Nonlinear Stability of Flows of Jeffreys Fluids at Low Weissenberg Numbers

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Research Report No. 94-NA-022

June 1994

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

Nonlinear stability of flows of Jeffreys fluids at low Weissenberg numbers

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Abstract

We consider the stability of steady flows of viscoelastic fluids of Jeffreys type. For sufficiently small Weissenberg numbers, but arbitrary Reynolds numbers, it is proved that the flow is stable to small disturbances if the spectrum of the linearized operator is in the left half plane.

1. Introduction

Viscoelastic fluids show many new instability phenomena different from those in Newtonian flows, and the stability of viscoelastic flows has been the subject of many recent studies. An excellent review is given by Larson [3]. In the study of stability, the following hypotheses are usually taken for granted:

1. A flow is stable to small disturbances if it is stable as a solution of the linearized equations.
2. The stability of the linearized equations can be decided from the spectrum.

Indeed, theorems to this effect are well known in Newtonian fluid mechanics. For viscoelastic fluids, however, these questions are unresolved.

Guillopé and Saut [1],[2] have given stability proofs for flows of Jeffreys fluids which are small perturbations of either the rest state or a stable Newtonian flow. The objective of the present study is somewhat different. The goal is not to prove stability, but to show that the most widely used criterion to assess stability (spectrum of the linearized operator) is actually valid. One would like to know this particularly in regimes where the flow cannot a priori be expected to be stable. In [5] and [6], the linear stability of parallel shear flows of Jeffreys fluids is considered and it is shown that if the velocity profile is strictly monotone, then linear stability is indeed determined by the spectrum.

In the present paper, we extend consideration to arbitrary flows and we consider nonlinear as well as linear stability. However, we need to restrict ourselves to sufficiently low Weissenberg numbers. Essentially, this means that instabilities, if present, are caused by inertia and would not be present if the Reynolds number were zero. Nevertheless, the situations allowed go beyond small perturbations of the Newtonian case.

2. Governing equations

We consider the motion of a Jeffreys type fluid. With $\mathbf{v}$ denoting the velocity, $\mathbf{T}$ the extra stress, p the pressure, and $\mathbf{f}$ a given body force, the governing equations are, in dimensionless form

$$\begin{aligned} R\Big(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) &= \epsilon\Delta\mathbf{v} - \nabla p + \operatorname{div}\mathbf{T} + \mathbf{f}, \\ \operatorname{div}\mathbf{v} &= 0, \\ W\Big(\mathbf{T}_t + (\mathbf{v}\cdot\nabla)\mathbf{T} + \mathbf{A}(\nabla\mathbf{v},\mathbf{T})\Big) + \mathbf{T} &= \nabla\mathbf{v} + (\nabla\mathbf{v})^T. \end{aligned} \tag{1}$$

Here $\mathbf{A}$ is a smooth nonlinear function with the property that $\mathbf{A}$ and its gradient vanish at the origin. The positive constants R, W and ϵ are knwon as the Reynolds number, the Weissenberg number and the retardation parameter. We are concerned with solutions in a bounded domain $\Omega \subset \mathbb{R}^3$, and we shall assume that Ω has a smooth boundary. On the boundary, we impose the Dirichlet condition

$$\mathbf{v} = \mathbf{0}. \tag{2}$$

We assume that (1) and (2) have a smooth steady solution $(\mathbf{v}_0, \mathbf{T}_0, p_0)$, and we are interested in the stability of this solution. In this context, we need to consider the linearized equations

$$\begin{aligned} R\Big(\mathbf{v}_t + (\mathbf{v}_0\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}_0\Big) &= \epsilon\Delta\mathbf{v} - \nabla p + \operatorname{div}\mathbf{T}, \\ \operatorname{div}\mathbf{v} &= 0, \\ W\Big(\mathbf{T}_t + (\mathbf{v}_0\cdot\nabla)\mathbf{T} + (\mathbf{v}\cdot\nabla)\mathbf{T}_0 + D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)\nabla\mathbf{v} &+ D_2\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)\mathbf{T}\Big) + \mathbf{T} \\ &= \nabla\mathbf{v} + (\nabla\mathbf{v})^T. \end{aligned} \tag{3}$$

Here $D_i\mathbf{A}$ denotes the derivative of $\mathbf{A}$ with respect to the ith argument.

3. Linear stability

We first reformulate the linearized equation (3) as an evolution problem in a Hilbert space. In doing so, we follow [6]. Let

$$X = (\mathbf{v}, \mathbf{T}) \in (L^2(\Omega))^3 \times (H^1(\Omega))^6 \mid \operatorname{div}\mathbf{v} = 0,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}. \tag{4}$$

We define Π to be the orthogonal projection from $(L^2(\Omega))^3$ onto the subspace of divergence-free vector fields with vanishing normal component on the boundary. The operator associated with (3) is

$$\begin{aligned} \mathcal{A}(\mathbf{v}, \mathbf{T}) = \Big(&\Pi(-(\mathbf{v}_0\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v}_0 + R^{-1}\epsilon\Delta\mathbf{v} + R^{-1}\operatorname{div}\mathbf{T}), \\ &-(\mathbf{v}_0\cdot\nabla)\mathbf{T} - (\mathbf{v}\cdot\nabla)\mathbf{T}_0 - D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)\nabla\mathbf{v} - D_2\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)\mathbf{T} \\ &- W^{-1}\mathbf{T} + W^{-1}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\Big). \end{aligned} \tag{5}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \{(\mathbf{v}, \mathbf{T}) \in X \mid \mathbf{v} \in (H^2(\Omega))^3,\ (\mathbf{v}_0\cdot\nabla)\mathbf{T} \in (H^1(\Omega))^6,\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}. \tag{6}$$

We shall first prove

Lemma 1: *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup in X.*

Proof: We follow the same procedure as in [6]. First, we can ignore bounded terms, since any bounded perturbation of a generator is a generator. This leads us to consider the simpler operator $\mathcal{B}$ given by

$$\begin{aligned}\mathcal{B}(\mathbf{v},\mathbf{T}) = \Big(&\Pi(-(\mathbf{v}_0\cdot\nabla)\mathbf{v} + R^{-1}\epsilon\Delta\mathbf{v}),\\ &-(\mathbf{v}_0\cdot\nabla)\mathbf{T} - (\mathbf{v}\cdot\nabla)\mathbf{T}_0 - D_1\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)\nabla\mathbf{v} + W^{-1}(\nabla\mathbf{v}+(\nabla\mathbf{v})^T)\Big).\end{aligned} \tag{7}$$

We can write $\mathcal{B}$ in the form

$$\mathcal{B}(\mathbf{v},\mathbf{T}) = (\mathcal{G}\mathbf{v}, -(\mathbf{v}_0\cdot\nabla)\mathbf{T} + \mathcal{H}\mathbf{v}). \tag{8}$$

We now define a transformation in X by

$$\Phi(\mathbf{v},\mathbf{T}) = (\mathbf{v}, \mathbf{T} - \mathcal{H}\mathcal{G}^{-1}\mathbf{v} + (\mathbf{v}_0\cdot\nabla)(\mathcal{H}\mathcal{G}^{-2}\mathbf{v})). \tag{9}$$

Since $\mathcal{H}$ is a first order differential operator and $\mathcal{G}$ is an elliptic operator of second order, this is indeed a bounded linear transformation in X. Now let $\mathcal{C} = \Phi\mathcal{B}\Phi^{-1}$. We find

$$\mathcal{C}(\mathbf{v},\mathbf{T}) = (\mathcal{G}\mathbf{v}, -(\mathbf{v}_0\cdot\nabla)\mathbf{T} + (\mathbf{v}_0\cdot\nabla)^2(\mathcal{H}\mathcal{G}^{-2}\mathbf{v})). \tag{10}$$

The last term on the right hand side is a bounded operator. Since $\mathcal{G}$ generates a C_0-semigroup (in fact an analytic semigroup) and $-(\mathbf{v}_0\cdot\nabla)$ also generates a C_0-semigroup, it follows that $\mathcal{C}$ (and hence $\mathcal{B}$) generates a C_0-semigroup.

∎

Our result on linear stability is the following.

Theorem 1: *Assume that the spectrum of $\mathcal{A}$ is contained entirely in the open left half plane. Assume in addition that the following quantity is sufficiently small:*

$$(W + \frac{W}{\sqrt{\epsilon}})\sup_{\Omega}(|\nabla\mathbf{v}_0| + |D_2\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)| + |D_1\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)|). \tag{11}$$

Then the type of the semigroup generated by $\mathcal{A}$ is negative (i.e. $\exp(\mathcal{A}t)$ decays to zero exponentially).

Proof: According to a theorem of Prüß [4], it suffices to show that the resolvent $(\mathcal{A}-\lambda)^{-1}$ is bounded uniformly in the right half plane. By assumption, the resolvent exists everywhere is the closed right half plane, and the only issue is to get a bound for large $|\lambda|$. Let us introduce the following operators:

$$\begin{aligned}\mathcal{P}\mathbf{v} &= \Pi\Big(-R(\mathbf{v}_0\cdot\nabla)\mathbf{v} - R(\mathbf{v}\cdot\nabla)\mathbf{v}_0 + \epsilon\Delta\mathbf{v}\Big),\\ \mathcal{Q}\mathbf{v} &= -(\mathbf{v}\cdot\nabla)\mathbf{T}_0 - D_1\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)\nabla\mathbf{v} + W^{-1}(\nabla\mathbf{v}+(\nabla\mathbf{v})^T),\\ \mathcal{R}\mathbf{T} &= -D_2\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)\mathbf{T}.\end{aligned} \tag{12}$$

With these notations, we get

$$\mathcal{A}(\mathbf{v},\mathbf{T}) = (R^{-1}(\mathcal{P}\mathbf{v} + \Pi\,\mathrm{div}\,\mathbf{T}), \mathcal{Q}\mathbf{v} + \mathcal{R}\mathbf{T} - (\mathbf{v}_0\cdot\nabla)\mathbf{T} - W^{-1}\mathbf{T}). \tag{13}$$

We consider the problem

$$\begin{aligned} \lambda\mathbf{v} + \mathbf{g} &= R^{-1}\mathcal{P}\mathbf{v} + R^{-1}\Pi\,\mathrm{div}\,\mathbf{T}, \\ \lambda\mathbf{T} + \mathbf{H} &= \mathcal{Q}\mathbf{v} + \mathcal{R}\mathbf{T} - (\mathbf{v}_0\cdot\nabla)\mathbf{T} - W^{-1}\mathbf{T}. \end{aligned} \tag{14}$$

With $\|\cdot\|_k$ denoting the norm in $H^k(\Omega)$, our goal is an estimate of the form

$$\|\mathbf{v}\|_2 + \|\mathbf{T}\|_1 \le C(\|\mathbf{g}\|_0 + \|\mathbf{H}\|_1), \tag{15}$$

with a uniform constant C for sufficiently large values of λ in the right half plane. We make the convention that in the following C denotes a generic constant which is independent of λ for λ sufficiently large. By C^* we denote a constant which is also independent of R, W, ϵ, $\mathbf{v}_0$ and $\mathbf{T}_0$ (although it may depend on these quantities how large $|\lambda|$ has to be for the estimate to hold).

Since the Stokes operator generates an analytic semigroup, we find from the first equation in (14) that for large $|\lambda|$ we have

$$\|\mathbf{v}\|_2 + \frac{R}{\epsilon}|\lambda|\|\mathbf{v}\|_0 \le C^*(\frac{1}{\epsilon}\|\Pi\,\mathrm{div}\,\mathbf{T}\|_0 + \frac{R}{\epsilon}\|\mathbf{g}\|_0), \tag{16}$$

and also, via an energy estimate

$$\|\mathbf{v}\|_1 + \sqrt{\frac{R|\lambda|}{\epsilon}}\|\mathbf{v}\|_0 \le C^*(\frac{1}{\epsilon}\|\mathbf{T}\|_0 + \frac{R}{\epsilon}\|\mathbf{g}\|_{-1}). \tag{17}$$

Next, we take the second equation in (14), multiply by $\mathbf{T}$ and integrate. With $(\cdot,\cdot)$ denoting the inner product in $L^2(\Omega)$, we find

$$(\mathrm{Re}\,\lambda + W^{-1})\|\mathbf{T}\|_0^2 = \mathrm{Re}\,(\mathcal{Q}\mathbf{v} + \mathcal{R}\mathbf{T}, \mathbf{T}) - \mathrm{Re}\,(\mathbf{T},\mathbf{H}). \tag{18}$$

On the right hand side, we find

$$|(\mathcal{R}\mathbf{T},\mathbf{T})| \le \|\mathbf{T}\|_0^2 \sup_\Omega |D_2\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)|, \tag{19}$$

and

$$(\mathcal{Q}\mathbf{v},\mathbf{T}) = -((\mathbf{v}\cdot\nabla)\mathbf{T}_0,\mathbf{T}) - (D_1\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)\nabla\mathbf{v},\mathbf{T}) + W^{-1}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T,\mathbf{T}). \tag{20}$$

The first term can be bounded by

$$C\|\mathbf{v}\|_0\|\mathbf{T}\|_0 \le C\frac{1}{\sqrt{|\lambda|}}(\|\mathbf{T}\|_0^2 + \|\mathbf{T}\|_0\|\mathbf{g}\|_{-1}). \tag{21}$$

For the second term, we have the bound

$$\|\mathbf{v}\|_1\|\mathbf{T}\|_0 \sup_{\Omega} |D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)|. \tag{22}$$

Finally, integration by parts transforms the third term to

$$-2W^{-1}(\mathbf{v}, \operatorname{div}\mathbf{T}) = 2W^{-1}(\mathbf{v}, \mathcal{P}\mathbf{v}) - 2W^{-1}R\lambda(\mathbf{v},\mathbf{v}) - 2W^{-1}R(\mathbf{g},\mathbf{v}). \tag{23}$$

We note that

$$(\mathbf{v}, \mathcal{P}\mathbf{v}) \leq -\epsilon\|\mathbf{v}\|_1^2 + C\|\mathbf{v}_1\|\|\mathbf{v}_0\|, \tag{24}$$

We can combine our estimates to find

$$\begin{aligned} W^{-1}\|\mathbf{T}\|_0^2 \leq \|\mathbf{T}\|_0^2 \sup_{\Omega} |D_2\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)| - 2W^{-1}\epsilon\|\mathbf{v}\|_1^2 + \|\mathbf{v}\|_1\|\mathbf{T}\|_0 \sup_{\Omega} |D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)| \\ + \|\mathbf{T}\|_0\|\mathbf{H}\|_0 + C\|\mathbf{g}\|_{-1}\|\mathbf{v}\|_1 + \frac{C}{\sqrt{|\lambda|}}(\|\mathbf{T}\|_0^2 + \|\mathbf{T}\|_0\|\mathbf{g}\|_{-1} + \|\mathbf{T}\|_0\|\mathbf{v}\|_1). \end{aligned} \tag{25}$$

Under the smallness assumption (11) this yields

$$\|\mathbf{T}\|_0 + \|\mathbf{v}\|_1 \leq C(\|\mathbf{H}\|_0 + \|\mathbf{g}\|_{-1}) \tag{26}$$

for sufficiently large $|\lambda|$.

We still need an estimate for higher order norms. Note that we can write the second equation of (14) in the form

$$\lambda\mathbf{T} + W^{-1}\mathbf{T} + (\mathbf{v}_0\cdot\nabla)\mathbf{T} = \mathcal{Q}\mathbf{v} + \mathcal{R}\mathbf{T} - \mathbf{H}, \tag{27}$$

and from this we obtain the estimate

$$\begin{aligned} W^{-1}\|\mathbf{T}\|_1 \leq C^*(\|\mathbf{H}\|_1 + \|\mathbf{T}\|_1 \sup_{\Omega} |\nabla\mathbf{v}_0| + W^{-1}\|\mathbf{v}\|_2 + \|\mathbf{v}\|_2 \sup_{\Omega} |D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)| \\ + \|\mathbf{T}\|_1 \sup_{\Omega} |D_2\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)|) + C(\|\mathbf{T}\|_0 + \|\mathbf{v}\|_1). \end{aligned} \tag{28}$$

By exploiting the smallness condition (11) and using (26), we find

$$\|\mathbf{T}\|_1 \leq C^*\|\mathbf{v}\|_2 + C(\|\mathbf{H}\|_1 + \|\mathbf{g}\|_0). \tag{29}$$

Our next task is to get an estimate for $\Pi \operatorname{div}\mathbf{T}$. For this purpose, we start out by applying the divergence operator to (27). This yields

$$\lambda \operatorname{div}\mathbf{T} + W^{-1}\operatorname{div}\mathbf{T} + (\mathbf{v}_0\cdot\nabla)\operatorname{div}\mathbf{T} - W^{-1}\Delta\mathbf{v} = \mathbf{d}, \tag{30}$$

where $\mathbf{d}$ satisfies an estimate of the form

$$\begin{aligned} \|\mathbf{d}\|_0 &\leq C^*(\sup_{\Omega} |\nabla\mathbf{v}_0|\|\mathbf{T}\|_1 + \|\mathbf{T}\|_1 \sup_{\Omega} |D_2\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)| \\ &\quad + \|\mathbf{v}\|_2 \sup_{\Omega} |D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)|) + C(\|\mathbf{H}\|_1 + \|\mathbf{g}\|_0) \\ &\leq C^*\|\mathbf{v}\|_2(\sup_{\Omega}(|\nabla\mathbf{v}_0| + |D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)| + |D_2\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)|) + C(\|\mathbf{H}\|_1 + \|\mathbf{g}\|_0). \end{aligned} \tag{31}$$

We now want to apply the Hodge projection to (29). We shall simplify notation by setting $\mathbf{u} = \operatorname{div}\mathbf{T}$. We note that the vectorfield

$$(\mathbf{v}_0 \cdot \nabla)(\Pi\mathbf{u}) - ((\Pi\mathbf{u}) \cdot \nabla)\mathbf{v}_0 \tag{32}$$

is divergence-free and its normal component vanishes on the boundary. Hence

$$(1-\Pi)[(\mathbf{v}_0 \cdot \nabla)(\Pi\mathbf{u})] = (1-\Pi)[((\Pi\mathbf{u}) \cdot \nabla)\mathbf{v}_0]. \tag{33}$$

We can write $\mathbf{u} - \Pi\mathbf{u}$ as a gradient, say ∇q, and we find

$$(\mathbf{v}_0 \cdot \nabla)\nabla q = \nabla((\mathbf{v}_0 \cdot \nabla)q) - (\nabla\mathbf{v}_0)^T\nabla q. \tag{34}$$

By combining these identities, we find

$$\Pi[(\mathbf{v}_0 \cdot \nabla)\mathbf{u}] = (\mathbf{v}_0 \cdot \nabla)\Pi\mathbf{u} + \Pi[-(\nabla\mathbf{v}_0)^T(\mathbf{u} - \Pi\mathbf{u})] - (1-\Pi)[((\Pi\mathbf{u} \cdot \nabla)\mathbf{v}_0]. \tag{35}$$

We write the right hand side as $(\mathbf{v}_0 \cdot \nabla)(\Pi\mathbf{u}) + \mathbf{e}$, and we find that $\mathbf{e}$ satisfies a bound of the form

$$\|\mathbf{e}\|_0 \leq C^*\|\mathbf{T}\|_1 \sup_\Omega |\nabla\mathbf{v}_0|. \tag{36}$$

Applying Π to (30) now yields

$$\lambda\Pi\mathbf{u} + (\mathbf{v}_0 \cdot \nabla)(\Pi\mathbf{u}) + W^{-1}\Pi\mathbf{u} - W^{-1}\Pi\Delta\mathbf{v} = \Pi\mathbf{d} - \mathbf{e}. \tag{37}$$

We multiply (37) by $\Pi\mathbf{u}$ and integrate. Taking real parts, we find

$$(\operatorname{Re}\lambda + W^{-1})(\Pi\mathbf{u}, \Pi\mathbf{u}) - W^{-1}\operatorname{Re}(\Pi\mathbf{u}, \Pi\Delta\mathbf{v}) = \operatorname{Re}(\Pi\mathbf{u}, \Pi\mathbf{d} - \mathbf{e}). \tag{38}$$

The right hand side can be estimated by

$$\begin{aligned}\|\Pi\mathbf{u}\|_0(\|\mathbf{d}\|_0 + \|\mathbf{e}\|_0) \leq \|\Pi\mathbf{u}\|_0(C^*\|\mathbf{v}\|_2(\sup_\Omega(|\nabla\mathbf{v}_0| + |D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)| + |D_2\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)|)\\ + C(\|\mathbf{H}\|_1 + \|\mathbf{g}\|_0)).\end{aligned} \tag{39}$$

On the left hand side of (38), we note that

$$(\Pi\mathbf{u}, \Pi\Delta\mathbf{v}) = R(\lambda\mathbf{v}, \Pi\Delta\mathbf{v}) + R(\mathbf{g} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}_0, \Pi\Delta\mathbf{v}) - \epsilon(\Pi\Delta\mathbf{v}, \Pi\Delta\mathbf{v}). \tag{40}$$

The first term on the right hand side of (40) has nonpositive real part for $\operatorname{Re}\lambda \geq 0$, and the second term can be estimated by

$$C\|\mathbf{v}\|_2(\|\mathbf{g}\|_0 + \|\mathbf{v}\|_1) \leq C\|\mathbf{v}\|_2(\|\mathbf{g}\|_0 + \|\mathbf{H}\|_0). \tag{41}$$

By using (39)-(41) in (38), we obtain an estimate of the form

$$\begin{aligned}W^{-1}\|\Pi\mathbf{u}\|_0^2 + W^{-1}\epsilon\|\mathbf{v}\|_2^2 \leq C^*\|\Pi\mathbf{u}\|_0\|\mathbf{v}\|_2 \sup_\Omega(|\nabla\mathbf{v}_0| + |D_1\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)|\\ + |D_2\mathbf{A}(\nabla\mathbf{v}_0, \mathbf{T}_0)|) + C(\|\Pi\mathbf{u}\|_0 + \|\mathbf{v}\|_2)(\|\mathbf{H}\|_1 + \|\mathbf{g}\|_0).\end{aligned} \tag{42}$$

By taking advantage of the smallness condition (11), we find

$$\|\Pi \mathbf{u}\|_0 + \|\mathbf{v}\|_2 \leq C(\|\mathbf{H}\|_1 + \|\mathbf{g}\|_0). \tag{43}$$

An application of (29) completes the proof.

■

4. Nonlinear stability

In the following $\mathbf{v}$ and $\mathbf{T}$ will denote the perturbations to the steady flow given by $\mathbf{v}_0$ and $\mathbf{T}_0$ so that the total velocity and stress are $\mathbf{v}_0 + \mathbf{v}$ and $\mathbf{T}_0 + \mathbf{T}$. We introduce the notation

$$\mathbf{N}(\nabla \mathbf{v}, \mathbf{T}) = \mathbf{A}(\nabla \mathbf{v}_0 + \nabla \mathbf{v}, \mathbf{T}_0 + \mathbf{T}) - D_1 \mathbf{A}(\nabla \mathbf{v}_0, \mathbf{T}_0)\nabla \mathbf{v} - D_2 \mathbf{A}(\nabla \mathbf{v}_0, \mathbf{T}_0)\mathbf{T}. \tag{44}$$

We can now rewrite equations (1) in the form

$$\begin{aligned} \mathbf{v}_t &= R^{-1}\mathcal{P}\mathbf{v} + R^{-1}\Pi \operatorname{div} \mathbf{T} - \Pi((\mathbf{v}\cdot\nabla)\mathbf{v}), \\ \mathbf{T}_t &= -((\mathbf{v}_0 + \mathbf{v})\cdot\nabla)\mathbf{T} + \mathcal{Q}\mathbf{v} + \mathcal{R}\mathbf{T} - W^{-1}\mathbf{T} - \mathbf{N}(\nabla \mathbf{v}, \mathbf{T}). \end{aligned} \tag{45}$$

Here $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are as defined in (12). The proof of nonlinear stability will be based on an iterative construction of the solution to (45):

$$\begin{aligned} \mathbf{v}_t^{n+1} &= R^{-1}\mathcal{P}\mathbf{v}^{n+1} + R^{-1}\Pi \operatorname{div} \mathbf{T}^{n+1} - \Pi((\mathbf{v}^n\cdot\nabla)\mathbf{v}^n), \\ \mathbf{T}_t^{n+1} &= -((\mathbf{v}_0 + \mathbf{v}^n)\cdot\nabla)\mathbf{T}^{n+1} + \mathcal{Q}\mathbf{v}^{n+1} + \mathcal{R}\mathbf{T}^{n+1} - W^{-1}\mathbf{T}^{n+1} - \mathbf{N}(\nabla \mathbf{v}^n, \mathbf{T}^n). \end{aligned} \tag{46}$$

In carrying this out, two main problems arise:

1. In order to deal with the nonlinearities, we need to get estimates for higher order norms of $\mathbf{v}$ and $\mathbf{T}$ than we did in the section on linear stability above.
2. In the second equation of (46), we have the term $-(\mathbf{v}^n\cdot\nabla)\mathbf{T}^{n+1}$. It is not possible to change this term to $-(\mathbf{v}^n\cdot\nabla)\mathbf{T}^n$. This means that the equations to be solved at each iteration step are slightly more general than the linearized equations.

In dealing with both of these issues, a transformation of the equations turns out to be very useful. This transformation is somewhat similar to that which we already used in the linearized problem in the proof of Lemma 1. Specifically, we define $\mathcal{P} =: \mathcal{P}_0 + \mathcal{P}_1$, where $\mathcal{P}_1 = \epsilon\Pi\Delta$, and $\mathcal{P}_0$ contains the terms of lower differential order. The transformation is given by

$$\mathbf{T} = \mathbf{S} + R\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{v}. \tag{47}$$

By making this substitution, we get the transformed equations

$$\begin{aligned} \mathbf{v}_t &= R^{-1}(\mathcal{P}_1\mathbf{v} + \mathcal{P}_0\mathbf{v} + \Pi \operatorname{div} \mathbf{S}) + \Pi \operatorname{div}(\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{v}) - \Pi(\mathbf{v}\cdot\nabla\mathbf{v}), \\ \mathbf{S}_t &= -((\mathbf{v}_0 + \mathbf{v})\cdot\nabla)\mathbf{S} + \mathcal{R}\mathbf{S} - W^{-1}\mathbf{S} - \mathcal{Q}\mathcal{P}_1^{-1}(\Pi \operatorname{div} \mathbf{S}) \\ &\quad - R((\mathbf{v}_0 + \mathbf{v})\cdot\nabla)(\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{v}) + R\mathcal{R}\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{v} - W^{-1}R\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{v} \\ &\quad - \mathcal{Q}\mathcal{P}_1^{-1}(\mathcal{P}_0\mathbf{v} + R\Pi \operatorname{div}(\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{v})) + R\mathcal{Q}\mathcal{P}_1^{-1}\Pi((\mathbf{v}\cdot\nabla)\mathbf{v}) - \mathbf{N}(\nabla\mathbf{v}, \mathbf{S} + R\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{v}). \end{aligned} \tag{48}$$

For the analysis that follows, we shall need some estimates relating to the equation

$$\mathbf{S}_t = -((\mathbf{v}_0 + \mathbf{v}) \cdot \nabla)\mathbf{S} + \mathcal{R}\mathbf{S} - W^{-1}\mathbf{S} - \mathcal{Q}\mathcal{P}_1^{-1}(\Pi \operatorname{div} \mathbf{S}) + \mathbf{F}, \ \mathbf{S}(0) = \mathbf{S}_0. \tag{49}$$

To state the estimate we want to establish for (49), we define

$$L^p_*((0,\infty);X) := \{\mathbf{F} \in L^p_{\mathrm{loc}}((0,\infty);X) \mid \sup_{i \in \mathbb{N}} \int_{i-1}^{i} \|\mathbf{F}(t)\|_X^p \, dt < \infty\}, \tag{50}$$

and we use the notation $\|\cdot\|_{n,p}$ to denote the norm in $L^p_*((0,\infty);H^n(\Omega))$. Later, we shall also use the space $W^{k,p}_*((0,\infty);X)$, which we naturally define as the space of functions which have k derivatives in $L^p_*((0,\infty);X)$. We write $\|\cdot\|_{n,k,p}$ for the norm in $W^{k,p}_*((0,\infty);H^n(\Omega))$. We shall establish the following result for solutions of (49).

Lemma 2: *Let n be any positive integer. Assume that the smallness condition (11) holds (with a specific bound that depends on n). Assume, moreover, that $\mathbf{v} \in L^\infty((0,\infty);H^n(\Omega) \cap W^{1,\infty}(\Omega))$ is divergence-free, vanishes on the boundary and*

$$\sup_{(0,\infty)\times\Omega} |\nabla \mathbf{v}| \tag{51}$$

is sufficiently small. Assume moreover, that $\mathbf{S}_0 \in H^n(\Omega)$ and $\mathbf{F}e^{\delta t} \in L^1_*((0,\infty);H^n(\Omega))$ for some $\delta > 0$. If δ is small enough, then $\mathbf{S}e^{\delta t} \in L^\infty((0,\infty);H^n(\Omega))$ and there is a bound of the form

$$\|\mathbf{S}e^{\delta t}\|_{n,\infty} \le C^*(\|\mathbf{S}_0\|_n + \|\mathbf{F}e^{\delta t}\|_{n,1}) + C\|\mathbf{S}e^{\delta t}\|_{n-1,\infty}. \tag{52}$$

Here C and C^* have the same meaning as in the previous section.

Proof: We set $\Pi \operatorname{div} \mathbf{S} = \mathbf{u}$, and rewrite (49) as

$$\mathbf{S}_t = -((\mathbf{v}_0 + \mathbf{v}) \cdot \nabla)\mathbf{S} + \mathcal{R}\mathbf{S} - W^{-1}\mathbf{S} - \mathcal{Q}\mathcal{P}_1^{-1}\mathbf{u} + \mathbf{F}. \tag{53}$$

We can multiply the equation by $\mathbf{S}$ and thus obtain an energy estimate for the L^2-norm of $\mathbf{S}$. Then we take derivatives of the equation and obtain energy estimates for derivatives of $\mathbf{S}$. For the nth derivatives, this yields an estimate of the form

$$\|\mathbf{S}e^{\delta t}\|_{n,\infty} \le C^*(\|\mathbf{S}_0\|_n + \|\mathbf{F}e^{\delta t}\|_{n,1} + W\|e^{\delta t}\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{u}\|_{n,\infty}) + C\|\mathbf{S}e^{\delta t}\|_{n-1,\infty}. \tag{54}$$

Next, we note that

$$\begin{aligned} \|\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{u}\|_n &\le C\|\mathcal{P}_1^{-1}\mathbf{u}\|_n + C^*W^{-1}\|\mathcal{P}_1^{-1}\mathbf{u}\|_{n+1} \\ &\le C\|\mathbf{u}\|_{n-2} + C^*W^{-1}\epsilon^{-1}\|\mathbf{u}\|_{n-1} \le C\|\mathbf{S}\|_{n-1} + C^*W^{-1}\epsilon^{-1}\|\mathbf{u}\|_{n-1}. \end{aligned} \tag{55}$$

Using this in (54), we get

$$\|\mathbf{S}e^{\delta t}\|_{n,\infty} \le C^*(\|\mathbf{S}_0\|_n + \|\mathbf{F}e^{\delta t}\|_{n,1} + \epsilon^{-1}\|e^{\delta t}\mathbf{u}\|_{n-1,\infty}) + C\|\mathbf{S}e^{\delta t}\|_{n-1,\infty}. \tag{56}$$

To proceed further, we apply the operator $\Pi\,\mathrm{div}$ to (49). This yields

$$\begin{aligned}\mathbf{u}_t = &-(\mathbf{v}+\mathbf{v}_0)\cdot\nabla\mathbf{u} - W^{-1}\mathbf{u} - \Pi\,\mathrm{div}\,(\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{u})\\ &+\Pi\,\mathrm{div}\,\mathbf{F} + \Pi\,\mathrm{div}\,(\mathcal{R}\mathbf{S}) + [(\mathbf{v}+\mathbf{v}_0)\cdot\nabla)(\Pi\,\mathrm{div}\,\mathbf{S}) - \Pi\,\mathrm{div}\,((\mathbf{v}_0+\mathbf{v})\cdot\nabla)\mathbf{S}].\end{aligned} \tag{57}$$

We have

$$\begin{aligned}\Pi\,\mathrm{div}\,(\mathcal{Q}\mathcal{P}_1^{-1}\mathbf{u}) = W^{-1}\epsilon^{-1}\mathbf{u} - \Pi\,\mathrm{div}\,(D_1\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)\nabla(\mathcal{P}_1^{-1}\mathbf{u}))\\ - \Pi\,\mathrm{div}\,[((\mathcal{P}_1^{-1}\mathbf{u})\cdot\nabla)\mathbf{T}_0].\end{aligned} \tag{58}$$

The second term on the right can be estimated as follows:

$$\|\Pi\,\mathrm{div}\,(D_1\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)\nabla(\mathcal{P}_1^{-1}\mathbf{u}))\|_{n-1} \le C^*\epsilon^{-1}\sup_\Omega|D_1\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)|\|\mathbf{u}\|_{n-1} + C\|\mathbf{u}\|_{n-2}, \tag{59}$$

and the third is bounded by $C\|\mathbf{u}\|_{n-2}$. Next, we estimate

$$\|\Pi\,\mathrm{div}\,\mathcal{R}\mathbf{S}\|_{n-1} \le C^*\|\mathcal{R}\mathbf{S}\|_n \le C^*\sup_\Omega|D_2\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)|\|\mathbf{S}\|_n + C\|\mathbf{S}\|_{n-1}. \tag{60}$$

Using arguments analogous to those in the previous section, we can also show that

$$\begin{aligned}\|((\mathbf{v}+\mathbf{v}_0)\cdot\nabla)(\Pi\,\mathrm{div}\,\mathbf{S}) - \Pi\,\mathrm{div}\,((\mathbf{v}_0+\mathbf{v})\cdot\nabla)\mathbf{S}\|_{n-1} \le C^*(\sup_\Omega|\nabla\mathbf{v}_0|\\ + \sup_{(0,\infty)\times\Omega}|\nabla\mathbf{v}|)\|\mathbf{S}\|_n + C\|\mathbf{S}\|_{n-1}.\end{aligned} \tag{61}$$

Combining these estimates and using them in (57), we find

$$\begin{aligned}\|\mathbf{u}e^{\delta t}\|_{n-1,\infty} \le C^*(\|\mathbf{S}_0\|_n + \|\mathbf{F}e^{\delta t}\|_{n,1} + \frac{W\epsilon}{1+\epsilon}\|\mathbf{S}e^{\delta t}\|_{n,\infty}(\sup_\Omega|D_2\mathbf{A}(\nabla\mathbf{v}_0,\mathbf{T}_0)| + |\nabla\mathbf{v}_0|\\ + \sup_{(0,\infty)\times\Omega}|\nabla\mathbf{v}|)) + C\|\mathbf{S}e^{\delta t}\|_{n-1,\infty}.\end{aligned} \tag{62}$$

The lemma follows by inserting (62) into (56).

∎

We now consider equation (46), which we rewrite in the form

$$\begin{aligned}\mathbf{v}_t^{n+1} &= R^{-1}\mathcal{P}\mathbf{v}^{n+1} + R^{-1}\Pi\,\mathrm{div}\,\mathbf{T}^{n+1} - \mathbf{g}^n,\\ \mathbf{T}_t^{n+1} &= -((\mathbf{v}_0+\mathbf{v}^n)\cdot\nabla)\mathbf{T}^{n+1} + \mathcal{Q}\mathbf{v}^{n+1} + \mathcal{R}\mathbf{T}^{n+1} - W^{-1}\mathbf{T}^{n+1} - \mathbf{H}^n.\end{aligned} \tag{63}$$

We prescribe initial data

$$\mathbf{v}^{n+1}(0) = \mathbf{v}_0,\ \mathbf{T}^{n+1}(0) = \mathbf{T}_0. \tag{64}$$

We shall derive the following estimate.

Lemma 3: *Assume that the smallness assumptions (11) and (51) hold, and that the steady solution is linearly stable in the sense of the previous section. Assume, moreover, that*

$$\begin{aligned} \mathbf{v}^n &\in L^2_*((0,\infty);H^3(\Omega)) \cap H^1_*((0,\infty);H^2(\Omega)) \cap H^2_*((0,\infty);L^2(\Omega)), \\ e^{\delta t}\mathbf{g}^n &\in L^2_*((0,\infty);H^1(\Omega)) \cap H^1_*((0,\infty);L^2(\Omega)), \\ e^{\delta t}\mathbf{H}^n &\in L^1_*((0,\infty);H^2(\Omega)) \cap W^{1,1}_*((0,\infty);H^1(\Omega)), \\ \mathbf{v}_0 &\in H^2(\Omega),\ \mathbf{T}_0 \in H^2(\Omega),\ \mathbf{v}_t^{n+1}(0) \in H^1(\Omega). \end{aligned} \tag{65}$$

Here $\mathbf{v}_t^{n+1}(0)$ is defined by the right hand side of the first equation in (63), with $\mathbf{v}^{n+1}$ and $\mathbf{T}^{n+1}$ replaced by their initial values. Finally, we assume that $\mathbf{v}_0$, $\mathbf{v}^n$ and $\mathbf{v}_t^{n+1}(0)$ are divergence-free and vanish on the boundary. Under these assumptions, and if $\delta > 0$ is small enough, equation (63) has a solution assuming the initial condition (64), with the regularity

$$\begin{aligned} e^{\delta t}\mathbf{v}^{n+1} &\in L^2_*((0,\infty);H^3(\Omega)) \cap H^1_*((0,\infty);H^2(\Omega)) \cap H^2_*((0,\infty);L^2(\Omega)), \\ e^{\delta t}\mathbf{T}^{n+1} &\in L^\infty((0,\infty);H^2(\Omega)) \cap W^{1,\infty}((0,\infty);H^1(\Omega)), \end{aligned} \tag{66}$$

and an estimate of the form

$$\begin{aligned} &\|e^{\delta t}\mathbf{v}^{n+1}\|_{3,2} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{2,1,2} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{0,2,2} + \|e^{\delta t}\mathbf{T}^{n+1}\|_{2,\infty} + \|e^{\delta t}\mathbf{T}^{n+1}\|_{1,1,\infty} \\ &\qquad \leq C(\|\mathbf{v}_0\|_2 + \|\mathbf{T}_0\|_2 + \|\mathbf{v}_t^{n+1}(0)\|_1 + \|e^{\delta t}\mathbf{g}^n\|_{1,2} + \|e^{\delta t}\mathbf{g}^n\|_{0,1,2} \\ &\qquad + \|e^{\delta t}\mathbf{H}^n\|_{2,1} + \|e^{\delta t}\mathbf{H}^n\|_{1,1,1}) \end{aligned} \tag{67}$$

holds. Here the constant C is uniform if $\mathbf{v}^n$ is chosen in a sufficiently small ball in $L^2_*((0,\infty);H^3(\Omega)) \cap H^1_*((0,\infty);H^2(\Omega)) \cap H^2_*((0,\infty);L^2(\Omega))$.

Proof: From the linear stability proof in the last section, we conclude that

$$\begin{aligned} \|e^{\delta t}\mathbf{v}^{n+1}\|_{0,\infty} + \|e^{\delta t}\mathbf{T}^{n+1}\|_{1,\infty} &\leq C(\|\mathbf{v}_0\|_0 + \|\mathbf{T}_0\|_1 + \|e^{\delta t}\mathbf{g}^n\|_{0,1} \\ &+ \|e^{\delta t}\mathbf{H}^n\|_{1,1} + \|e^{\delta t}(\mathbf{v}^n\cdot\nabla)\mathbf{T}^{n+1}\|_{1,1}). \end{aligned} \tag{68}$$

For the last term in (68), we have

$$\|e^{\delta t}(\mathbf{v}^n\cdot\nabla)\mathbf{T}^{n+1}\|_{1,1} \leq C \sup_{(0,\infty)\times\Omega} |\nabla\mathbf{v}^n| \|e^{\delta t}\mathbf{T}^{n+1}\|_{2,1}. \tag{69}$$

We use the transformation (47) and Lemma 2 to find

$$\begin{aligned} \|e^{\delta t}\mathbf{T}^{n+1}\|_{2,1} \leq \|e^{\delta t}\mathbf{T}^{n+1}\|_{2,\infty} &\leq C(\|e^{\delta t}\mathbf{S}^{n+1}\|_{2,\infty} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{1,\infty}) \\ &\leq C(\|\mathbf{S}_0\|_2 + \|e^{\delta t}\mathbf{H}^n\|_{2,1} + \|e^{\delta t}\mathbf{g}^n\|_{1,1} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{2,1} \\ &+ \|e^{\delta t}\mathbf{v}^{n+1}\|_{1,\infty} + \|e^{\delta t}\mathbf{S}^{n+1}\|_{1,\infty}). \end{aligned} \tag{70}$$

Using standard estimates for the Stokes problem, we find from the first equation of (63)

$$\|e^{\delta t}\mathbf{v}^{n+1}\|_{2,2}+\|e^{\delta t}\mathbf{v}^{n+1}\|_{0,1,2} \leq C(\|\mathbf{v}_0\|_1+\|e^{\delta t}\mathbf{g}^n\|_{0,2}+\|e^{\delta t}\mathbf{T}^{n+1}\|_{1,2}+\|e^{\delta t}\mathbf{v}^{n+1}\|_{0,2}). \tag{71}$$

By combining the estimates, we find

$$\|e^{\delta t}\mathbf{v}^{n+1}\|_{2,2} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{0,1,2} + \|e^{\delta t}\mathbf{T}^{n+1}\|_{2,\infty} \leq C(\|\mathbf{v}_0\|_1 + \|\mathbf{T}_0\|_2 + \|e^{\delta t}\mathbf{g}^n\|_{0,2} + \|e^{\delta t}\mathbf{g}^n\|_{1,1} + \|e^{\delta t}\mathbf{H}^n\|_{2,1}). \tag{72}$$

From the second equation of (48), we estimate

$$\|e^{\delta t}\mathbf{S}_t^{n+1}\|_{1,\infty} \leq C(\|e^{\delta t}\mathbf{S}^{n+1}\|_{2,\infty} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{1,\infty} + \|e^{\delta t}\mathbf{g}^n\|_{0,\infty} + \|e^{\delta t}\mathbf{H}^n\|_{1,\infty}). \tag{73}$$

Hence we get a bound for $\|e^{\delta t}\mathbf{S}^{n+1}\|_{1,1,\infty}$. Moreover, from (47) we have

$$\|e^{\delta t}\mathbf{T}^{n+1}\|_{1,1,\infty} \leq C(\|e^{\delta t}\mathbf{S}^{n+1}\|_{1,1,\infty} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{0,1,\infty}). \tag{74}$$

We differentiate the first equation of (63) with respect to time and use estimates for the Stokes problem to obtain

$$\|e^{\delta t}\mathbf{v}^{n+1}\|_{0,2,2} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{2,1,2} \leq C(\|e^{\delta t}\mathbf{g}^n\|_{0,1,2} + \|\mathbf{v}_t^{n+1}(0)\|_1 + \|e^{\delta t}\mathbf{T}^{n+1}\|_{1,1,2} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{0,1,2}). \tag{75}$$

We use (74) to bound the norm of $\mathbf{T}^{n+1}$ on the right hand side of (75), and we use the interpolation inequality

$$\|e^{\delta t}\mathbf{v}^{n+1}\|_{0,1,\infty} \leq \epsilon(\|e^{\delta t}\mathbf{v}^{n+1}\|_{0,2,2} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{2,1,2}) + C(\epsilon)\|e^{\delta t}\mathbf{v}^{n+1}\|_{0,1,2}, \tag{76}$$

where ϵ can be chosen arbitrarily small. An application of elliptic estimates on the first equation of (63) finally yields

$$\|e^{\delta t}\mathbf{v}^{n+1}\|_{3,2} \leq C(\|e^{\delta t}\mathbf{v}^{n+1}\|_{1,1,2} + \|e^{\delta t}\mathbf{v}^{n+1}\|_{1,2} + \|e^{\delta t}\mathbf{T}^{n+1}\|_{2,2} + \|e^{\delta t}\mathbf{g}^n\|_{1,2}). \tag{77}$$

The lemma now follows by combining these estimates.

■

Using Lemma 3 and straightforward estimates for the nonlinearities in (63), we can find an a priori bound for all iterates provided that the norms of the initial data are sufficiently small, and provided sufficiently small data are used to start the iteration. Once it is known that all iterates are a priori bounded, it can be shown that the iteration converges in a weaker norm by considering the differences between successive iterates. We omit this fairly routine argument. As a consequence, we find that the solution of (45) decays exponentially if the initial data are sufficiently small. We state this as a theorem.

Theorem 2: *Let the assumptions of Theorem 1 hold. In addition, let the initial data be such that* $\|\mathbf{v}(0)\|_2 + \|\mathbf{T}(0)\|_2 + \|\mathbf{v}_t(0)\|_1$ *is sufficiently small, and in addition* $\mathbf{v}(0)$ *and* $\mathbf{v}_t(0)$ *are divergence-free and vanish on the boundary. Then there exists* $\delta > 0$ *such that the solution of (45) satisfies*

$$\begin{aligned} &e^{\delta t}\mathbf{v} \in L^2_*((0,\infty); H^3(\Omega)) \cap H^1_*((0,\infty); H^2(\Omega)) \cap H^2_*((0,\infty); L^2(\Omega)), \\ &e^{\delta t}\mathbf{T} \in L^\infty((0,\infty); H^2(\Omega)) \cap W^{1,\infty}((0,\infty); H^1(\Omega)). \end{aligned} \tag{78}$$

Acknowledgement

This research was supported by the National Science Foundation under Grant DMS-9306635 and by the Office of Naval Research under Grant N00014-92-J-1664.

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