a large collection of possible minimizers or minimizing 'microstructures'.

[2] Or, more generally: does the dynamics favour particular geometries or length scales?

[3] The emphasis here is on 'precise'. As for (i): For nonconvex variational problems the choice of function space in which to seek minimizers is a delicate issue and may not be 'physically obvious'. Just how dramatically the class of minimizers may depend on this choice was demonstrated in recent work of Müller, Šverák and Dolzmann [MS], [DM] on the two-dimensional two-well-problem (n=N=2 in (1.1), (1.3)): While in [MS] a large class of minimizers $y \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ with complicated geometry is constructed (consisting of infinite rank laminates), in [DM] it is shown (taking up earlier ideas of Fonseca [Fo]) that under the seemingly mild additional assumption $Dy \in BV(\Omega; M^{2\times 2})$ each minimizer must locally be a simple laminate. – Regarding (ii): For example, are the limit states 'absolute' or 'relative' minimizers?

(F) Do the limiting patterns selected by the dynamics depend in a hysteretic manner on the initial states? If so, can this be used to explain the hysteretic behaviour of macroscopic quantities in response to changing control variables (like: stress versus strain) as universally observed in specimen?

These are hard questions. Answering them seems to lie beyond the scope of existing mathematical methods – even in the case of the mathematically simplest models like the equations of viscoelasticity of Kelvin-Voigt type in three dimensions [Ry], [SH]

$$\rho y_{tt} - Div\Big(\frac{\partial \Phi}{\partial A}(Dy) + \beta Dy_t\Big) = 0 \quad (\Omega \subset \mathbb{R}^3,\ y : \Omega \times [0,\infty) \to \mathbb{R}^3). \tag{1.4}$$

Here $\rho$, $\beta$ are positive constants (the density in the reference configuration and the viscosity). These equations, while ignoring e.g. temperature fluctuations and the requirement of dynamic frame indifference, are perhaps the mathematically simplest evolution equations associated with the energy functional (1.1) which are thermodynamically and mechanically consistent, in the sense that (i) the total free energy $E[y, y_t] = \int_\Omega \frac{\rho}{2} y_t^2 + I[y]$ decreases along solutions and (ii) the equations are derived from the law of balance of linear momentum $\rho y_{tt} = Div\, T$ by a constitutive assumption for the stress tensor $T$, $T = \frac{\partial \Phi}{\partial A}(Dy) + \beta Dy_t$ (which we rewrite here for later purposes as

$$T = \frac{\delta I[y]}{\delta(Dy)} + \beta Dy_t \tag{1.5}$$

where $\delta/\delta(Dy)$ denotes the functional derivative with respect to the $L^2(\Omega; M^{n\times n})$ inner product). While the initial-value-problem for (1.4), (1.2) is well-posed under reasonable hypotheses on $\Phi$ consistent with the potential well structure (1.3) [Ry], for these equations it is not even known whether solutions converge in any sense to a limit state as time $t \to \infty$, not even if the minimum of $I[y]$ were attained. This situation led Ball et al. [BHJPS] to introduce a simplified version of (1.4) in which the space dimension is reduced to one but the key qualitative feature of finer and finer phase mixtures being energetically favourable is captured. We briefly recall the derivation of this model in [BHJPS]: prescribing the boundary condition (1.2) in weakened form via an energy penalty, the variational integral (1.1) becomes

$$I[y] = \int_\Omega \Phi(Dy(x))\, dx + \frac{1}{\epsilon}\int_{\partial\Omega} |y(x) - F x|^2 \quad (\epsilon > 0 \text{ small}),$$

and by considering boundary data $F = id_{\mathbb{R}^n}$, a domain $\Omega = [0,1] \times \tilde\Omega$ ($\tilde\Omega \subset \mathbb{R}^{n-1}$), and deformations independent of $x_2, ..., x_n$ ($y(x_1, ..., x_n) = (x_1 + u(x_1), x_2, ..., x_n)$) which satisfy the boundary condition

$$u(0) = u(1) = 0, \tag{1.6}$$

one calculates

$$I[y] = \int_0^1 \Big(W(u_{x_1}(x_1)) + \frac{\alpha}{2} u^2(x_1)\Big)\, dx_1 =: J[u] \tag{1.7}$$

with $\alpha = \frac{2|\partial\tilde\Omega|}{\epsilon}$ and $W(u_{x_1}) = |\tilde\Omega|\, \Phi(Dy(x))$. Correspondingly, after replacing the constitutive assumption (1.5) by its analogon $T = \frac{\delta J[u]}{\delta(u_x)} + \beta u_{xt}$, (1.4) becomes

$$\rho u_{tt} - \Big(W'(u_{x_1}) + \beta u_{x_1 t}\Big)_{x_1} + \alpha u = 0. \tag{1.8}$$

(1.8), (1.7), (1.6) are the evolution equation, elastic energy and boundary condition of Ball et al. Note that the above reduction procedure has preserved thermodynamic consistency as the one-dimensional free energy $E[u, u_t] = \int_0^1 \frac{\rho}{2} u_t^2 + J[u]$ decreases along solutions. For the details of why the one-dimensional stored-energy function $W$ becomes a double-well-potential like $\frac{1}{4}(u_{x_1}^2 - 1)^2$ if $x_1$ is chosen in a particular crystallographic direction, e.g. orthogonal to a martensitic twin plane, we refer to [BHJPS, Sec. 7]. Without the boundary term $\frac{\alpha}{2} u^2$ in (1.7), the evolution equation would reduce to the equation of one-dimensional viscoelasticity (studied notably in [Da], [An], [AB], [Pe]) but the feature of finer and finer phase mixtures being energetically favourable would be lost. However, thanks to this boundary term the minimum of $J[u]$ is not attained, minimizing sequences striving simultaneously to achieve $u_{x_1} \in \{\pm 1\}$ and $u \equiv 0$ which is impossible. The functional $J[u]$ is perhaps the simplest example exhibiting this behaviour, and is well-known in the mathematical literature (see Young who studies an almost identical example in his monograph [Yo]).

[BHJPS] showed that the initial-value-problem for (1.8), (1.6) is well-posed in the natural phase space $(u, u_t) \in W_0^{1,\infty}(0,1) \times L^2(0,1)$, and proved the interesting result that no solution minimizes the total free energy $E[u, u_t]$ globally as time $t \to \infty$. However, the issues listed above remained largely open. In particular, while the above result suggested that solutions may fail to form finer and finer microstructure and while the article [BHJPS] has since its appearance inspired several related investigations[1], the central issue (B) remained unanswered.

Our analysis is limited to the dynamics at low energy and in particular excludes the interesting issue of nucleation of phases from the unstable equilibrium state $u_x \equiv 0$; however, at low energy (i.e. $u_x \approx \pm 1$ in most of the interval $(0,1)$) we obtain a fairly complete picture including nucleation (or rather: the failure thereof) and an answer to (B), layer dynamics, and the important question of variational stability of limit states. In detail: regarding questions (A) and (B) see Thm 3.1, and for contributions to (C)-(F) see Thm 4.1.

### 1.2 Mathematical peculiarities of the evolution equation of Ball et al. and related models, and some comments on our methods of analysis.

For evolution equations with nonelliptic free energy like the model problem (1.8), standard techniques for analyzing the dynamics seem difficult to apply. Here we comment on two aspects, the behaviour as time $t \to \infty$ and the evolution of transition layers.

**1.2.1 Behaviour as $t \to \infty$.** From the viewpoint of the geometric theory of dynamical systems prominent in this journal, (1.8) is a dynamical system with a Lyapunov function and so it is natural to conjecture that solutions should in some sense settle to equilibrium as time $t \to \infty$. More precisely, understanding the large time dynamics should reduce to understanding the underlying

- equilibrium states

and their

- linear stability properties

(and perhaps their connecting orbits for a complete picture).[2] But here we have already arrived exactly at what is the crux for models like (1.8). In the absence of regularizing contributions to the free energy from interfacial energy, on top of the issue of possible microstructure formation the set of equilibria of nonconvex variational integrals is typically geometrically very complicated (and form a noncompact, infinite-dimensional, non-smooth, multiply connected set), and associated evolution equations are not $C^1$ with respect to the natural topologies (in fact the dynamics of the linearized and the nonlinear equation are typically topologically different).

For the evolution equation (1.8), (1.6) studied here, let us state these peculiarities precisely. For simplicity we take $W = \frac{1}{4}(u_x^2 - 1)^2$ and $\alpha = 0$, and drop the subscript of $x_1$.

**Remark 1.** (See [AB]) The set of 'stable' equilibria, $\{u \in W_0^{1,\infty}(0,1) : W'(u_x(x)) \equiv const,$ $W''(u_x(x)) \geq 0$ a.e.$\}$, and the set of 'unstable' equilibria, $\{u \in W_0^{1,\infty}(0,1) : W'(u_x(x)) \equiv const,$

[1] Swart & Holmes [SH] present careful numerical studies of (1.8) suggesting convergence to 'relative minimizers' with finitely many interfaces; Lin & Pence [LP] study the issue of dynamic energy minimization for a simpler, Riemann type problem involving energy dissipation through phase boundary propagation; Kalies & Holmes [KH] analyse (1.8) with a strain gradient term $\frac{\gamma}{2}u_{x_1x_1}^2$ added to the free energy density and show – following earlier work of [HM] – how the global attractor becomes increasingly complicated as $\gamma \to 0$; Brandon, Fonseca & Swart [BFS] investigate the evolution of Young measures under (1.8).

[2] A well-known example where this can be made rigorous are one-dimensional parabolic equations $u_t - \epsilon^2 u_{xx} + f(u) = 0$ [He2], [BF1], [BF2].

$W''(u_x(x)) \not\geq 0$ a.e.$\}$, are noncompact in $W_0^{1,p}(0,1)$ $(1 \leq p \leq \infty)$. Even worse, the weak* closure in $W_0^{1,\infty}(0,1)$ of either set contains the unit ball in $W_0^{1,\infty}(0,1)$.

To see this, note that the set of stable (and resp. unstable) equilibria contains – among other elements – all states with strain $u_x(x) \in \{\pm 1\}$ a.e. (resp. $u_x(x) \in \{\pm 1, 0\}$ a.e. with $|\{x : u_x(x) = 0\}| > 0$), arranged in arbitrary spatial patterns with arbitrarily many jumps in $u_x$, only subject to the minimal constraint that the proportion of the phases $\pm 1$ must be equal, to ensure $u(0) = u(1) = 0$. (In the case $\alpha \neq 0$, the set of equilibria behaves qualitatively the same, the only difference being that inbetween jumps the strain $u_x$ of equilibria is no longer constant but satisfies the ODE $(W'(u_x))_x = \alpha u$, see [BHJPS, Sec. 2.2]).

In the language of dynamical systems, this lack of compactness and regularity of the equilibrium problem prevents the equation from being 'asymptotically smooth' in the sense of Hale [Ha] or from possessing finite-dimensional attracting sets or inertial manifolds. In the language of partial differential equations, Remark 1 implies that one cannot pass to the limit $t \to \infty$ in (1.8) via a 'soft' argument: to pass to the limit would require precompactness of orbits $u(t)$ in a topology with respect to which the nonlinearity $W'(u_x)$ is continuous (like the strong topology of $W^{1,p}$), but the available a-priori estimates [BHJPS] only give precompactness in the weak* topolgy of $W^{1,\infty}$.

A further difficulty arises from the fact that the equilibrium states are not isolated but form a multiply connected set in $W^{1,p}$: even if precompactness of an orbit $u(t)$ in $W^{1,p}$ were known, it is not clear whether it stabilizes to a unique limiting pattern. More technically speaking, it would not be clear whether its $\omega$-limit-set $\{u \in W_0^{1,p}(0,1) : \exists\, t_j \to \infty$ such that $u(t_j) \to u\}$ must consist of a single equilibrium. As a well-known example loosely akin to equation (1.8) where such connected continua naturally arise, even when the equilibrium problem is elliptic, we mention the one-dimensional continuum of travelling waves for nonlinear diffusion equations $u_t - \epsilon^2 u_{xx} + f(u) = 0$ on the whole real line, arising from the translation invariance of the equation. As shown in [FM1], [FM2], [FM3], typical solutions nevertheless converge exponentially fast to a unique travelling wave (or standing wave in the Allen-Cahn case $f(u) = u^3 - u$). More generally [HaM], [HaR], [Pe2], convergence to a unique equilibrium remains true in dynamical sysems whose set of equilibria forms a smooth hyperbolic manifold; however, problems like (1.8) lie beyond the scope of these results, due to Remark 1 and due to the lack of differentiability of the dynamics to which we turn now.

While the evolution via (1.8) is $C^1$ with respect to the $W^{1,\infty}$ norm (allowing dynamic stability analyses via linearization in small $W^{1,\infty}$ neighbourhoods of equilibria [Pe1], [BHJPS]), such information is of limited interest: most solutions do not converge in $W^{1,\infty}$ (i.e. the strain $u_x(t)$ does not converge uniformly) as time $t \to \infty$. This is a consequence exactly of one of the oustanding 'physical' aspects of (1.8), its modelling of the formation of discontinuous patterns of the strain $u_x$ from smooth initial data. On the other hand, in weaker norms like $W^{1,p}$ $(1 \leq p < \infty)$ in which typical solutions converge as $t \to \infty$, the evolution is no longer $C^1$ (the mapping $u_x \mapsto W'(u_x)$ or more generally mappings $w \mapsto \sigma(w)$ from $L^p \to L^p$ are nowhere Fréchet-differentiable, for every $\sigma : \mathbb{R} \to \mathbb{R}$ which is not affine). And more:

**Remark 2.** Let $u_0$ be an equilibrium state with $(u_0)_x \in \{\pm 1\}$ a.e. Then under the dynamics of the linearized evolution equation in $W_0^{1,p}(0,1)$ $(1 \leq p < \infty)$, $u_0$ is globally asymptotically stable (Lemma 4.1 below). However, every $W_0^{1,p}(0,1)$-neighbourhood of $u_0$ contains other equilibria of the nonlinear equation (1.8). In particular, in every neighbourhood of $u_0$ the dynamics of (1.8) is not topologically equivalent to the linearized dynamics.

To summarize, when wanting to prove results like Thm 3.1 (strong convergence in $W^{1,p}$ to a unique limit state as $t \to \infty$) and Thm 4.1(P6) (exponentially fast convergence in $W^{1,p}$), even for a single initial configuration with a 'nice' transition layer structure, one has to overcome the following obstacles:

- to prove that $u(t)$ stays in a compact set of $W^{1,p}$ as $t \to \infty$ (which is, as already emphasized several times, energetically unfavourable)
- to prove that the $\omega$-limit-set of $u(t)$, $\{u \in W_0^{1,p}(0,1) : \exists\, t_j \to \infty$such that $u(t_j) \to u\}$, consists only of a single equilibrium, without being able to appeal to geometric results like

those of [HM], [HR], [Pe2]

- to prove that the convergence is exponential without being able to 'linearize' the dynamics.

The first two difficulties are overcome here largely by patience and attention to detail, and our arguments appear to be of limited use in a more-dimensional situation. The proofs build upon the techniques developed in [AB], [Pe1], [BHJPS] and involve in addition: (i) sharpened versions of various a-priori estimates of [BHJPS] (Thm 2.1); (ii) an appropriate parametrization of the set of equilibrium patterns via a 'phase function' and a first integral of the equilibrium equation; (iii) a comparison principle for solutions of the nonelliptic equilibrium equation $(W'(u_x))_x = \alpha u$; (iv) an argument related to the lock-in of transition layers discussed below which gives orbit precompactness for all initial data with low energy (Lemmas 3.2, 3.3).

Regarding the third difficulty we present a more general tool, which may perhaps also be of use in different situations like the full 3D equation (1.4) or the viscous Cahn-Hilliard equation studied in [NP]: a new lemma on the abstract parabolic equation $z_t + Az = f(t,z)$ which extends a conclusion of Henry [He1] to nonlinearities which are not Fréchet-differentiable (Lemma A2 in the Appendix), thus allowing to analyse the speed of convergence of smooth functions to discontinuous patterns (Remark 5 in Section 4).

**1.2.2 Layer dynamics.** In contrast with the above complications, the behaviour of transition layers under (1.8) turns out to be 'simpler' than in problems with a regularizing interfacial energy contribution to the free energy. In problems like the one-dimensional Allen-Cahn or Cahn-Hilliard equation with small $O(\epsilon)$ interfacial energy, the energetic favourability of 'coarser' equilibria manifests itself as a tiny $O(e^{-d/\epsilon})$ driving traction on transition layers exerted by neighbouring layers with distance $O(d)$ (for careful studies of the resulting dynamics see [CP], [BX2]). By contrast, in equation (1.8) the energetic favourability of 'finer' eqilibria does not lead to any traction on layers. Instead, the behaviour of neighbouring layers decouples as time $t \to \infty$ and each of the layers simply 'locks in' and steepens up exponentially fast, converging to a stationary sharp interface close to its initial position.

Regarding the proof: While tracking layer positions via constructing approximate invariant manifolds and estimating the error appears to be forbidden by Remarks 1 and 2, it is possible to conclude here via the construction of a dynamically invariant region in which transition layers are trapped (Claim 1 in Section 4).

For a more detailed discussion including connections with earlier work of Pego [Pe1] on one-dimensional viscoelasticity and the work of Abeyaratne & Knowles [AK] in the mechanics literature, we refer to Section 4.

## 2 Existence and a-priori estimates

We study the initial-value problem

$$u_{tt} - \sigma(u_x)_x - \beta u_{xxt} + \alpha u = 0 \qquad (x \in (0,1),\, t \in (0,\infty)) \qquad (2.1a)$$

$$u|_{x=0} = u|_{x=1} = 0 \qquad (t \in [0,\infty)) \qquad (2.1b)$$

$$u|_{t=0} = u_0, \;\; u_t|_{t=0} = v_0 \qquad (x \in [0,1]) \qquad (2.1c)$$

where $\beta > 0$ and $\alpha \geq 0$ are constants, $\sigma = W'$, and $W$ (the stored-energy function) is throughout this paper required to satisfy the following hypotheses:

(H1) $W \in C^2(\mathbb{R})$

(H2) $W'(z) \to \pm\infty$ as $z \to \pm\infty$ (superlinear growth).

Throughout Sections 3 and 4 we will also assume

(H3) $W$ is a double-well potential, that is, there exist $z_- < z_1 < 0 < z_2 < z_+$ such that $W(z_\pm) = 0$, $W > 0$ elsewhere, $W'(0) = 0$, $W''|_{(z_1,z_2)} < 0$, $W''|_{\mathbb{R}\setminus[z_1,z_2]} > 0$.

We remark that any nonmonotonic, cubic-like stress-strain function $\sigma = W'$ satisfies (H3) if the co-ordinate system is chosen so that the Maxwell stress and the unstable strain state corresponding to the Maxwell stress are zero. Prototypical is the choice $\sigma(z) = z^3 - z$ studied in [BHJPS].

As pointed out in the Introduction, in the special case $\alpha = 0$ (2.1a) becomes the equation of 1D nonlinear viscoelasticity, and $u(x,t)$ can be interpreted physically as the displacement at time $t$ of a reference point $x$ on a bar with reference configuration $[0,1]$ of unit mass density. In this case, the result of Section 3 (strong convergence for low-energy initial data), though nowhere stated in the literature, is a more or less obvious consequence of the work of Andrews & Ball [AB] combined with the work of Pego [Pe1] (see [Fr1] for details). However, the results presented in Section 4 of this paper (transition layer dynamics, variational stability of limit states, exponential convergence) are new and interesting even if $\alpha = 0$.

Following [BHJPS], we study (2.1) in the phase space $W_0^{1,\infty}(0,1) \times L^2(0,1) \ni (u, u_t)$; note that this rather "weak" space allows to admit equilibria with discontinuous strain as data.

**Definition 2.1** *Given initial data* $(u_0, v_0) \in W_0^{1,\infty} \times L^2$, *by a weak solution of* (2.1) *on* $[0,T)$ *we mean a pair of functions*

$$\begin{aligned}(u,v) &\in C([0,T); W_0^{1,\infty} \times L^2) \\ &\cap C^1((0,T); W_0^{1,\infty} \times L^2))\end{aligned}$$

*which satisfies* (2.1c) *and for which the equation*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v \\ \sigma(u_x)_x + \beta u_{xxt} - \alpha u \end{pmatrix} \tag{2.2}$$

*holds for all* $t \in (0,T)$ *as an identity in* $L^2 \times W^{-1,\infty}$ *(note that for* $(u,v)$ *in the above space and every* $t \in (0,T)$*, each of the three terms* $v_t$, $\sigma(u_x)_x$, $\beta u_{xxt}$ *is well-defined as an element of* $W^{-1,\infty}$*).*

One of the important features of (2.1) is the fact that solutions dissipate energy: due to the viscous damping term $\beta u_{xxt}$, the total (kinetic + elastic) energy

$$E(u,v) := \int_0^1 \left(\frac{1}{2}v^2 + W(u_x) + \frac{\alpha}{2}u^2\right)$$

(which is conserved along smooth solutions of the undamped, purely mechanical equation of motion associated with the above energy) decreases along solutions,

$$E(u(t), v(t)) = E(u(t_0), v(t_0)) - \beta \int_{t_0}^{t} ||u_{xt}||_{L^2}^2 \qquad \text{for all } t \geq t_0 \geq 0. \tag{2.3}$$

This will be the starting point for our discussion of the large time behaviour of solutions in Sections 3 and 4.

The aim of the present, "preparatory" section is to summarize and slightly extend the existence results and a-priori estimates of [BHJPS, Theorem 3.1]. The sharper a-priori estimates derived here, in particular estimate (c), will be needed for our study of transition layer dynamics in Section 4.

Notation. Here and below, by $C^0$, $C^1$ and $C^2$ we do not mean the sets of $k$ times continuously differentiable functions $f : (0,1) \to \mathbb{R}$, but the Banach spaces of $k$ times continuously differentiable functions with finite norm $||f||_{C^k} = \sum_{i=0}^{k} \sup_{x\in(0,1)} |\frac{d^i}{dx^i} f(x)|$.

**Theorem 2.1** *(Existence and a-priori estimates)*
*Assume* $W$ *satisfies (H1) and (H2).*
(a) *Given any initial data* $(u_0, v_0) \in W_0^{1,\infty} \times L^2$ *there exists a unique weak solution* $(u,v)$ *of (2.1) on* $(0,\infty)$.

(b) *Given $M, \tau > 0$ there exist constants $K(M)$, $K(M,\tau) > 0$ such that whenever $\|(u_0, v_0)\|_{W^{1,\infty}\times L^2} \leq M$, then*

$$\sup_{t\geq 0} \|(u(t), v(t)\|_{W^{1,\infty}\times L^2} \leq K(M), \tag{2.4}$$

$$\sup_{t\geq \tau} \|(u_t(t), v_t(t)\|_{W^{1,\infty}\times L^2} \leq K(M,\tau). \tag{2.5}$$

(c) *If in addition $v_0 \in W_0^{1,2}$, then $v \in C([0,\infty); C^0)$, and moreover given $M > 0$ there exists $K(M) > 0$ such that if $\|(u_0, v_0)\|_{W^{1,\infty}\times W^{1,2}} \leq M$, then*

$$\sup_{t\geq 0} \|v(t)\|_{C^0} \leq K(M).$$

(d) *If in addition $\sigma \in C^2(\mathbb{R})$ and $(u_0, v_0) \in C^2 \times W_0^{1,2}$, then $(u,v) \in C([0,\infty); C^2 \times C^0) \cap C^1((0,\infty); C^2 \times C^0)$, and equation (2.1a) holds classically for all $t > 0$.*

The proof is a straightforward adaptation of the work of [BHJPS] (who prove (a) and (d) and derive the first estimate of (b) for the standard nonlinearity $\sigma(z) = z^3 - z$), combined with an estimate on the abstract parabolic equation $z_t + Az = f(z,t)$ (Lemma A1 in the Appendix). Following [BHJPS], we work with a transformed equation: for every $(u,v) \in W_0^{1,1}(0,1) \times L^1(0,1)$ one defines $\mathcal{P}(u,v) = (p,q) \in W_a^{1,1}(0,1) \times L_a^1(0,1)$ by

$$\begin{aligned} p(x) &:= \int_0^x v(x')dx' - \int_0^1 \int_0^x v(x')dx'dx \\ q &:= \beta u_x - p \end{aligned}$$

(where here and below $(\ )_a$ denotes the subspace of functions with zero average). Then $(u,v)$ can be recovered from $(p,q)$ via

$$\begin{aligned} u(x) &= \int_0^x \frac{p(x') + q(x')}{\beta} dx' \\ v &= p_x. \end{aligned}$$

Since a change of variables of this kind was to the best of our knowledge first introduced by R. Pego [Pe1], we take the liberty of heretoafter referring to $\mathcal{P}$ as the *Pego transform*. The following fact will be used frequently below:

**Lemma 2.1** *The Pego transform $(u,v) \mapsto \mathcal{P}(u,v)$ is a continuous isomorphism of Banach spaces $W_0^{1,p} \times L^2 \xrightarrow{\sim} W_a^{1,2} \times L_a^p$ $(1 \leq p \leq \infty)$.*

The proof of the lemma is elementary.

An elementary formal calculation, made rigorous below, suggests that through the above change of variables, the linear part of (2.2) diagonalizes and (2.1) becomes

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = \begin{pmatrix} \beta\frac{d^2}{dx^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \pi_a \begin{pmatrix} \sigma(\frac{p+q}{\beta}) - \alpha\int_0^x \int_0^{x'} \frac{p+q}{\beta} \\ -\sigma(\frac{p+q}{\beta}) + \alpha\int_0^x \int_0^{x'} \frac{p+q}{\beta} \end{pmatrix} \quad (x \in (0,1),\ t \in (0,\infty)) \tag{2.6a}$$

$$p_x|_{x=0} = p_x|_{x=1} = 0 \quad (t \in (0,\infty)) \tag{2.6b}$$

$$p|_{t=0} = p_0, \quad q|_{t=0} = q_0 \quad (x \in (0,1)) \tag{2.6c}$$

where $\pi_a f = f - \int_0^1 f$, that is, $\pi_a$ denotes the orthogonal projection in $L^2(0,1)$ onto the subspace of functions with zero average.

**Proof of (a) and (b).** For convenience of the reader, we include the relevant details from [BHJPS] who treat (2.6) as an abstract parabolic equation $z_t + Az = f(z)$ on a Banach space $X$ and appeal to results of Henry [He1]. Let

$$z = \begin{pmatrix} p \\ q \end{pmatrix}, \quad A = \begin{pmatrix} -\beta\frac{d^2}{dx^2} & 0 \\ 0 & 0 \end{pmatrix}, \quad f(z) = \pi_a\Big(\sigma(\tfrac{p+q}{\beta}) - \alpha\int_0^x \int_0^{x'} \tfrac{p+q}{\beta}\Big) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$X = L^2_a \times L^\infty_a$, $\mathcal{D}(A) = \{p \in W^{2,2}_a \ : \ p_x|_{\{0,1\}} = 0\} \times L^\infty_a$. Clearly $A$ is a sectorial operator, the fractional power space $X^{1/2}$ is simply $W^{1,2}_a \times L^\infty_a$, and the nonlinear term $f(z)$ is locally lipschitz as a mapping $X^{1/2} \to X$. Now given initial data $(p_0, q_0) \in X^{1/2}$, by [He1, Theorem 3.3.3] there exists a unique solution $(p, q)$ of (2.6) on some time interval $[0, t_1)$ – provided, of course, we adopt Henry's definition of "solution":

**Definition 2.2** *(See [He1, Chapter 3.3]) Given $(p_0, q_0) \in X^{1/2}$, a solution $(p, q)$ of (2.6) on $[0, T)$ is a pair of functions*

$$\begin{aligned}(p,q) \ &\in \ C([0,T); X^{1/2}) \\ &\cap \ C^1((0,T); X) \\ &\cap \ C((0,T); \mathcal{D}(A)) \ =: Y\end{aligned}$$

*which satisfies (2.6c) and for which equation (2.6a) holds for every $t \in (0,T)$ as an identitiy in $X$ (note that for $(p,q)$ in the above space and every $t \in (0,T)$, each of the three terms in (2.6a) is well-defined as an element of $X$).*

Before proceeding with the proof of global existence, let us point out

**Lemma 2.2** *With the notions of solution as specified in Definitions 2.1 and 2.2, $(u,v)$ is a solution of (2.1) with initial data $(u_0, v_0)$ if and only if its Pego transform $(p,q) = \mathcal{P}(u,v)$ is a solution of (2.6) with initial data $(p_0, q_0) = \mathcal{P}(u_0, v_0)$.*

**Proof of Lemma 2.2.** The "only if" direction is immediate via Lemma 2.1. The "if" direction is less immediate since $(p,q) \in Y$ only gives $(u,v) \in C([0,T); W^{1,\infty}_0 \times L^2)) \cap C^1((0,T); W^{1,2}_0 \times W^{-1,2}) \cap C((0,T); W^{1,\infty}_0 \times W^{1,2}_0)$. However, by [He1, Theorem 3.5.2], if $(p,q)$ is a solution of (2.6) it must have the additional regularity $(p,q) \in C^1((0,T); X^\gamma)$ for all $\gamma \in [0,1)$, in particular (by taking $\gamma = 1/2$ and using Lemma 2.1) $(u,v) \in C^1((0,T); W^{1,\infty}_0 \times L^2)$.

**Proof of (a) and (b), ctd.** Let $z = (p,q)$ be the local solution constructed above, and let $T \in (0, \infty]$ be the maximal time of existence. By a continuation theorem of Henry [He1, Corollary 3.3.5], in order to establish $T = \infty$ it is enough to show that

$$\frac{||f(z(t))||}{1 + ||z(t)||_{1/2}} \text{ stays bounded as } t \to T, \tag{2.7}$$

where $||\cdot||$ and $||\cdot||_{1/2}$ denote, respectively, the norm in $X$ and in $X^{1/2}$. We shall prove more, namely

Claim 1: Given $M > 0$ there exists $K(M) > 0$ such that if $||(p_0, q_0)||_{1/2} \le M$, then the solution of (2.6) satisfies $||(p(t), q(t)||_{1/2} \le K(M)$ for all $t \in [0,T)$.

To prove Claim 1, we first of all note that by Lemma 2.2 the inverse Pego transform $\mathcal{P}^{-1}(p,q)$ is smooth enough to be a solution of (2.1) in the sense of Definition 2.1, and hence smooth enough to satisfy the dissipation identity (2.3). This together with the continuity of $E \circ \mathcal{P}^{-1} \ : \ X^{1/2} \to \mathbb{R}$ implies

$$||p_x(t)||^2_{L^2} \le 2(E \circ \mathcal{P}^{-1})(p_0, q_0) \le K_1(M) \qquad \text{for all } t \in (0,T),$$

giving in particular $||p(t)||_{L^\infty} \le K_1(M)^{1/2}$, and

$$||\alpha \int_0^x \int_0^{x'} \tfrac{p(t)+q(t)}{\beta}||_{L^\infty} \le \left(2\alpha (E \circ \mathcal{P}^{-1})(p_0, q_0)\right)^{1/2} \le K_2(M).$$

Now we show $||q(t)||_{L^\infty} \le K_3(M)$, as follows. (This is the only part in the proof of (a) and (b) where the arguments of [BHJPS] do not straightforwardly generalize to the more general nonlinearities admitted here.) Arguing as in [Pe1, Corollary 3.4], there exists a subset $\tilde{\Omega} \subseteq (0,1)$ of full measure (independent of $t$) and a pointwise representative $\bar{q}$ of $q \in C([0,T); L^\infty_a) \cap C^1((0,T); L^\infty_a)$ such that for all $x \in \tilde{\Omega}$, $\bar{q}(x, \cdot)$ is continuous on $[0,T)$ and continuously differentiable on $(0,T)$, $(\bar{q})_t(\cdot, t)$ is a pointwise representative of $q_t(\cdot, t)$ for all $t \in (0,T)$, $\bar{q}(x, \cdot)$ is a classical solution of the ODE given by the second component of (2.6a) on $(0,T)$, and $|\bar{q}(x,t)| \leq \|q(t)\|_{L^\infty}$ for all $t \in [0,T)$. Now by (H2) there exists $R(M) > 0$ such that $\sigma(\frac{p_1+q_1}{\beta}) - \sigma(\frac{p_2+q_2}{\beta}) > 2K_2(M)$ for all $p_1, p_2 \in [-K_1, K_1]$ and all $q_1, q_2$ with either $q_2 \leq 0, q_1 \geq R$ or $q_2 \leq -R, q_1 \geq 0$. Thus since for $x_1, x_2 \in \bar{\Omega}$

$$(\bar{q}(x_1) - \bar{q}(x_2))_t = -\Big(\sigma(\tfrac{p(x_1)+\bar{q}(x_1)}{\beta}) - \sigma(\tfrac{p(x_2)+\bar{q}(x_2)}{\beta})\Big) + \alpha \int_0^{x_1}\int_0^{x'} \tfrac{p+\bar{q}}{\beta} - \alpha \int_0^{x_2}\int_0^{x'} \tfrac{p+\bar{q}}{\beta},$$

we have

$$(\bar{q}(x_1) - \bar{q}(x_2))_t\,(\bar{q}(x_1) - \bar{q}(x_2)) \leq 0 \quad (x_1, x_2 \in \bar{\Omega})$$

whenever either $\bar{q}(x_2) \leq 0$, $\bar{q}(x_1) \geq R$ or $\bar{q}(x_2) \leq -R$, $\bar{q}(x_1) \geq 0$. Since $\int_0^1 q \equiv 0$, this implies $\sup_{x_1\in\bar{\Omega}} \bar{q}(x_1,t) - \inf_{x_2\in\bar{\Omega}} \bar{q}(x_2,t) \leq 2\max\{\|q_0\|_{L^\infty}, R\}$ and hence (because the sup is $\geq 0$ and the inf $\leq 0$) $\|q(t)\|_{L^\infty} \leq 2\max\{\|q_0\|_{L^\infty}, R\}$. This proves Claim 1.

Now Claim 1 has three corollaries: First, (2.7) holds, i.e. $T = \infty$. Second, thanks to Lemma 2.2 we obtain global existence for problem (2.1): given $(u_0, v_0) \in W_0^{1,\infty} \times L^2$ and letting $(p,q)$ be the solution of (2.6) with initial data $\mathcal{P}(u_0,v_0)$, its inverse Pego transform $(u,v) = \mathcal{P}^{-1}(p,q)$ is a solution of (2.1). Third, Claim 1 combined with Lemma 2.1 and with the fact that $T = \infty$ implies the first estimate of (b). Finally, the second estimate of (b) is via Lemma 2.1 equivalent to

Claim 2: Given $M, \tau > 0$ there exists $K(M,\tau) > 0$ such that if $\|(p_0,q_0)\|_{1/2} \leq M$ then the solution of (2.6) satisfies $\|p_t(t), q_t(t)\|_{1/2} \leq K(M,\tau)$ for all $t \geq \tau$.

This follows from Claim 1 and a Harnack type inequality of Pego [Pe1, Lemma A3] for the abstract parabolic equation $z_t + Az = f(z)$ which implies that for $\tau > 0$

$$\|(p_t(t), q_t(t)\|_{1/2} \leq K(\tau)\Big(\|p(t-\tau), q(t-\tau))\|_{1/2} + \sup_{s\in[t-\tau,t]} \|f(p(t), q(t))\|\Big).$$

This completes the proof of (a) and (b).

Next, as an intermediate step towards proving (c) we show

Claim 3: Given $\gamma \in [\frac{1}{2}, 1)$ there exists $K(M,\gamma)$ such that if $(p_0,q_0) \in X^\gamma$ and $\|(p_0,q_0)\|_\gamma \leq M$, then the solution $(p,q)$ of (2.6) satisfies $(p,q) \in C([0,\infty); X^\gamma)$ and $\sup_{t\geq 0} \|(p(t),q(t))\|_\gamma \leq K(M,\gamma)$.

Here the fact that $(p,q) \in C([0,\infty); X^\gamma)$ follows from [He1, Theorem 3.3.3]. To deduce the a-priori estimate, write $(p,q) = z$ and use the fact that by Claim 1, $\sup_{t\geq 0} \|z(t)\| \leq C(M)$, $\sup_{t\geq 0} \|f(z)\| \leq K(M)$. Hence by Lemma A1 in the Appendix $\sup_{t\geq 0} \|A^\gamma z(t)\| \leq K(M,\gamma)$, proving Claim 3.

**Proof of (c).** If $v_0 \in W_0^{1,2}$, then $(p_0,q_0) = \mathcal{P}(u_0,v_0) \in \mathcal{D}(A)$, thus $\in X^\gamma$ for all $\gamma < 1$. Hence $(p,q) = \mathcal{P}(u,v) \in C([0,\infty); X^\gamma)$ for all $\gamma < 1$. Moreover if $\|v_0\|_{W^{1,2}} \leq M$ then $\|A(p_0,q_0)\| \leq K(M)$, hence by Claim 3 $\|A^\gamma(p(t),q(t))\| \leq K(M,\gamma)$ for all $t \geq 0$. Now choose $\gamma > \frac{3}{4}$ and use the fact that $X^\gamma \hookrightarrow C^{1,\nu}$ for $\gamma > \frac{\nu}{2} + \frac{3}{4}$, giving $p \in C([0,\infty); C^1)$ and $\sup_{t\geq 0} \|p(t)\|_{C^1} \leq K(M)$, hence $u_t \in C([0,\infty); C)$ and $\sup_{t\geq 0} \|u_t\|_C \leq K(M)$.

**Proof of (d).** As carried out in [BHJPS] for $\sigma(z) = z^3 - z$, (d) can be proved by simply replacing the function spaces used in the proof of (a). One works with the space $X = L^2_a \times C^1_a$, lets $\mathcal{D}(A) = \{p \in W^{2,2}_a \,:\, p_x|_{x=0,1} = 0\} \times C^1_a$, and takes initial data $(p_0,q_0) \in X^\delta$ ($\delta \in (\frac{3}{4}, 1)$), noting that thanks to the imbedding $X^\delta \hookrightarrow C^1 \times C^1$ and the assumption $\sigma \in C^2$, $f$ is locally lipschitz from $X^\delta$ to $X$. The proof of Theorem 2.1 is complete.

## 3 Strong convergence for low-energy initial data

The static problem underlying equation (2.1), that of minimizing $E(u,v)$, does not attain its infimum: every minimizing sequence $(u^j, v^j) \subset W_0^{1,\infty} \times L^2$ of $E$ converges weakly (in $W_0^{1,1} \times L^2$) but not strongly (in $W_0^{1,1} \times L^2$) to the nonminimizing state $(u,v) = (0,0)$, while the strain $u_x$ forms finer and finer microstructure ($u^j_x \to \nu$ in the sense of Young measures, where $\nu = \{\nu_x\}_{x\in(0,1)}$ is the homogeneous Young measure $\nu_x = \lambda\delta_{z_-} + (1-\lambda)\delta_{z_+}$ with $\lambda = z_+/(z_+ - z_-)$).

Remarkably, solutions $(u(t), v(t))$ to the evolution equation (2.1), despite the energy $E(u(t), v(t))$ decreasing strictly with time unless $(u_0, v_0)$ is an equilibrium, do NOT mimick this behaviour as time $t \to \infty$. Instead, dynamics acts as a mechanism preventing the formation of finer and finer microstructure in $u_x$ and all solutions with low initial energy get stuck in equilibrium states which are not absolute minimizers of $E$ but at best 'relative minimizers'.

To state this precisely, we recall that the equilibrium states are the time-independent solutions of (2.1) (that is, the functions $(u, v) \in W_0^{1,\infty} \times L^2$ such that $\sigma(u_x(x)) - \alpha \int_0^x u \equiv const$, $v \equiv 0$), and we assume throughout this section that the stored-energy function $W$ satisfies (H1), (H2) and (H3).

**Theorem 3.1** *There exists $\epsilon > 0$ such that given any initial data $(u_0, v_0) \in W_0^{1,\infty} \times L^2$ with low energy*

$$E(u_0, v_0) < \epsilon,$$

*the corresponding solution $(u(t), v(t))$ of (2.1) converges strongly in $W_0^{1,p} \times L^2$ $(1 \le p < \infty)$ to some limit state $(u_\infty, 0)$ as $t \to \infty$. Moreover this limit state lies in $W_0^{1,\infty} \times L^2$ and is an equilibrium state of (2.1).*

**Remarks.** 1) We will in fact show convergence in a slightly stronger topology: the strain $u_x(t)$ converges not only in $L^p$, but boundedly-almost-everywhere as $t \to \infty$. However, in general $u_x(t)$ does not converge in $L^\infty$: initial data with smooth strain typically approach limits with discontinuous strain. See Section 4 below.

2) In contrast to this delicate behaviour of the strain $u_x(t)$, nothing interesting happens to the velocity $v(t)$, and the above statement that $v(t) \to 0$ in $L^2$ is in fact an elementary consequence of the energy dissipation identity (2.3) and the a-priori estimates in the previous section: From Poincaré's inequality and (2.3) we see

$$\int_0^\infty ||u_t(t)||_{L^2}^2 dt \le \int_0^\infty ||u_{xt}(t)||_{L^2}^2 dt < \infty,$$

thus $\int_{t-1}^t ||u_t||_{L^2}^2 \to 0$ as $t \to \infty$, thus (since $\sup_{t \ge T} \frac{d}{dt} ||u_t(t)||_{L^2}^2 < \infty$ by Theorem 2.1(b))

$$||u_t(t)||_{L^2}^2 \to 0 \quad \text{as } t \to \infty.$$

3) In particular, from Theorem 3.1 we recover the interesting result of Ball, Holmes, James, Pego and Swart [BHJPS, Theorem 4.1, proved there by contradiction] that no solution of (2.1) minimizes energy globally as time $t \to \infty$: since $E > 0$ on $W_0^{1,\infty} \times L^2$ and since the mapping $(u, v) \mapsto E(u, v)$ is continuous with respect to $L^2$-convergence of $v$ and convergence boundedly-almost-everywhere of $u_x$, we have (assuming without loss $E(u_0, v_0) < \epsilon$)

$$\lim_{t \to \infty} E(u(t), v(t)) \;=\; E(u_\infty, 0) \;>\; 0 \;=\; \inf_{W_0^{1,\infty} \times L^2} E.$$

4) The result of Theorem 3.1 was conjectured in [BHJPS], even without any restriction on the initial energy, on the basis of the above nonminimization result and numerical simulations by Swart and Holmes [SH]. We do not know whether the restriction on the initial energy is necessary.

The three subsections below are devoted to proving Theorem 3.1.

## 3.1 Parametrizing the set of equilibria

As pointed out in the Introduction, (2.1) possesses an infinite-dimensional continuum of different equilibria. As a first step towards proving that each low-energy solution selects exactly one of these equilibria as its final state (as claimed in Theorem 3.1 above), we present a "parametrization" of the set of equilibria and show its "stability" under an appropriate notion of convergence.

Notation. Throughout the rest of this paper, we fix $r > 0$ such that for $\lambda \in [-r, r]$ the equation $\sigma(z) - \lambda = 0$ has exactly three roots $z_1(\lambda) < z_2(\lambda) < z_3(\lambda)$. Moreover, we let

$$\begin{aligned} m &:= \min_{z \in \sigma^{-1}([-r,r])} |\sigma'(z)|, \\ M &:= \max_{z \in \sigma^{-1}([-r,r])} |\sigma'(z)|. \end{aligned}$$

**Definition 3.1** *We introduce the function*

$$\psi(z) := \begin{cases} j, & z \in \bigcup_{\lambda \in [-r,r]} z_j(\lambda) \\ +\infty, & \sigma(z) \notin [-r, r]. \end{cases}$$

*We will refer to $\psi$ as the "phase function", since given a strain state $z$, $\psi(z)$ indicates its "phase" (1, 2 or 3).*

Besides the phase function, the second quantity needed to characterize equilibrium states will be the function

$$\sigma(u_x(x)) - \alpha \int_0^x u.$$

Since equation (2.1a) can be rewritten as

$$u_{tt} - \left( (\sigma(u_x) - \alpha \int_0^x u) + \beta u_{xt} \right)_x = 0$$

and corresponds to balance of linear momentum, it is natural to refer to $\sigma(u_x) - \alpha \int_0^x u$ as the "elastic stress". (The "viscous stress" $\beta u_{xt}$ does not play a role below as it decays to zero as time $t \to \infty$).

For reasons that will become clear at the beginning of Section 3.2, in the lemma below we can not afford to be cavalier about the distinction between elements of $L^\infty(0,1)$ and their pointwise representatives. This makes the statement of the lemma a little technical.

**Lemma 3.1** *(Convergence of elastic stress and of the phase function implies convergence of strain) Let $\{u^k\} \subset W_0^{1,\infty}$, and assume there exists a subset $\bar{\Omega} \subseteq (0,1)$ of full measure and pointwise representatives $\bar{w}^k$ of $u_x^k$ such that*

(a) $\sup_k ||u_x^k||_{L^\infty} \le K$

(b) $\sigma(\bar{w}^k(x)) - \alpha \int_0^x u^k =: \lambda_k(x) \to \lambda$ *($k \to \infty$) for some $\lambda \in (-r, r)$ and all $x \in \bar{\Omega}$*

(c) $\lim_{k\to\infty} \psi(\bar{w}^k(x))$ *exists and is finite for all $x \in \bar{\Omega}$.*

*Then $\lim_{k\to\infty} \bar{w}^k(x) =: \bar{w}(x)$ exists for all $x \in \bar{\Omega}$ (note in particular that we neither have to pass to a subsequence, nor to exclude a subset of $\bar{\Omega}$ of measure zero). Moreover, its equivalence class $w$ satisfies $||w||_{L^\infty} \le K$, and $u(x) := \int_0^x w$ satisfies: $u \in W_0^{1,\infty}$ (i.e. in particular $u(1) = 0$), $\sigma(u_x(x)) - \alpha \int_0^x u \equiv \lambda$ a.e. (i.e. in particular $(u, 0)$ is an equilibrium state of (2.1)), $\psi(u_x(x)) = \lim_{k\to\infty} \psi(u_x^k(x))$ a.e., and*

$$\begin{aligned} u_x^k &\to u_x \quad \textit{boundedly-a.e., in particular} \\ u^k &\to u \quad \textit{in } W_0^{1,p}\ (1 \le p < \infty). \end{aligned}$$

**Remark.** In particular, by taking $u^k \equiv u$ for some equilibrium state $(u, 0)$ of (2.1), we obtain the following corollary: each equilibrium state with $\sigma(u_x(x)) - \alpha \int_0^x u \in (-r, r)$ can be uniquely recovered from

(a) the phase function $\psi(u_x) \in L^\infty(0,1)$

(b) the elastic stress $\sigma(u_x) - \alpha \int_0^x u \equiv const \in \mathbb{R}$.

**Proof.** First we show that $\{\int_0^x u^k\}$ is convergent in $C([0,1])$, as follows. Define

$$\begin{aligned} \chi_i^k(x) &:= \begin{cases} 1, & x \in \bar{\Omega} \text{ and } \psi(\bar{w}^k(x)) = i \in \{1,2,3\} \\ 0, & \text{else} \end{cases} \\ \chi_0^k(x) &:= \begin{cases} 1, & x \in \bar{\Omega} \text{ and } \psi(\bar{w}^k(x)) = \infty \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Then in $\bar{\Omega}$

$$\bar{w}^k - \bar{w}^l = \sum_{i=1}^{3}[\chi_i^k \cdot z_i(\alpha \int_0^x u^k + \lambda_k) - \chi_i^l \cdot z_i(\alpha \int_0^x u^l + \lambda_l)] + \chi_0^k \cdot \bar{w}^k - \chi_0^l \cdot \bar{w}^l. \tag{3.1}$$

Thus letting $\xi(x) := \int_0^x |u^k - u^l|$ and using the fact that $z_j'(\lambda) = \frac{1}{\sigma'(z_j(\lambda))}$ and hence

$$|z_j(a) - z_j(b)| \leq \tfrac{1}{m}|a-b| \qquad (a, b \in [-r, r]),$$

we can estimate:

$$\begin{aligned}
0 \ \leq \ \tfrac{d}{dx}\xi(x) \ &= \ |u^k(x) - u^l(x)| \\
&= \ |\textstyle\int_0^x (u_x^k - u_x^l)| \qquad \text{(since } u^k(0) = u^l(0) = 0) \\
&= \ \left| \int_0^x \left[ \sum_{i=1}^{3} \chi_i^l(x')\{z_i(\alpha \textstyle\int_0^{x'} u^k + \lambda_k) - z_i(\alpha \int_0^{x'} u^l + \lambda_l)\} \right.\right. \\
&\quad \left.\left. + \sum_{i=1}^{3} (\chi_i^k(x') - \chi_i^l(x'))\, u_x^k + \chi_0^k u_x^k - \chi_0^l u_x^l \right] dx' \right| \\
&\leq \ \textstyle\int_0^x \frac{1}{m}|\alpha \int_0^{x'} u^k + \lambda_k - (\alpha \int_0^{x'} u^l + \lambda_l)| + 2K|\{x \in (0,1) \,:\, \psi(\bar{w}^k(x)) \neq \psi(\bar{w}^l(x))\}| \\
&\leq \ 1 \cdot \tfrac{1}{m} \cdot \left(\textstyle\int_0^1 |\lambda_k - \lambda_l| + \alpha \int_0^x |u^k - u^l|\right) + 2K|\{x \in (0,1) \,:\, \psi(\bar{w}^k(x)) \neq \psi(\bar{w}^l(x))\}| \\
&= \ \underbrace{\tfrac{1}{m}||\lambda_k - \lambda_l||_{L^1} + 2K|\{x \in (0,1) \,:\, \psi(\bar{w}^k(x)) \neq \psi(\bar{w}^l(x))\}|}_{=:\epsilon_{k,l}} + \tfrac{\alpha}{m}\xi(x)
\end{aligned}$$

and thus

$$\xi(x) \leq \epsilon_{k,l} \cdot \tfrac{m}{\alpha} \cdot (e^{\frac{\alpha}{m}x} - 1) \qquad \text{for all } x \in [0,1].$$

Now $\lambda_k \to \lambda$ boundedly-a.e. (and hence in $L^1$) as $k \to \infty$ and $|\{x \in (0,1) \,:\, \psi(\bar{w}^k(x)) \neq \psi(\bar{w}^l(x))\}| \to 0$ as $\min\{k,l\} \to \infty$ by hypothesis, and hence $\epsilon_{k,l} \to 0$ as $\min\{k,l\} \to \infty$. Consequently

$$||\textstyle\int_0^x u^k - \int_0^x u^l||_{C([0,1])} \to 0 \qquad (\min\{k,l\} \to \infty).$$

Hence by combining this with (b) and (c), we deduce that the right hand side of (3.1) converges to zero for all $x \in \bar{\Omega}$, thus

$$\bar{w}^k - \bar{w}^l \to 0 \ \text{ for all } x \in \bar{\Omega} \qquad (\min\{k,l\} \to \infty).$$

Consequently

$$\begin{aligned}
\lim_{k\to\infty} \textstyle\int_0^x u^k \ &=: \ U \qquad \text{exists for all } x \in [0,1], \\
\lim_{k\to\infty} \bar{w}^k \ &=: \ \bar{w} \qquad \text{exists for all } x \in \bar{\Omega}.
\end{aligned}$$

Combining this with (a) and the fact that convergence boundedly-a.e. implies convergence in $L^1$, we have in particular

$$u_x^k \to w \qquad \text{in } L^1,$$

where $w$ is the equivalence class of $\bar{w}$. Thus (letting $u(x) := \int_0^x w$)

$$u(1) = \int_0^1 u_x = \lim_{k\to\infty} \int_0^1 u_x^k = \lim_{k\to\infty} u^k(1) = 0$$

(i.e. the boundary condition is preserved as $k \to \infty$), and

$$u^k = \int_0^x u_x^k \to \int_0^x w = u \qquad \text{in } C([0,1]),$$

hence $U = \int_0^x u$. To verify the remaining claimed properties of $u$ is now trivial.

## 3.2 Reduction to convergence of mean elastic stress

According to our parametrization of equilibria introduced in Lemma 3.1, in order to prove Theorem 3.1 we have to show that along low-energy solutions of (2.1)

(a) $\sigma(u_x(x,t)) - \alpha \int_0^x u(t) \to \lambda \in (-r, r)$ a.e. as $t \to \infty$

(b) $\lim_{t\to\infty} \psi(u_x(x,t))$ exists and is finite a.e.

In this subsection we reduce the problem of proving convergence of the above infinite-dimensional set of parameters to the problem of proving convergence of a single one-dimensional parameter, namely the mean elastic stress $\int_0^1 (\sigma(u_x(x,t)) - \alpha \int_0^x u(x',t))dx$.

Before proceeding with this reduction, we have to address a minor technical issue regarding the meaning of convergence a.e. of a noncountable family $\{w(t)\}_{t\in(0,\infty)}$ of $L^1$ functions:

**Definition 3.2** *Let $\{w(t)\}_{t\in(0,\infty)} \subset L^1(0,1)$, and let $\phi : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be any function. We say that $\phi(w(t))$ converges a.e. as $t \to \infty$ if there exist pointwise representatives $\bar{w}(t)$ of $w(t)$ such that $\lim_{t\to\infty} \phi(\bar{w}(x,t))$ exists for a.e. $x \in (0,1)$.*

It is obvious that unlike for countable families of $L^1$ functions, one can always violate the conclusion "$\lim_{t\to\infty} \phi(\bar{w}(x,t))$ exists a.e." by a bad choice of representatives $\bar{w}(t)$, for any family $\{w(t)\}$ and any nonconstant function $\phi$. Thus when wanting to deduce pointwise information like (a) or (b) above for a solution of (2.1), some care has to be taken as to picking "good" pointwise representatives of the strain $u_x$ at each $t$.

**Lemma 3.2** *(Convergence of mean elastic stress implies convergence of elastic stress)*
*Let $(u,v)$ be a solution of (2.1), and assume*

$$\lim_{t\to\infty} \int_0^1 \Big(\sigma(u_x(x,t)) - \alpha {\textstyle\int_0^x} u(t)\Big) dx =: \lambda \ \text{ exists.}$$

*Then*

$$\sigma(u_x(x,t)) - \alpha {\textstyle\int_0^x} u(t) \to \lambda \ \text{ a.e. as } t \to \infty.$$

**Lemma 3.3** *(Convergence of the phase function)*
*Let $(u,v)$ be a solution of (2.1), and assume*

$$\limsup_{t\to\infty} \left| \int_0^1 \Big(\sigma(u_x(x,t)) - \alpha {\textstyle\int_0^x} u(t)\Big) dx \right| < \frac{2r}{3},$$
$$\sqrt{2\alpha E(u_0, v_0)} < \frac{r}{3}.$$

*Then (b) holds true.*

**Lemma 3.4** *(Small initial energy implies small mean elastic stress)*
*Given $\delta_0 > 0$ there exists $\epsilon_0(\delta_0)$ (independently of $\alpha$) such that every solution $(u,v)$ of (2.1) with $E(u_0, v_0) < \epsilon_0$ satisfies*

$$\limsup_{t\to\infty} \left| \int_0^1 \Big(\sigma(u_x(x,t)) - \alpha {\textstyle\int_0^x} u(t)\Big) dx \right| < \delta_0.$$

Combining these three lemmas while taking $\delta_0 = \frac{2r}{3}$ in Lemma 3.4 gives the following

**Proposition 3.1** *Let $(u,v)$ be a solution of (2.1) and assume that*

$$E(u_0, v_0) < \min\{\epsilon_0(\tfrac{2r}{3}), \tfrac{r}{3\sqrt{2\alpha}}\} \quad (\epsilon_0 \text{ as in Lemma 3.4}),$$
$$\lim_{t\to\infty} \int_0^1 \Big(\sigma(u_x(x,t)) - \alpha {\textstyle\int_0^x} u(t)\Big) dx \ \text{ exists.}$$

*Then (a) and (b) hold true. In particular (by Lemma 3.1), $(u(t), v(t))$ converges strongly in $W^{1,p} \times L^2$ $(1 \le p < \infty)$ to some equilibrium state $(u_\infty, 0)$ as $t \to \infty$, $u_\infty \in W_0^{1,\infty}$, and $u_x(t) \to (u_\infty)_x$ boundedly-a.e.*

**Proof of Lemma 3.2.** We will prove that for any solution $(u,v)$ of (2.1)

$$\Big(\sigma(u_x(x,t)) - \alpha\textstyle\int_0^x u(t)\Big) - \displaystyle\int_0^1 \Big(\sigma(u_x(x,t)) - \alpha\textstyle\int_0^x u(t)\Big)\,dx \;\rightarrow\; 0 \text{ a.e. as } t \rightarrow \infty. \tag{3.2}$$

We write $(p,q) = \mathcal{P}(u,v)$. Arguing as in [Pel, Corollary 3.4] we see that there exists a subset $\tilde{\Omega} \subseteq (0,1)$ of full measure (independent of $t$) and a pointwise representative $\bar{q}$ of $q \in C([0,\infty); L^\infty_a)$ such that for all $x \in \tilde{\Omega}$, $\bar{q}(x,\cdot)$ is continuous on $[0,\infty)$, continuously differentiable on $(0,\infty)$, a classical solution of the ODE

$$\bar{q}_t = -\pi_a\Big(\sigma(\tfrac{p+\bar{q}}{\beta}) - \alpha\textstyle\int_0^x u\Big), \tag{3.3}$$

$|\bar{q}(x,t)| \leq ||q(t)||_{L^\infty}$ for all $(x,t) \in \tilde{\Omega} \times (0,\infty)$, and $(\bar{q})_t$ is a pointwise representative of $q_t \in C((0,\infty); L^\infty_a)$. By (3.3) and the fact that $\frac{p+\bar{q}}{\beta} =: \bar{v}$ is a pointwise representative of $u_x$, it is enough to show that

$$\bar{q}_t \rightarrow 0 \text{ as } t \rightarrow \infty \text{ for a.e. } x \in (0,1). \tag{3.4}$$

Consider the following modification of the Lyapunov function $E$:

$$\tilde{E}(t) := \int_0^1 \Big(W(u_x(t)) + \tfrac{\alpha}{2}u^2 + \tfrac{1}{\beta}pq_t\Big).$$

Compute

$$\tfrac{d}{dt}\tilde{E}(t) = -\tfrac{1}{\beta}\int_0^1 q_t^2 + \tfrac{1}{\beta}\int_0^1 pq_{tt}.$$

Integrate over $t \in (0,T)$ and observe that the left hand side $\tilde{E}(T) - \tilde{E}(0)$ stays bounded as $T \rightarrow \infty$ by Theorem 2.1(b), and that the second term on the right hand side $\int_0^T\int_0^1 pq_{tt}$ stays bounded since

$$\begin{aligned}
\Big|\textstyle\int_0^1 pq_{tt}\Big| &\leq ||p||_{L^2}||q_{tt}||_{L^2} \\
&\leq ||p_{xx}||_{L^2}||q_{tt}||_{L^2} \\
&= ||u_{xt}||_{L^2}\Big(||\pi_a(\sigma'(u_x)u_{xt})||_{L^2} + ||\pi_a(\alpha\textstyle\int_0^x u_t)||_{L^2}\Big) \\
&\leq K||u_{xt}||^2_{L^2}
\end{aligned}$$

for some positive constant $K$ and $\int_0^T ||u_{xt}||^2_{L^2} = \frac{1}{\beta}\big(E(u_0,v_0) - E(u(T),v(T))\big)$ by (2.3). Consequently

$$\int_0^\infty\int_0^1 q_t^2\,dx\,dt < \infty$$

and thus

$$\int_t^\infty\int_0^1 q_t^2\,dx\,dt \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Hence by Fubini's theorem

$$\int_0^1 \Big(\int_t^\infty q_t^2\,dt\Big)\,dx \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Monotone convergence and the fact that $\bar{q}_t$ is a pointwise representative of $q_t$ implies

$$\int_t^\infty \bar{q}_t^2\,dt \rightarrow 0 \text{ a.e. as } t \rightarrow \infty. \tag{3.5}$$

In addition, by differentiating (3.3) with respect to $t$ and using Theorem 2.1(b) which implies $\sup_{t\geq\tau}||p_t(t)||_{W^{1,2}} < \infty$ for $\tau > 0$, we have the a-priori estimate

$$\sup_{t\geq\tau}|\bar{q}_{tt}(x,t)| < \infty \text{ for } x \in \tilde{\Omega},\ \tau > 0. \tag{3.6}$$

Combining (3.5) and (3.6) proves (3.4) and (3.2) and completes the proof of Lemma 3.2.

**Proof of Lemma 3.3.** By hypothesis and (3.2)

$$\limsup_{t\to\infty} \left| \sigma(\bar{w}(x,t)) - \alpha \textstyle\int_0^x u(t) \right| < \frac{2r}{3} \quad a.e., \tag{3.7}$$

and by (2.3) and by hypothesis

$$\|\alpha \textstyle\int_0^x u(t)\|_{L^\infty} \leq \alpha \|u(t)\|_{L^2} \leq \sqrt{2\alpha E(u(t),v(t))} \leq \sqrt{2\alpha E(u_0,v_0)} < \frac{r}{3}. \tag{3.8}$$

Combining (3.7) and (3.8), we thus have

$$\limsup_{t\to\infty} |\sigma(\bar{w}(x,t))| < r \text{ a.e.}$$

Since $\bar{w}(x,\cdot)$ is continuous in $t$ for a.e. $x$, for a.e. $x$ there exists $T(x)$ such that $\bigcup_{t\geq T(x)} \bar{w}(x,t)$ (since it is connected) must be contained in a connected component of $\sigma^{-1}([-r,r])$. Because the connected components are the sets $\bigcup_{\lambda\in[-r,r]} z_j(\lambda)$ ($j = 1, 2, 3$), for a.e. $x$ there exist $T(x)$, $j(x)$ such that

$$\bar{w}(x,t) \in \bigcup_{s\in[-r,r]} z_j(\lambda) \text{ for all } t \geq T,$$

i.e. $\lim_{t\to\infty} \psi(\bar{w}(x,t))$ exists and is finite a.e.

**Proof of Lemma 3.4.** The gist of the argument lies in Maxwell's equal area rule: the stress $\sigma(u_x)$ of minimizers of $I[u] = \int_0^1 W(u_x(x))dx$ subject to linear boundary values $u(x)|_{\{0,1\}} = L\cdot x$ must be identically equal to the Maxwell stress $\sigma_M = (W^{**})'(L)$ (where $W^{**}$ is the convex envelope of $W$), which in our case is $\sigma_M = 0$. See e.g. [Fr2] for further information. To prove the "averaged" and "perturbed" version of this statement claimed in the lemma, we argue by contradiction. Suppose there exists $\delta_0 > 0$ and a sequence $(u^k, v^k)$ of solutions of (2.1) such that

$$\lim_{k\to\infty} E(u^k(0), v^k(0)) = 0,$$
$$\limsup_{t\to\infty} \left| \int_0^1 \left( \sigma(u_x^k(x)) - \alpha \textstyle\int_0^x u^k \right) dx \right| > \delta_0.$$

Since by (3.3), (3.4) and Theorem 2.1(b)

$$\left\| \pi_a \left( \sigma(u_x^k(t)) - \alpha \textstyle\int_0^x u^k(t) \right) \right\|_{L^1} \to 0 \ (t\to\infty) \text{ for all } k,$$

and since $E(u^k(t), v^k(t))$ decreases in $t$ for all $k$, we can choose $t_k$ such that for $\bar{u}^k := u^k(t_k)$, $\bar{v}^k := v^k(t_k)$

$$E(\bar{u}^k, \bar{v}^k) \to 0 \text{ as } k \to \infty \tag{3.9}$$

$$\left| \int_0^1 \left( \sigma(\bar{u}_x^k) - \alpha \textstyle\int_0^x \bar{u}^k \right) \right| \geq \delta_0 \text{ for all } k \tag{3.10}$$

$$\left\| \pi_a \left( \sigma(\bar{u}_x^k(t)) - \alpha \textstyle\int_0^x \bar{u}^k(t) \right) \right\|_{L^1} \to 0 \text{ as } k \to \infty. \tag{3.11}$$

But by (3.9) $\bar{u}_x^k \to \{z_\pm\}$ in measure, hence $\sigma(\bar{u}_x^k) \to 0$ in measure, and by (3.9) and (3.8) $\|\alpha \int_0^x \bar{u}^k\|_{L^\infty} \to 0$, hence

$$\sigma(\bar{u}_x^k) - \alpha \textstyle\int_0^x \bar{u}^k \to 0 \text{ in measure},$$

contradicting (3.10) and (3.11).

## 3.3 Convergence of mean elastic stress

Here we complete the proof of Theorem 3.1 by showing

**Lemma 3.5** *(Convergence of mean elastic stress)*
*Let $\epsilon_0$ be as in Lemma 3.4, $\epsilon_1$ as specified below in (3.24), and let $(u,v)$ be a solution of (2.1) with*

$$E(u_0,v_0) < \min\{\epsilon_0(\tfrac{2r}{3}), \tfrac{r}{3\sqrt{2\alpha}}, \epsilon_1\} =: \epsilon.$$

*Then*

$$\lim_{t\to\infty} \underbrace{\int_0^1 \Big(\sigma(u_x(x,t)) - \alpha \textstyle\int_0^x u(t)\Big)dx}_{=:c(t)} \quad \textit{exists.} \tag{3.12}$$

By Proposition 3.1 in the previous section, this establishes Theorem 3.1 with the above choice of $\epsilon = \min\{\epsilon_0(\frac{2r}{3}), \frac{r}{3\sqrt{2\alpha}}, \epsilon_1\}$.

**Proof of Lemma 3.5.** This is the most delicate step in the proof of Theorem 3.1, and can be proved as follows. Suppose (3.12) fails. Then the $\omega$-limit-set of $c$, $\{\lambda \in I\!R : \exists t_k \to \infty$ such that $c(t_k) \to \lambda\}$, which is *connected* since $c(t)$ is a continuous function of $t$, must contain a nonempty interval $I \subset (-\frac{2r}{3}, \frac{2r}{3})$. Now given $\lambda \in I$ choose $t_k \to \infty$ such that $\lim_{t_k\to\infty} c(t_k) = \lambda$. By (3.5), this means

$$\sigma(u_x(x,t_k)) - \alpha \textstyle\int_0^x u(t_k) \to \lambda \text{ a.e. as } t_k \to \infty.$$

(Note that here and from now on, we no longer have to distinguish between elements of $L^p$ and their pointwise representatives, since we are dealing with countable sequences of functions.) Also, by the choice of $\epsilon$, Lemma 3.3 and Lemma 3.4, the phase function converges:

$$\lim_{t_k\to\infty} \psi(u_x(t_k)) \text{ exists and is finite a.e.}$$

Consequently, letting $t_k \to \infty$, by Lemma 3.1 we obtain the existence of a function $u^\lambda \in W_0^{1,\infty}$ such that

$$\begin{aligned} \sigma(u_x^\lambda) - \alpha \textstyle\int_0^x u^\lambda &\equiv \lambda \text{ a.e.,} \\ \psi(u_x^\lambda) = \lim_{t_k\to\infty} \psi(u_x(t_k)) &=: \psi_\infty \text{ a.e.} \end{aligned} \tag{3.13}$$

Recall that $\lambda \in$ was arbitrary, i.e. such a $u^\lambda$ exists for all $\lambda \in I$; note also that the phase function of the $u^\lambda$ is independent of $\lambda$. Now let us investigate how the family of equilibria $\{u^\lambda\}_{\lambda\in I}$ varies with $\lambda$. First, for the sake of motivation consider the case where

$$\psi_\infty(x) = \lim_{t_k\to\infty} \psi(u_x(x,t_k)) \in \{1,3\} \text{ a.e.,} \tag{3.14}$$

or equivalently $\sigma'(u_x^\lambda) = W''(u_x^\lambda) > 0$ a.e., that is to say all the $u_x^\lambda$ consist only of locally stable phases. We claim that we now arrive at a contradiction, by appealing to the following

**Comparison principle for weak solutions of the ODE $\sigma(u_x)_x = \alpha u$.**
*Assume $u$, $\bar u \in W^{1,\infty}$ satisfy*

$$\begin{aligned} \sigma(u_x(x)) - \alpha \textstyle\int_0^x u &\equiv \lambda \text{ a.e.,} \\ \sigma(\bar u_x(x)) - \alpha \textstyle\int_0^x \bar u &\equiv \bar\lambda \text{ a.e.,} \end{aligned}$$

*$\lambda < \bar\lambda$, $u(0) = \bar u(0) = 0$, $\sigma(u_x(x))$ and $\sigma(\bar u_x(x)) \in [-r,r]$ a.e., $\psi(u_x) = \psi(\bar u_x)$ a.e., $\psi(u_x) \in \{1,3\}$ a.e. ("no unstable phases"). Then*

$$u(x) < \bar u(x) \text{ for all } x \in (0,1].$$

**Proof of Lemma 3.5, ctd.** Pick $\lambda$, $\bar\lambda \in I$, $\lambda < \bar\lambda$, and then the equilibria $u^\lambda$, $u^{\bar\lambda}$ constructed in (3.13) satisfy the assumptions of the Comparison Principle, hence

$$u^\lambda(1) < u^{\bar\lambda}(1),$$

contradicting the boundary condition $u^\lambda(1) = u^{\bar\lambda}(1) = 0$ and completing the proof of Lemma 3.5 in the special case of when (3.14) holds.

**Proof of the Comparison Principle.** Under the assumption $\psi(u_x) = \psi(\bar u_x) \in \{1,3\}$ a.e., the ODE solved by $u$ and $\bar u$ becomes elliptic, so comparison principles do not come as a surprise even though the one needed here is somewhat nonstandard. Let $\chi_i(x) := \begin{cases} 1, & \psi(u_x(x)) = i \\ 0, & \text{else} \end{cases}$. Then

$$\begin{aligned} u_x &= \chi_1 \cdot z_1(\alpha \textstyle\int_0^x u + \lambda) + \chi_3 \cdot z_3(\alpha \textstyle\int_0^x u + \lambda) \text{ a.e.,} \\ \bar u_x &= \chi_1 \cdot z_1(\alpha \textstyle\int_0^x \bar u + \bar\lambda) + \chi_3 \cdot z_3(\alpha \textstyle\int_0^x \bar u + \bar\lambda) \text{ a.e..} \end{aligned} \tag{3.15}$$

As $x \to 0^+$, $\int_0^x u$ and $\int_0^x \bar u$ tend to zero, hence (since $\lambda < \bar\lambda$) for small enough $x > 0$ we have

$$\alpha \textstyle\int_0^x u + \lambda \ < \ \alpha \textstyle\int_0^x \bar u + \bar\lambda.$$

Thus, since $z_j(\bar\lambda)$ $(j = 1,3)$ are strictly increasing functions of $\bar\lambda$ (note $\frac{d}{d\bar\lambda} z_j(\bar\lambda) = \frac{1}{\sigma'(z_j(\bar\lambda))} > 0$ for $\bar\lambda \in [-r, r]$),

$$z_j(\alpha \textstyle\int_0^x u + \lambda) \ < \ z_j(\alpha \textstyle\int_0^x \bar u + \bar\lambda) \ \text{ for } x > 0 \text{ small and } j = 1, 3. \tag{3.16}$$

Now integrating (3.15) and using (3.16) gives

$$u(x) \ = \ \int_0^x u_x \ < \ \int_0^x \bar u_x \ = \ \bar u(x) \ \text{ for } x > 0 \text{ small.} \tag{3.17}$$

Now suppose $\{x \in (0,1] : u(x) \geq \bar u(x)\} =: \Omega_+ \neq \emptyset$, and let $x^* := \inf \Omega_+$. By (3.17), $x^* > 0$. Hence

$$\begin{aligned} & u < \bar u \ \text{ for } 0 < x < x^* \\ \Longrightarrow \quad & \textstyle\int_0^x u < \int_0^x \bar u \ \text{ for } 0 < x < x^* \\ \Longrightarrow \quad & \alpha \textstyle\int_0^x u + \lambda < \alpha \int_0^x \bar u + \bar\lambda \ \text{ for } 0 < x < x^* \\ \Longrightarrow \quad & z_j(\alpha \textstyle\int_0^x u + \lambda) < z_j(\alpha \int_0^x \bar u + \bar\lambda) \ \text{ for } 0 < x < x^* \text{ and } j = 1, 3 \\ \Longrightarrow \quad & u_x < \bar u_x \ \text{ a.e. for } 0 < x < x^* \ \text{ (by (3.15))} \\ \Longrightarrow \quad & u(x^*) = \textstyle\int_0^{x^*} u_x < \int_0^{x^*} \bar u_x = \bar u(x^*), \end{aligned}$$

contradicting the fact that $u(x^*) = \bar u(x^*)$ by the definition of $x^*$. The proof of the Comparison Principle is complete.

**Proof of Lemma 3.5, ctd.** It remains to prove the Lemma with the assumption (3.14) replaced by the assumption of small initial energy; this means that $\psi_\infty(x)$ is now allowed to take the "unstable" value 2 on a set of small measure. This more complicated situation, in which the ODE under study is no longer elliptic, will be handled by means of the following

**Refined Comparison Principle for weak solutions of the ODE $\sigma(u_x)_x = \alpha u$.**
*There exists $\epsilon_1 > 0$ with the following property: Whenever $u$, $\bar u \in W^{1,\infty}$ satisfy*

$$\begin{aligned} \sigma(u_x(x)) - \alpha \textstyle\int_0^x u &\equiv \lambda \ \text{ a.e.,} \\ \sigma(\bar u_x(x)) - \alpha \textstyle\int_0^x \bar u &\equiv \bar\lambda \ \text{ a.e.,} \end{aligned}$$

*$\lambda < \bar\lambda$, $u(0) = \bar u(0) = 0$, $\sigma(u_x(x))$ and $\sigma(\bar u_x(x)) \in [-r, r]$ a.e., $\psi(u_x) = \psi(\bar u_x)$ a.e., and*

$$\int_0^1 W(u_x) < \epsilon_1$$

*"the proportion of unstable phases is small", then*

$$u(1) < \bar u(1). \tag{3.18}$$

**Remark.** That is, roughly speaking: when violating the assumption "$\psi(u_x) = \psi(\bar{u}_x) \in \{1, 3\}$ a.e." in the ordinary Comparison Principle (which we had seen to imply $u(x) < \bar{u}(x)$ for all $x > 0$) on a set of small measure, an averaged form of the Comparison Principle survives. This is by no means obvious, as not only $\bar{u} - u$ may now decrease on $\{x : \psi(u_x(x)) = 2\}$, but also on $\{x : \psi(u_x(x)) \in \{1,3\}\}$: If, say, $\psi(u_x) = 2$ on some interval $(0, x_1)$, $\psi(u_x) = 1$ on some interval $(x_1, x_2)$, and $\bar{\lambda} - \lambda$ is small compared to $x_1$, then

$$\int_0^{x_1} \bar{u} - \int_0^{x_1} u < -(\bar{\lambda} - \lambda),$$

and hence

$$(\bar{u} - u)_x = \chi_1 \cdot \Big(z_1(\textstyle\int_0^x \bar{u} + \bar{\lambda}) - z_1(\int_0^x u + \lambda)\Big) < 0 \;\text{ for } x \geq x_1,\ x \text{ near } x_1,$$

that is to say $u_x$ *decreases further* for $x \geq x_1$, $x$ near $x_1$. This example suggests at first sight that inequality (3.18) should only be expected to hold if $\bar{\lambda} - \lambda$ is large compared to $\epsilon_1$. However, the proof below shows that (3.18) in fact holds without such a restriction.

**Proof of Lemma 3.5, ctd.** The lemma now follows immediately from combining the Refined Comparison Principle with the proof in the case when (3.14) holds.

**Proof of the Refined Comparison Principle.** Let $\chi_i$ ($i = 1, 2, 3$) be as defined in the proof of the Comparison Principle, and define

$$\begin{aligned}\Omega_u &:= \{x \in (0,1) : \psi(x) = 2\}\\ \Omega_s &:= \{x \in (0,1) : \psi(x) \in \{1,3\}\}.\end{aligned}$$

The main idea in the proof is a blow-up argument with respect to $\lambda$. Namely, we consider the difference quotient

$$w := \frac{\bar{u} - u}{\bar{\lambda} - \lambda}.$$

This difference quotient satisfies a "nice" linear nonautonomous ODE, namely

$$\begin{aligned}
w_x(x) &= \frac{\bar{u}_x(x) - u_x(x)}{\bar{\lambda} - \lambda}\\
&= \sum_{i=1}^{3} \chi_i(x) \cdot \frac{z_i(\alpha\int_0^x \bar{u} + \bar{\lambda}) - z_i(\alpha\int_0^x u + \lambda)}{\bar{\lambda} - \lambda}\\
&= \sum_{i=1}^{3} \chi_i(x) \cdot z_i'(\zeta(x)) \cdot \frac{(\alpha\int_0^x \bar{u} + \bar{\lambda}) - (\alpha\int_0^x u + \lambda)}{\bar{\lambda} - \lambda}\\
&= \sum_{i=1}^{3} \chi_i(x) \cdot \frac{1}{\sigma'(\zeta(x))} \cdot \Big(1 + \alpha\textstyle\int_0^x w\Big) \qquad (3.19)
\end{aligned}$$

for some $\zeta(x)$ between $\sigma(u_x(x))$ and $\sigma(\bar{u}_x(x))$. (In particular, $\zeta(x) \in [-r, r]$ a.e.) We would like to prove $w(1) > 0$, and do this in three steps:

$$\alpha \left|\int_0^x w\right| \leq e^{\frac{\alpha}{m} x} - 1 \leq e^{\frac{\alpha}{m}} - 1 =: C \;\text{ for all } x \in [0,1] \qquad (3.20)$$

$$|\Omega_u| < \frac{m}{2\alpha(1+C)} \implies \alpha \int_0^x w \geq -\frac{1}{2} \;\text{ for all } x \in [0,1] \qquad (3.21)$$

$$|\Omega_u| < \min\left\{\frac{m}{2\alpha(1+C)}, \left(1 + \frac{2M(1+C)}{m}\right)^{-1}\right\} \implies w(1) > 0. \qquad (3.22)$$

To prove (3.20), note that $\xi(x) := \int_0^x |w|$ satisfies

$$\begin{aligned} 0 \;&\le\; \frac{d}{dx}\xi(x) \;=\; |w(x)| \;\le\; \int_0^x |w_x| \\ &\le\; \frac{1}{m}\Big(1+\alpha\int_0^x |w|\Big) \;=\; \frac{1}{m}\Big(1+\alpha\xi(x)\Big) \end{aligned}$$

and so (3.20) follows from Gronwall's inequality and the fact that $\xi(0)=0$.

To show (3.21), note first that the conclusion is clear for $x$ near zero. Suppose for contradiction that $\{x\in(0,1) : \alpha\int_0^x w < -\frac{1}{2}\} \neq \emptyset$, and let $x^* := \inf\{x\in(0,1) : \alpha\int_0^x w < -\frac{1}{2}\}$. Then $x^*>0$, and

$$\begin{aligned} &\alpha\int_0^{x^*} w = -\frac{1}{2}, \\ &\alpha\int_0^{x} w \ge -\frac{1}{2} \text{ for all } x\in[0,x^*]. \end{aligned}$$

Hence by (3.20)

$$w_x = \frac{1}{\sigma'(\zeta(x))}\Big(1+\alpha\textstyle\int_0^x w\Big) \ge \begin{cases} \frac{1}{2M}, & x\in[0,x^*]\cap\Omega_s \\ -\frac{1+C}{m}, & x\in[0,x^*]\cap\Omega_u. \end{cases} \tag{3.23}$$

Thus

$$\begin{aligned} -\frac{1}{2} \;&=\; \alpha\int_0^{x^*} w \;\ge\; \alpha\, x^* \inf_{x\in[0,x^*]} w(x) = \alpha\, x^* \inf_{x\in[0,x^*]}\int_0^x w_x \\ &\ge\; \alpha\, x^*\Big(\frac{1}{2M}|[0,x]\cap\Omega_s| - \frac{1+C}{m}|[0,x]\cap\Omega_u|\Big) \;\ge\; -\frac{\alpha(1+C)}{m}|\Omega_u| \end{aligned}$$

and consequently

$$|\Omega_u| \ge \frac{m}{2\alpha(1+C)},$$

contradicting the hypothesis of (3.21). This proves (3.21).

Finally, to deduce (3.22) we use (3.21) and (3.23) to estimate

$$w_x \ge \begin{cases} \frac{1}{2M}, & x\in\Omega_s \\ -\frac{1+C}{m}, & x\in\Omega_u, \end{cases}$$

consequently

$$w(1) \;=\; \int_0^1 w_x \;\ge\; \frac{1}{2M}|\Omega_s| - \frac{1+C}{m}|\Omega_u| \;=\; \frac{1}{2M} - \Big(\frac{1+C}{m}+\frac{1}{2M}\Big)|\Omega_u|,$$

and the quantity on the right hand side is positive by the hypothesis of (3.22). This establishes (3.22). Finally, letting

$$W_0 := \min_{\lambda\in[-r,r]} W(z_2(\lambda))$$

(note $W_0>0$) we have

$$|\Omega_u| \le \frac{\int_{\Omega_u} W(u_x)}{W_0}.$$

Hence if

$$\int_0^1 W(u_x) \;<\; W_0\cdot\min\left\{\frac{m}{2\alpha(1+C)}, \Big(1+\frac{2M(1+C)}{m}\Big)^{-1}\right\} =: \epsilon_1, \tag{3.24}$$

then the hypothesis of (3.22) holds. Thus the Refined Comparison Principle holds true with the above choice of $\epsilon_1$. The proof of Theorem 3.1 is finally complete.

# 4 Layer dynamics

In the previous section we proved that each solution $(u(t), v(t))$ of (2.1) with low initial energy approaches some equilibrium state in the limit $t \to \infty$.

We now focus on a restricted class of initial data – those with a transition layer structure – and investigate more closely how the evolution towards equilibrium occurs, and in particular why solutions fail to form microstructure and how the limit state "selected" by the evolution law (2.1) depends on the initial state.

Throughout this Section, the stored-energy function $W$ is assumed to satisfy (H1), (H2) and (H3), and the initial data $(u_0, v_0) \in W_0^{1,\infty} \times L^2$ will be required to satisfy

(A1) (smoothness and a-priori bounds)
$u_0 \in C^2$, $v_0 \in W_0^{1,2}$, $||(u_0)_x||_{L^\infty} + ||v_0||_{W^{1,2}} \leq M$

(A2) (low initial energy)
$E(u_0, v_0) < \epsilon$

(A3) (no transition layers at the boundary of $(0,1)$)
$\mathcal{L}_\rho(0) := \{x \in [0,1] : |(u_0)_x(x)| \leq \rho\} \subset (0,1)$

(A4) (steepness of transition layers)
$|(u_0)_{xx}(x)| \geq K$ in $\mathcal{L}_\rho(0)$

for suitable constants $M$, $\epsilon$, $\rho$, $K > 0$. Here the crucial conditions are (A4) and (A2).

(A4) means that $|(u_0)_{xx}|$ must be large (i.e. the graph of $(u_0)_x$ steep) whenever $(u_0)_x$ is near the unstable state 0. Note that by (A4), (A3) and (A1) (which in particular requires $||u_0||_{C^2} < \infty$), $\mathcal{L}_\rho(0)$ consists of a finite number of *connected components* or *"transition layers"* $[a_i^0, b_i^0]$ $(0 < a_1^0 < b_1^0 < \ldots < a_N^0 < b_N^0 < 1)$, in each of which $(u_0)_x$ is strictly monotone and has exactly one zero $x_i^0$.

Since the constant $\epsilon$ will be required to be small, (A2) means that the strain $(u_0)_x$ must be close to $z_-$ or $z_+$ inbetween layers. See Figure 1.

**Theorem 4.1** *Given an arbitrary constant $M > 0$ and given any interval $[-\rho, \rho]$ in which $\sigma' < 0$, there exist $\epsilon(M, \rho)$, $K(M, \rho) > 0$ such that all solutions $(u, v)$ of (2.1) whose initial data satisfy (A1)-(A4) have the following properties:*

(P1) *(preservation of the number of phases / no nucleation or coarsening)*
*The number of zeroes of $u_x(\cdot, t)$ is finite, positive, and independent of $t$. Moreover, denoting the zeroes by $0 < x_1(t) < \ldots < x_N(t) < 1$, each $x_i(t)$ depends continuously on $t \in [0, \infty)$.*

(P2) *(preservation of transition layer structure)*
*The number of connected components of $\mathcal{L}_{\rho/2}(t) := \{x \in (0,1) : |u_x(x,t)| \leq \rho/2\}$ is finite, positive, and independent of $t$, and in each connected component, $u_x(\cdot, t)$ is strictly monotone and has exactly one zero. Moreover, denoting the connected components by $[a_i(t), b_i(t)]$ $(0 < a_1(t) < b_1(t) < \ldots < a_N(t) < b_N(t) < 1)$, the $a_i(t)$, $b_i(t)$ depend continuously on $t \in [0, \infty)$.*

(P3) *(lock-in and steepening of transition layers)*
*For all $t \geq 0$,*

$$[a_i(t), b_i(t)] \subset [a_i^0, b_i^0] \quad \forall i, \text{ in particular } x_i(t) \in [a_i^0, b_i^0]$$
$$|u_{xx}(x,t)| \geq \tfrac{K}{2} e^{\gamma t} \quad \forall x \in \textstyle\bigcup_{i=1}^N [a_i(t), b_i(t)] = \mathcal{L}_{\rho/2}(t)$$
$$|b_i(t) - a_i(t)| \leq \tfrac{2\rho}{K} e^{-\gamma t} \quad \forall i,$$

*where $\gamma = \frac{1}{2\beta} \min_{[-\rho,\rho]} |\sigma'| > 0$.*

(P4) *(exponential convergence of transition layer positions)*
*$\lim_{t\to\infty} x_i(t) =: x_i^*$ exists for all $i$, $x_i^* \in [a_i^0, b_i^0]$ (thus in particular $0 < x_1^* < \ldots < x_N^* < 1$), and $|x_i(t) - x_i^*| \leq \min\{|b_i^0 - a_i^0|, \frac{3\rho}{K} e^{-\gamma t}\}$.*

(P5) *(dependence of the limit state on the initial state)*
$\lim_{t\to\infty} u_x(t) =: (u^*)_x$ *(which exists as an $L^p$-limit by Theorem 3.1) is continuous on* $(0,1)\backslash\{x_1^*, ..., x_N^*\}$ *but discontinuous at every* $x_i^*$, *and (letting* $x_0^* = x_0^0 = 0$, $x_{N+1}^* = x_{N+1}^0 = 1$*)*

$$\psi(u_x^*)|_{(x_i^*,x_{i+1}^*)} = \begin{cases} 1, & (u_0)_x|_{(x_i^0,x_{i+1}^0)} < 0 \\ 3, & (u_0)_x|_{(x_i^0,x_{i+1}^0)} > 0, \end{cases} \quad (i = 0, ..., N)$$

*where* $\psi(z)$ *is the phase function introduced in Definition 3.1.*

(P6) *(exponential convergence of the solution)*
*There exist* $C, \gamma' > 0$ *independently of* $p$ *such that for* $p \in [2,\infty)$,

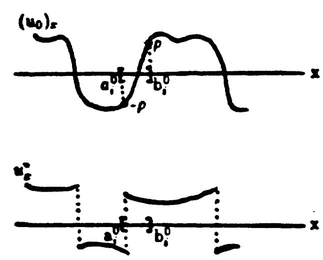
$$\|(u(t), v(t)) - (u^*, 0)\|_{W^{1,p}\times L^2} \le Ce^{-\frac{\gamma'}{p}t} \quad \forall t \ge 0.$$

*However,*

$$\|(u(t), v(t)) - (u^*, 0)\|_{W^{1,\infty}\times L^2} \not\to 0 \quad (t \to \infty).$$

(P7) *(variational properties of the limit state)*
$(u^*, 0)$ *is a local minimizer of* $E(u,v)$ *in* $W_0^{1,\infty} \times L^2$. *However, if the coefficient* $\alpha$ *in equation (2.1a) is nonzero then* $(u^*, 0)$ *is neither a global minimizer of* $E(u,v)$ *in* $W_0^{1,\infty} \times L^2$, *nor a local minimizer of* $E(u,v)$ *in* $W_0^{1,p} \times L^2$ *(*$1 \le p < \infty$*).*

**Figure 1.** A typical initial state and its large time limit as described by Theorem 4.1.



**Remarks.** 1) The lock-in and sharpening of strain interfaces as captured by (P3) was observed numerically by Swart and Holmes [SH], whose numerics in fact motivated us to investigate this issue analytically. For similar numerical observations in a related model (whose underlying free energy does not contain a displacement penalty term $\alpha u^2$ but a strain gradient term $(u_{xx})^2$) see [BG], [GB]. The proof of (P1), (P2), (P3) proceeds by explicit construction of an invariant region (see Claim 1 below) in which transition layers are trapped; some of the underlying ideas are adopted from the work of Pego [Pe1, Proposition 6.2] on one-dimensional nonlinear viscoelasticity.

An inkling of why an equation like (2.1) may exhibit a lock-in effect like (P3) can be had from the behaviour of a special class of solutions to (2.1a) in the special case $\alpha = 0$: Pego [Pe1] shows that two strain phases $w_-$ and $w_+$ can be connected by a travelling wave $u_x(x,t) = w(x - ct)$ ($w(x') \to w_\pm$ as $x' \to \pm\infty$) if and only if either $c = 0$ (that is, $(u, 0)$ is an equilibrium) or the "chord criterion" is satisfied:

The chord connecting the points $(w_-, \sigma(w_-))$ and
$(w_+, \sigma(w_+))$ does not intersect the graph of $\sigma$.

We emphasize that the chord criterion forbids the existence of travelling waves whose phases $w_-$, $w_+$ are close to the minimizing phases $z_-$, $z_+$, unless the wave speed $c$ is exactly zero. Note also that this lock-in-effect for travelling waves underlies (and is reflected by) the 'kinetic relation' obtained in the limit of vanishing viscosity by Abeyaratne & Knowles [AK] for the motion of a single interface shock in the system of conservation laws $\alpha = \beta = 0$: For a special piecewise linear choice of $\sigma$, these authors show [AK, Figure 13] that the resulting kinetic relation represents a stick-slip-system, that is, the interface is locked in until the 'driving traction' $W(w_+) - W(w_-) - \frac{1}{2}(\sigma(w_+) + \sigma(w_-))(w_+ - w_-)$ (which is zero at $(w_+, w_-) = (z_+, z_-)$ by Maxwell's equal area rule) exceeds a critical level. (For a general discussion of 'kinetic relations' and forces on interfaces in two-phase materials, the reader may also consult [Gu, Sec. 5].)

2) Even though initial data satisfying (A1)-(A4) can have only a finite number of transition layers, provided $M > \max\{|z_-|, |z_+|\}$ the class $\mathcal{A}_{M,\epsilon,\rho,K}$ of such data contains elements with an *arbitrarily large* number of transition layers, since no bound from above is imposed on the steepness of the layers and since the condition $E(u_0, v_0) < \epsilon$ can be met by letting $(u_0)_x \approx z_+$ or $z_-$ inbetween layers. The fact that patterns with an arbitrarily large number of phases are stable under the dynamics underlines the genuinely infinite-dimensional nature of the system (2.1).

3) The behaviour of transition layers as captured by (P1)-(P5) is "local", in the following sense: provided assumption (A3) and its definition of $\mathcal{L}_\rho(0)$ are changed to

$$\text{(A3)}' \qquad \mathcal{L}_\rho(0) := \{x \in \bar{\Lambda} : |(u_0)_x(x)| \le \rho\} \subset \Lambda$$

where $\bar{\Lambda}$ is the closure of some open subset $\Lambda$ of $(0,1)$, conclusions (P1)-(P5) remain true for the solution restricted to $\Lambda$, as is clear from the proof below. Note, however, that in the interesting situation of initial data $(u_0)_x \approx 0$ outside $\Lambda$, the nonlocal hypothesis (A2) imposes the constraint that the measure of $(0,1)\backslash\Lambda$ must be small.

4) The nonconvergence result in (P6) is clear from (P5): since $u_x(\cdot,t)$ is continuous for all $t$ and its limit is discontinuous, the convergence cannot be uniform.

5) By contrast, the first part of (P6), exponential convergence of $u_x(\cdot,t)$ in $L^p$, is rather less obvious. It clarifies how the competition between the exponentially stabilizing behaviour of equation (2.1a) away from the transition layers (see the stability result in [BHJPS] for small $W^{1,\infty}$-perturbations of weak relative minimizers $u^*$ of $J[u]$) and the exponentially destabilizing behaviour in the layers themselves (see (P3)) is decided: the interaction between these two effects results in overall exponential stability.

Both this result – and the proof – is in these authors' opinion more interesting than the usual "exponential stability by linearization" results in the PDE or dynamical systems literature. Namely, in the $L^p$-norm, in which the strain field $u_x(t)$ approaches its limit state, the nonlinear term $\pi_a(\sigma(u_x(t)))$ in equation (2.6a) is not locally dominated by the linear terms, or more technically speaking: the mapping $f \mapsto \sigma(f)$ from $L^p \to L^p$ $(1 \le p < \infty)$ is *nowhere* Fréchet-differentiable, for *every* $\sigma : I\!R \to I\!R$ which is not affine. However, in the $L^\infty$-norm (or any stronger norms like $W^{1,p}$-norms), in which the mapping is Fréchet-differentiable so that the nonlinear terms are locally dominated by the linear terms, we know that $u_x(t)$ does not converge as $t \to \infty$.

This situation will be handled by combining the lock-in result (P3) with a new lemma for the abstract parabolic equation $z_t + Az = f(z)$ (Lemma A2 in the Appendix), which extends the work of Henry [He1] from Fréchet-differentiable situations to a more general scenario that allows to analyse the speed of convergence of smooth functions to discontinuous patterns.

6) The variational stability result in (P7) can be viewed as a contribution towards the fundamental issue (C) from the Introduction, raised in a much more general context in [Ba]. For the specific model studied here, the conclusion of (P7) was conjectured in [BHJPS], for "generic" initial data. Regarding the involved Sobolev norms, let us remark that $(u^*, 0)$ is a local minimizer in $W_0^{1,\infty} \times L^2$ of $E(u,v)$ if and only if $u^*$ is a local minimizer in $W_0^{1,\infty}$ of $J[u]$ ($J$ as defined in (1.7)), which is in turn equivalent to $u^*$ being a 'weak relative minimizer' of $J[u]$ in the language of the calculus of variations [Ce].

Also, this result parallels in an interesting manner the behaviour of a numerical minimization algorithm for the functional $J[u]$ proposed recently by Ma & Walkington [MW], where the authors observe convergence to equilibria of the discretized functional which are not global minimizers but discrete analoga of the local minimizers described in (P7). Note that their algorithm forbids interfaces of the limit state to move as the computation is done on a fixed mesh and the strain of typical limit states jumps at (almost) every grid point [MW, Figure 3].

7) We emphasize the asymmetry in the appearance of the $W^{1,\infty}$- and the $W^{1,p}$-norms in statements

(P6) and (P7):

$(u(t), v(t))$ converges to $(u^*, 0)$ in $W_0^{1,p} \times L^2$, but not in $W_0^{1,\infty} \times L^2$;

$(u^*, 0)$ is a local minimizer of $E(u,v)$ in $W_0^{1,\infty} \times L^2$, but not in $W_0^{1,p} \times L^2$.

From the viewpoint of the geometric theory of dynamical systems, this is a genuinely infinite-dimensional subtlety, with no analogon in finite-dimensional systems. However, on the level of layer dynamics there is a simple interpretation: the limit states are only minimizers of the Lyapunov function subject to an *asymptotic constraint imposed by dynamics*, namely the lock-in of strain interfaces.

**Proof of (P1) and (P2).** We fix a solution $(u,v)$ and write $(p,q) = \mathcal{P}(u,v)$. First, let us collect together the various a-priori estimates needed later. From (A1) and Theorem 2.1, there exists $K_1(M)$ such that

$$\sup_{t\geq 0} ||u_x(t)||_{L^\infty} \leq K_1(M), \tag{4.25}$$

$$\sup_{t\geq 0} ||q(t)||_{L^\infty} \leq K_1(M), \tag{4.26}$$

$$\sup_{t\geq 0} ||\frac{v(t)}{\beta}||_{L^\infty} = \sup_{t\geq 0} ||\frac{p_x(t)}{\beta}||_{L^\infty} \leq K_1(M), \tag{4.27}$$

$$\sup_{t\geq 0} ||\alpha u(t) - \frac{\sigma'(u_x(t))}{\beta} p_x(t)||_{L^\infty} \leq K_1(M). \tag{4.28}$$

From (A2), the definition of $E$ and (2.3),

$$||\pi_a \alpha\left(\int_0^x u(t)\right)||_{L^\infty} \leq 2||\alpha\left(\int_0^x u(t)\right)||_{L^\infty} \leq 2\sqrt{2\alpha E(u_0,v_0)} \leq 2\sqrt{2\alpha\epsilon} \tag{4.29}$$

(as already used in (3.7)), and

$$||\frac{p(t)}{\beta}||_{L^\infty} \leq \frac{1}{\beta}||v(t)||_{L^1} \leq \frac{1}{\beta}\sqrt{2E(u_0,v_0)} \leq \frac{1}{\beta}\sqrt{2\epsilon}. \tag{4.30}$$

Furthermore, letting

$$K_2(M) := \inf_{z\in[-K_1(M),K_1(M)]\setminus\{z_-,z_+\}} \frac{|\sigma(z)|}{\sqrt{W(z)}} > 0$$

we have

$$\left|\int_0^1 \sigma(u_x(t))\right| \leq K_2(M)\sqrt{E(u_0,v_0)} \leq K_2(M)\sqrt{\epsilon}. \tag{4.31}$$

Finally, define

$$\sigma_0 := \inf_{[-\rho,\rho]} |\sigma'|,$$
$$\sigma_- := \inf_{\sigma^{-1}(\sigma([-\rho,\rho]))} |\sigma'|,$$
$$\sigma_+ := \sup_{\sigma^{-1}(\sigma([-\rho,\rho]))} |\sigma'|,$$

let $\eta \in (0, \rho/4)$ ($\eta$ to be specified later), and set

$$\rho_0 := \rho - \eta \; (> \tfrac{3}{4}\rho).$$

After all these estimates and definitions, we now introduce the crucial technical tool for proving (P1)-(P5), the set

$$\tilde{\mathcal{L}}(t) := \{x \in (0,1) : |\tfrac{q(x,t)}{\beta}| \leq \rho_0\}.$$

Claim 1. Assume $K \geq (1+\frac{2}{\sigma_0})K_1(M)$, $\epsilon \leq \min\{\frac{\beta^2\eta^2}{2}, \frac{\sigma_0^2\eta^2}{(K_2(M))^2}, \frac{\sigma_0^2\eta^2}{8\alpha}\}$. Then

(i) $\tilde{\mathcal{L}}(t) \subseteq \tilde{\mathcal{L}}(t_0)$ $\forall t \geq t_0 \geq 0$ (monotonicity),

and for all $x \in \tilde{\mathcal{L}}(t)$

(ii) $|q_x(x,t)| \geq e^{\gamma t}|q_x(x,0)|$ (exponential growth).

First, let us prove (i). By (4.30), (4.31), (4.29) and the choice of $\epsilon$

$$||\frac{p(t)}{\beta}||_{L^\infty} \leq \eta \quad \text{for all } t \geq 0, \tag{4.32}$$

$$||\int_0^1 \sigma(u_x(t))|| \leq \sigma_0\eta \quad \text{for all } t \geq 0,$$

$$||\pi_a\alpha\Big(\textstyle\int_0^x u(t)\Big)||_{L^\infty} \leq \sigma_0\eta \quad \text{for all } t \geq 0.$$

Now the interval $[-\beta\rho_0, \beta\rho_0]$ is, at each point $x \in (0,1)$, a negatively invariant interval for the nonautonomous one-dimensional ODE

$$\frac{d}{dt}q(x,t) = -\sigma\Big(\tfrac{q(x,t)}{\beta} + \tfrac{p(x,t)}{\beta}\Big) + \int_0^1 \sigma(u_x(t)) + \pi_a\Big(\alpha\textstyle\int_0^x u(t)\Big) \tag{4.33}$$

$$=: f^{(x)}(q(x,t),t), \tag{4.34}$$

because at $|q| = \beta\rho_0$ we have (using $|u_x| = |\frac{p}{\beta} + \frac{q}{\beta}| \leq \eta + \rho_0 = \rho$ and $\text{sign}(u_x) = \text{sign}(q)$)

$$\begin{aligned}
\frac{d}{dt}|q(x,t)| &= \text{sign}(q(x,t))\Big(\sigma(0) - \sigma(u_x(x,t)) + \textstyle\int_0^1\sigma(u_x(t)) + \pi_a\Big(\alpha\int_0^x u(t)\Big)\Big) \\
&\geq \sigma_0|u_x(x,t)| - \Big|\textstyle\int_0^1\sigma(u_x(t))\Big| - ||\pi_a\Big(\alpha\int_0^x u(t)\Big)||_{L^\infty} && (4.35)\\
&\geq \sigma_0\Big(|\tfrac{q(x,t)}{\beta}| - |\tfrac{p(x,t)}{\beta}|\Big) - \Big|\textstyle\int_0^1\sigma(u_x(t))\Big| - ||\pi_a\Big(\alpha\int_0^x u(t)\Big)||_{L^\infty} && (4.36)\\
&\geq \sigma_0(\rho_0 - \eta - \eta - \eta) = \sigma_0(\rho - 4\eta) \ > \ 0. && (4.37)
\end{aligned}$$

This proves (i). To show (ii), fix $t$ and fix $x \in \tilde{\mathcal{L}}(t)$. By (i), $x \in \tilde{\mathcal{L}}(s)$ for all $s \in [0,t]$, and thus in particular $|u_x(x,s)| \leq \rho$ for all $s \in [0,t]$. Consequently from (4.28)

$$\begin{aligned}
\frac{d}{ds}|q_x(x,s)| &= \text{sign}(q_x(x,s))\left(\frac{-\sigma'(u_x(x,s))}{\beta}q_x(x,s) + \alpha u(x,s) - \frac{\sigma'(u_x(x,s))}{\beta}p_x(x,s)\right) \\
&\geq \frac{\sigma_0}{\beta}|q_x(x,s)| - K_1(M) \\
&\geq \frac{\sigma_0}{\beta}\Big(|q_x(x,s)| - \frac{\beta}{\sigma_0}K_1\Big) \quad \text{for all } x \in \tilde{\mathcal{L}}(t) \text{ and all } s \in [0,t].
\end{aligned}$$

Thus because $|q_x(x,s)| \geq \beta(K - K_1) \geq \frac{2\beta}{\sigma_0}K_1$ at $s = 0$, $|q_x(x,s)| \geq \frac{2\beta}{\sigma_0}K_1$ for all $s \in [0,t]$, and consequently

$$\frac{d}{ds}|q_x(x,s)| \geq \frac{\sigma_0}{2\beta}|q_x(x,s)| = \gamma|q_x(x,s)| \quad \text{for all } x \in \tilde{\mathcal{L}}(t) \text{ and all } s \in [0,t].$$

Integrating this differential inequality gives (ii), completing the proof of Claim 1.

Claim 2. Assume $K > K_1(M)\max\{1+\frac{2}{\sigma_0}, 4\}$, and let $\epsilon$ be as in Claim 1. Then $\mathcal{L}_{\rho/2}(t) \subseteq \tilde{\mathcal{L}}(t) \subseteq \mathcal{L}_\rho(0)$, and $|u_{xx}(x,t)| \geq \frac{K}{2}e^{\gamma t}$ in $\mathcal{L}_{\rho/2}(t)$, for all $t \geq 0$.

Indeed, if $x \in \mathcal{L}_{\rho/2}(t)$, then $|u_x(x,t)| \leq \rho/2$, hence (by (4.32)) $|\frac{q(x,t)}{\beta}| = |u_x - \frac{p}{\beta}| \leq \rho/2 + \eta \leq \rho_0$, hence $x \in \tilde{\mathcal{L}}(t)$, and furthermore by (4.32), (4.27), Claim 1 and the fact that $K \geq 4K_1(M)$

$$|u_{xx}(x,t)| \ \geq \ |\frac{q_x(x,t)}{\beta}| - |\frac{p_x(x,t)}{\beta}|$$

$$
\begin{aligned}
&\geq e^{\gamma t}\left|\frac{q_x(x,0)}{\beta}\right| - K_1(M)\\
&\geq e^{\gamma t}\left(|u_{xx}(x,0)| - \left|\frac{p_x(x,0)}{\beta}\right|\right) - K_1(M)\\
&\geq e^{\gamma t}\Big(|u_{xx}(x,0)| - 2K_1(M)\Big)\\
&\geq e^{\gamma t}\tfrac{K}{2}.
\end{aligned}
$$

This establishes Claim 2.

Next, we show that the number of zeros of $u_x(\cdot,t)$ and the number of connected components of $\mathcal{L}_{\rho/2}(t)$ is conserved, and that the zeros and the connected components depend continuously on $t$. From Claim 2 and the fact that $||u(\cdot,t)||_{C^3} < \infty$ for all $t \geq 0$, it is clear that at every $t \in [0,\infty)$ $\mathcal{L}_{\rho/2}(t)$ possesses a finite number of connected components $[a_i(t), b_i(t)]$ $(0 < a_1(t) < b_1(t) < \ldots < a_{N(t)} < b_{N(t)} < 1)$, in each of which $u_x(\cdot,t)$ is strictly monotone and has exactly one zero $x_i(t)$. Also, $N(t) \geq 1$ because the boundary condition $u(0,t) = u(1,t) = 0$ implies that $u_x(\cdot,t)$ must have at least one zero. We would like to prove that $N(t)$ is constant and that the $a_i$, $b_i$, $x_i$ depend continuously on $t \in [0,\infty)$. To this end, we apply the implicit function theorem, respectively, to the equations

$$
g(x,t) := \left\{ \begin{array}{l} u_x(x,t) - \rho/2 \\ u_x(x,t) - \rho/2 \\ u_x(x,t) \end{array} \right. = 0,
$$

noting that by Theorem 2.1(d), $g$, $g_x \in C((0,1)\times[0,\infty))$ and that at each zero $(x_0,t_0) \in (0,1)\times[0,\infty)$ of $g$, $|g_x| \geq K/2$ by Claim 2. That is, for each $t_0$ the set $\{g(x_0,t_0) : (x_0,t_0) \text{ is a zero of } g\}$ does not contain a critical value (with respect to $x$) of $g(\cdot,t_0)$. Hence by the implicit function theorem, the number of zeros of $g(\cdot,t)$ is independent of $t$ (i.e. $N(t) \equiv \mathit{const} =: N$), and the zeros $a_i(t)$, $b_i(t)$, $x_i(t)$ depend continuously on $t$. This proves (P1) and (P2).

**Proof of (P3).** From Claim 1 and from arguing as in our above analysis of $\mathcal{L}_{\rho/2}(t)$, we can write $\tilde{\mathcal{L}}(t) = \bigcup_{i=1}^{N}[\alpha_i(t), \beta_i(t)]$ for $0 < \alpha_1(t) < \beta_1(t) < \ldots < \alpha_N(t) < \beta_N(t) < 1$, where the $\alpha_i$, $\beta_i$ depend continuously on $t$. Now from $\mathcal{L}_{\rho/2}(t) \subseteq \tilde{\mathcal{L}}(t) \subseteq \mathcal{L}_\rho(0)$ we deduce

$$
[a_i(t), b_i(t)] \subseteq [\alpha_i(t), \beta_i(t)] \subseteq [a_i^0, b_i^0] \quad \text{for all i}, \tag{4.38}
$$

proving the first part of (P3).

The second part of (P3) was already proved in Claim 2.

The third part of (P3) is a trivial consequence of the second part: For all $i$, $t$ we have

$$
\rho = |u_x(b_i(t),t) - u_x(a_i(t),t)| = \int_{a_i(t)}^{b_i(t)} |u_x x(\cdot,t)| \geq |b_i(t) - a_i(t)|\frac{K}{2}e^{\gamma t}. \tag{4.39}
$$

**Proof of (P4).** The key to understanding (P4) lies in the monotonicity property stated as Claim 1(i) above, which implies $[\alpha_i(t), \beta_i(t)] \subseteq [\alpha_i(t_0), \beta_i(t_0)]$ for all $t \geq t_0 \geq 0$. That is, the $[\alpha_i(t), \beta_i(t)]$ form a *nested* family of intervals. Since

$$
x_i(t) \in [a_i(t), b_i(t)] \subseteq [\alpha_i(t), \beta_i(t)] \subseteq [a_i^0, b_i^0],
$$

it is enough to prove

$$
|\beta_i(t) - \alpha_i(t)| \leq \frac{3\rho}{K}e^{-\gamma t}. \tag{4.40}
$$

But this follows readily as in (4.39):

$$
\begin{aligned}
2\rho &> 2\rho_0 = \left|\frac{q(\beta_i(t),t)}{\beta} - \frac{q(\alpha_i(t),t)}{\beta}\right| = \int_{\alpha_i(t)}^{\beta_i(t)}\left|\frac{q_x(\cdot,t)}{\beta}\right| \geq |\beta_i(t) - \alpha_i(t)|e^{\gamma t}\inf_{x\in(0,1)}\left|\frac{q(x,0)}{\beta}\right|\\
&\geq |\beta_i(t) - \alpha_i(t)|e^{\gamma t}(K - K_1(M)) \geq \frac{3}{4}K|\beta_i(t) - \alpha_i(t)|e^{\gamma t}.
\end{aligned}
$$

**Proof of (P5).** We introduce the sets

$$\tilde{\mathcal{L}}^i(t) := \{x \in (0,1) : \tfrac{q(x,t)}{\beta} \in z_i(\sigma([-\rho_1, \rho_1]))\} \quad (i = 1,2,3),$$

where $\rho_1 \in (3\eta, \rho_0] = (3\eta, \rho - \eta]$ (further restrictions on $\rho_1$ will be imposed later). Note that if $\rho_1 = \rho_0$, then $\tilde{\mathcal{L}}^2(t) = \tilde{\mathcal{L}}(t)$. As proved in Claim 1, $\tilde{\mathcal{L}}(t) \subseteq \tilde{\mathcal{L}}(t_0)$ for $t \geq t_0 \geq 0$, since the unstable interval $[-\beta\rho_0, \beta\rho_0]$ is negatively invariant for the 1D nonautonomous ODE (4.34). Similarly:

<u>Claim 3.</u> Assume $\epsilon \leq \sigma_-^2 \eta^2 \cdot \min\{\frac{\beta^2}{2\sigma_+^2}, \frac{1}{(K_2(M))^2}, \frac{1}{8\alpha}\}$. Then:

(i) $\tilde{\mathcal{L}}^i(t) \supseteq \tilde{\mathcal{L}}^i(t_0)$ for all $t \geq t_0 \geq 0$ and $i = 1,3$.

(ii) $x \notin \tilde{\mathcal{L}}^2(t),\ u_x(x,t) \begin{cases} < 0 \\ > 0 \end{cases} \implies x \in \begin{cases} \tilde{\mathcal{L}}^1(t+\tau) \\ \tilde{\mathcal{L}}^3(t+\tau) \end{cases}$ where $\tau = \frac{K_1(M)}{\sigma_-(\rho_1 - 3\eta)} (> 0)$.

The proof of (i) is very similar to that of Claim 1(i). By (4.30), (4.31), (4.29) and the choice of $\epsilon$

$$\|\frac{p(t)}{\beta}\|_{L^\infty} \leq \frac{\sigma_-}{\sigma_+}\eta, \tag{4.41}$$

$$\left|\int_0^1 \sigma(u_x(\cdot,t)\right| \leq \sigma_-\eta, \tag{4.42}$$

$$\|\pi_a\Big(\alpha \textstyle\int_0^x u(\cdot,t)\Big)\|_{L^\infty} \leq \sigma_-\eta \tag{4.43}$$

for all $t \geq 0$. We claim that if $q(x,t) = \beta z_j(\sigma(\pm\rho_1))$, then $u_x(x,t)$ lies between $z_j(\sigma(\rho_1 + \eta))$ and $z_j(\sigma(\rho_1 - \eta))$ (hence in particular $\text{sign}(u_x(x,t) - z_j(0)) = \text{sign}(q(x,t) - \beta z_j(0))$ and $\sigma(u_x(x,t)) \in \sigma([-\rho, \rho])$). Indeed, by (4.41)

$$\begin{aligned} |u_x(x,t) - z_j(\sigma(\rho_1))| &= |\frac{p(x,t)}{\beta}| \leq \frac{\sigma_-}{\sigma_+}\eta = \frac{\sigma_-}{\sigma_+}((\rho_1 + \eta) - \rho_1) \\ &\leq \frac{1}{\sigma_+}|\sigma(\rho_1 + \eta) - \sigma(\rho_1)| \leq |z_j(\sigma(\rho_1 + \eta)) - z_j(\sigma(\rho_1))| \end{aligned}$$

and similarly

$$|u_x(x,t) - z_j(\sigma(\rho_1))| \leq \frac{\sigma_-}{\sigma_+}\eta = \frac{\sigma_-}{\sigma_+}(\rho_1 - (\rho_1 - \eta)) \leq |z_j(\sigma(\rho_1)) - z_j(\sigma(\rho_1 - \eta))|.$$

Consequently, the interval $I_j = [\beta z_j(\sigma(\rho_1)), \beta z_j(\sigma(-\rho_1))]$ is for $j = 1,3$ a positively invariant interval for the ODE (4.34), since for $q(x,t) \in \partial I_j$

$$\begin{aligned} \frac{d}{dt}|q(x,t) - \beta z_j(0)| &= \text{sign}(q(x,t) - \beta z_j(0))\Big(-\sigma(\tfrac{q(x,t)}{\beta}) + (\sigma(\tfrac{q(x,t)}{\beta}) - \sigma(u_x(x,t))) \\ &\qquad + \textstyle\int_0^1 \sigma(u_x(\cdot,t)) + \pi_a(\alpha \int_0^x u(\cdot,t))\Big) \\ &\leq -\sigma_-\rho_1 + \sigma_+|\tfrac{p(x,t)}{\beta}| + \left|\textstyle\int_0^1 \sigma(u_x(\cdot,t))\right| + \|\pi_a(\alpha \textstyle\int_0^x u(\cdot,t))\|_{L^\infty} \\ &\leq \sigma_-(-\rho_1 + \eta + \eta + \eta) = \sigma_-(-\rho_1 + 3\eta) < 0. \end{aligned}$$

This proves (i). Now we prove (ii). Without loss of generality, assume $u_x(x,t) < 0$. If $x \in \tilde{\mathcal{L}}^1(t)$, then (ii) follows from (i), for any $\tau \geq 0$. So assume $x \notin \tilde{\mathcal{L}}^1(t)$. Hence for all $s \geq t$ with $x \in \tilde{\mathcal{L}}^1(s)$

$$\frac{d}{ds}|q(x,s) - \beta z_1(0)| \leq -\sigma_-(\rho_1 - 3\eta),$$

by estimating as above. Also, $\text{dist}(q(x,t), [\beta z_1(\sigma(\rho_1)), \beta z_1(\sigma(-\rho_1))]) \leq K_1(M)$ by (4.26). Hence if $x \notin \tilde{\mathcal{L}}^1(s)$ for all $s \in [t, t+\tau]$ and some $\tau > 0$, then

$$\text{dist}\Big(q(x,t+\tau), [\beta z_1(\sigma(\rho_1)), \beta z_1(\sigma(-\rho_1))]\Big) \leq K_1(M) - \tau\sigma_-(\rho_1 - 3\eta).$$

Since the left hand side is greater or equal to zero, this implies $\tau \leq \frac{K_1(M)}{\sigma_-(\rho_1-3\eta)}$, proving (ii).

Now assume that $\rho_1$, $\eta$ are so small that the interval $[-r,r]$ from Definition 3.1 contains $\sigma([-\rho_1-\eta,\rho_1+\eta])$. Then since $\sigma(u_x(x,t)) \in \sigma([-\rho_1-\eta,\rho_1+\eta])$ whenever $x \in \tilde{\mathcal{L}}^j(t)$ $(j=1,3)$, we have

$$x \in \tilde{\mathcal{L}}^j(t) \implies \psi(u_x(x,t)) = j.$$

This together with Claim 3 allows to calculate $\psi(u_x^*)$: Assume without loss $(u_0)_x|_{(x_i^0,x_{i+1}^0)} < 0$. Then $(u_0)_x|_{(\beta_i(t),\alpha_{i+1}(t))} < 0$ for all $t \geq 0$, hence $\psi(u_x(\cdot,t+s))|_{(\beta_i(t),\alpha_{i+1}(t))} = 1$ for all $s \geq \tau$, by Claim 3. Consequently,

$$\psi(u_x^*)|_{\bigcup_{t\geq 0}(\beta_i(t),\alpha_{i+1}(t))} = 1.$$

Since $\bigcup_{t\geq 0}(\beta_i(t),\alpha_{i+1}(t)) = (x_i^*,x_{i+1}^*)$, this proves the last part of (P5). Now it is clear that $u_x^*$ is discontinuous at the $x_i^*$ (since $\psi(u_x^*)$ jumps), and continuity on $(x_i^*,x_{i+1}^*)$ follows from the equilibrum equation: $\sigma(u_x^*(x)) = \alpha\int_0^x u^* + \lambda$ for some $\lambda \in \mathbb{R}$, and hence

$$u_x^* = z_j(\alpha\textstyle\int_0^x u^* + \lambda) \quad \text{in } (x_i^*,x_{i+1}^*) \text{ for some } j \in \{1,3\} \text{ independent of } x,$$

consequently $u \in C^2((x_i^*,x_{i+1}^*))$ ($\in C^\infty$ if $\sigma$ is $C^\infty$). (P5) is proved.

**Proof of (P6).** We will apply Lemma A2 (or more precisely: Corollary A1) with $A$, $f$, $x(t) = (p(t),q(t))$, $\alpha = 1/2$ as in Section 2, but $X = L_a^2 \times L_a^2$ and $\mathcal{D}(A) = \{p \in W_a^{2,2} : p_x|_{\{0,1\}} = 0\} \times L_a^2$ unlike in Section 2. Unless $\sigma$ is globally lipschitz – which we have no intention to assume –, $f$ is now not defined on the whole of $X^{1/2}$, but $x$ is a global solution $\in C([0,\infty);X^{1/2}) \cap C^1((0,T);X) \cap C((0,T);\mathcal{D}(A))$, as required. We set $x_0 := (0,q^*) := (0,\beta u_x^*)$ so that $x(t) = (p(t),q(t)) \to (0,q^*) = x_0$ in $W_a^{1,2} \times L_a^2 = X^{1/2}$ as $t \to \infty$, and let $B$ be the Gateaux derivative of $f$ at $x_0$. Let us write out what $x(t) - x_0$, $B$ and $g(x(t))$ are:

$$\begin{aligned}
x(t) - x_0 &= \begin{pmatrix} p(t) \\ q(t) - q^* \end{pmatrix}, \\
B\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} &= \pi_a\Big(\sigma'(u_x^*)\Big(\tfrac{\tilde{p}+\tilde{q}}{\beta}\Big) - \alpha\textstyle\int_0^x\int_0^{x'}\tfrac{\tilde{p}+\tilde{q}}{\beta}\Big)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
g(x(t)) &= \pi_a\Big(\sigma(\tfrac{p(t)+q(t)}{\beta}) - \sigma(\tfrac{0+q^*}{\beta}) - \sigma'(u_x^*)(\tfrac{(p(t)-0)+(q(t)-q^*)}{\beta})\Big)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{aligned}$$

We verify hypotheses (i), (ii), (iii) of Corollary A1.

(i) is clear: in fact, $B$ is a bounded linear map from $X$ to $X$.

(ii) is taken care of by (P5) (which implies $\inf_{x\in(0,1)}\sigma'(u_x^*(x)) > 0$) together with a Lemma of Ball, Holmes, James, Pego and Swart:

**Lemma 4.1** (BHJPS, Theorem 3.3) *With $A$, $B$ as above and $X = L_a^2 \times L_a^p$, $\mathcal{D}(A) = \{p \in W_a^{2,2} : p_x|_{\{0,1\}} = 0\} \times L_a^p$ $(1 \leq p \leq \infty)$, the spectrum of $A - B$ lies in*

$$\{\lambda \in \mathbb{C} : Re\,\lambda \geq \min\{\tfrac{1}{\beta}\inf_{x\in(0,1)}\sigma'(u^*(x)),\ \tfrac{\beta}{2}\}\}.$$

(In [BHJPS] the result is stated with $p = \infty$, but their proof in fact does not require any knowledge about $p$.)

It remains to verify (iii). Since $||(\tilde{p},\tilde{q})|| = ||\tilde{p}||_{L^2} + ||\tilde{q}||_{L^2}$,

$$\begin{aligned}
||g(x(t))||^2 &= 4||\pi_a(\sigma(u_x(t)) - \sigma(u_x^*) - \sigma'(u_x^*)(u_x(t) - u_x^*))||_{L^2}^2 \\
&\leq 4\int_0^1 |\sigma(u_x(t)) - \sigma(u_x^*) - \sigma'(u_x^*)(u_x(t) - u_x^*)|^2 dx.
\end{aligned}$$

Now at each time $t$, we split the domain of integration into two parts, a "stable" and an "unstable" region:

$$\begin{aligned}
\Omega_s(t) &:= \tilde{\mathcal{L}}^1(t) \cup \tilde{\mathcal{L}}^3(t) \\
\Omega_u(t) &:= (0,1)\backslash\Omega_s(t).
\end{aligned}$$

First, let us deal with the stable region.

Since $\sigma$ is continuously differentiable, there exists $\bar{\epsilon}(\lambda) \to 0$ $(\lambda \to 0)$ such that

$$\left|\frac{\sigma(z)-\sigma(z_0)-\sigma'(z_0)(z-z_0)}{z-z_0}\right| \le \bar{\epsilon}(\lambda) \quad \text{for all } z,\, z_0 \in [-K_1(M), K_1(M)] \text{ with } |z-z_0| \le \lambda.$$

Now if $x \in \Omega_s(t)$, then $u_x(x,t)$ lies between $z_j(\sigma(\rho_1+\eta))$ and $z_j(\sigma(-\rho_1-\eta))$ for some $j \in \{1,3\}$. By the monotonicity of the $\Omega_s(t)$ (by Claim 3, $\Omega_s(t) \supseteq \Omega_s(t_0)$ for $t \ge t_0 \ge 0$), so also must $u_x^*(x)$, with the same $j$, hence

$$|u_x(x,t)-u_x^*(x)| \le |z_j(\sigma(\rho_1+\eta)) - z_j(\sigma(-\rho_1-\eta))| \le \frac{\sigma_+}{\sigma_-}|\rho_1+\eta-(-\rho_1-\eta)| = \frac{\sigma_+}{\sigma_-}\cdot 2(\rho_1+\eta).$$

Now choose $\rho_1 \le 4\eta$, and then

$$\int_{\Omega_s(t)} |\sigma(u_x(t)) - \sigma(u_x^*) - \sigma'(u_x^*)(u_x(t)-u_x^*)|^2 dx \le \underbrace{\bar{\epsilon}^2(\tfrac{\sigma_+}{\sigma_-}\cdot 10\eta)}_{=:\bar{\epsilon}(\eta)} ||u_x(t)-u_x^*||_{L^2}^2 \tag{4.44}$$

where $\bar{\epsilon}(\eta) \to 0$ as $\eta \to 0$.

In the unstable region, the integrand is not small, but the unstable region itself shrinks exponentially: By Claim 3, $\Omega_u(t) \subseteq \tilde{\mathcal{L}}(t-\tau)$ for $t \ge \tau$, hence by (4.40)

$$|\Omega_u(t)| \le \frac{3\rho N}{K} e^{-\gamma t}$$

and thus by (4.25)

$$\int_{\Omega_u(t)} |\sigma(u_x(t) - \sigma(u_x^*) - \sigma'(u_x^*)(u_x(t)-u_x^*)|^2 dx \le \frac{K_3(M)\cdot 3\rho N}{K} e^{-\gamma t} =: \tilde{K}^2 e^{-\gamma t}. \tag{4.45}$$

Consequently by combining (4.44) and (4.45)

$$\begin{aligned}
||g(x(t))|| &\le \left(4\bar{\epsilon}^2(\eta)||u_x(t)-u_x^*||_{L^2}^2 + 4\tilde{K}^2 e^{-\gamma t}\right)^{1/2} \\
&\le 2\bar{\epsilon}(\eta)||u_x(t)-u_x^*||_{L^2} + 2\tilde{K}e^{-\frac{\gamma}{2}t} \\
&\le 2\bar{\epsilon}(\eta)(||\frac{p-0}{\beta}||_{L^2} + ||\frac{q-q^*}{\beta}||_{L^2}\Big) + 2\tilde{K}e^{-\frac{\gamma}{2}t} \\
&= \frac{2\bar{\epsilon}(\eta)}{\beta}\Big(||x(t)-x_0|| + 2\tilde{K}e^{-\frac{\gamma}{2}t}.
\end{aligned}$$

Now note that $||x(t)-x_0|| \le ||x(t)-x_0||_{1/2}$, and choose $\eta$ small enough so that $\frac{2\bar{\epsilon}(\eta)}{\beta} < r_0$, where $r_0$ is the constant supplied by Corollary A1. Then Corollary A1 applies and we obtain:

$$||(p(t),q(t)) - (0,q^*)||_{W^{1,2}\times L^2} \le \tilde{M}e^{-\frac{1}{2}\gamma' t}$$

for some constant $\tilde{M}$ and all $\gamma' < \min\{\gamma, 2\delta\} = \min\{\gamma, \beta, \frac{2\inf_{x\in(0,1)}\sigma'(u^*(x))}{\beta}\}$ and hence in particular for

$$\gamma' = \min\{\beta, \tfrac{1}{2\beta} \inf_{\sigma^{-1}(\sigma([-\rho,\rho]))} |\sigma'|\}.$$

This establishes (P6) in the case $p=2$, because

$$||(u(t),v(t)) - (u^*,0)||_{W^{1,2}\times L^2} = ||\mathcal{P}^{-1}(p(t),q(t)) - \mathcal{P}^{-1}(0,q^*)||_{W^{1,2}\times L^2}$$

and the inverse Pego transform $\mathcal{P}^{-1} : W_a^{1,2} \times L_a^2 \to W_0^{1,2} \times L^2$ is continuous (see Lemma 2.1). Finally, for $p \in (2,\infty)$ we can estimate using (4.26) and Hölder's inequality

$$\begin{aligned}
||q(t)-q^*||_{L^p} &\le ||q(t)-q^*||_{L^2}^{2/p} ||q(t)-q^*||_{L^\infty}^{1-2/p} \\
&\le \tilde{M}^{2/p} e^{-\frac{\gamma'}{p}t}(2K_1(M))^{1-2/p} \\
&\le \max\{\tilde{M},1\}\cdot\max\{2K_1(M),1\}\cdot e^{-\frac{\gamma'}{p}t}.
\end{aligned}$$

A straightforward calculation shows that the operator norm of $\mathcal{P}^{-1}$ in $\mathcal{L}(W_a^{1,2} \times L_a^p, W_0^{1,p} \times L^2)$ is bounded independently of $p$, precisely:

$$||\mathcal{P}^{-1}(\tilde{p}, \tilde{q})||_{W^{1,p}\times L^2} \leq (\frac{2}{\beta} + 1)||(\tilde{p}, \tilde{q})||_{W^{1,2}\times L^p} \quad \text{for all } (\tilde{p}, \tilde{q}) \in W_a^{1,2} \times L_a^p,$$

and (P6) follows.

**Proof of (P7).** The local minimization property of $(u^*, 0)$ is a consequence of the (non-elementary) fact (P5), which implies $\sigma'(u_x^*(x)) \geq \inf\{\sigma'(z) : z \in z_1(\sigma([-\rho, \rho])) \cup z_3(\sigma([-\rho, \rho]))\} > 0$ for all $x \in (0,1)\backslash\{x_1^*, ..., x_N^*\}$, together with the following (elementary) lemma on the variational integral

$$J(u) = \int_0^1 (W(u_x(x)) + \frac{\alpha}{2}u^2(x))\, dx.$$

**Lemma 4.2** *Let $u_0 \in W_0^{1,\infty}$ be a stationary point of $J$ (that is to say $\frac{d}{d\epsilon}|_{\epsilon=0} J(u_0 + \epsilon\phi) = 0$ for all $\phi \in W_0^{1,\infty}$) with $\sigma'((u_0)_x(x)) \geq \bar{\sigma} > 0$ a.e. Then $u_0$ is a local minimizer of $J$ in $W_0^{1,\infty}$.*

**Proof of the Lemma.** This is a standard fact from the calculus of variations, but for convenience of the reader we include a proof. Let $u \in W_0^{1,\infty}$, $||u_x - (u_0)_x||_{L^\infty} < \epsilon$, $\epsilon \in (0,1)$. Since $W$ is $C^2$, $W(u_x) = W((u_0)_x) + \sigma((u_0)_x))(u_x - (u_0)_x) + \frac{\sigma'(\xi)}{2}(u_x - (u_0)_x)^2$ for some function $\xi$ (which can be chosen to depend measurably on $x$) with values $\xi(x)$ strictly between $u_x(x)$ and $(u_0)_x(x)$ . Now if $\epsilon$ is small enough, then $\sigma'(\xi(x)) \geq \bar{\sigma}/2 > 0$. Also, since $u_0$ is a stationary point of $J$ (or equivalently: $(u_0, 0)$ is an equilibrium of (2.1)), there exists $\lambda \in I\!R$ such that $\sigma((u_0)_x(x)) - \alpha \int_0^x u_0 \equiv \lambda$ a.e. Hence we compute

$$\begin{aligned} J(u) - J(u_0) &= \int_0^1 \Big(\sigma((u_0)_x)(u - u_0)_x + \frac{\sigma'(\xi)}{2}((u - u_0)_x)^2 + \alpha u_0(u - u_0) + \frac{\alpha}{2}(u - u_0)^2\Big) \\ &= \lambda \underbrace{\int_0^1 (u - u_0)_x}_{=0} + \int_0^1 \Big(\frac{\sigma'(\xi)}{2}((u - u_0)_x)^2 + \frac{\alpha}{2}(u - u_0)^2\Big) \\ &\geq 0. \end{aligned}$$

The last part of (P7) is taken care of by another elementary lemma:

**Lemma 4.3** *Let $\alpha > 0$. Then $J$ does not possess any local minimizers in $W_0^{1,p}$ $(1 \leq p < \infty)$.*

**Proof.** (For $W(z) = \frac{(z^2-1)^2}{4}$ the argument is sketched in [BHJPS, p. 23].)
Fix $p \in [1, \infty)$. First, we prove that $u_0 \equiv 0$ is not a local minimizer. Let $\lambda = \frac{z_+}{z_+ - z_-}$, $w = \begin{cases} z_-, & 0 < x \leq \lambda \\ z_+, & \lambda < x < 1, \end{cases}$, $u(x) = \int_0^x w$ extended periodically to the whole of $I\!R$, $u^n(x) = \frac{1}{n}u(nx)$, $\tilde{u}^n(x) = \begin{cases} u^n(x), & 0 < x \leq 1/n \\ 0, & \text{else.} \end{cases}$ Then

$$J(\tilde{u}^n) - J(u_0) = -\frac{1}{n}W(0) + \frac{1}{n^3}\frac{\alpha}{2}\int_0^1 u^2 < 0 \quad \text{for } n \text{ large enough,}$$

but $\tilde{u}^n \longrightarrow u_0$ in $W^{1,p}$ as $n \longrightarrow \infty$.

Now we prove that $u_0 \not\equiv 0$ is not a local minimizer. To this end, let $x_0 \in (0,1)$ be a point where $|u_0|$ achieves its global maximum. Since $u_0 \not\equiv 0$, $u_0(x_0) \neq 0$. Without loss of generality, assume $u_0(x_0) > 0$. Set $\bar{u}^n = \min\{u_0, u_0(x_0) + \tilde{u}^n(\cdot + \frac{\lambda}{n} - x_0)\}$. Then for $n$ large enough,

$$J(\bar{u}^n) - J(u_0) < 0,$$

but $\bar{u}^n \longrightarrow u_0$ in $W^{1,p}$ as $n \longrightarrow \infty$. The proof of the lemma is complete.

This concludes the proof of Theorem 4.1 and our discussion of layer dynamics.

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# Appendix: Two results on the abstract parabolic equation $x_t + Ax = f(t,x)$.

Our first result is a simple a-priori estimate which follows immediately from standard estimates (Henry [He1, 1.4.3]) for fractional powers of sectorial operators.

Our second result generalizes a conclusion of Henry [He1, Theorem 5.1.1] on exponential convergence to equilibrium via linearization. Namely, we extend Henry's result to nonlinearities $f(t,x)$ which are not Fréchet-differentiable at the limiting equilibrium but only possess directional derivatives. This more complicated situation is not an esoteric peculiarity of the partial differential equation studied in this paper, but occurs whenever one wants to study the approach of smooth solutions of a partial differential equation to discontinuous patterns as time $t$ tends to infinity; see Remark 5 in Section 4.

Notation. As in Henry [He1, Section 5.1], $A$ denotes a sectorial operator on a Banach space $X$, $\mathcal{D}(A)$ denotes the domain of $A$, $X^\beta$ $(0 \le \beta < 1)$ are the associated fractional power spaces, $||\cdot||$ and $||\cdot||_\beta$ denote the norm in $X$ and, respectively, in $X^\beta$, $V$ is an open subset of $X^\alpha$ (for some fixed $\alpha \in [0,1)$), and $f : [0,\infty) \times V \to X$ is a mapping which is locally lipschitz continuous in $x$ and locally Hölder continuous in $t$, that is, given $(t_1, x_1) \in [0,\infty) \times V$ there exists a neighbourhood $\tilde{V}$ of $(t_1, x_1)$ in $[0,\infty) \times V$ and constants $\theta$, $L > 0$ such that for all $(t,x)$, $(s,y) \in \tilde{V}$,

$$||f(t,x) - f(s,y)|| \le L\Big(|t-s|^\theta + ||x-y||_\alpha\Big).$$

Note that by [He1, Thm 3.3.3], for such $A$ and $f$ the initial value problem

$$\begin{cases} x_t + Ax = f(t,x) \\ x(0) = x_1 \end{cases} \tag{A1}$$

possesses for all $x_1 \in V$ a unique local solution $x \in C([0,T); X^\alpha) \cap C^1((0,T); X) \cap C((0,T); \mathcal{D}(A))$ on some time interval $[0, T(x_1))$.

**Lemma A1** *Let $\gamma \in [0,1)$ and $||A^\gamma z_0|| \le M_{z_0}$, let $T \in (0,\infty]$, and assume $z(t)$ is a solution on $[0,T)$ of (A1) such that $\sup_{t\in[0,T)} ||z(t)|| \le M_z$, $\sup_{t\in[0,T)} ||f(z(t),t))|| \le M_f$. Then there exists $C(\gamma, M_{z_0}, M_z, M_f)$ independent of $T$ such that*

$$\sup_{t\in[0,T)} ||A^\gamma z(t)|| \le C.$$

**Proof.** Let $T(t)$ denote the semigroup generated by $A$. Fix $\tau \in (0,T)$ and estimate differently in the intervals $[0,\tau]$ and $[\tau,\infty)$:

$$\begin{aligned} ||A^\gamma z(t)|| &= ||A^\gamma T(t) z_0 + \textstyle\int_0^t A^\gamma T(t-s) f(z(s),s)\, ds|| \\ &\le C(\tau)||A^\gamma z_0|| + \int_0^t \frac{C(\tau)}{(t-s)^\gamma} M_f\, ds \\ &\le C(\tau)\Big(M_{z_0} + \frac{M_f \tau^{1-\gamma}}{1-\gamma}\Big) \qquad \forall t \in [0,\tau] \end{aligned}$$

and

$$\begin{aligned} ||A^\gamma z(t)|| &= ||A^\gamma T(\tau) z(t-\tau) + \textstyle\int_0^\tau A^\gamma T(\tau-s) f(z(t-\tau+s), t-\tau+s)\, ds|| \\ &\le \frac{C(\tau)}{\tau^\gamma}||z(t-\tau)|| + \int_0^\tau \frac{C(\tau)}{(\tau-s)^\gamma} M_f\, ds \\ &\le C(\tau)\Big(\frac{M_z}{\tau^\gamma} + \frac{M_f \tau^{1-\gamma}}{1-\gamma}\Big) \qquad \forall t \in [\tau, T). \end{aligned}$$

The proof of the Lemma is complete.

**Lemma A2** *Let $A$, $\alpha$, $V$, $f$ be as above, let $x_0 \in V$ be an equilibrium point of* (A1) *(i.e. $x_0 \in \mathcal{D}(A)$, and $Ax_0 = f(t, x_0)$ $\forall t > 0$), and let $x : [0, \infty) \to V$ be a solution of* (A1). *Suppose in addition that*

$$f(t, x) - f(t, x_0) = B(x - x_0) + g(t, x)$$

*where*

(i) $B : X^\alpha \to X$ *is a bounded linear map,*

(ii) *the spectrum of $A - B$ lies in $\{Re\,\lambda > \delta\}$ for some $\delta > 0$, and*

(iii) $||g(t, x(t))|| = o(||x(t) - x_0||_\alpha) + O(e^{-\gamma t})$ *as* $||x(t) - x_0||_\alpha \to 0$, *for some* $\gamma > 0$.

*Then there exist $\rho$, $M > 0$ such that provided*

$$||x(t) - x_0||_\alpha \le \rho \qquad \forall t \ge 0,$$

*then*

$$||x(t) - x_0||_\alpha \le M e^{-\min\{\gamma, \delta\} t} \qquad \forall t \ge 0.$$

**Proof.** Clearly $A - B$ is a sectorial operator. Pick $\delta'$ such that $0 < \delta < \delta' < Re\,Spec(A - B)$, and let $T(t)$ be the semigroup generated by $A - B$. By standard estimates [He1, Sections 1.3, 1.4],

$$\begin{aligned} ||T(t)z||_\alpha &\le Ce^{-\delta' t}||z||_\alpha \qquad && \forall z \in X_\alpha,\ \forall t > 0 \\ ||T(t)z||_\alpha &\le Ct^{-\alpha}e^{-\delta' t}||z|| && \forall z \in X,\ \forall t > 0 \end{aligned}$$

with some constant $C$. Let $m := \min\{\gamma, \delta\}$, and choose $r_0 > 0$ so small that

$$r_0 \int_0^\infty s^{-\alpha} e^{-(\delta' - m)s} ds < \frac{1}{2C}. \tag{A2}$$

Choose $\rho > 0$ so small that

$$||g(t, x)|| \le r_0 ||x - x_0||_\alpha + Ke^{-\gamma t} \tag{A3}$$

for $||x - x_0||_\alpha \le \rho$, $t \ge 0$. Let $z(t) = x(t) - x_0$. Then $z$ solves

$$z_t + (A - B)z = g(t, x_0 + z),$$

hence

$$\begin{aligned} ||z(t)||_\alpha &= ||T(t)z(0) + \int_0^t T(t - s)g(s, x_0 + z(s))ds||_\alpha \\ &\le Ce^{-\delta' t}||z(0)||_\alpha + C\int_0^t (t - s)^{-\alpha} e^{-\delta'(t - s)}\Big(r_0||z(s)||_\alpha + Ke^{-\gamma s}\Big)\, ds. \end{aligned} \tag{A4}$$

Now define $u(t) := \sup_{s \in [0,t]} e^{ms}||z(s)||_\alpha$, let $\tau \in (0, t]$ and compute using (A4) and (A2)

$$\begin{aligned} e^{m\tau}||z(\tau)||_\alpha &\le C||z(0)||_\alpha + C\int_0^\tau (\tau - s)^{-\alpha} e^{-(\delta' - m)(\tau - s)}\Big(r_0 e^{ms}||z(s)||_\alpha + K\Big)\, ds \\ &\le C||z(0)||_\alpha + C\Big(r_0\, u(t) + K\Big)\int_0^\infty \xi^{-\alpha} e^{-(\delta' - m)\xi} d\xi \\ &\le C||z(0)||_\alpha + \frac{1}{2}u(t) + \frac{K}{2r_0}, \end{aligned}$$

hence by taking the supremum over $\tau \in (0, t]$

$$u(t) \le 2C||z(0)||_\alpha + \frac{K}{r_0}.$$

The proof of the Lemma is complete. In fact, since the assumptions (iii) and $||x(t) - x_0||_\alpha \le \rho$ were only needed to obtain (A3), the above proof yields

**Corollary A1** *Assume all hypotheses of Lemma A2 except* (iii) *hold. Then there exist* $r_0$, $M > 0$ *such that provided*

$$||g(t, x(t))|| \leq r_0 ||x(t) - x_0||_\alpha + Ke^{-\gamma t} \quad \textit{for all } t \geq 0 \textit{ and some } K, \gamma > 0$$

*then*

$$||x(t) - x_0||_\alpha \leq Me^{-\min\{\gamma,\delta\}t} \qquad \forall t \geq 0.$$

# References


[AB] G. Andrews & J. M. Ball. Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity. *J. Diff. Eqns.* 44:306-341, 1982

[ABF] N. Alikakos, P. W. Bates & G. Fusco. Slow motion for the Cahn-Hilliard equation in one space dimension. *J. Diff. Eqns.* 90:81-135, 1991

[AK] R. Abeyaratne & J. Knowles. Nucleation kinetics and admissibility criteria for propagating phase boundaries. IMA preprint No. 1044, University of Minnesota, 1992

[An] G. Andrews. On the existence of solutions to the equation $u_{tt} = u_{xxt} + \sigma(u_x)_x$. *J. Diff. Eqns.* 35:200-231, 1980

[Ba] J. M. Ball. Dynamics and minimizing sequences. In: *Problems involving change of type*, ed. K. Kirchgässner, Lecture Notes in Physics 359:3-16, Springer-Verlag, 1990

[BCJ] J. M. Ball, C. Chu & R. D. James. Metastability and hysteresis in elastic crystals. To appear.

[BF] P. W. Bates & P. C. Fife. The dynamics of nucleation for the Cahn-Hilliard equation. *SIAM J. Appl. Math.* To appear.

[BF1] P. Brunovský & B. Fiedler. Connecting orbits in scalar reaction diffusion equations. *Dynamics reported* Vol. 1. U. Kirchgraber & H. O. Walther (eds.), John Wiley & Sons and B. G. Teubner, 1988

[BF2] P. Brunovský & B. Fiedler. Connecting orbits in scalar reaction diffusion equations II. The complete solution. *J. Diff. Eqns.* 81:106-135, 1989

[BFJK] K. Bhattacharya, N. B. Firoozye, R. D. James & R. V. Kohn. Restrictions on microstructure. *Proc. R. Soc. Edinb.* To appear.

[BFS] D. Brandon, I. Fonseca & P. J. Swart. The creation and propagation of oscillations in a dynamical model of displacive phase transformations. To appear.

[BG] G. S. Bales & R. J. Gooding. Interfacial dynamics at a first-order phase transition involving strain: dynamical twin formation. *Phys. Rev. Lett.* 67-24: 3411-3415, 1991

[Bh] K. Bhattacharya. Self-accomodation in martensite. *Arch. Rat. Mech. Analysis* 120:201-244, 1992

[BHJPS] J. M. Ball, P. J. Holmes, R. D. James, R. L. Pego & P. J. Swart. On the dynamics of fine structure. *J. Nonlinear Sci.* 1:17-70, 1991

[BJ1] J. M. Ball & R. D. James. Fine phase mixtures as minimizers of energy. *Arch. Rat. Mech. Analysis* 100: 13-52, 1987

[BJ2] J. M. Ball & R. D. James. Proposed experimental tests of a theory of fine microstructure and the two-well problem. *Phil. Trans. R. Soc. London A*, 338: 389-450, 1992

[BX1] P. W. Bates & P. J. Xun. Metastable patterns for the Cahn-Hilliard Equation: Part I. *J. Diff. Eqns.* To appear.

[BX2] P. W. Bates & P. J. Xun. Metastable patterns for the Cahn-Hilliard-Equation: Part II. Layer dynamics and slow invariant manifold. *J. Diff. Eqns.* To appear.

[Ce] L. Cesari. *Optimization-theory and applications.* Springer-Verlag, 1983

[CFNT] P. Constantin, C. Foias, B. Nicolaenko & R. Temam. *Integral manifolds and inertial manifolds for dissipative partial differential equations.* Applied mathematical sciences 70. Springer-Verlag, 1989

[CK] M. Chipot & D. Kinderlehrer. Equilibrium configurations of crystals. *Arch. Rat. Mech. Analysis* 103:237-277, 1988

[CP] J. Carr & R. L. Pego. Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$. *Comm. Pure Appl. Math.* XLII:523-576, 1989

[Da] C. M. Dafermos. The mixed initial-boundary value problem for the equations of nonlinear one dimensional viscoelasticity. *J. Diff. Eqns.* 6:71-86, 1969

[DM] G. Dolzmann & S. Müller. Microstructures with finite surface energy: the two-well problem. To appear.

[FH] G. Fusco & J. K. Hale. Slow-motion manifolds, dormant instability, and singular perturbations. *J. Dyn. Diff. Eqns.* 1:75-94, 1989

[FM1] P. C. Fife & J. B. McLeod. The approach of solutions of nonlinear diffusion equations to travelling wave solutions. *Bull. Amer. Math. Soc.* 81:1076-1078, 1975

[FM2] P. C. Fife & J. B. McLeod. The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Rat. Mech. Analysis* 65:335-361, 1977

[FM3] P. C. Fife & J. B. McLeod. A phase plane discussion of convergence to travelling fronts for nonlinear diffusion. *Arch. Rat. Mech. Analysis* 75:281-314, 1981

[Fo] I. Fonseca. Phase transitions for elastic solid materials. *Arch. Rat. Mech. Analysis* 107:195-223, 1989

[Fr1] G. Friesecke. *Static and dynamic problems in nonlinear mechanics.* PhD-thesis, Heriot-Watt University, Edinburgh, 1993

[Fr2] G. Friesecke. A necessary and sufficient condition for nonattainment and formation of microstructure almost everywhere in scalar variational problems. *Proc. R. Soc . Edinb. A.* In the press.

[Fr3] G. Friesecke. A dynamical approach to hysteresis in a bar undergoing a martensitic transformation. In: H. T. Banks (ed.), *Mathematics and Control in Smart Structures* 2192:88-93, 1994

[GB] R. J. Gooding & G. S. Bales. Boosted kink-type solitary waves in nonlinear elastic media. *Physica D*, 55: 251-258, 1992

[Gu] M. E. Gurtin. The nature of configurational forces. Research report No. 94-NA-013, Carnegie Mellon University, 1994

[Ha] J. K. Hale. *Asymptotic behavior of dissipative systems.* Mathematical surveys and monographs 25. AMS, Providence, 1988

[HaM] J. K. Hale & P. Massat. Asymptotic behavior of gradient-like systems. In: *Dynamical Systems II*, eds. A. R. Bednarek & L. Cesari, Academic Press, 1982

[HaR] J. K. Hale & G. Raugel. Convergence in gradient-like systems with applications to PDE. *Z. angew. Math. Phys.* 43:63-124, 1992

[He1] D. Henry. Geometric theory of semilinear parabolic equations. *Springer Lecture Notes in Mathematics* 840. Springer-Verlag, 1981

[He2] D. Henry. Some infinite-dimensional Morse-Smale systems defined by parabolic partial differential equations. *J. Diff. Eqns.*, 59: 165-205, 1985

[HM] H. Hattori & K. Mischaikow. A dynamical systems approach to a phase transition problem. *J. Diff. Eqns.* 94:340-378, 1991

[HS] K. H. Hoffmann & Z. Songmu. Uniqueness for structural phase transitions in shape memory alloys. *Math. Methods in the Applied Sci.* 10:145-151, 1988

[JK] R. D. James & D. Kinderlehrer. Theory of magnetostriction with applications to $Tb_xDy_{1-x}Fe_2$. *Phil. Mag. B*, 68:237-274, 1993

[KH] W. D. Kalies & P. J. Holmes. On a dynamical model for phase transformation in nonlinear elasticity. Preprint, 1993

[KM] R. V. Kohn and S. Müller. Branching of twins near an austenite/twinned-martensite interface. *Phil. Mag. A*, 66: 697-715, 1992

[KP] D. Kinderlehrer & P. Pedregal. Characterizations of Young measures generated by gradients. *Arch. Rat. Mech. Analysis* 115:329-365, 1991

[LP] J. Lin & T. J. Pence. On the dissipation due to wave ringing in nonelliptic elastic materials. *J. Nonlinear Sci.* 3:269-305, 1993

[Ma] J. P. Matos. Young measures and the absence of fine microstructures in a class of phase transitions. *Euro. J. Applied Math.* 3:31-54, 1992

[MS] S. Müller & V. Šverák. Surprising attainment results for the two-well problem. To appear.

[MW] L. Ma & N. Walkington. On algorithms for non-convex optimization. Research Report No. 93-NA-010, Carnegie Mellon University, 1993

[NP] A. Novick-Cohen & R. L. Pego. Stable patterns in a viscous diffusion equation. *Trans. Amer. Math. Soc.* 324:331-351, 1991

[NS] M. Niezgodka & J. Sprekels. Existence of solutions of a mathematical model of structural phase transitions in shape memory alloys. *Math. Methods in the Applied Sci.* 10:197-223, 1988

[Pe1] R. L. Pego. Phase transitions in one-dimensional nonlinear viscoelasticity: Admissibility and Stability. *Arch. Rat. Mech. Analysis* 97: 353-394, 1987

[Pe2] R. L. Pego. Stabilization in a gradient system with a conservation law. *Proc. AMS* 114:1017-1024, 1992

[RG] A. C. E. Reid & R. J. Gooding. Elastic hydrodynamics and dynamical nucleation in first-order strain transitions. *Physica D*, 66: 180-186, 1993

[Ry] P. Rybka. Dynamical modeling of phase transitions by means of viscoelasticity in many dimensions. *Proc. R. Soc. Edinb.* A, 121:101-138, 1992

[SH] P. J. Swart & P. J. Holmes. Energy minimization and the formation of microstructure in dynamic anti-plane shear. *Arch. Rat. Mech. Analysis*, 121: 37-85, 1992

[Sv1] V. Šverák. On Tartar's conjecture. *Inst. H. Poincaré – Anal. Nonl.* To appear.

[Sv2] On the problem of two wells. In: *Microstructure and phase transition*, D. Kinderlehrer et al. (eds.). Springer-Verlag, 1992

[Te] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*. Applied mathematical sciences 68. Springer-Verlag, 1988

[Yo] L. C. Young. *Lectures on the calculus of variations and optimal control theory*. Chelsea, 1980

[Zh] K. Zhang. Rank-one connections and the three-well problem. Preprint.