The Volume Preserving Motion By Mean Curvature As An Asymptotic Limit Of Reaction Diffusion Equations

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Research Report No. 94-NA-023

June 1994

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U.S. Army Research Office
Research Triangle Park
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Washington, DC 20550

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The volume preserving motion by mean curvature as an asymptotic limit of reaction - diffusion equations

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May 30, 1994

ABSTRACT

We study the asymptotic limit of the reaction-diffusion equation

$$u_t^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon) + \frac{1}{\varepsilon} g(u^\varepsilon) \lambda^\varepsilon$$

as ε tends to zero in a radially symmetric domain in $\mathbf{R}^n$ subject to the constraint $\int_\Omega h(u^\varepsilon)\, dx = const.$ The energy estimates and the signed distance function approach are used to show that a limiting solution can be characterized by moving interfaces. The interfaces evolve by nonlocal (volume preserving) mean curvature flow. Possible interactions between the interfaces are discussed as well.

1. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with an outward normal vector $\boldsymbol{n}(x)$, $x \in \partial\Omega$ and consider the following nonlocal reaction-diffusion equation:

$$\begin{cases} u_t^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon) + \lambda^\varepsilon, & x \in \Omega, t > 0, \\ \left.\frac{\partial u^\varepsilon}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = 0, & x \in \partial\Omega, \\ u^\varepsilon(x, 0) = \phi(x), & x \in \Omega, t = 0. \end{cases} \tag{1}$$

Here $|\Omega|$ is a volume of Ω, $f(u) = W'(u)$ and $W(u)$ is a double-well potential, $\phi \in C^\infty(\Omega)$ is a function satisfying compatibility condition $\left.\frac{\partial \phi}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = 0$ while

$$\lambda_\varepsilon = \frac{1}{\varepsilon^2 |\Omega|} \int_\Omega f(u^\varepsilon)\, dx. \tag{2}$$

This problem was formally studied by Rubinstein and Sternberg in [1]. They observed that (1) is a particular regime of the viscous Cahn - Hillard equation, introduced by Novick - Cohen in [2]:

$$\alpha u_t = \Delta\, (f(u) - \beta \Delta\, u + \nu u_t) \tag{3}$$

Then (1) corresponds to the case when α is much less than β, ν and 1. Thus, by assuming that u^ε is a concentration of one component in a binary mixture, (1) can be viewed as a model of phase separation in that mixture with (2) representing the mass conservation law:

$$\int_\Omega u^\varepsilon dx = const. \tag{4}$$

The small parameter ε was introduced in such a way so that the diffusion term in (1) is negligible compared to the reaction term except in a narrow ($\sim \varepsilon$) transition layer where gradients are large. Rubinstein and Sternberg analyzed (1) using the method of matched asymptotic expansions and multiple time scales. They found that the phases rapidly separate as $\varepsilon \to 0$ and that the propagation of the interfaces between these phases is a coarsening process depending in nonlocal manner on mean curvature. Namely, the limiting equation for (1) as $\varepsilon \to 0$ is:

$$V = \textit{mean curvature} + \lambda(t) \text{ in } \Omega, \tag{5}$$

where $\lambda(t)$ is such that the volume of the region enclosed by the interfaces Γ_t, corresponding to a limiting solution of (1), is constant in time and V is the normal velocity of Γ_t. The pair $(\Gamma_t, \lambda(t))$ is called a volume preserving mean curvature flow. The associated geometric PDE is:

$$u_t - \left(\Delta u - \frac{(D^2 u Du, Du)}{|Du|^2}\right) - \lambda(t)|Du| = 0 \quad \text{in } \boldsymbol{R}^n \times (0, \infty).$$

In 1992 Barles, Soner and Souganidis [3] developed a rigorous approach for obtaining the asymptotic limits of reaction-diffusion equations using the notion of signed distance function to the front. They also conjectured that this approach can be successfully used for studying the asymptotic limit of (1) under certain restrictions on the behavior of the Lagrange multiplier $\lambda(t)$ as $\varepsilon \to 0$.

In our work we provide a rigorous version of [1] for a model (7) similar to (1), following the technique developed in [3]. First, in Section 3 we show that, as in [1], model (7) represents a gradient flow for the energy functional

$$E^{\varepsilon}[u^{\varepsilon}] := \int_{\Omega} \left(\frac{\varepsilon}{2}|Du^{\varepsilon}|^2 + \frac{1}{\varepsilon} W(u^{\varepsilon})\right) dx, \tag{6}$$

then following [6] find that as $\varepsilon \to 0$, the functions $u^{\varepsilon} \to v$ a.e. in $\Omega \times [0, T]$, where $v(x, t)$ assumes values $+1$ or -1 almost everywhere in $(x, t) \in \Omega \times [0, T]$ (phase separation is achieved). Then concentrating our attention on a radial domain ($\Omega = \{\, |x| \le R \,\}$) and special choice of initial data, using monotonicity of $E^{\varepsilon}[u^{\varepsilon}](\cdot)$ and estimates found in [12] and [3], we establish the asymptotic behavior of the Lagrange multiplier λ^{ε} (Section 5), the limiting equation governing the motion of the interfaces and the validity of the mass conservation law for v (Section 6). Finally, in Section 7, we extend some of these results to the initial data more general than the one assumed in Section 4 and discuss possible interactions between the interfaces.

After this work was completed I found that the similar results were obtained at the same time by Bronsard and Stoth [4].

I would like to express a deep gratitude to my advisor, Prof. H.Mete Soner for his encouragement, patience and advice. I also wish to thank Prof. Barbara Stoth for her comments on a draft of this paper.

2. Formulation of the Phase Field Model.

As we have already mentioned we work with the model slightly different from (1). This is motivated by the use of a travelling wave representation of solutions of (1) in the distance function approach for finding the asymptotic limit of the reaction-diffusion equation. Suppose that:

$$W(u) = \frac{1}{2}(u^2 - 1)^2, \qquad f(u) = W'(u) = 2u(u^2 - 1),$$

$$h(u) = 2\left(\frac{u^3}{3} - u\right), \qquad g(u) = h'(u) = 2(u^2 - 1).$$

The choice of $W(u)$ and $h(u)$ is motivated purely by the simplicity of calculations. Our results can be easily extended to the arbitrary double-well potential W. Consider now the following problem:

$$\begin{cases} u_t^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon) + \frac{1}{\varepsilon} g(u^\varepsilon)\lambda^\varepsilon, & x \in \Omega,\ t > 0, \\ \left.\frac{\partial u^\varepsilon}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = 0, & x \in \partial\Omega, \\ u^\varepsilon(x, 0) = \phi^\varepsilon(x), & x \in \Omega,\ t = 0. \end{cases} \tag{7}$$

Here Ω and $\boldsymbol{n}$ are as before, $\phi^\varepsilon \in C^\infty(\Omega)$ for all $\varepsilon > 0$ and λ^ε is chosen in such a way that

$$\int_\Omega h(u^\varepsilon)\, dx = const. \tag{8}$$

The initial data are assumed to satisfy:

$$E^\varepsilon[\phi^\varepsilon] \le M \quad \text{for all } \varepsilon > 0, \tag{9.a}$$

$$\phi^\varepsilon \to \phi \quad \text{in } L^1(\Omega), \tag{9.b}$$

$$|\phi^\varepsilon(x)| \le 1\ ,\ x \in \Omega, \tag{9.c}$$

where $M > 0$ is a constant independent of ε and E^ε is the energy functional defined in (6).

Multiplying equation (7) by $g(u^\varepsilon)$, integrating over Ω and using (8) we obtain that:

$$\lambda^\varepsilon(t) = \varepsilon \cdot \frac{\int_\Omega \left[g'(u^\varepsilon)|Du^\varepsilon|^2 + \frac{1}{\varepsilon^2} f(u^\varepsilon) g(u^\varepsilon) \right] dx}{\int_\Omega g^2(u^\varepsilon)\, dx}. \tag{10}$$

Then using (10) and the fact that:

$$g^2(u) = 4(u^2 - 1)^2 = 8W(u),$$
$$g'(u) = 4u,$$
$$f(u)\, g(u) = 4u(u^2 - 1)^2 = 8uW(u)$$

we have:

$$\lambda^\varepsilon(t) = \frac{\int_\Omega u^\varepsilon \left[\frac{\varepsilon}{2}|Du^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right] dx}{\int_\Omega W(u^\varepsilon)\, dx}. \tag{11}$$

The next theorem allows for a travelling wave representation of u^ε:

Theorem 1. *Suppose that u^ε is a solution of (7) and that (9.c) is satisfied. Then $|u^\varepsilon(x, t)| \le 1$ on $\bar{\Omega} \times [0, T]$.*

Proof: The proof easily follows by the standard application of the maximum principle as

$u_0 = \pm 1$ are solutions of (7). Indeed, suppose for example, that (x_0, t_0) is such that $\max_{x \in \bar{\Omega}} u^\varepsilon(x, t_0) = u^\varepsilon(x_0, t_0) = 1$ and $u_t^\varepsilon(x_0, t_0) > 0$, then $Du^\varepsilon(x_0, t_0) = 0$ and $\Delta u^\varepsilon(x_0, t_0) \le 0$. Substituting into (7) and using (9.c) we obtain a contradiction.

■

As in [1] we remark that since a local solution to (7) exists by a fixed point argument, it can be extended to a global one using the previous theorem.

Now set $q := \tanh$ and let $u^\varepsilon =: q(z^\varepsilon / \varepsilon)$. By the properties of the hyperbolic tangent $q' = 1 - q^2$ then $q'' = -2qq'$. Hence, as $q' > 0$, (7) and (11) take the form:

$$z_t^\varepsilon - \Delta z^\varepsilon + 2\lambda^\varepsilon(t) + \frac{2u^\varepsilon}{\varepsilon}(|Dz^\varepsilon|^2 - 1) = 0, \tag{12}$$

$$\lambda^\varepsilon(t) = \frac{1}{\varepsilon} \cdot \frac{\int_\Omega q(z^\varepsilon/\varepsilon)\,(q'(z^\varepsilon/\varepsilon))^2\,(1 + |Dz^\varepsilon|^2)\,dx}{\int_\Omega (q'(z^\varepsilon/\varepsilon))^2\,dx}, \tag{13}$$

where z^ε satisfies Neumann boundary conditions while $z^\varepsilon(\cdot, 0) =: \alpha^\varepsilon \in C^\infty(\Omega)$ and $\phi^\varepsilon(x) = q(\alpha^\varepsilon(x)/\varepsilon)$. We also assume in addition to (9.a) - (9.c) that:

$$|D\alpha^\varepsilon| \le 1 , \tag{14}$$

$$\alpha^\varepsilon \to d_0(\cdot) \text{ uniformly in } \Omega,$$

where $d_0(\cdot)$ is the signed distance to the set $\partial\{\phi(x) = 1\}$.

3. Energy estimates

Recall that

$$E^\varepsilon[u^\varepsilon] = \int_\Omega \left(\frac{\varepsilon}{2}|Du^\varepsilon|^2 + \frac{1}{\varepsilon}W(u^\varepsilon)\right) dx. \tag{15}$$

Then multiplying equation (7) by u_t^ε , integrating over Ω and using (8) we have that:

$$E^\varepsilon[u^\varepsilon](t) = E^\varepsilon[u^\varepsilon](0) - \varepsilon \int_0^t \int_\Omega (u_t^\varepsilon)^2\,dx\,dt. \tag{16}$$

As a direct consequence of this equality we obtain:

Theorem 2. *Assume* (7), *(9.a) - (9.b). Then:*

(a) $sup_{t\geq 0}E^{\varepsilon}[u^{\varepsilon}](t) \leq M,$ (17)

(b) $sup_{t\geq 0}\int_{\Omega}((u^{\varepsilon})^2-1)^2 dx \leq 2M\varepsilon,$ (18)

(c) $\varepsilon\int_{t_1}^{t_2}\int_{\Omega}(u_t^{\varepsilon})^2 dxdt = E^{\varepsilon}[u^{\varepsilon}](t_1) - E^{\varepsilon}[u^{\varepsilon}](t_2)$ *for* $t_2 > t_1$. (19)

We also adopt the following result from Bronsard and Kohn [6]:

Theorem 3. *Assume that (7), (9.a) - (9.b) are satisfied. Then for any sequence of ε's tending to zero there is a subsequence ε_j such that the limit $\lim_{\varepsilon_j\to 0} u^{\varepsilon_j}(x,t) = v(x,t)$ exists for a.e. $(x,t) \in \Omega\times(0,\infty)$. The function v takes only the values ± 1 and there are positive constants C and C_1 depending only on M such that:*

$$\int_{\Omega}|v(x,t_1) - v(x,t_2)|dx \leq C|t_2 - t_1|^{\frac{1}{2}} \quad \text{for any } t_2,\ t_1 > 0, \tag{20}$$

$$sup_{t\geq 0}\int_{\Omega}|Dv(x,t)|dx \leq C_1, \tag{21}$$

and its initial value is the limit of the initial data for u^{ε} a.e. in x:

$$lim_{t\to 0}v(x,t) = \phi(x). \tag{22}$$

Denote

$$E^{0}[u] = \begin{cases} K\cdot Per_{\Omega}(\{u=1\}), & \text{if } u(x)\in\{-1,1\} \text{ a.e. in } \Omega\,, \\ \infty, & \text{otherwise}\,, \end{cases} \tag{23}$$

where $K = \int_{-1}^{1}\sqrt{2W(s)}\,ds$ (In our case K=4/3) and $Per_{\Omega}(A)$ is a perimeter of A in Ω (For the definition of perimeter see e.g. [5]) .Then we find that E^0 is a $\Gamma(L^1(\Omega))$-limit of E^{ε} (See e.g. [7], [8] or [9]). In other words the following holds:

Theorem 4. *Let E^{ε} and E^{o} be as above. Then:*

(1) *If $v^{\varepsilon}\to v^0$ in $L^1(\Omega)$ then $\liminf_{\varepsilon\to 0} E^{\varepsilon}[v^{\varepsilon}] \geq E^0[v^0]$.* (24)

(2) *For any $v^0 \in L^1(\Omega)$ there exists a family (v^{ε}) such that $v^{\varepsilon}\to v^0$ in $L^1(\Omega)$ and*

$$\lim_{\varepsilon\to 0} E^{\varepsilon}[v^{\varepsilon}] = E^{0}[v^{0}]. \tag{25}$$

We can also obtain some additional information on a limit of $E^{\varepsilon}[u^{\varepsilon}]$ as a function of t. Fix $T > 0$. Then, being monotone on $[0,T]$, each of the functions $E^{\varepsilon}[u^{\varepsilon}]$ also belongs to $BV([0,T])$ and by (9.a) and (16) the set $\{E^{\varepsilon}[u^{\varepsilon}]\}_{\varepsilon>0}$ is uniformly bounded in BV-norm and thus is relatively compact in $L^1([0,T])$. Moreover, the following theorem holds ([17]):

Theorem 5. *Let F be an infinite family of increasing functions, defined on an interval* $[0, T]$. *If all functions of the family are bounded by the same number,*

$$|f(x)| \le K, \quad f \in F, \quad 0 \le x \le T,$$

then there is a sequence of functions $\{f_n\}_{n\in \mathbf{N}}$ *in F which converges to an increasing function g at every point of* $[a, b]$.

Therefore, there exists a subsequence $\{E^{\varepsilon_j}\left[u^{\varepsilon_j}\right]\}_{j\in \mathbf{N}}$ and a bounded monotone decreasing nonnegative function E such that $E^{\varepsilon_j}\left[u^{\varepsilon_j}\right]$ converges to E pointwise at every point of $[0,T]$. It will be shown later that $E^{0}[v](t) = E(t)$ everywhere except finitely many t's.

Since E is monotone, it is well known that a set of points where E is discontinuous is at most countable and that both left- and right-hand limits of E exist at every point of $(0, T)$. This enables us to prove the next theorem.

Theorem 6. *Let* $E^{\varepsilon_j}\left[u^{\varepsilon_j}\right]$ *and E be as above. Then if t is a continuity point of E*

$$\overline{\lim_{\substack{\varepsilon\to 0\\ s\to t}}} \left|E^{\varepsilon_j}\left[u^{\varepsilon_j}\right](s) - E(t)\right| = 0. \tag{26}$$

Proof: Let t be a continuity point for E. Set $a = \overline{\lim_{\substack{\varepsilon\to 0\\ s\uparrow t}}} \left|E^{\varepsilon_j}\left[u^{\varepsilon_j}\right](s) - E(t)\right|$. Suppose that $a > 0$. Then for sufficiently small ε we can choose two subsequences $\{s_j\}_{j\in\mathbf{N}}$ and $\{\varepsilon_j\}_{j\in\mathbf{N}}$ such that $\left|E^{\varepsilon_j}\left[u^{\varepsilon_j}\right](s_j) - E(t)\right| \ge \frac{a}{2}$ where $s_j \uparrow t$ and $\varepsilon_j \to 0$ as $j\to\infty$. Since both $E^{\varepsilon_j}\left[u^{\varepsilon_j}\right]$ and E are monotone decreasing we obtain for any fixed $j \ge 1$ that:

$$E(s_j) = \lim_{k\to\infty} E^{\varepsilon_k}\left[u^{\varepsilon_k}\right](s_j) \ge \overline{\lim}_{k\to\infty} E^{\varepsilon_k}\left[u^{\varepsilon_k}\right](s_k) \ge E(t) + \frac{a}{2}.$$

Therefore for any $j \ge 1$, we find that $E(s_j) \ge E(t) + a/2$. But since E is continuous at t, $\lim_{j\to\infty} E(s_j)$ exists and is greater than $E(t) + a/2$. Therefore E has a jump at t which contradicts the fact that t is a continuity point of E. We can show that $\overline{\lim_{\substack{\varepsilon\to 0\\ s\downarrow t}}} \left|E^{\varepsilon_j}\left[u^{\varepsilon_j}\right](s) - E(t)\right| = 0$ as well, then (26) follows.

■

4. Additional estimates. The radial case.

We now restrict our attention to the radial case $\Omega = \{ |x| \le \mathbf{R}^n \}$. We will work in two dimensions for the reason of notational simplicity only - our results can be immediately extended to any dimension. All functions from now on will be assumed radially symmetric. Then the phase-field model (6) takes the following form:

$$u_t^\varepsilon = u_{rr}^\varepsilon + \frac{1}{r} u_r^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon) + \frac{1}{\varepsilon} g(u^\varepsilon) \lambda^\varepsilon, \qquad (r, t) \in \Omega_T, \tag{27}$$

$$u_r^\varepsilon(t, 0) = u_r^\varepsilon(t, R) = 0 \tag{28}$$

for $u^\varepsilon(r, t)$;

$$z_t^\varepsilon - z_{rr}^\varepsilon - \frac{1}{r} z_r^\varepsilon + 2\lambda^\varepsilon(t) + \frac{2u^\varepsilon}{\varepsilon} \left(\left| z_r^\varepsilon \right|^2 - 1 \right) = 0, \qquad (r, t) \in \Omega_T, \tag{29}$$

$$z_r^\varepsilon(t, 0) = z_r^\varepsilon(t, R) = 0 \tag{30}$$

for $z^\varepsilon(r, t)$. Here $\Omega_T := [0, R] \times [0, T]$. For now we will suppose in addition to (14) that:

$d_0(\cdot)$ has a finite number N of zeroes $r_i(0)$, $i = 1, \dots, N$ in $(0, R)$, (31.a)

$z_r^\varepsilon(\cdot, 0)$ has $N - 1$ zeroes $s_i^\varepsilon(0)$, $i = 1, \dots, N - 1$ in $(0, R)$, (31.b)

$z_{rr}^\varepsilon(s_i^\varepsilon(0), 0) \ne 0$, $i = 1, \dots, N - 1$ for any $\varepsilon > 0$, (31.c)

$E^\varepsilon[\phi^\varepsilon] \to \frac{4}{3} Per_\Omega(\{\phi = 1\})$ as $\varepsilon \to 0$. (31.d)

We make these assumptions in order to simplify the presentation. They will be removed later in Section 7 when collisions between the interfaces will be discussed. Until then we will use the following definition:

Definition 1. *Let $\alpha \in BV([0, R])$ be such that $\alpha(r) \in \{-1, 1\}$ a.e. A point $r_0 \in (0, R)$ will be called an interface provided there exists $\delta > 0$ such that $\alpha(r) = +1$ (-1) a.e. for $r \in (r_0 - \delta, r_0)$ and $\alpha(r) = -1$ $(+1)$ a.e. on $r \in (r_0, r_0 + \delta)$. We denote the set of these points as Γ.*

Observe that r_0 is an interface only if it belongs to the support of the Radon measure $|D\alpha|$ where $D\alpha$ is a gradient of α in the sense of distributions. Also, when (a) - (d) hold, the set $\partial\{\phi(x) = 1\}$ consists of N circular interfaces with radii $r_i(0)$, $i = 1, \dots, N$.

Our assumptions lead us to the another series of estimates:

Theorem 7. *Assume that (14) holds. Then $\left| z_r^\varepsilon(r, t) \right| \le 1$ on Ω_T for $\varepsilon > 0$.*

Proof: From (29) we have that $w^{\varepsilon} := z_r^{\varepsilon}$ satisfies the following problem:

$$w_t^{\varepsilon} - w_{rr}^{\varepsilon} - \frac{1}{r} w_r^{\varepsilon} + \frac{1}{r^2} w^{\varepsilon} + \frac{4q}{\varepsilon} w^{\varepsilon} w_r^{\varepsilon} + \frac{2q'}{\varepsilon^2} w^{\varepsilon} ((w^{\varepsilon})^2 - 1) = 0, \tag{32.a}$$

where q and q' are evaluated at $z^{\varepsilon}/\varepsilon$. At the same time:

$$w^{\varepsilon}(0,t) = w^{\varepsilon}(R,t) = 0, \qquad t \in (0,T), \tag{32.b}$$

$$\left| w^{\varepsilon}(r,0) \right| \leq 1, \qquad r \in (0,R). \tag{32.c}$$

Then the proof immediately follows by the maximum principle.

■

Corollary 1. *For all* $t \in [0,T]$ *we have* $\left|\varepsilon\lambda^{\varepsilon}(t)\right| \leq 2$.

Proof: From the previous theorem and by (13) we find that:

$$\left|\varepsilon\lambda^{\varepsilon}(t)\right| = \frac{\left| \int_{\Omega} q(z^{\varepsilon}/\varepsilon)\,(q'(z^{\varepsilon}/\varepsilon))^2 (1 + |Dz^{\varepsilon}|^2)\, dx \right|}{\int_{\Omega} (q'(z^{\varepsilon}/\varepsilon))^2 dx} \leq \frac{2\int_{\Omega} (q'(z^{\varepsilon}/\varepsilon))^2 dx}{\int_{\Omega} (q'(z^{\varepsilon}/\varepsilon))^2 dx} \leq 2.$$

■

Theorem 8. *Suppose that* $w^{\varepsilon}(\cdot,0)$ *has* $N-1$ *zeroes in* $(0,R)$ *for any* $\varepsilon > 0$. *Then* $w^{\varepsilon}(\cdot,t)$ *has at most* $N-1$ *zeroes in* $(0,R)$ *for all* $t \in [0,T]$ *and* $\varepsilon > 0$.

The proof is based on the following lemma:

Lemma 1. *Denote as* $s_i^{\varepsilon}(0)$, $i = 1,\ldots,N-2$ *the position of the* i^{th} *interior zero of* $w^{\varepsilon}(\cdot,0)$. *Then:*

(a) *For each* $i = 1,\ldots,N-2$, *there exists a continuous curve* $s_i^{\varepsilon}(t)$ *such that* $w^{\varepsilon}(s_i^{\varepsilon}(t),t) = 0$ *and* $w^{\varepsilon}(r,t) \neq 0$ *if* $r \in (s_i^{\varepsilon}(t), s_{i+1}^{\varepsilon}(t))$ *for all* $t \leq t_0^{\varepsilon}$. *Here* t_0^{ε} *is such that either* $s_i^{\varepsilon}(t_0^{\varepsilon}) = 0$ *or* $s_i^{\varepsilon}(t_0^{\varepsilon}) = R$ *or* $s_i^{\varepsilon}(t_0^{\varepsilon}) = s_{i\pm1}^{\varepsilon}(t_0^{\varepsilon})$.

(b) *Suppose that two or more curves intersect at* $(r,t) = (s_0^{\varepsilon}, t_0^{\varepsilon})$ *If the number of the intersecting curves is odd, then for* $t > t_0^{\varepsilon}$ *there is a single continuous curve originating at the point of intersection. If the number of the intersecting curves is even, there exist* $\delta, \gamma > 0$ *dependent on* ε *such that* $w^{\varepsilon}(r,t) \neq 0$ *for all* $(r,t) \in (s_0 - \delta, s_0 + \delta) \times (t_0, t_0 + \gamma)$.

(c) *Let* $s_0^{\varepsilon}(t) \equiv 0$ *and* $s_N^{\varepsilon}(t) \equiv R$. *Then the result of part (a) can be extended for* $i = 0,\ldots,N-1$. *Also if* τ_i^{ε} *is such that* $s_i^{\varepsilon}(\tau_i^{\varepsilon}) = 0$, *then there exist* $\delta, \gamma > 0$ *depending on* ε *such that* $w^{\varepsilon}(r,t) > 0\ (< 0)$ *where* $(r,t) \in (0,\delta) \times (\tau_i^{\varepsilon}, \tau_i^{\varepsilon} + \gamma)$. *The similar result holds if* $s_i^{\varepsilon}(\tau_i^{\varepsilon}) = R$.

A more general result for a solution of a parabolic equation was obtained by Angenent in [10].

Proof of Lemma 1: The proof is based on a maximum principle.

(a) As a solution of (32.a) - (32.c), w^{ε} is smooth. Then from our assumption on $w^{\varepsilon}(\cdot, 0)$, we obtain by implicit function theorem that smooth $s_i^{\varepsilon}(t)$, $i = 1, \dots, N-1$ exist for some small time, say τ^{ε}. By the strict maximum principle and the smoothness of w^{ε}, the curves s_i^{ε} must also remain continuous for $i = 1, \dots, N-2$ as functions of t after τ^{ε}. The sign of w^{ε} follows by the strict maximum principle as well. Observe that since w^{ε} is smooth, then for every $i = 1, \dots, N-2$ the curves s_i^{ε} cannot terminate without intersecting either adjacent curve s_i^{ε} or hitting the boundary $r = 0$ or $r = R$.

(b) Suppose that the even number of curves, for example s_i^{ε} and s_{i+1}^{ε}, intersect at $(r, t) = (s_0^{\varepsilon}, t_0^{\varepsilon})$ for some $i \in \{1, \dots, N\}$. We have for $t = t_0$ that $w^{\varepsilon}(r, t) > 0$ (< 0) on $(s_{i-1}^{\varepsilon}(t_0^{\varepsilon}), s_i^{\varepsilon}(t_0^{\varepsilon})) \cup (s_i^{\varepsilon}(t_0^{\varepsilon}), s_{i+2}^{\varepsilon}(t_0^{\varepsilon}))$ and $w^{\varepsilon}(s_i^{\varepsilon}(t_0^{\varepsilon}), t_0^{\varepsilon}) = 0$. Then if $t_1^{\varepsilon} > t_0^{\varepsilon}$ is such that $s_{i-1}^{\varepsilon}(t_1^{\varepsilon}) \neq s_{i+2}^{\varepsilon}(t_1^{\varepsilon})$, the strict maximum principle implies that $w^{\varepsilon}(r, t) > 0$ (< 0) for $t \in (t_0^{\varepsilon}, t_1^{\varepsilon})$ and $r \in (s_{i-1}^{\varepsilon}(t), s_{i+2}^{\varepsilon}(t))$. The similar proof works when the odd number of curves intersect at one point.

(c) Consider, for example, the endpoint $r = 0$. Let $t = \tau_1^{\varepsilon}$ be such a time that $s_1^{\varepsilon}(\tau_1^{\varepsilon}) = 0$. Assume $t < \tau_1^{\varepsilon}$. Then one can easily check, following e.g. [11], that even though equation (32.a) has a singularity at $r = 0$, the strict maximum principle applies inside the parabolic region bounded by $r = s_1^{\varepsilon}(t)$ from the right $r = s_0^{\varepsilon}(t)$ from the left and $t = 0$ from below. Then, as in (a), we have that $w^{\varepsilon}(r, t)$ is strictly positive (negative) for $r \in (s_0^{\varepsilon}(t), s_1^{\varepsilon}(t))$ if $w^{\varepsilon}(r, 0)$ is strictly positive (negative) for $r \in (s_0^{\varepsilon}(0), s_1^{\varepsilon}(0))$. After the time τ_1^{ε}, we can apply an analysis similar to the one in part (b) to show that the curve s_i^{ε} disappears when it crosses the boundary $r = 0$ and no additional curves are created. The same results hold for $r = R$.

■

Proof of Theorem 8 10: Follows immediately from the previous lemma.

■

5. Asymptotic behavior of the Lagrange multiplier

We will extensively use the following estimates, due to Stoth [12], that are obtained by:

(a) Multiplying the equation (27) by $\varepsilon r u_r^{\varepsilon}$ and integrating in r by parts from y to y_1 where y

and $y_1 \in [0,R]$:

$$\varepsilon \int_y^{y_1} u_r^\varepsilon u_t^\varepsilon r dr - \left[\frac{1}{2\varepsilon} r\left[(q')^2((z_r^\varepsilon)^2 - 1)\right]\right]\Bigg|_y^{y_1}$$

$$-\frac{1}{2\varepsilon}\int_y^{y_1}\left[(q')^2((z_r^\varepsilon)^2 + 1)\right]dr \tag{33}$$

$$-\left[y_1 h(u^\varepsilon(y_1)) - y h(u^\varepsilon(y)) - \int_y^{y_1} h(u^\varepsilon(r))\,dr\right]\cdot \lambda^\varepsilon(t) = 0.$$

b). Differentiating the equation (27) in t, multiplying the result by $\varepsilon r u_t^\varepsilon$ and integrating in r by parts over $[0, R]$:

$$\frac{1}{2}\frac{d}{dt}\left(\varepsilon\int_0^R (u_t^\varepsilon)^2 r dr\right)$$
$$= -\varepsilon\int_0^R \left|\nabla u_t^\varepsilon\right|^2 r dr - \frac{1}{\varepsilon}\int_0^R [f'(u^\varepsilon) - \varepsilon\lambda^\varepsilon g'(u^\varepsilon)]\,(u_t^\varepsilon)^2 r dr. \tag{34}$$

Remark 1. Since "$z^\varepsilon(x, t) = z^\varepsilon(r(x), t)$" and $z^\varepsilon(x, t)$ is smooth in $\Omega\times[0, T]$ as a solution of (12) then z_{rr}^ε and $\frac{1}{r}z_r^\varepsilon$ are bounded in Ω_T for every $\varepsilon > 0$.

Remark 2. Multiplying (19) by ε^α for any $\alpha > 0$, using (17) and passing, if necessary, to a subsequence in ε, we find that

$$\varepsilon^{1+\alpha}\int_0^R (u_t^\varepsilon)^2 r dr \to 0 \quad a.e.\ t\in[0, T]. \tag{35}$$

Remark 3. After multiplying estimate (34) by $2\varepsilon^2$, integrating it over $[t, t_1]$ for any $t, t_1 \in [0, T]$ and using (17), (19) and Corollary 1 we get:

$$\varepsilon^3\int_0^R (u_t^\varepsilon)^2 r dr\Big|_t^{t_1} \le C\left[E^\varepsilon(t_1) - E^\varepsilon(t)\right], \tag{36}$$

where C is a constant independent of ε.

Remark 4. Now multiply the estimate (33) by ε. Then the first term can be bounded by

$$\varepsilon^2\left|\int_y^{y_1} u_r^\varepsilon u_t^\varepsilon r dr\right| \le \varepsilon^2\left|\int_0^R u_r^\varepsilon u_t^\varepsilon r dr\right| \le \left(\int_0^R \varepsilon (u_r^\varepsilon)^2 r dr\right)^{\frac{1}{2}}\left(\int_0^R \varepsilon^3 (u_t^\varepsilon)^2 r dr\right)^{\frac{1}{2}}$$
$$\le M^{\frac{1}{2}}\left(\int_0^R \varepsilon^3 (u_t^\varepsilon)^2 r dr\right)^{\frac{1}{2}} \downarrow 0 \ a.e.\ t\in [0,T]. \tag{37}$$

The third term in (33) when multiplied by ε, converges to zero by (17). Moreover, if the constant in (8) is not equal to zero, then for each $t \in [0, T]$ we can choose y and $y_1 \in (0, R)$ arbitrarily close to each other, such that, for example $u^\varepsilon(y, t) \to -1$ and $u^\varepsilon(y_1, t) \to +1$ (If the constant in (8) is zero then by (20) we can find $\tau > 0$ such that there exist y and $y_1 \in (0, R)$ with the same properties as above for $t < \tau$.) Choosing y and y_1 to be the limits of integration in (33), one can easily see that the coefficient of $\varepsilon\lambda^\varepsilon$ approaches a nonzero constant as $\varepsilon \to 0$ while the term

$$\frac{1}{2}r\left[(q')^2((z_r^\varepsilon)^2 - 1)\right]\Big|_y^{y_1}$$

converges to zero. This implies that $\varepsilon\lambda^\varepsilon \downarrow 0$ as $\varepsilon \to 0$ a.e. $t \in [0, T]$.

Theorem 9. $\varepsilon\lambda^\varepsilon \downarrow 0$ *as* $\varepsilon \to 0$ *at the continuity points of* E.

Proof: Suppose that t is a continuity point of E. Then according to Remark 2 we can pass to a subsequence (independent of t) such that in any small neighborhood of t we can find t_1 for which $\varepsilon^3\int_0^R (u_t^\varepsilon)^2 r dr \to 0$ at t_1 as $\varepsilon \downarrow 0$. At the same time by Remark 3 we have:

$$\left(\varepsilon^3\int_0^R (u_t^\varepsilon)^2 r dr\right)(t) \le \left(\varepsilon^3\int_0^R (u_t^\varepsilon)^2 r dr\right)(t_1) + C\left[E^\varepsilon(t_1) - E^\varepsilon(t)\right]$$

hence $\overline{\lim}_{\varepsilon\to 0}\left(\varepsilon^3\int_0^R (u_t^\varepsilon)^2 r dr\right)(t) \le C\,[E(t_1) - E(t)]$. The right-hand side of this expression can be made arbitrarily small by choosing t_1 close enough to t which proves that the $\lim_{\varepsilon\to 0}\left(\varepsilon^3\int_0^R (u_t^\varepsilon)^2 r dr\right)(t)$ exists and equal to zero. Then by the same procedure as in Remark 4, $\lim_{\varepsilon\to 0} \varepsilon\lambda^\varepsilon(t) = 0$.

■

Theorem 10. $\lim_{\substack{\varepsilon\to 0\\ s\to t}} \varepsilon\,\lambda^\varepsilon(s) = 0$ *at the continuity points of* E

Proof: Assume again that t is a continuity point of E. Then following the proof of the previous theorem and using t as t_1 and s as t we obtain by Theorem 6 and Theorem 9 that:

$$\overline{\lim_{\substack{\varepsilon \to 0 \\ s \to t}}} |\varepsilon \lambda^{\varepsilon}(s)| \le C \overline{\lim_{\substack{\varepsilon \to 0 \\ s \to t}}} \left[\left(\varepsilon^3 \int_0^R (u_t^{\varepsilon})^2 r dr \right)(t) + C_1 [E^{\varepsilon}(s) - E^{\varepsilon}(t)] \right] = 0.$$

■

6. Asymptotic behavior of reaction - diffusion equations. Small time.

We will start by considering the asymptotic limit of the reaction - diffusion equations for the short time and then in the next section will investigate the limiting behavior of our model as time progresses, while allowing for disappearance of the interfaces and their interactions.

a. Asymptotic limit of z^{ε}.

By (24), (31.d) and Theorem 4.Theorem 5 we have that:

$$\begin{aligned} \frac{8\pi}{3}\sum_{i=1}^{N} r_i(0) &= \frac{4}{3}Per_{\Omega}(\{\phi = 1\}) = E(0) \ge E(t) \qquad \text{for all } t > 0, \\ &= \lim_{\varepsilon \to 0} E^{\varepsilon}(t) \ge E^{0}(t) = \frac{4}{3}Per_{\Omega}(\{\upsilon(\cdot, t) = 1\}) \end{aligned} \tag{38}$$

where $r_i(0)$, $i = 1, \ldots, N$ are the positions of the interfaces at $t = 0$. At the same time by Theorem 3:

$$\int_0^R |\upsilon(r, t) - \phi(r)| r dr \le Ct^{1/2}, \text{ for all } t > 0. \tag{39}$$

Then, as υ and ϕ take the values ± 1 a.e., we can prove the following simple lemma:

Lemma 2. *Assume that for $t > 0$*

$$\int_0^R |\upsilon(r, t) - \phi(r)| r dr < l$$

where $l = min\{r_1(0);\ R - r_n(0);\ r_{i+1}(0) - r_i(0),\ 1 \le i \le N-1\}$ is the smallest distance between two interfaces or the interface and the boundary at time zero. Then there are at least N interfaces at time t.

Remark 5. The similar result will hold in $\mathbf{R}^n$, $n > 2$ although with different constant.

Proof of Lemma 2: Let $\Theta := \{\alpha \in BV([0, R]) : \int_0^R |\alpha^2 - 1| dr = 0,\ \alpha \text{ has } N-1 \text{ interfaces}\}$.

We want to show that for $\alpha \in \Theta$:

$$\int_0^R |\alpha(r) - \phi(r)| r dr > l. \tag{40}$$

The proof can be done by using a simple geometric argument. By varying the positions of the interfaces for α, relative the ones for ϕ and taking into account that $|\alpha(r) - \phi(r)|$ is equal to 0 or 2 a.e. on $[0, R]$ we find that:

$$\inf_{\alpha \in \Theta} \int_0^R |\alpha(r) - \phi(r)| r dr = 2l > l. \tag{41}$$

Suppose now that $v(\cdot, t)$ has $N-1$ interfaces, then $v(\cdot, t) \in \Theta$ and thus satisfies (40). But this contradicts the assumption of the lemma. Furthermore, it is easy to observe that if Θ contains α's with less then $N-1$ interfaces, (41) will still hold. Then there are at least N interfaces at time t.

■

Denote $\tau = l^2/C^2$ with C as in (39), then (39) and the above lemma show that if $t < \tau$, then there exist at least as many interfaces at time t, as there are at time zero.

Theorem 11. *For any $t < \tau$ there are exactly N interfaces, $0 < r_1(t) < \dots < r_N(t) < R$, at the time t. Furthermore, $u^\varepsilon(\cdot, t)$ converges to $+1$ or -1 uniformly on $[r_i(t) + \delta, r_{i+1}(t) - \delta]$, as $\varepsilon \to 0$ for each $i = 1, \dots, N$ and any $\delta < 0$.*

Proof: (a) By Lemma 2 we know that there are at least N interfaces. Suppose now that their number exceeds N and is $N+K$. Let $r_i(t)$, where $1 \le i \le N+K$, be a position of the i^{th} interface at time $t < \tau$. Fix i and choose small $\delta > 0$. We want to show that for ε small enough, $u^\varepsilon(\cdot, t)$ has a zero inside $(r_i(t) - \delta, r_i(t) + \delta)$. Indeed, if this is not the case, then we can choose a subsequence $\{\varepsilon_j\}_{j \in \mathbf{N}}$ such that $u^{\varepsilon_j}(\cdot, t) > 0$ (< 0) on $(r_i(t) - \delta, r_i(t) + \delta)$ for every j large enough. Since $u^{\varepsilon_j}(\cdot, t)$ converges to $v(\cdot, t)$ in $L^1([0, R])$ we obtain that $v(\cdot, t) > 0$ a.e. on $(r_i(t) - \delta, r_i(t) + \delta)$ which contradicts our definition of the interface. Thus, for $\varepsilon > 0$ small, $u^\varepsilon(\cdot, t)$ has at least one zero near each of the $N+K$ interfaces and $u^\varepsilon_r(\cdot, t)$ has at least $N+K-1$ zeroes inside $(0, R)$ by the Rolle's theorem. This contradicts Theorem 8. Therefore, there are exactly N interfaces at time t.

(b) Fix $\delta > 0$ and $t < \tau$. By (18),

$$\int_\Omega (1 - |u^\varepsilon|)\, dx \le 8M\varepsilon \text{ for any } t \ge 0. \tag{42}$$

By (a) there exists $\varepsilon_0 > 0$ small, such that $u^\varepsilon(\cdot, t)$ has a zero inside $(r_i(t) - \delta/2, r_i(t) + \delta/2)$ for each $i = 1, \dots, N$ and $\varepsilon < \varepsilon_0$. It is easy to observe that $|u^\varepsilon|$ does not have local minima on an interval $[\, r_i(t), r_{i+1}(t)\,]$ for every $i = 1, \dots, N$. Then for every $\varepsilon < \varepsilon_0$ the minimum of $|u^\varepsilon|$ on each of the intervals $[\, r_i(t) + \delta, r_{i+1}(t) - \delta\,]$ occurs at one of the endpoints. Hence if $|u^\varepsilon|$ does not converge to $+1$ uniformly on $[\, r_i(t) + \delta, r_{i+1}(t) - \delta\,]$ for some $i = 1, \dots, N$, then $\liminf_{\varepsilon \to 0} |u^\varepsilon(r, t)| =: a < 1$ where either $r = r_i(t) + \delta$ or $r = r_i(t) - \delta$. Therefore,

$$\frac{\delta(1-a)}{2} \le \overline{\lim_{\varepsilon \to 0}} \int_\Omega (1 - |u^\varepsilon|)\, dx.$$

This contradicts (42).

■

We are now in a position to find the asymptotic limit of z^ε:

Theorem 12. *Let τ be as above. Then $z^\varepsilon(\cdot, t_0) \to d(\cdot, t_0)$ uniformly as $\varepsilon \to 0$ if $t_0 \in [\, 0, \tau\,]$ is a continuity point of E. Here $d(\cdot, t_0)$ is a signed distance function to the set Γ_t of interfaces.*

Proof: The proof is based on the method in [3] to which we refer the reader for a more detailed treatment. Let us first make the following definitions:

$$z_*(r, t) = \liminf_{\substack{\varepsilon \to 0 \\ s \to t}} z^\varepsilon(r, s), \tag{43}$$

$$z^*(r, t) = \limsup_{\substack{\varepsilon \to 0 \\ s \to t}} z^\varepsilon(r, s). \tag{44}$$

Then z^* is an upper semicontinuous and z_* is a lower semicontinuous functions on $\Omega_\tau := [\, 0, R\,] \times [\, 0, \tau\,]$. Moreover, by Theorem 7, both $z^*(\cdot, t)$ and $z_*(\cdot, t)$ are Lipschitz continuous for all $t > 0$ with Lipschitz constant $L = 1$.

(a) Let $\Psi \in C^1([\, 0, R\,])$ be such that $(z^* - \Psi)(\cdot, t_0)$ has a maximum at $r = r_0$, where $r_0 \in (\, r_i(t_0) + \delta, r_{i+1}(t_0) - \delta\,)$ for some fixed i and δ and suppose that $u^\varepsilon(\cdot, t_0)$ converges to -1 on this interval. As $q' > 0$ then $z^\varepsilon(\cdot, t_0) < 0$ on the same interval if ε is sufficiently small; therefore $z^*(\cdot, t_0) \le 0$ by Theorem 11 and Theorem 3 .

Let $g_\alpha(r, t) = \Psi(r) + \dfrac{(r - r_0)^2}{2\gamma} + \dfrac{(t - t_0)^2}{2\alpha}$ for $\alpha > 0$, $\gamma > 0$ small. Then, as z^* is an upper semicontinuous on a bounded domain, $z^* - g_\alpha$ has a global maximum at some $(\, r_\alpha, t_\alpha\,) \in \Omega_\tau$. By the construction of g_α and the boundedness of z^* we have that $(\, r_\alpha, t_\alpha\,) \notin \{\, 0, R\,\} \times [\, 0, \tau\,] \cup [\, 0, R\,] \times \{\, 0, \tau\,\}$ for α and γ sufficiently small. Furthermore, even though this maximum might not be strict, it is attained inside some small region, determined by the values of α and γ.

If we keep γ fixed and let $\alpha \to 0$ then:

$$(z^* - \Psi)(r_0, t_0) = (z^* - g_\alpha)(r_0, t_0)$$

$$\leq (z^* - g_\alpha)(r_\alpha, t_\alpha) \leq z^*(r_\alpha, t_\alpha) - \Psi(r_\alpha) - \frac{(r_\alpha - r_0)^2}{2\gamma} - \frac{(t_\alpha - t_0)^2}{2\alpha}.$$

As (r_α, t_α) is contained in a bounded domain, we can choose a subsequence $\{\alpha_k\}_{k \in N}$ such that $(r_{\alpha_k}, t_{\alpha_k}) \to (r_1, t_1) \in \Omega_t$. Then immediately $t_1 = t_0$ and as

$$(z^* - \Psi)(r_0, t_0) \leq z^*(r_{\alpha_k}, t_{\alpha_k}) - \Psi(r_{\alpha_k}, t_{\alpha_k}),$$

by passing to the limit in α_k we obtain:

$$(z^* - \Psi)(r_0, t_0) \leq limsup_{\alpha_k \to 0} z^*(r_{\alpha_k}, t_{\alpha_k}) - \Psi(r_1, t_0)$$

$$\leq z^*(r_1, t_0) - \Psi(r_1, t_0) \leq (z^* - \Psi)(r_0, t_0).$$

Therefore, $r_1 = r_0$ and since any sequence contains a subsequence convergent to the same limit $(r_\alpha, t_\alpha) \to (r_0, t_0)$ as $\alpha \to 0$.

(b) As $z^* - g_\alpha$ attains its maximum on some small subset A of Ω_τ, by definition of z^* we know that $z^\varepsilon - g_\alpha$ must attain a local maximum $(r_\alpha^\varepsilon, t_\alpha^\varepsilon)$ in some neighborhood of A for sufficiently small ε's. Notice, that passing, if necessary, to a subsequence, $(r_\alpha^\varepsilon, t_\alpha^\varepsilon) \to (r_\alpha, t_\alpha) \in \bar{A}$. Then by (12) we have at $(r_\alpha^\varepsilon, t_\alpha^\varepsilon)$ that

$$\frac{1}{\alpha}(t_\alpha^\varepsilon - t_0) - \Psi_{rr} - \frac{1}{\gamma} + 2\lambda^\varepsilon + \frac{2u^\varepsilon}{\varepsilon}\left(\left|\Psi_r + \frac{1}{\gamma}(r_\alpha^\varepsilon - r_0)\right|^2 - 1\right) \leq 0.$$

Multiplying this estimate by ε and letting $\varepsilon \to 0$ we obtain that by Theorem 11 and Theorem 3

$$2 \liminf_{\substack{\varepsilon \to 0 \\ s \to t_\alpha}} \varepsilon\lambda^\varepsilon - 2\left(\left|\Psi_r + \frac{1}{\gamma}(r_\alpha - r_0)\right|^2 - 1\right) \leq 0 \quad \text{at } (r_\alpha, t_\alpha).$$

Now if we send α to zero, use (a) and Theorem 10, we arrive at $1 - \Psi_r^2(r_0, t_0) \leq 0$. On the other hand, by Theorem 7, if $\Psi \in C^1([0, R])$ is such that $(z^* - \Psi)(\cdot, t_0)$ has a minimum at $r = r_0$ then $1 - \Psi_r^2(r_0, t_0) \geq 0$. Hence that $z^*(\cdot, t_0)$ is a viscosity solution of $1 - |Dz|^2 = 0$ in $(r_i(t_0) + \delta, r_{i+1}(t_0) - \delta)$ for arbitrary $\delta > 0$ (Since $z^*(\cdot, t_0)$ is Lipschitz continuous for fixed t_0.)

(c) As

$$z^*(r, t_0) \geq z_*(r, t_0) \tag{45}$$

the fact that the interface is located at $r_i(t_0)$ requires that $z^*(r_i(t_0), t_0) = z_*(r_i(t_0), t_0)$ since $z^*(\cdot, t_0)$ and $z_*(\cdot, t_0)$ are continuous. Then (b) and comparison result for viscosity solutions of the equation $1 - |Dz|^2 = 0$ imply that $z^*(\cdot, t_0) = d(\cdot, t_0)$ on $\{z^* < 0\}$. Similarly,

$z_*(\cdot, t_0) = d(\cdot, t_0)$ on $\{z_* > 0\}$ by the uniqueness of a viscosity solution for the problem

$$|Dz|^2 - 1 = 0 \text{ on } \{z_* > 0\},$$
$$z|_{\partial\{z_* > 0\}} = 0.$$

Finally by (45) and the restriction on the growth of $z^*(\cdot, t_0)$, $z_*(\cdot, t_0)$ we obtain that $z^*(\cdot, t_0) = z_*(\cdot, t_0) = d(\cdot, t_0)$, that is $z^\varepsilon(\cdot, t_0) \to d(\cdot, t_0)$ uniformly.

■

b. Asymptotic limit of $E^\varepsilon[u^\varepsilon]$.

Let τ again be the time until which the initial number of interfaces is preserved. We want to prove that E is continuous on $[0, \tau)$ and that $E = E^0$ (See Theorem 4 and remarks before it for definitions).

Remark 6. Since by (17) and (19) $\left\{\varepsilon^{\frac{1}{2}} u_t^\varepsilon\right\}_{\varepsilon > 0}$ is bounded in $L^1([0, R] \times [0, T])$ for any $T > 0$, Fatou's lemma implies that:

$$\int_0^T \liminf_{\varepsilon \to 0} \left(\varepsilon \int_0^R (u_t^\varepsilon)^2 r dr\right) dt \le M,$$

and thus $\liminf_{\varepsilon \to 0} \left(\varepsilon \int_0^R (u_t^\varepsilon)^2 r dr\right)$ is bounded for a.e. $t \in [0, T]$. Denote the set of such t's as A.

Then, by the definition of a lower limit, for every $t \in A$ there exists a subsequence $\{\varepsilon_{n_k(t)}\}_{k=1}^\infty$ such that $\lim_{k \to \infty} \left(\varepsilon_{n_k} \int_0^R \left(u_t^{\varepsilon_{n_k}}\right)^2 r dr\right) < \infty$. Denote the bound on $\{\varepsilon_{n_k} \int_0^R \left(u_t^{\varepsilon_{n_k}}\right)^2 r dr\}_{k=1}^\infty$ at time t as a_t. Observe that since both A and the set of continuity points of E have a full measure, so does their intersection.

Lemma 3. *Let $t \in A$ be a continuity point of E. Then, passing to a subsequence $\{\varepsilon_{n_k(t)}\}_{k=1}^\infty$, we have $\overline{\lim}_{\varepsilon \to 0} |\lambda^\varepsilon(t)| \le C(M, a_t)$, where C depends on uniform in ε bounds on the energy and its time derivative at a time t (We suppress index n_k for notational simplicity.)*

Proof: Multiplying equation (27) by $\varepsilon u_r^\varepsilon$ and integrating it in r from y to y_1 we have that:

$$\varepsilon \int_y^{y_1} u_r^\varepsilon u_t^\varepsilon dr - \left[\frac{1}{2\varepsilon}(q'(z^\varepsilon/\varepsilon))^2((z_r^\varepsilon)^2 - 1)\right]\Bigg|_y^{y_1} - \varepsilon \int_y^{y_1} (u_r^\varepsilon)^2 \frac{dr}{r} - \lambda^\varepsilon h(u^\varepsilon)\Big|_y^{y_1} = 0. \tag{46}$$

By the previous theorem we can choose y, $y_1 \in (0, R)$ (depending on t) such that:

$$\left.\begin{array}{c} \left[\frac{1}{2\varepsilon}(q'(z^{\varepsilon}/\varepsilon))^2((z_r^{\varepsilon})^2-1)\right]\Big|_y^{y_1} \to 0 \\ h(u^{\varepsilon}(y_1,t)) \to \pm\frac{2}{3} \\ h(u^{\varepsilon}(y,t)) \to \mp\frac{2}{3} \end{array}\right\} \quad \text{as } \varepsilon \to 0.$$

The third term in (46) can be bounded by:

$$\varepsilon \int_y^{y_1} (u_r^{\varepsilon})^2 \frac{dr}{r} \le \frac{2M}{y^2}$$

and the first by:

$$\varepsilon \int_y^{y_1} \left|u_r^{\varepsilon} u_t^{\varepsilon}\right| dr \le \frac{1}{y}\left(\varepsilon \int_y^{y_1} (u_r^{\varepsilon})^2 r dr\right)^{\frac{1}{2}} \left(\varepsilon \int_y^{y_1} (u_t^{\varepsilon})^2 r dr\right)^{\frac{1}{2}} \le \frac{(2Ma_t)^{\frac{1}{2}}}{y}.$$

Then the proof follows from (46).

■

Lemma 4. *Let t be as in Lemma 3. Then on a subsequence* $\{\varepsilon_{n_k(t)}\}_{k=1}^{\infty}$:

$$\frac{1}{2\varepsilon}\int_0^R (q'(z^{\varepsilon}/\varepsilon))^2 (1-(z_r^{\varepsilon})^2) r dr \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{47}$$

Proof: (a) Let r_i, $i = 1, \dots, N$ be the positions of the interfaces at time t (For notational simplicity we omit variable t). For any fixed $\delta > 0$, denote $C_\delta := [0, R] \setminus \bigcup_i^N [r_i - \delta, r_i + \delta]$. As $z^{\varepsilon}(\cdot, t) \to d(\cdot, t)$ uniformly, we immediately obtain that:

$$\frac{1}{2\varepsilon}\int_{C_\delta} (q'(z^{\varepsilon}/\varepsilon))^2 (1-(z_r^{\varepsilon})^2) r dr \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for any } \delta > 0 \text{ small}.$$

b). Now along with δ, fix $i \in \{1, \dots, N\}$ and integrate (33) in y over $[r_i - \delta, r_i + \delta]$ while $y_1 = r_i + \delta$. Then:

$$\varepsilon \int_{r_i-\delta}^{r_i+\delta} \int_y^{r_i+\delta} u_r^{\varepsilon} u_t^{\varepsilon} r dr dy + \frac{\delta}{\varepsilon}\left[(r_i+\delta)(q'((z^{\varepsilon}(r_i+\delta))/\varepsilon))^2 (1-(z_r^{\varepsilon}(r_i+\delta, t))^2)\right]$$

$$-\frac{1}{2\varepsilon}\int_{r_i-\delta}^{r_i+\delta} (q'(z^{\varepsilon}/\varepsilon))^2((z_r^{\varepsilon})^2-1) r dr - \int_{r_i-\delta}^{r_i+\delta} \int_y^{r_i+\delta} \left[\varepsilon \frac{(u_r^{\varepsilon})^2}{2} + \frac{W(u^{\varepsilon})}{\varepsilon}\right] dr dy$$

$$- \lambda^{\varepsilon} \int_{r_i - \delta}^{r_i + \delta} \left[\left(r h(u^{\varepsilon}) \Big|_y^{r_i + \delta} \right) - \int_y^{r_i + \delta} h(u^{\varepsilon})\, dr \right] dy = 0.$$

Here the terms admit the following bounds:

$$(1) \quad \left| \varepsilon \int_{r_i - \delta}^{r_i + \delta} \int_y^{r_i + \delta} u_r^{\varepsilon} u_t^{\varepsilon} r\, dr\, dy \right| \le \int_{r_i - \delta}^{r_i + \delta} \left(\varepsilon \int_y^{r_i + \delta} (u_r^{\varepsilon})^2 r\, dr \right)^{\frac{1}{2}} \left(\varepsilon \int_y^{r_i + \delta} (u_t^{\varepsilon})^2 r\, dr \right)^{\frac{1}{2}} dy$$

$$\le 2\delta (2 M a_t)^{\frac{1}{2}}$$

by Hölder 's inequality, definition of A and (19);

(2) The term

$$\frac{\delta}{\varepsilon} \left[(r_i + \delta)\, (q'(z^{\varepsilon}(r_i + \delta) \;/\; \varepsilon))^2 (1 - (z_r^{\varepsilon}(r_i + \delta, t))^2) \right]$$

converges to zero when $\varepsilon \to 0$ (as in (a));

$$(3) \quad \int_{r_i - \delta}^{r_i + \delta} \int_y^{r_i + \delta} \left[\varepsilon \frac{(u_r^{\varepsilon})^2}{2} + W(u^{\varepsilon}) \right] dr\, dy \le \frac{4 M \delta}{r_i} \quad \text{by (19);}$$

$$(4) \quad \overline{\lim_{\varepsilon \to 0}} \left| \lambda^{\varepsilon} \int_{r_i - \delta}^{r_i + \delta} \left[\left(r h(u^{\varepsilon}) \Big|_y^{r_i + \delta} \right) - \int_y^{r_i + \delta} h(u^{\varepsilon})\, dr \right] dy \right| \le \frac{8}{3} C(M, a_t) R \delta \quad \text{by definition of } h(u^{\varepsilon})$$

and the previous lemma.

Thus:

$$\overline{\lim_{\varepsilon \to 0}} \left[\frac{1}{2\varepsilon} \int_{r_i - \delta}^{r_i + \delta} (q'(z^{\varepsilon} / \varepsilon))^2 (1 - (z_r^{\varepsilon})^2)\, r\, dr \right] \le C(M, a_t, r_i)\, \delta.$$

For a given $\delta > 0$ similar estimates hold for all $i = 1, \ldots, N$.

(c) From (a) and (b):

$$\overline{\lim_{\varepsilon \to 0}} \left[\frac{1}{2\varepsilon} \int_0^R (q'(z^{\varepsilon} / \varepsilon))^2 (1 - (z_r^{\varepsilon})^2)\, r\, dr \right] \le C(M, a_t, \min_{1 \le i \le N} r_i)\, \delta$$

for any $\delta > 0$. Letting $\delta \to 0$ we deduce that:

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^R (q'(z^{\varepsilon} / \varepsilon))^2 (1 - (z_r^{\varepsilon})^2)\, r\, dr = 0.$$

■

Lemma 5. *Let t and r_i, $i = 1, \ldots, N$ be the same as in the previous lemma. Then on a subsequence $\{\varepsilon_{n_k(t)}\}_{k=1}^{\infty}$ we have (suppressing index n_k for notational simplicity):*

$$E^{\varepsilon}[u^{\varepsilon}](t) \to \frac{8\pi}{3}\sum_{i=1}^{N} r_i(t). \tag{48}$$

$$\tag{49}$$

Proof: Fix $\delta > 0$ small. Then by (15) we have:

$$E^{\varepsilon}[u^{\varepsilon}](t) = \frac{\pi}{\varepsilon}\int_0^R (q')^2 (1 + (z_r^{\varepsilon})^2)\, r dr \le \frac{\pi}{\varepsilon}\int_0^R (q')^2 (1 - (z_r^{\varepsilon})^2)\, r dr$$

$$+ \frac{2\pi}{\varepsilon}\int_{[0,R]\setminus C_\delta} (q')^2 |z_r^{\varepsilon}|\, r dr + \sum_{i=1}^{N}\left[\frac{2\pi}{\varepsilon}\int_{r_i-\delta}^{r_i+\delta} (q')^2 |z_r^{\varepsilon}|\, r dr\right],$$

where C_δ is as in Lemma 4. Observe that the first and the second integrals on the right-hand side converge to zero by the previous lemma and the uniform convergence of $z^{\varepsilon}(\cdot, t) \to d(\cdot, t)$. Also by Theorem 12, z_r^{ε} has a constant sign on each of $[r_i - \delta, r_i + \delta]$ if $\varepsilon > 0$ is small. Then by computing the integral in the third term by parts and letting $\varepsilon \to 0$ we obtain:

$$\lim_{\varepsilon \to 0} \sum_{i=1}^{N}\left[\frac{2\pi}{\varepsilon}\int_{r_i-\delta}^{r_i+\delta} (q')^2 |z_r^{\varepsilon}|\, r dr\right] = \frac{8\pi}{3}\sum_{i=1}^{N} r_i(t)$$

which together with Theorem 4. implies (48).

■

Theorem 13. *E is continuous on $[0, \tau)$ and*

$$E^{\varepsilon}[u^{\varepsilon}](t) \to \frac{8\pi}{3}\sum_{i=1}^{N} r_i(t) \tag{50}$$

for every $t \in [0, \tau)$.

Proof: Since for every continuity point t of E in A, we have that $E^{\varepsilon}[u^{\varepsilon}](t) \to E(t)$ on the whole sequence, $E(t) = \frac{8\pi}{3}\sum_{i=1}^{N} r_i(t)$ by the previous lemma. By Remark 6, the set of points for which Lemma 5 holds, is dense in $[0, \tau)$. Then, since the positions of the interfaces are continuous in time (see Theorem 3) and since E is monotone, we conclude that E is continuous on $[0, \tau)$ and thus $E = E^0[v]$ on $[0, \tau)$.

■

Remark 7. As E is continuous on $[0, \tau)$ using Theorem 12 we obtain that $z^{\varepsilon} \to d$ locally uniformly on $[0, R] \times [0, \tau)$.

Theorem 14. *If E is continuous on $(0, \tau)$ then $E^{\varepsilon}[u^{\varepsilon}] \to E$ uniformly on compact subsets* *of* $(0, \tau)$ *as* $\varepsilon \to 0$.

Proof: More generally, suppose that there is a family $\{f_n\}_{n \in \boldsymbol{N}}$ of uniformly bounded monotone continuous functions on an interval (a, b), converging to a monotone continuous function f in $L^1((a, b))$. For all $\delta > 0$ and $n \in \boldsymbol{N}$ denote $\Delta_\delta^n := \{t: |f(t) - f_n(t)| > \delta\}$. Then $meas(\Delta_\delta^n) \to 0$ as $n \to \infty$ for any $\delta > 0$, where $meas(A)$ is a Lebesgue measure of the set A. Choose $[t_1, t_2] \subset (a, b)$ and suppose that f_n does not converge to f uniformly on $[t_1, t_2]$. Then there exists $\delta > 0$ and a subsequence $\{n_k\}_{k \in \boldsymbol{N}}$ such that $[t_1, t_2] \cap \Delta_\delta^{n_k} \neq \varnothing$ for all $k \in \boldsymbol{N}$. For every $k \in \boldsymbol{N}$ select $s_{n_k} \in [t_1, t_2] \cap \Delta_\delta^{n_k}$ then passing yet to another subsequence we have that $s_{n_k} \to s \in [t_1, t_2]$ as $k \to \infty$.

In what follows we will omit index k for notational simplicity. Suppose that $\{f_n\}_{n \in \boldsymbol{N}}$ and f are monotone decreasing. Since f is continuous on (a, b), there exists $h > 0$ such that $|f(s+h) - f(s-h)| < \delta/4$. As $meas(\Delta_{\delta/2}^n) \to 0$, we can find $l \in \boldsymbol{N}$ such that $s_l \in (s-h, s+h)$ and there exist $p_1 \in (s-h, s_l) \cap \Delta_{\delta/2}^l$ and $p_2 \in (s_l, s+h) \cap \Delta_{\delta/2}^l$. Assume at first that $f(s_l) - f_l(s_l) \geq 0$ then $f(s_l) - \delta \geq f_l(s_l)$. At the same time:

$$f(s_l) = f(p_2) + (f(s_l) - f(p_2)) \leq f(p_2) + \delta/4 \leq f_l(p_2) + 3\delta/4,$$

then

$$f_l(p_2) - \delta/4 \geq f_l(s_l),$$

- a contradiction since f_l is monotone decreasing. If $f(s_l) - f_l(s_l) \leq 0$ then the same procedure works with p_1 substituted by p_2. The case when $\{f_n\}_{n \in \boldsymbol{N}}$ and f are monotone increasing can be handled in a similar manner.

■

Theorem 15. *E is locally Lipschitz continuous on* $(0, \tau)$.

Proof: As in the proof of (20) (See [6]) we have by Hölder's inequality that for any $t_1, t_2 \in [0, T]$:

$$\int_{t_1}^{t_2}\int_\Omega |h_t^\varepsilon(x,t)|\,dxdt \leq \left(\int_{t_1}^{t_2}\int_\Omega \varepsilon(u_t^\varepsilon(x,t))^2\,dxdt\right)^{1/2}\left(\int_{t_1}^{t_2}\int_\Omega \frac{(W(u^\varepsilon))^2}{2\varepsilon}\,dxdt\right)^{1/2}$$

$$\leq C(t_2 - t_1)^{1/2}\left(E^\varepsilon[u^\varepsilon](t_1) - E^\varepsilon[u^\varepsilon](t_2)\right)^{1/2},$$

where $h^\varepsilon(x,t) = h(u^\varepsilon(x,t))$ and C is independent of ε.
At the same time:

$$\int_{t_1}^{t_2}\int_\Omega |h_t^\varepsilon(x,t)|\,dxdt \geq \int_\Omega |h^\varepsilon(x,t_2) - h^\varepsilon(x,t_1)|\,dx.$$

Then, by Dominated Convergence Theorem, we have for any $t_1, t_2 \in (0,\tau)$ that as $\varepsilon \to 0$

$$\int_\Omega | h(x, t_2) - h(x, t_1)| dx \le C(t_2 - t_1)^{1/2} (E(t_1) - E(t_2))^{1/2}.$$

Here $h(x, t) = h(v(x, t))$. Since v takes values +1 or -1 a.e.,

$$| h(x, t_2) - h(x, t_1)| = \frac{2}{3} | v(x, t_2) - v(x, t_1)| \text{ a.e.}$$

Choose compact $A \in (0,\tau)$. Then as $v(x, t)$ is known explicitly on $(0,\tau)$,

$$\int_\Omega \frac{2}{3} | v(x, t_2) - v(x, t_1)| dx = \frac{2}{3} \sum_{i=1}^{N} (r_i^2(t_2) - r_i^2(t_1))$$

$$\ge \frac{4}{3} \min_{t \in A} r_1(t) \sum_{i=1}^{N} |r_i(t_2) - r_i(t_1)| \ge C_1 (E(t_1) - E(t_2))$$

for any $t_1, t_2 \in A$, where C_1 depends only on A. Therefore for any $t_1, t_2 \in A$:

$$E(t_1) - E(t_2) \le C(t_2 - t_1)^{1/2} (E(t_1) - E(t_2))^{1/2}.$$

Dividing both sides by $(E(t_1) - E(t_2))^{1/2}$, we conclude that E is locally Lipschitz continuous on $(0,\tau)$ with Lipschitz constant depending on the energy bound M and the set A.

■

This result is expected, since each of the interfaces move by its mean curvature and the limiting energy depends linearly on a position of every interface. Then, if at least one of them shrinks to zero at a time t, the limiting energy, while remaining continuous, cannot be differentiable at t.

Remark 8. By slightly changing the proof of the previous theorem, one can show that r_i are locally Lipschitz continuous on $(0,\tau)$ for all $i = 1, \dots, N$.

Remark 9. Multiplying (46) by ε and choosing y_1 appropriately we have that for any $[t_1, t_2] \subset (0,\tau)$

$$(q'(z^\varepsilon/\varepsilon))^2 ((z_r^\varepsilon)^2 - 1) \to 0$$

uniformly on $[0, R] \times [t_1, t_2]$.

Remark 10. Choose a sequence $\{h_n\}_{n \in \mathbf{N}} \subset \mathbf{R}$, such that $h_n \to 0$ as $n \to \infty$. Integrating equation (46) in t for the appropriate y and y_1, using the previous remark and passing if necessary to a subsequence, we obtain that the averages:

$$\lambda_{h_n}^\varepsilon(t) := \frac{2}{h_n} \int_t^{t+h_n} \lambda^\varepsilon(s)\, ds \tag{51}$$

converge uniformly to a limit λ_{h_n} for each $n \in \mathbf{N}$. Moreover, by Theorem 15, the sequence

$\{ \lambda_{h_n} \}_{n \in \mathbf{N}}$ is locally uniformly bounded on $(0, \tau)$.

c. Asymptotic behavior of phase - field equations.

Our next goal is to determine the dynamics of the moving interfaces on an interval $[0, \tau)$. We will use the following strategy. First, as in [3], we are going to show that the distance function d satisfies certain differential inequalities in a viscosity sense. Then using these inequalities we will determine that the interface position functions solve a system of the ODE's (again in a viscosity sense). Finally, we will prove that given the initial positions of interfaces the uniqueness of viscosity solution to latter system of ODE's implies that r_i, $i = 1, \dots, N$ is also a classical solution of the same system.

We can prove the following refinement of Remark 10:

Lemma 6. *Let λ_h^ε be as in (51). Then there exist $\lambda_h \in C([0, \tau-\gamma])$ and $\lambda \in C([0, \tau-\gamma])$ such that:*

a. $\lambda_h^\varepsilon \to \lambda_h$ *uniformly on* $[0, \tau-\gamma]$ *as* $\varepsilon \to 0$;

b. $\lambda_h \to \lambda$ *uniformly on* $[0, \tau-\gamma]$ *as* $h \to 0$.

Moreover,

$$\lambda(t) = \frac{\alpha_N(t)}{\sum_{i=1}^{N} r_i(t)}.$$

Here N is the number of interfaces and α_N is a function depending on the geometry of the problem:

$$\alpha_N(t) = \frac{v(R,t) - v(0,t)}{2} \tag{52}$$

Proof: Multiply the equation (27) by $\varepsilon r |u_r^\varepsilon|$ and integrate over $[0, R]$ by parts. Then for $\varepsilon > 0$ small we obtain:

$$\int_0^R |u_r^\varepsilon| u_t^\varepsilon r\,dr - \frac{\varepsilon}{2}\int_0^R |u_r^\varepsilon| u_r^\varepsilon\,dr + \frac{1}{\varepsilon}\int_0^R W'(u^\varepsilon)|u_r^\varepsilon| r\,dr - \lambda^\varepsilon \int_0^R h'(u^\varepsilon)|u_r^\varepsilon| r\,dr = 0. \tag{53}$$

Since $z^\varepsilon \to d$ uniformly on $[0, R] \times [0, \tau - \gamma]$ and by our condition on the number of zeroes of z_r^ε, we immediately obtain that zeroes of z_r^ε are uniformly on a distance of order 1 away from the interfaces if $\varepsilon > 0$ is small enough. It is also easy to observe that the coefficient of λ^ε,

$$B^\varepsilon := -\int_0^R h'(u^\varepsilon)|u_r^\varepsilon| r\,dr = \frac{1}{2\pi}\int_\Omega |Dh(u^\varepsilon)|\,dx \to \frac{4}{3}\sum_{i=1}^{N} r_i$$

uniformly as $\varepsilon > 0$ since $z^\varepsilon \to d$ uniformly on $[0, R] \times [0, \tau - \gamma]$. As r_i are uniformly away from zero on $[0, \tau - \gamma]$ for $i = 1, \dots, N$, then for sufficiently small $\varepsilon > 0$ and $h > 0$ we can divide (53) by

B_ε and integrate the result over $(t, t+h)$ to obtain

$$
\begin{aligned}
&-\varepsilon \int_t^{t+h} B_\varepsilon^{-1}(s)\int_0^R \left|u_r^\varepsilon\right| u_t^\varepsilon r\,dr\,ds + \int_t^{t+h} B_\varepsilon^{-1}(s)\left(\sum_{i\in A_+}\left\{\int_{r_i-\delta}^{r_i+\delta}\left(\frac{\varepsilon}{2}(u_r^\varepsilon)^2+\frac{1}{\varepsilon}W(u^\varepsilon)\right)dr\right\}\right. \\
&\left. + \sum_{i\in A_-}\left\{\int_{r_i-\delta}^{r_i+\delta}\left(\frac{\varepsilon}{2}(u_r^\varepsilon)^2+\frac{1}{\varepsilon}W(u^\varepsilon)\right)dr\right\}\right)ds + o(1) = \int_t^{t+h}\lambda^\varepsilon(s)\,ds
\end{aligned}
\tag{54}
$$

for any small $\delta > 0$, where $o(1)$ denotes the terms uniformly convergent to zero on $[0, t-\gamma]$ as $\varepsilon \to 0$ while A_+ is a set of indices of the interfaces with $d_r > 0$ and A_- is a set of indices of the interfaces with $d_r < 0$. Note that:

$$
\frac{1}{2\varepsilon}\int_t^{t+h}\int_0^R (q'(z^\varepsilon/\varepsilon))^2(1-\left|z_r^\varepsilon\right|)^2 r\,dr\,ds \to 0 \quad \text{as } \varepsilon \to 0
$$

uniformly on $[0, t-\gamma]$. From (8) and since $q' = 1 - q^2 = g(q)$ we have:

$$
0 = \frac{d}{dt}\int_0^R h(u^\varepsilon)\,r\,dr = \int_0^R g(u^\varepsilon)\,u_t^\varepsilon r\,dr = \frac{1}{\varepsilon}\int_0^R (q'(z^\varepsilon/\varepsilon))^2 z_t^\varepsilon r\,dr, \tag{55}
$$

then by (55) and Hölder's inequality:

$$
\begin{aligned}
&\left|\varepsilon\int_t^{t+h} B_\varepsilon^{-1}(s)\int_0^R \left|u_r^\varepsilon\right| u_t^\varepsilon r\,dr\,ds\right| = \left|\frac{1}{\varepsilon}\int_t^{t+h} B_\varepsilon^{-1}(s)\int_0^R (q'(z^\varepsilon/\varepsilon))^2\left|z_r^\varepsilon\right| z_t^\varepsilon r\,dr\,ds\right| \\
&= \left|\frac{1}{\varepsilon}\int_t^{t+h} B_\varepsilon^{-1}(s)\int_0^R (q'(z^\varepsilon/\varepsilon))^2(1-\left|z_r^\varepsilon\right|)\,z_t^\varepsilon r\,dr\,ds\right| \\
&\le \sup_{s\in[t,t+h]} B_\varepsilon^{-1}(s)\left(\int_t^{t+h}\int_0^R (q'(z^\varepsilon/\varepsilon))^2\frac{(1-\left|z_r^\varepsilon\right|)^2}{\varepsilon}r\,dr\,ds\right)^{\frac{1}{2}}\left(\varepsilon\int_t^{t+h}\int_0^R (u_t^\varepsilon)^2 r\,dr\,ds\right)^{\frac{1}{2}} \to 0
\end{aligned}
$$

as $\varepsilon \to 0$ uniformly on $[0, t-\gamma]$ by the remark above. It is easy to show that the remaining term in (54) converges uniformly to:

$$
\int_t^{t+h}\frac{\alpha_N(s)}{\sum_{i=1}^{N} r_i(s)}\,ds
$$

where α_N defined in (52) is either $+1$, -1 or 0 depending on the relative number of the interfaces of A_+ and A_- types. Thus for any $h > 0$ small:

$$\lambda_h^\varepsilon \to \lambda_h = \frac{1}{h}\int_t^{t+h} \frac{\alpha_N(s)}{\sum_{l=1}^{N} r_l(s)}\,ds$$

uniformly on $[0, t-\gamma]$ as $\varepsilon \to 0$. Now as $\dfrac{\alpha_N(\cdot)}{\sum_{l=1}^{N} r_l(\cdot)}$ is continuous on a compact set $[0, t-\gamma]$,

$$\lambda_h \to \frac{\alpha_N(\cdot)}{\sum_{l=1}^{N} r_l(\cdot)}$$

uniformly on $[0, t-\gamma]$ as $h \to 0$.

■

Theorem 16. *Assume that* $\phi \in C^\infty([0, R] \times [0, \tau))$ *is such that* $d - \phi$ *has a maximum at* $(r_0, t_0) \in (0, R) \times (0, t)$ *and* $d(r_0, t_0) < 0$. *Then:*

$$\phi_t - \Delta\phi + \lambda \le 0 \quad at \ (r_0, t_0), \tag{56}$$

where $\lambda(t)$ *is as in Lemma 6.*

Remark 11. Without loss of generality we can assume that the maximum of $d - \phi$ is strict. Indeed, this can be achieved by replacing ϕ with $\phi_{\alpha\beta} = \dfrac{(r-r_0)^4}{\alpha} + \dfrac{(t-t_0)^2}{\beta}$ and choosing α and β small enough.

Proof: Fix $\gamma > 0$ and let $\gamma > h > 0$. Suppose $\Omega_\gamma := [0, R] \times [0, \tau - \gamma]$. Denote:

$$z_h^\varepsilon = \frac{1}{h}\int_t^{t+h} z^\varepsilon(x, s)\,ds$$

$$d_h = \frac{1}{h}\int_t^{t+h} d(x, s)\,ds$$

$$\phi_h = \frac{1}{h}\int_t^{t+h} \phi(x, s)\,ds$$

As $z^\varepsilon \to d$ uniformly on $[0, R] \times [0, \tau)$ we have that:

(a) $z_h^\varepsilon \to d_h$ uniformly as $\varepsilon \to 0$ on Ω_γ;

(b) $d_h \to d$, $\dfrac{\partial^i}{\partial r^i}\phi_h \to \dfrac{\partial^i}{\partial r^i}\phi$, $\dfrac{\partial}{\partial t}\phi_h \to \dfrac{\partial}{\partial t}\phi$ for $i = 1, 2$, uniformly on as $h \to 0$ on Ω_γ since both d and ϕ together with its derivatives are continuous on compact $[0, R] \times [0, \tau - \gamma + h]$.

As $d_h - \phi_h$ converges to $d - \phi$ uniformly for $h > 0$ small, $d_h - \phi_h$ achieves its local maximum in a small neighborhood $A_h := [r_0 - a_h, r_0 + a_h] \times [t_0 - b_h, t_0 + b_h]$ of (r_0, t_0) such that $a_h, b_h \downarrow 0$ as $h \to 0$. Similarly as $z_h^\varepsilon \to d_h$ uniformly, for $\varepsilon > 0$ small $z_h^\varepsilon - \phi_h$ has a maximum in a small neighborhood $A_{\varepsilon h} := [\, r_0 - a_{\varepsilon h},\, r_0 + a_{\varepsilon h}\,] \times [\, t_0 - b_{\varepsilon h},\, t_0 + b_{\varepsilon h}\,]$ of $(\, r_0,\, t_0\,)$ and such that $a_{\varepsilon h} \downarrow a_h$, $b_{\varepsilon h} \downarrow b_h$ as $\varepsilon \to 0$. Observe also that if h and ε are given, $|A_{\varepsilon h}|$ can be made arbitrarily small by choosing α and β small enough, where α and β are as in Remark 11. This implies that we can find h_1, $\varepsilon_1 > 0$ such that for any $\varepsilon_1 > \varepsilon > 0$ and $h_1 > h > 0$,

$$z^{\varepsilon}(\, r, t\,) < 0 \quad \text{if} \quad (\, r, t\,) \in [\, r_0 - a_{\varepsilon h},\, r_0 + a_{\varepsilon h}\,] \times [\, t_0 - b_{\varepsilon h},\, t_0 + b_{\varepsilon h} + h\,].$$

By integrating equation (12) in t we obtain:

$$z^{\varepsilon}_{ht} - \Delta z^{\varepsilon}_{h} + \frac{2}{h}\int_t^{t+h} \lambda^{\varepsilon}(s)\, ds + \frac{2}{\varepsilon h}\int_t^{t+h} u^{\varepsilon}\left(\frac{z^{\varepsilon}(r,s)}{\varepsilon}\right)\left(\left(z^{\varepsilon}_{r}(r,s)\right)^2 - 1\right) ds = 0 \tag{57}$$

on Ω_γ. Then by definition of u^{ε} and Theorem 7, the last integral is positive in $A_{\varepsilon h}$. Fix any $(\, r_{\varepsilon h},\, t_{\varepsilon h}\,) \in A_{\varepsilon h}$ where $z^{\varepsilon}_{h} - \phi_h$ has a local maximum, then by (57) we have:

$$\phi_{ht} - \Delta \phi_h + \lambda^{\varepsilon}_{h} \le 0 \quad at \quad (r_{\varepsilon h}, t_{\varepsilon h}),$$

where $\lambda^{\varepsilon}_{h}(t)$ is as in (51). By Lemma 6, $\lambda^{\varepsilon}_{h} \to \lambda_h$ uniformly as $\varepsilon \to 0$ and, in turn, $\lambda_h \to \lambda$ uniformly as $h \to 0$. Let $\varepsilon \to 0$. Then passing to a subsequence, we obtain that $(\, r_{\varepsilon h},\, t_{\varepsilon h}\,) \to (\, r_h,\, t_h\,) \in \bar{A}_h$ and that:

$$\phi_{ht} - \Delta \phi_h + \lambda_h \le 0 \quad at \quad (r_h, t_h).$$

Letting now $h \to 0$ and repeating the above procedure for h we obtain:

$$\phi_t - \Delta \phi + \lambda \le 0 \quad at \quad (r_0, t_0). \tag{58}$$

■

Suppose now that $\phi \in C^{\infty}(\,[\,0, R\,] \times [\,0, \tau\,)\,)$ is such that $d - \phi$ has a maximum at $(\, r_i(\, t_0\,),\, t_0\,)$ and i^{th} interface is such that $d_r(\, t_0,\, r_i(\, t_0\,)\,) > 0$. Since in the neighborhood of $(\, r_i(\, t_0\,),\, t_0\,)$ the distance function $d(\, r, t\,) \equiv r - r_i(\, t\,)$ and r_i is continuous on $[\,0, \tau - \gamma\,]$ we have that for $d > 0$ small:

$$\phi_\delta := \phi(\, r + \delta,\, t\,) + \delta \in C^{\infty}(\,[\,0, R\,] \times [\,0, \tau\,)\,)$$

is such that $d - \phi_\delta$ has a maximum at $(\, r_i(\, t_0\,) - \delta,\, t_0\,)$. Then by (56):

$$\phi_{\delta t} - \Delta \phi_\delta + \lambda \le 0 \quad \text{at} \quad (r_i(t_0) - \delta, t_0),$$

and thus:

$$\phi_t - \Delta \phi + \lambda \le 0 \quad \text{at} \quad (r_i(t_0), t_0).$$

Therefore, d is a viscosity subsolution of:

$$z_t - \Delta z + \lambda = 0 \quad \text{at} \quad (r_i(t_0), t_0). \tag{59}$$

Similarly, we can show that d is also a viscosity supersolution and thus a viscosity solution of (59) at $(\, r_i(\, t_0\,),\, t_0\,)$ for any $t_0 \in [0, \tau - \gamma]$. Since d is known explicitly for any $t \in [\,0, \tau\,)$ and constructing a test function in a way , similar to that of Theorem 12, we find that r_i is a viscosity solution of:

$$\dot{r}_i + \frac{1}{r_i} - \frac{\alpha_N}{\sum_{i=1}^{N} r_i} = 0. \tag{60}$$

The same procedure for the interfaces with $d_r(t_0, r_i(t_0)) < 0$ shows that r_i satisfies:

$$\dot{r}_i + \frac{1}{r_i} + \frac{\alpha_N}{\sum_{i=1}^{N} r_i} = 0 \tag{61}$$

in a viscosity sense. Given the initial locations of the interfaces, the system of ODE's (60) and (61) has a unique classical solution. Then due to the uniqueness of viscosity solution for the same system, $\bar{r} = (r_1, \ldots, r_N)$ is a classical solution of (60-61) on $[0, \tau - \gamma]$ and, since γ is arbitrary, on $[0, \tau)$. The mass preservation property of the limiting flow follows directly from (60) and (61) or by considering an asymptotic limit of (8).

Any subsequence of our original sequence in ε contains a subsequence such that u^ε converges in $L^1([0, R] \times [0, \tau))$ to a limit v (that may, in general, depend on the subsequence) satisfying:

(a) $v(x, t) \in \{-1, 1\}$ a. e. on $[0, R] \times [0, \tau)$,

(b) $lim_{t \to 0} v(x, t) = \phi(x)$ a. e. on $[0, R]$,

with interfaces moving according to (60-61). However, since a function with these properties is unique in $L^1([0, R] \times [0, \tau))$, then v is the same for all subsequences. Therefore, v is an asymptotic limit of u^ε on the whole sequence.

Observe that if N is even, then λ is zero and therefore interfaces simply move by their mean curvature and our solution exists at least up to the time when the innermost interface shrinks to zero. No interfaces can collide with each other in this case. However, latter may occur if N is odd. We consider these situations in the next section.

7. Asymptotic behavior of reaction-diffusion equations.

Interactions.

Observe first, that if two interfaces of the opposite sign collide and so called "ghost" interface is formed, then the assumptions (31) become useless. Therefore, in order to continue our analysis, we have to generalize the results of Section 6 to a wider class of initial data.

Suppose that (31) is no longer valid. First, we have to prove the analog of Theorem 12 in a new setting. This theorem was demonstrated using the facts that the interfaces, defined in Section 4, move continuously in time and that u^ε is uniformly negative (correspondingly, positive) between the interfaces. We want to extend this result to the "ghost" interfaces (See Definition 2.)

Let:

$$N_t := int\ cl\{r \in (0, R) : u^\varepsilon(r, t) \to -1\},$$

$$P_t := int\ cl\{r \in (0,R) : u^{\varepsilon}(r,t) \to +1\}, \tag{62}$$

where $int\ A$ is an interior of the set A and $cl\ A$ is its closure. N_t and P_t are clearly non-empty for all $t > 0$ (except when $v(r,0) = +1$ a.e or $v(r,0) = -1$ a.e. $r \in (0,R)$). For each $\delta > 0$ denote:

$$\begin{aligned} N_t^{\delta} &:= \{r \in N_t : \overline{\lim_{\substack{\varepsilon \to 0 \\ s \to r}}}\ u^{\varepsilon}(s,t) \geq -\delta\}, \\ P_t^{\delta} &:= \{r \in P_t : \overline{\lim_{\substack{\varepsilon \to 0 \\ s \to r}}}\ u^{\varepsilon}(s,t) \leq +\delta\}. \end{aligned} \tag{63}$$

Remark 12. Fix $t_0 > 0$ and $\delta > 0$. Since the energy is bounded, passing, if necessary, to a subsequence, one can assume that there are finitely many points in $N_{t_0}^{\delta} \cap [r_0, R]$ and in $P_{t_0}^{\delta} \cap [r_0, R]$ for any $r_0 \in (0,R)$.

We are now in a position to prove the following technical lemma that is somewhat similar to the clearing - out lemma (See [13], [14], [15] or [16]):

Lemma 7. *Let $T > 0$ be given. Then there exist a $\delta > 0$ and a set $\Pi_\delta \subset [0, T]$ containing the finitely many points such that the following holds:*

Let $t_0 \in [0,T] \setminus \Pi_\delta$ and $A \subset N_{t_0} \setminus N_{t_0}^{\delta}$ be an open set where $N_{t_0}^{\delta}$ is as in Remark 12. Then there exists an $\alpha(A, t_0) > 0$ for which

$$N_t^{\delta} \cap [r_0, R] \cap \bar{A} = \varnothing \tag{64}$$

for any $t \in [t_0, t_0 + \alpha(A, t_0))$ and any $r_0 \in (0, R)$. The same result holds for P_t^{δ}.

Proof: Let $T > 0$ be given. Fix $\theta > 0$. Since the "limiting energy" E is a bounded, monotone decreasing function of t, there are at most finitely many points $t \in [0,T]$ such that $E(t^-) - E(t^+) \geq \theta/2$. Denote the set of the such points as Π_δ. Then $[0,T] \setminus \Pi_\delta$ consists of the finitely many intervals on which the variation of E does not exceed $\theta/2$:

$$\begin{aligned} (0,T) \setminus \Pi_\delta &= \bigcup_{i=1}^{n} (t_{i-1}, t_i), \\ E(t_{i-1}^{+}) - E(t_i^{-}) &\geq \frac{\theta}{2} \quad \text{for } i = 1, \dots, n, \end{aligned}$$

where $E(t_i^{\pm}) = \lim_{t \to t_i^{\pm}} E(t)$. Then, by choosing $[\hat{t}_{i-1}, \hat{t}_i] \subset (t_{i-1}, t_i)$ for $i = 1, \dots, n$, we have for ε sufficiently small that the variation of $E^{\varepsilon}[u^{\varepsilon}]$ on each of these intervals does not exceed θ. Applying the procedure of Theorem 9, we find that for every point $t_0 \in [0,T] \setminus \Pi_\delta$ one can choose $\gamma(t_0)>0$ such that

$$\left|\varepsilon\lambda^{\varepsilon}(t)\right| \le C(M)\,\theta^{\frac{1}{2}}+o(\varepsilon) \tag{65}$$

uniformly on $(t_0, t_0+\gamma(t_0))$. Fix now $t_0\in [0,T]\setminus\Pi_\delta$ and $\varepsilon_0>0$. Denote

$$\frac{\delta}{2}=\frac{\delta(\theta,\varepsilon_0)}{2} := C(M)\,\theta^{\frac{1}{2}}+o(\varepsilon_0)\ . \tag{66}$$

Let A be an open set such that $\bar{A}\subset N_{t_0}\setminus\ N^{\delta}_{t_0}$ and suppose for simplicity that A lies between two consecutive points in $N^{\delta}_{t_0}$ (which is possible by Remark 12.) Suppose that for every $t\in (t_0, t_0+\gamma(t_0))$:

$$N^{\delta}_{t}\cap\bar{A}\ \neq\emptyset. \tag{67}$$

This implies that for every $t\in (t_0, t_0+\gamma(t_0))$ there exists an $r_t\in\bar{A}$ and a subsequence $\{\varepsilon_{j_k(t)}\}_{k\in\mathbf{N}}$ such that

$$\lim_{\varepsilon\to 0}\ u^{\varepsilon}(r_t\,,t)\ \ge\ -\,\delta. \tag{68}$$

Fix $t\in (t_0, t_0+\gamma(t_0))$. We can choose an open set B such that $\bar{B}\subset N_{t_0}\setminus\ N^{\delta}_{t_0}$ and $\bar{A}\subset B$. By our assumption, $u^{\varepsilon}(r,t_0)<-\ \delta$ uniformly on B for $\varepsilon>0$ small. Let $\partial B=\{\partial B^{-},\partial B^{+}\}$ (assuming again that B lies entirely between two consecutive points in $N^{\delta}_{t_0}$). Fix $\varepsilon>0$ small. Suppose that $u^{\varepsilon}(r,t_0)<-\ \delta$ on $\partial B\times(0,t)$. Then the remark above and (68) imply that for some $\tau\in(0,t)$ there exists an interior maximum on $B\times(0,\tau]$ at (r_τ, t) with $u^{\varepsilon}(r_\tau,\tau)\le-\ \delta$. At the point of the maximum we must have (following the usual proof of the maximum principle):

$$0<-\ \frac{\delta}{2}+\delta\le\varepsilon\lambda^{\varepsilon}(\tau)-u^{\varepsilon}(r_\tau\,,\tau)\le 0$$

- contradiction. Thus for some $\tau_\varepsilon\in(0,T)$ we have that, for example, $u^{\varepsilon}(\partial B^{+},\tau_\varepsilon)\approx-\ \delta$ and there must exist a continuous curve $r^{\varepsilon}(t)$ such that $u^{\varepsilon}(r^{\varepsilon}(t),t)\approx-\ \delta$ for $t\in[\tau_\varepsilon\,,t]$ and $r^{\varepsilon}(\tau_\varepsilon)=\partial B^{+}$, $r^{\varepsilon}(t)=r_t$. In other words, for any $r\in(r_t\,,\partial B^{+})$ there exists $t_\varepsilon{}^{r}\in[t_0,t]$ such that $u^{\varepsilon}(r,t_\varepsilon{}^{r})\approx-\ \delta$. Consequently,

$$\int_{t_0}^{t}\left|h^{\varepsilon}_t(r,s)\right|ds\ge\left|h^{\varepsilon}(t_\varepsilon{}^{r})-h^{\varepsilon}(t_0)\right|\ \approx\ h^{\varepsilon}(t_0)-h(-\delta)\,. \tag{69}$$

Also, following the proof of Theorem 3, we obtain the following estimate:

$$\int_{\Omega} \int_{t_1}^{t_2} \left| h_t^{\varepsilon}(x, s) \right| ds dx \le C(M) (t_2 - t_1)^{\frac{1}{2}}. \qquad (70)$$

Then:

$$\int_{r_t}^{\partial B^+} (h^{\varepsilon}(t_0) - h(-\delta))\, ds \le \int_{r_t}^{\partial B^+} \int_{t_0}^{t} \left| h_t^{\varepsilon}(x, s) \right| ds dx \le C(M) (t - t_0)^{\frac{1}{2}},$$

and as $h^{\varepsilon} = \frac{(u^{\varepsilon})^3}{3} - u^{\varepsilon}$ with $u^{\varepsilon} \to -1$ a.e. on B:

$$\left(\frac{2}{3} - \delta + \frac{\delta^3}{3}\right) \times (\partial B^+ - \partial A^+) \le \left(\frac{2}{3} - \delta + \frac{\delta^3}{3}\right) \times (\partial B^+ - r_t) \le C(M) (t - t_0)^{\frac{1}{2}}. \qquad (71)$$

As ε_0 in (66) can be chosen arbitrarily small, $\delta = 2C(M)\theta^{\frac{1}{2}}$. Similarly, θ can be chosen initially so small, that $\delta < 1$. Then the L.H.S. in (71) is strictly positive while the R.H.S. can be made arbitrarily small by choosing t sufficiently close to t_0. This is a contradiction. Thus there exists $\alpha(A, t_0) > 0$ such that $N_t^{\delta} \cap \bar{A} = \varnothing$ for any $t \in [t_0, t_0 + \alpha(A, t_0))$. The generalization to (64) is obvious.

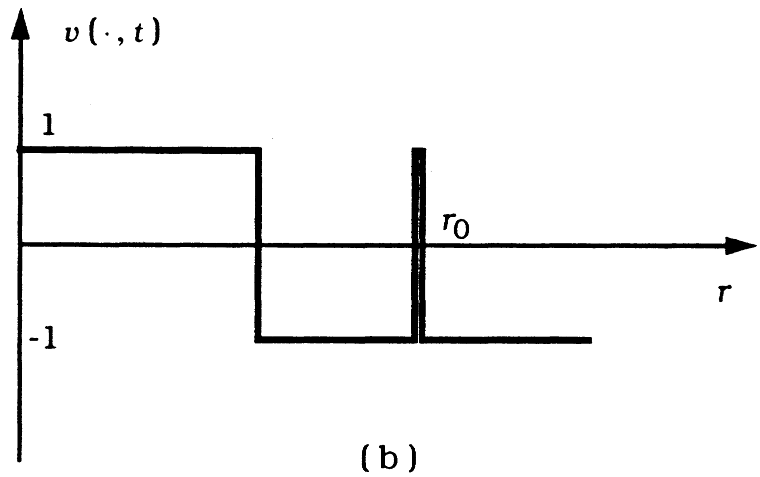
■

From now on we will assume that δ is as in the previous lemma.

Definition 2. *Fix $t \ge 0$. Then every point $r_0 \in N_t^{\delta} \cup P_t^{\delta}$ will be called a "ghost" interface at the time t. We will denote the set $N_t^{\delta} \cup P_t^{\delta}$ of the "ghost" interfaces as Γ_t'.*

The geometrical interpretation of this definition is shown on Fig. 1. In what follows we will refer to the interfaces from Γ_t (See Def. 1) as "regular" interfaces. Observe that our definitions of the "ghost" and "regular" interfaces does not use any information about their inner structure. In particular, each may have multiplicity higher than one.

Remark 13. Let $[t_1, t_2] \subset [0, T] \setminus \Pi_{\delta}$ and choose an open set $A \subset [0, R] \times [t_1, t_2]$ such that for every $s \in [t_1, t_2]$, the set $A_s := A \cap \{t = s\} \subset N_s \setminus N_s^{\delta}$ and $dist(A_s, N_s^{\delta}) \ge a$ for some $a > 0$ (Here $dist(A, B)$ is a distance between two sets $A, B \subset [0, R]$.) Then following the proof of Lemma 7, one can see that $\alpha(A_t, t)$ is uniformly bounded from below on $[t_1, t_2]$. The same result holds for P_t^{δ}.



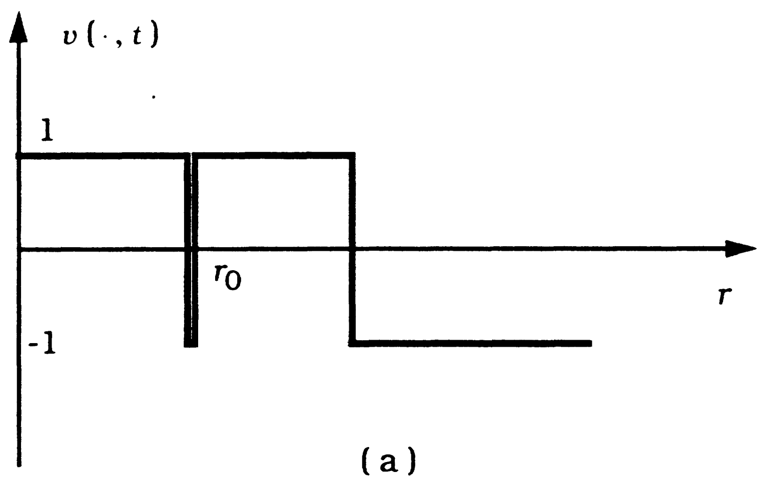


Figure 1: The "ghost" interfaces: (a) $r_0 \in N_t^\delta$; (b) $r_0 \in P_t^\delta$

The next three corollaries are immediate consequences of Lemma 7.

Corollary 1. *The position functions of the "ghost" interfaces are continuous on their respective intervals of existence, except maybe the points from the set* Π_δ.

Corollary 2. *The "ghost" interfaces can nucleate only at exceptional times* $t \in \Pi_\delta$.

Corollary 3. *For each* $t \in [0, T] \setminus \Pi_\delta$ *and each closed set* $A \subset N_t \setminus N_t^\delta$ *there exists an* $h > 0$ *such that for* $\varepsilon > 0$ *small* u^ε *is uniformly negative on* $A \times [t, t+h]$. *The same result holds for* P_t^δ.

Then repeating the proof of Theorem 12 we obtain:

Theorem 17. *Fix* $r_0 \in (0, R)$. *Then* $z^\varepsilon(\cdot, t_0) \to d(\cdot, t_0)$ *uniformly on* $[r_0, R]$ *if* $t_0 \in [0, T] \setminus \Pi_\delta$ *is a continuity point of E. Here* $d(\cdot, t_0)$ *is a signed distance function to the set* $(\Gamma_t \cup \Gamma_t') \cap [r_0, R]$.

Suppose now that a time $t_0 \in [0, T] \setminus \Pi_\delta$ there are finitely many interfaces of either kind - "regular" or "ghost" - and that they are all located away from zero. In a view of Theorem 8 this will hold, for example, if it is assumed that ϕ_r^ε have uniformly bounded in ε number of zeroes in $[0, R]$. Of course, the "ghost" interfaces depend, in general, on a subsequence chosen in Lemma 7. We would like to show first, that the evolution of every "regular" interface depends in a nonlocal fashion on its mean curvatures. By our assumption on t_0, there exists a number $\alpha(t_0)$ such that the positions of all interfaces are continuous functions $r_i(\cdot)$ on $[t_0, t_0 + \alpha(t_0)]$ for $i = 1, \ldots, N+K$. Here N is a number of the "regular" interfaces and K is a number of the "ghost" interfaces. Furthermore, since there are finitely many interfaces (the set $\Gamma_t \cup \Gamma_t'$ is finite) for every $t \in [t_0, t_0 + \alpha(t_0)]$ then r_0 can be set equal to zero in Theorem 17. Then, with minor changes, the results of Section 6b still hold on $(t_0, t_0 + \alpha(t_0))$. In particular, Remark 9 enables us to describe an inner structure of every interface on $(t_0, t_0 + \alpha(t_0))$. Geometrically, each of the interfaces consists of finitely many jumps between + 1 and − 1 clustered at a given point. On a scale $\sim \varepsilon$ these jumps are located infinitely far away from each other. The number of jumps constitute a multiplicity of the interface. If the total variation of v across the interface is + 1 or − 1 we have a "regular" interface while, if it is equal to zero, we obtain a "ghost" interface.

Recall that for any $h > 0$

$$d_h = \frac{1}{h} \int_t^{t+h} d(r, s)\, ds.$$

Then as d is known explicitly we have for h small enough, that near the "regular" interfaces either

$$d_h(r,t) = r - \frac{1}{h}\int_t^{t+h} r_i(s)\,ds, \text{ if } d_r(r_i(t),t) > 0 \text{ for } t \in (t_0, t_0+\alpha(t_0)-\gamma) \tag{72}$$

or

$$d_h(r,t) = \frac{1}{h}\int_t^{t+h} r_i(s)\,ds - r, \text{ if } d_r(r_i(t),t) < 0 \text{ for } t \in (t_0, t_0+\alpha(t_0)-\gamma), \tag{73}$$

for some small $\gamma \in (0, \alpha(t_0))$ and $i = 1, \ldots, N$. Denote

$$r_{ih}(t) := \frac{1}{h}\int_t^{t+h} r_i(s)\,ds.$$

Following the proof of Theorem 16 we obtain

Theorem 18. *Choose $\{h_n\}_{n \in \mathbf{N}}$ for which the conclusion of Remark 10 holds. For $\gamma \in (0, \alpha(t\ ))$ small, denote $\Omega_\alpha = [0, R] \times (t_0, t_0+\alpha(t_0)-\gamma)$. Assume that $\phi \in C^\infty\ (\Omega_\alpha)$ is such that $d_{h_n} - \phi$ has a maximum at (r_0, t_0) and $d_{h_n}(r_0, t_0) < 0$. Then:*

$$\phi_t - \Delta\phi + \lambda_{h_n} \le 0 \text{ at } (r_0, t_0),$$

where λ_{h_n} is as in Remark 10.

Finally, by repeating the arguments following Theorem 17, we conclude that for every h small (omitting the index n in h_n):

(a) If $d_r(r_i(t), t) > 0$ for $t \in (t_0, t_0+\alpha(t_0)-\gamma)$, then r_{ih} is a viscosity solution of

$$\dot{r}_{ih} + \frac{1}{r_i^h} - \lambda_h = 0 \quad \text{on } (t_0, t_0+\alpha(t_0)-\gamma), \tag{74}$$

(b) If $d_r(r_i(t), t) < 0$ for $t \in (t_0, t_0+\alpha(t_0)-\gamma)$, then r_{ih} is a viscosity solution of

$$\dot{r}_{ih} + \frac{1}{r_i^h} + \lambda_h = 0 \quad \text{on } (t_0, t_0+\alpha(t_0)-\gamma), \tag{75}$$

where $\lambda_h \in C([t_0, t_0+\alpha(t_0)])$. Recall that, by Remark 10, the set $\{\lambda_h\}_{n \in \mathbf{N}}$ is bounded in $C([t_0 + \gamma, t_0+\alpha(t_0)-\gamma])$ for $\gamma \in (0, \alpha(t_0))$ small enough.

Since the positions of all interfaces are known at the time $t = t_0$, then so are the initial conditions on r_{ih} for every $i = 1, \ldots, N$. Then, given λ_h, the equations (74) and (75) have the unique classical and, therefore, viscosity solutions on $[\ t_0, t_0+\alpha(t_0)-\gamma)$. We assume for simplicity that no "ghost" interface "opens up" on $(t_0, t_0+\alpha(t_0)-\gamma)$, creating one or more "regular" interfaces. If this had happened for some time $t_1 \in (t_0, t_0+\alpha(t_0)-\gamma)$, we would have two different systems (74 - 75) on two consecutive time intervals, (t_0, t_1) and $(t_1, t_0+\alpha(t_0)-\gamma)$. Then both systems can be solved separately and their solutions "glued" together at $t = t_1$.

Now we can prove the following

Theorem 19. *Fix* $\gamma \in (0, \alpha(t_0))$. *There exists* $\lambda \in L^{\infty}([t_0 + \gamma, t_0 + \alpha(t_0) - \gamma])$ *such that on* $[t_0 + \gamma, t_0 + \alpha(t_0) - \gamma]$ *the function* r_i *is a weak (in a sense of distributions) solution of*

$$\dot{r}_i + \frac{1}{r_i} - \lambda = 0 \quad \text{if } d_r(r_i(t), t) > 0, \tag{76}$$

and

$$\dot{r}_i + \frac{1}{r_i} + \lambda = 0 \quad \text{if } d_r(r_i(t), t) > 0, \text{ where } i = 1, \dots, N. \tag{77}$$

Moreover, λ *satisfies*

$$\int_{t_1}^{t_2} \left(\sum_{i=1}^{N} r_i(s) \right) \lambda(s)\, ds = \alpha_N (t_2 - t_1), \quad \text{for any } t_1, t_2 \in [t_0 + \gamma, t_0 + \alpha(t_0) - \gamma] \tag{78}$$

where α_N *is as in (51).*

Proof: Let $A := [t_0 + \gamma, t_0 + \alpha(t_0) - \gamma]$. Since $\|\lambda_{h_n}\|_{L^{\infty}(A)} \le C$ for every $n \in \mathbf{N}$, then on a subsequence $h \to 0$ (omitting the index n in h_n):

$$\lambda_h \overset{*}{\rightharpoonup} \lambda \quad \text{weakly - * in } L^{\infty}(A) \text{ for some } \lambda \in L^{\infty}(A).$$

We also know that when $h \to 0$,

$$r_{ih} \to r_i \quad \text{uniformly on } A,$$

by definition of r_{ih} for $i = 1, \dots, N$. Then, as r_{ih} are uniformly bounded away from zero on A for every $i = 1, \dots, N$,

$$\frac{1}{r_{ih}} \to \frac{1}{r_i} \quad \text{uniformly on } A \text{ for } i = 1, \dots, N.$$

Hence, by (77) and (78)

$$\dot{r}_{ih} \overset{*}{\rightharpoonup} \rho_i \quad \text{weakly - * in } L^{\infty}(A) \text{ for some } \rho \in L^{\infty}(A).$$

At the same time, since r_i is locally Lipschitz on A, the function $r_i \in W^{1,\infty}(A)$ for every $i = 1, \dots, N$. Therefore, $\rho_i = \dot{r}_i$ for every $i = 1, \dots, N$, where $\dot{r}_i$ is the derivative of r_i in the sense of distributions. Passing to a limit in the equations (77) and (78), we find that r_i is a weak solution of

$$\dot{r}_i + \frac{1}{r_i} - \lambda = 0 \quad \text{if } d_r(r_i(t), t) > 0,$$

and

$$\dot{r}_i + \frac{1}{r_i} + \lambda = 0 \quad \text{if } d_r(r_i(t), t) > 0,$$

where $i = 1, \dots, N$.

Multiplying the equations (77) by $-r_i$ and the equations (78) by r_i for their respective i's and adding the resulting equations together, we have for every $h>0$ that

$$-\sum_{d_r>0} r_{ih}\dot{r}_{ih}+\sum_{d_r<0} r_{ih}\dot{r}_{ih}+\left(\sum_{i=1}^{N} r_{ih}\right)\lambda_h = \alpha_N,$$

where α_N is as in (51). Integrating this expression over $[t_1, t_2]$ for any $t_1, t_2 \in A$ and using the mass preservation property of the limiting flow:

$$\int_\Omega v(t,x)\,dx = \int_\Omega v(t,0)\,dx \text{ for all } t\geq 0,$$

we conclude that for every $h>0$ and any $t_1, t_2 \in A$:

$$\int_{t_1}^{t_2}\left(\sum_{i=1}^{N} r_{ih}\right)\lambda_h dt = \int_{t_1}^{t_2}\alpha_N dt = \alpha_N(t_2-t_1),$$

since α_N is constant on A. Then as $h\to 0$

$$\int_{t_1}^{t_2}\left(\sum_{i=1}^{N} r_i\right)\lambda dt = \alpha_N(t_2-t_1) \quad \text{for any } t_1, t_2 \in A.$$

■

Remark 14. For the "ghost" interfaces our procedure would only identify the bounds on the positions of every "ghost" interface at any time $t \in A$ (Assuming, of course, that their position at the time t_0 is known.)

In order to identify λ explicitly one has to use the procedure of Lemma 6. Unfortunately, in general, this requires to identify an asymptotic limit of the quantity:

$$\frac{1}{\varepsilon}\int_0^R W'(u^\varepsilon)\left|u_r^\varepsilon\right| r dr = \frac{1}{\varepsilon^2}\int_0^R q(z^\varepsilon/\varepsilon)\,(q'(z^\varepsilon/\varepsilon))^2\left|z_r^\varepsilon\right| r dr \tag{79}$$

This can be achieved if the rate of the uniform convergence of z^ε to its asymptotic limit is of the order ε. The task of proving this fact is rather difficult and is beyond the scope of this paper. However, in some special cases the necessity of estimating (79) can be avoided. We return now to the problem from the previous section.

Recall that τ is the first time when two "regular" interfaces collide or one "regular" interface shrinks to zero. Assume the former and suppose that $\tau \in [0,T]\setminus\Pi_\delta$. Suppose without loss of generality that $\alpha_N > 0$ and that the "ghost" interface, appearing as a result of the collision, does not disappear instantly. Then, by our assumption on τ, there exists a number τ_1 such that every interface is continuous on $[\tau, \tau_1]$. By Theorem 19 for any $t_1, t_2 \in [\tau, \tau_1]$:

$$\lim_{\iota\to 0}\int_{t_1}^{t_2}\left(\sum_{i=1}^{N-2} r_{ih}\right)\lambda_h dt = \int_{t_1}^{t_2}\left(\sum_{i=1}^{N-2} r_i\right)\lambda dt = \alpha_N(t_2-t_1). \tag{80}$$

where N is a number of the "regular" interfaces at the time $t = 0$.
On the other hand we can estimate λ_h using Lemma 6 and the fact that the "bad" term in (79) is positive. Then

$$\lambda_h \geq \frac{\alpha_N}{h} \int_t^{t+h} \left(p(s) + \sum_{i=1}^{N-2} r_i(s) \right)^{-1} ds, \tag{81}$$

where $r_i(t)$ is a position of the i^{th} "regular" interface for $i = 1, \ldots, N$ and $p(t)$ is a position of the "ghost" interface. Substituting (81) to the L.H.S. of (80), using the fact that $r_i(t)$ and $p(t)$ are bounded away from zero on $[\tau, \tau_1]$ for $i = 1, \ldots, N$ and letting $h \to 0$ we find that

$$\alpha_N(t_2 - t_1) < \int_{t_1}^{t_2} \left(\sum_{i=1}^{N-2} r_i(s) \right) \left(p(s) + \sum_{i=1}^{N-2} r_i(s) \right)^{-1} dt \leq \alpha_N(t_2 - t_1)$$

– contradiction. Therefore, there is no "ghost" interface past the time $t = \tau$. Moreover, as $\tau \in [0, T] \setminus \Pi_\delta$ then by the maximum principle for z^ε, "extra" zeroes of z_r^ε will disappear as well. Hence the procedure and the conclusions of Section 6c apply. This analysis can be continued until we reach the time $\hat{t}$ when two "regular" interfaces collide and $\hat{t} \in \Pi_\delta$ or until one "regular" interface shrinks to zero.

We can analyze the case when at $t = t_0$ the set $\Gamma_{t_0}' = \varnothing$ and one of the "regular" interfaces shrinks to zero in a similar manner after we notice that the limiting energy remains continuous at t_0.

We summarize this discussion in the following

Theorem 20. *Suppose that $\tau \geq 0$ is such that:*

(a) $\alpha_N(\tau) > 0$,

(b) $P_\tau^\delta = \varnothing$,

(c) $\tau \in [0, T] \setminus \Pi_\delta$,

(d) *The set $\Gamma_\tau \cup \Gamma_\tau'$ is finite (There are finitely many "ghost" and "regular" interfaces at the time τ) and the multiplicity of every interface does not exceed one. Then there exists $\hat{\tau} > \tau$ such that the set of the "ghost" interfaces $\Gamma_t' = \varnothing$ for every $t \in (\tau, \hat{\tau})$ except, maybe, finitely many points. The "regular" interfaces move by (60) and (61) on $(\tau, \hat{\tau})$. The same conclusions hold if (a) and (b) are substituted by:*

(a_1) $\alpha_N(\tau) < 0$,

(b_1) $N_\tau^\delta = \varnothing$.

Finally we conjecture that, in general, on time intervals where the limiting energy is continuous:

(i) The "ghost" interfaces do not exist (except, maybe, the origin),

(ii) The multiplicity of the "regular" interfaces does not exceed one.

However to prove that, at least in the framework of our method, one has to show that the limit of (79) as $\varepsilon \to 0$ is equal to zero.

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