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1. Introduction.

The purpose of this note is to prove uniqueness of the positive radial solution to the Dirichlet problem for

$$\Delta u - u + u^3 = 0, \tag{1.1}$$

on the annulus $\Omega = \{x \in \mathbf{R}^3 : R_1 < |x| < R_2\}$ where $0 < R_1 < R_2 \leq \infty$; (the Dirichlet condition at $R_2 = \infty$ is interpreted, as usual, to mean that the solution belongs to $L^2(\Omega)$). In [2], uniqueness on a ball and uniqueness to within translation on $\mathbf{R}^3$ were proved for the positive radial solution of the Dirichlet problem for this equation.

The proof in [2] made use of features peculiar to the case of dimension 3 and to the cubic non-linearity. (However, contrary to a statement made there, the proof does still apply if the cubic non-linearity is replaced by u^p with $1 < p \leq 3$; this was first brought to my attention by George Hanna.) In [11], McLeod and Serrin generalized the results of [2] both in regard to the dimension and to the form of the non-linearity. For the case of the power non-linearity in $\mathbf{R}^3$ these results still gave uniqueness only for powers in the range $1 < p \leq 3$, while one has existence for $1 < p < 5$. The situation in [11] was similar for other dimensions. Subsequently, Kwong, [8], proved uniqueness, in all dimensions, for the equation with power non-linearity and for all values of the exponent for which there is existence. McLeod, [12], further generalized these results to a larger class of non-linearities. A phase-space geometric proof has been given by Clemons and Jones, [1].

For the equations to which they apply, any of proofs described above can be adapted to yield uniqueness for the positive *radial* solution, on an annulus or *radial* exterior domain, for the mixed boundary value problem with Neumann condition on the inner boundary and Dirichlet condition on the outer boundary. This appears to have first been noticed by Kwong,[8], see also [9]. The Dirichlet problem on the annulus seems to require further analysis. It is the purpose of this note to provide that analysis.

It should be remarked that the well-known results of Gidas, Ni and Nirenberg, [4],[5], imply that any positive solution to the Dirichlet problem for (1.1) on the ball or on $\mathbf{R}^n$ must be radial. On an annulus, on the other hand, it is known that non-radial positive solutions exist, [3], [10].

2. Method of proof.

The problem of course is an ordinary differential equations problem, the equation to be studied,

$$y'' + \frac{2}{x}y' - y + y^3 = 0. \tag{2.1}$$

The basic method is common to the earlier papers [2], [8], [9], [11], [12]. It uses the "shooting method" applied to the initial value problem for (2.1)

$$y(x_o) = 0, \;\; y'(x_o) = a, \tag{2.2}$$

($x_o > 0, a > 0$) together with Sturm comparison of the solutions of (2.1) to solutions of the initial value problem

$$\delta'' + \frac{2}{x}\delta' - \delta + 3y^2\delta = 0, \;\; \delta\,(x_o) = 0, \;\; \delta'\,(x_o) = 1, \tag{2.3}$$

for the variational equation corresponding to (2.1). In (2.3), $y = y(x, a, x_0)$ denotes the solution to (2.1), (2.2). By standard results (see e.g [6], Ch. V) the solution $\delta = (x, a, x_o)$ to (2.3) satisfies

$$\begin{aligned} \delta(x,a,x_o) &= \frac{\partial}{\partial a}y(x,a,x_o), \\ \delta'(x,a,x_o) &= \frac{\partial}{\partial a}y'(x,a,x_o). \end{aligned} \tag{2.4}$$

The basic ideas have their origin in Kolodner, [7], (although the equation treated there involves a different sort of non-linearity).

If we think of x_o as fixed then, depending on $a > 0$, solutions to (2.1), (2.2) do one of three things: (i) remain positive on (x_o, ∞) and oscillate about the line

$y = 1$ as $x \to \infty$; (ii) vanish at some $x_4 > x_o$; (iii) remain positive on (x_o, ∞) and tend to 0 as $x \to \infty$ with the asymptotic behaviory

$$y(x) = cx^{-1}e^{-x} + o(x^{-1}e^{-x}), \tag{2.5}$$

$$y'(x) = -cx^{-1}e^{-x} + o(x^{-1}e^{-x}).$$

We shall denote by $A_1(x_o)$ the set of values $a > 0$ for which the solution to (2.1), (2.2) has the behavior (i) and by $A_2(x_o)$ the set of $a > 0$ for which the solution has the behavior (ii); $A_3(x)$ will denote the set of a for which the solution has the behavior (iii). It is clear that $A_1(x_o)$ and $A_2(x_1)$ are open subsets of $(0, \infty)$.

Our object is to show the following.

Proposition 2.1. *Let $y(x, a, x_o)$ and $\delta(x, a, x_o)$ be as above. If $y(x, a, x_o)$ has the behavior (ii) above, with $y(x, a, x_o) > 0$ on (x_o, x_4) and $y(x_4) = 0$, then $\delta(x, a, x_o)$ vanishes exactly once in (x_o, x_4) and, if $x_4 < \infty, \delta(x_4, a, x_o) < 0$. If $y\,(x, a, x_o)$ has the behavior (iii), then $\delta(x, a, x_o)$ vanishes exactly once in (x_o, ∞) and is unbounded there with $|\delta(x, a, x_o)|$ growing exponentially as $x \to \infty$.*

From Proposition 2.1 it follows that for $a \in \ A_2(x_o)$ and x_4 as above, x_4 moves strictly monotonically to the left as a increases and $A_2(x_o) \subseteq (0, \infty)$ is a semi-infinite interval. For a such that $y(x, a)$ has behavior (iii) we must have $a \in \overline{A_1(x_o)} \cap \overline{A_2(x_o)}$. The uniqueness assertions then follow. Further details can be found in the papers quoted above.

In what follows we shall think of $x_o > 0$ in (2.2) as *fixed* but arbitrary, the arguments a, x_o will be omitted henceforth. We denote by $x_1 < x_2 < x_3$ (depending on a) the points defined implicitly by

$$0 < y(x) < 1 \text{ on } (x_o, x_1), \quad 1 < y(x) \text{ on } (x_1, x_3),$$

$$y'(x_2) = 0, \quad y(x_3) = 1.$$

If $a \in A_2(x_o), x_4$ is defined by

$$0 < y(x) < 1, \text{ on } (x_3, x_4), \quad y(x_4) = 0;$$

for $a \in A_3(x_o)$ we take $x_4 = \infty$.

We define z_1 and z_2 by

$$0 < \delta'(x) \quad \text{on} \quad (x_o, z_1), \qquad \delta'(z_1) = 0,$$

$$0 < \delta(x) \quad \text{on} \quad (x_o, z_2), \qquad \delta(z_2) = 0.$$

The first step of the proof is to show that

$$z_1 > x_1.$$

3. Proof that $z_1 > x_1$.

Associated to a solution $y = y(x, a)$ of (2.1), (2.2) we introduce the auxiliary function Φ defined by

$$\Phi(x) = (y'(x))^2 - \left(y^2(x) - \frac{1}{2}y^4(x)\right), \tag{3.1}$$

and to the pair (y, δ) we associate the function Ψ defined by

$$\Psi(x) = y'(x)\,\delta'(x) - \delta(x)\left(y(x) - y^3(x)\right). \tag{3.2}$$

Note that

$$\frac{\partial}{\partial a}\Phi(x) = 2\Phi(x).$$

Proposition 3.1. *We have*

$$\Phi'(x) = -\frac{4}{x}\left(y'(x)\right)^2, \tag{3.3}$$

and

$$\Psi'(x) = -\frac{4}{x} y'(x)\,\delta'(x), \tag{3.4}$$

Proof. This is the result of a straightforward computation. □

Lemma 3.2. *We have*

$$0 < \delta'(x) < a^{-1}y'(x), \quad \text{on} \quad (x_o, z_1). \tag{3.5}$$

Proof. If we put $v(x) = a^{-1}y(x) - \delta(x)$ then v satisfies the differential equation

$$v'' + \frac{2}{x}v' - v + 3y^2 v = 2a^{-1}y^3,$$

where $2a^{-1}y^3 > 0$ on (x_o, z_1). The result follows by Sturm comparison of this equation with the differential equation in (2.3). □

Lemma 3.3. *If $a \in A_2(x_o) \cup A_3(x_o)$ then*

$$\Psi(x) > 0 \quad \text{on} \quad [x_o, z_1],$$

consequently

$$z_1 > x_1. \tag{3.6}$$

Proof. From integration of (3.4), using the boundary conditions in (2.2) and (2.3)

$$\Psi(x) = a - 4\int_{x_o}^{x} y'(t)\,\delta'(t)\,\frac{dt}{t}. \tag{3.7}$$

For $a \in A_2(x_o)$ there is an $x_4 > x_1$ such that $y(x_4) = 0$, hence $\Phi(x_4) > 0$, while for $a \in A_3(x_o)$,

$$\lim_{x \to \infty} \Phi(x) = 0.$$

In either case, since Φ is a decreasing function of x,

$$\Phi(x_1) > 0$$

and hence it follows from (2.2) and (3.3) that

$$4\int_{x_o}^{z_1} (y'(t))^2\,\frac{dt}{t} < a^2.$$

From Lemma 3.2 it then follows that

$$4\int_{x_o}^{z_1} (\delta'(t))^2\,\frac{dt}{t} < 1,$$

whence from (3.7) and the Schwarz inequality

$$\Psi(z_1) > 0. \tag{3.8}$$

Since

$$\Psi(z_1) = -\delta(z_1)\left(y(z_1) - y^3(z_1)\right),$$

the inequality (3.8) implies (clearly $\delta(z_1) > 0$) that $y(z_1) > 1$, hence $z_1 > x_1$. □

4. Completion of the proof.

The completion of the proof makes use of the following two identities (for y, δ satisfying (2.1), (2.2) and (2.3) respectively)

(4.1) $\Psi_1'(x) = \left[(xy' + y - 1)(x\delta' + \delta) - x^2\left(y - y^3\right)\delta\right]' = -x\delta\,(y-1)^2\,(2y+1)\,,$

(4.2) $\Psi_2'(x) = \left[x^3\left(y'\delta' - \left(y - y^3\right)\delta\right) + x^2\,(y-1)\,\delta'\right]' = x^2\delta\,(y-1)\,(3y+1)\,,$

(here $\Psi_1(x)$ and $\Psi_2(x)$ are to be understood as *identical* to the terms in the square brackets in (4.1) and (4.2) respectively) cf. formulas (4.29) and (4.20) of [2].

Let ξ be such that

$$x_2 < \xi < x_3 \quad \text{and} \quad \xi y'(\xi) + y(\xi) = 1.$$

It follows from (3.6) that

$$\Psi_2(x_1) = {x_1}^3 y'(x_1)\delta'(x_1) > 0,$$

thus, by integration of (4.2),

$$\Psi_2(\xi) = \xi^3(y^3 - y)\delta > 0$$

from which it follows that $\delta(\xi) > 0$ and hence

$$z_2 > \xi > x_2. \tag{4.3}$$

Putting $\omega = y - 1$ we have

$$\omega'' + \frac{2}{x}\omega' - \omega + \left(y^2 + y + 1\right)\omega = 0 \tag{4.4}$$

Since $3y^2 \geq y^2 + y + 1$ for $y \geq 1$ it follows from Sturm comparison of (4.4) and the differential equation in (2.3) that

$$z_2 < x_3.$$

Lemma 4.1. *The function*

$$xy'(x) + y(x)$$

is decreasing in (x_1, x_3)*; if* $a \in A_2(x_o) \cup A_3(x_o)$ *then this function is negative on* (x_3, x_4).

Proof. We have

$$(xy'(x) + y(x))' = x(y - y^3),$$

so the function is decreasing on (x_1, x_3) as claimed and increasing on (x_3, x_4). The function is negative at x_4 if $x_4 < \infty$ and tends to zero at ∞ otherwise and thus must be negative on (x_3, x_4). □

Suppose that $a \in A_2(x_o) \cup A_3(x_o)$. From Lemma 4.1, the definition of ξ and (4.3) it follows that

$$\Psi_1(z_2) = x(xy' + y - 1)\delta'(x)|_{x=z_2} > 0. \tag{4.5}$$

If we suppose that δ vanishes at $z_3 \in (z_2, x_4)$ and is negative on (z_2, z_3) then integration of (4.1) from z_2 to z_3 gives $\Psi_1(z_3) > 0$ and this contradicts the fact that $\delta'(z_3)$ must be positive. When $x_4 < \infty$ the same holds if z_3 coincides with x_4.

Finally, suppose that $a \in A_3(x_o)$. We know from what has just been demonstrated that δ is negative on (z_2, ∞). If δ remains bounded then it has the asymptotic behavior

$$\begin{aligned} \delta(x) &= cx^{-1}e^{-x} + o(x^{-1}e^{-x}), \\ \delta'(x) &= -cx^{-1}e^{-x} + o(x^{-1}e^{-x}), \end{aligned}$$

where $c < 0$. This would imply that $\Psi_1(x)$ tends to 0 as x tends to ∞. Integrating (4.1) and using (4.5) however implies that this limit must be positive. Thus our assumption has led to a contradiction and δ must be unbounded. This completes the proof of Proposition 2.1.

Concluding remarks.

1. The conclusion of the proof, as presented in this section, is a slightly compressed version of the argument given in [2] for the case of the ball.

2. The argument here applies also if y^3 is replaced by y^p with $1 < p \leq 3$. The limitation arises with the use of the formula (4.2); the positivity of the derivative for $y > 1$ fails for $p > 3$.

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