NAMT
94-017

Convergence of the Phase Field Equations to the Mullins-Sekerka Problem With Kinetic Undercooling

H. Mete Soner
Carnegie Mellon University

Research Report No. 94-NA-017

May 1994

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

Convergence of the Phase Field Equations to the Mullins-Sekerka Problem with kinetic undercooling [1]

dedicated to Mort Gurtin on the occasion of his sixtieth birthday

H. Mete Soner
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213-3890

May 19, 1994

[1]This work was partially supported by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis and under the National Science Foundation grant DMS-9200801.

Abstract

I prove that the solutions of the phase field equations, on a subsequence, converge to a weak solution of the Mullins-Sekerka problem with kinetic undercooling. The method is based on energy estimates, a monotonicity formula, and the equipartition of the energy at each time. I also show that the limiting interface is $(d-1)$- rectifiable for almost all t with a square integrable mean curvature vector.

Key Words: phase transitions, monotonicity formula, Ginzburg-Landau equation, mean curvature flow, phase-field equations.

AMS Classifications: 35A05, 35K57

1 Introduction.

Phase field equations for solidification were introduced by Caginalp [7, 8], Collins & Levine [14], Fix [18] and Langer [23], and studied by Caginalp [8], to treat several phenomena not covered by the classical Stefan problem. These equations are for the temperature deviation θ and the phase field φ, consist of a heat equation

$$c\theta_t + \ell\varphi_t = k\Delta\theta, \tag{1.1}$$

and a Ginzburg-Landau equation

$$\beta\varphi_t = \lambda\Delta\varphi - \nu W'(\varphi) + \ell\theta, \tag{1.2}$$

where $c, \ell, k, \beta, \lambda$ and ν are positive constants and W is a double-well potential whose wells, of equal depth, correspond to the solid and liquid phases.

Recently several thermodynamically consistent models have been developed in Fried & Gurtin [19], Penrose & Fife [26], Wang *et.al.* [31] and in references therein. In particular, [19], [26], and [31] allow the latent heat ℓ to depend on the order parameter φ.

The main goal here is to rigorously study the global-time asymptotics of (1.1) and (1.2) in the limit $\epsilon \downarrow 0$ for

$$c, k = 1, \quad \beta = \lambda = \epsilon, \quad \nu = \frac{1}{\epsilon}, \quad \ell = \ell(\varphi). \tag{1.3}$$

For specificity, I will use the following functions:

$$W(\varphi) = \frac{1}{2}(1-\varphi^2)^2, \quad \ell = \sqrt{2W(\varphi)}. \tag{1.4}$$

However, my analysis can be modified to analyze any potential W with two wells of equal depth and any function ℓ of the form,

$$\ell(\varphi) = \sqrt{2W(\varphi)}H(\varphi)$$

where$H \geq 0$ is an arbitrary smooth function. In particular, the choice $\ell = W$ would simplify some of the analysis, see Remark 4.1 below.

In [7], [14], and [18], it is formally argued that the solutions of the Ginzburg-Landau equation (1.2) form a sharp interface whose normal velocity depends linearly on the mean curvature of the interface and the temperature deviation at the interface. To describe this result precisely, let $(\theta^\epsilon, \varphi^\epsilon)$ be the solution of the phase field equations with parameters consistent with (1.3) and assume that $(\theta^\epsilon, \varphi^\epsilon)$ converges to (θ, φ). Since the two minima of W are ± 1, it is easy to prove that $|\varphi| = 1$ almost everywhere. Let $\Gamma(t)$ be the interface separating the two regions $\Omega(t) = \{\varphi = -1\}$ and $\{\varphi = 1\}$. Then (θ, Ω) solves the heat equation,

$$\theta_t - \Delta\theta = -(h(\varphi))_t = \frac{4}{3}(\chi_{\Omega(t)})_t, \tag{1.5}$$

coupled with the geometric equation at the interface $\Gamma(t)$,

$$V = -K - \theta, \tag{1.6}$$

where χ_Ω is the indicator of the set Ω, while V and K are the normal velocity and mean curvature of the interface $\Gamma(t)$, respectively. A derivation of these sharp interface equations from thermodynamics as well as an exhaustive list of earlier references are given in Gurtin's book [20, Chapter 3]. In 1964, Mullins & Sekerka [25] studied the linear stability of a related system of equations obtained by replacing (1.6) by the Gibbs-Thompson condition: $\theta = -K$. They showed that planar interfaces are unstable under some perturbations, thus explaining the dentritic growth observed in solidification. I refer to equations (1.5), (1.6) as the *Mullins-Sekerka problem* with kinetic undercooling.

My chief result is that, in the limit, $\theta, \Omega = \{\varphi = -1\}$ solve the Mullins-Sekerka problem with kinetic undercooling. This result is global in time and I do not assume the existence of a solution of (1.5), (1.6). Therefore I also provide an existence result for this limit problem, extending a previous result of Chen & Reitich [12] for local-time existence. To the best of my knowledge, the only other global results are due to Almgren & Wang [3], and Luckhaus [24]. They proved the global existence of weak solutions for the heat equation (1.5) coupled with the Gibbs-Thompson condition: $\theta = -K$.

There are two essential difficulties in the analysis of (1.5), (1.6): a solution θ, Ω of (1.5), (1.6) can start out smooth and yet, in finite time, the boundary of Ω may develop geometric singularities, and θ may blow up pointwise (see remark 3.1 below). These difficulties also complicate the analysis of convergence. Since θ is unbounded, θ^ϵ does not converge to θ uniformly. Hence I cannot use the convergence results of [4], which discusses the convergence of (1.2) with a given continuous temperature field. Also, the approach of [17] is not directly applicable to the phase field equations, as they do not have maximum principle and there is no a-priori weak theory for the limit equations. [17] studies the asymptotics of the Cahn-Allen equation, obtained by setting $\ell = 0$ in (1.2), via sub and supersolutions constructed from the weak solutions of the mean curvature flow.

I overcome these difficulties by utilizing the energy estimates in Section 2.2, and a monotonicity result in Section 5. This monotonicity result is an extension of the Ilmanen's result [22] for the Cahn-Allen equation which originates from the Huisken's result for smooth mean-curvature flows [21]. My main observation is that the geometric equation (1.6) is not simply a perturbation of the mean-curvature flow and, therefore the monotonicity result should use the energy

$$\int_{\mathcal{R}^d} \frac{\epsilon}{2}|\nabla\varphi^\epsilon|^2 + \frac{1}{\epsilon}W(\varphi^\epsilon) + \frac{1}{2}(\theta^\epsilon)^2 dx \, ,$$

related to the system (1.5) and (1.6). The main technical difficulty is then to show that, in the limit, the discrepancy measure

$$\xi^\epsilon(t:A) = \int_A \frac{\epsilon}{2}|\nabla\varphi^\epsilon|^2 - \frac{1}{\epsilon}W(\varphi^\epsilon)dx ,$$

is non-positive. For the Cahn-Allen equation, the negativity of ξ^ϵ follows immediately from the maximum principle. For the phase field equations, however, it follows from a series of estimates obtained in Section 4. In later sections, following Ilmanen [22], I prove that the weak* limit of ξ^ϵ is indeed equal to zero.

I close this introduction with a brief survey of related results. (1.1), (1.2) with $c = \beta = 0$, $\ell = 1$ is the Cahn-Hilliard equation. Recently the convergence of the Cahn-Hilliard equation to the Hele-Shaw problem was proven by Alikakos, Bates & Chen [2] using a spectral estimate of Chen [11]. In contrast to this paper, they assume the existence of a smooth solution to the limiting problem. Briefly their method is to construct approximate solutions for the "ϵ-problem" that are close to the smooth solution of the limit problem. They then use the spectral estimates to bound the error terms. Also, Stoth [29] studied the asymptotic limit of the phase field equations with radial symmetry. Independently, a radially symmetric problem in an annular domain with one interface was studied in [10]. Asymptotics of the Cahn-Allen equation, obtained by setting ℓ to zero in (1.2), have been studied extensively. An exhaustive list of references related to the Cahn-Allen equation can be found in my paper [28].

This paper is organized as follows. In the next section I outline the background and state the main results. In Section 3, several elementary estimates are obtained. A gradient estimate is proven in Section 4; this estimate implies that ξ^ϵ is non-positive in the limit. In Section 5 I derive a monotonicity result which I use in Section 6 to prove a clearing-out lemma. I then establish the equipartition of energy in Section 7. In that section I also show that the Hausdorff dimension of the interface is $d-1$. I complete the proofs in Section 8.

Acknowledments.

Part of this work was done during my visit, supported by SFB256, to the University of Bonn. I would like to thank Tom Ilmanen and Stefan Luckhaus for valuable discussions during that visit and to express my deepest gratitude to Mort Gurtin for numerous conversations.

2 Prelimenaries

2.1 Equations

For a scalar u, set

$$W(u) = \frac{1}{2}(u^2 - 1)^2 , \qquad f(u) = W'(u) = 2u(u^2 - 1) , \tag{2.1}$$

and

$$h(u) = u - \frac{1}{3}u^3 , \qquad g(u) = h'(u) = (1 - u^2) = \sqrt{2W(u)} . \tag{2.2}$$

In this paper, we study the phase field equations with the above functions. Then the heat equation (1.1) and the order parameter equation (1.2) with parameters as in (1.3), (1.4) are,

$$\varphi_t^\epsilon - \Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon) - \frac{1}{\epsilon}g(\varphi^\epsilon)\theta^\epsilon = 0 , \quad \text{in } (0, \infty) \times \mathcal{R}^d , \tag{OPE}$$

$$\theta_t^\epsilon - \Delta\theta^\epsilon + g(\varphi^\epsilon)\varphi_t^\epsilon = 0 , \quad \text{in } (0. \infty) \times \mathcal{R}^d . \tag{HE}$$

For $\epsilon > 0$, let $(\varphi^\epsilon, \theta^\epsilon)$ be the unique, smooth, bounded solution of the phase field equations satifying the initial data

$$\varphi^\epsilon(x, 0) = \varphi_0^\epsilon(x) , \quad \theta^\epsilon(x, 0) = \theta_0^\epsilon(x) , \quad x \in \mathcal{R}^d. \tag{IC}$$

We assume that

$$|\, \varphi_0^\epsilon(x) \,| \le 1 , \quad \forall x \in \mathcal{R}^d . \tag{A1}$$

Then since $W'(\pm 1) = g(\pm 1) = 0$, by the maximum principle we have

$$|\, \varphi^\epsilon(t, x) \,| < 1 , \quad \forall (t, x) \in (0, \infty) \times \mathcal{R}^d .$$

For a real number r, let $q(r) = \tanh(r)$. Then

$$q'' = W'(q) , \quad q' = \sqrt{2W(q)} = g(q) , \tag{2.3}$$

and q is the standing wave associated to the reaction diffusion equation with nonlinearity W'. Since $|\varphi^\epsilon| < 1$, we can define z^ϵ by

$$\varphi^\epsilon(t, x) = q(\frac{z^\epsilon(t, x)}{\epsilon}) \iff z^\epsilon = \epsilon q^{-1}(\varphi^\epsilon). \tag{2.4}$$

Then z^ϵ solves (observe that $g(\varphi^\epsilon) = g(q(z^\epsilon/\epsilon)) = q'(z^\epsilon/\epsilon)$ and $q'' = 2\varphi^\epsilon q'$),

$$z_t^\epsilon - \Delta z^\epsilon - \theta^\epsilon + \frac{2\varphi^\epsilon}{\epsilon}\,(|\nabla z^\epsilon|^2 - 1) = 0 . \tag{ZE}$$

2.2 Energy

For a Borel subset $A \subset \mathcal{R}^d$, define

$$\mu^\epsilon(t;A) = \int_A \frac{\epsilon}{2}|\nabla\varphi^\epsilon|^2 + \frac{1}{\epsilon}W(\varphi^\epsilon)dx \ ,$$

$$\hat{\mu}^\epsilon(t;A) = \mu^\epsilon(t;A) + \int_A \frac{1}{2}(\theta^\epsilon)^2 dx \ .$$

Then by direct differentiation and integration by parts we obtain,

$$\frac{d}{dt}\hat{\mu}^\epsilon(t;\mathcal{R}^d) = -\int_{\mathcal{R}^d} \epsilon(\varphi_t^\epsilon)^2 + |\nabla\theta^\epsilon|^2 dx \, .$$

If we assume that

$$\hat{\mu}^\epsilon(0;\mathcal{R}^d) \ \le C_1^*, \quad \epsilon > 0\,. \tag{A2}$$

then we have

$$\hat{\mu}^\epsilon(t;\mathcal{R}^d) + \int_0^t \int_{\mathcal{R}^d} \epsilon(\varphi_t^\epsilon)^2 + |\nabla\theta^\epsilon|^2 dx dt \ \le C_1^* \ , \quad \epsilon, t \ge 0. \tag{2.5}$$

(Assumption (A2) can be relaxed as in [28]). We localize the above estimate in the following way. Let ψ be any *positive*, smooth, compactly supported function. Then,

$$\begin{aligned}
\frac{d}{dt}\int \psi(x)\hat{\mu}^\epsilon(t;dx) = & \ -\int_{\mathcal{R}^d} \epsilon\psi\left[(\varphi_t^\epsilon + \frac{\nabla\varphi^\epsilon\cdot\nabla\psi}{2\psi})^2 + |\nabla\theta^\epsilon + \frac{\theta^\epsilon\nabla\psi}{2\psi}|^2\right] \\
& + \int_{\mathcal{R}^d} \frac{\epsilon}{4\psi}(\nabla\varphi^\epsilon\cdot\nabla\psi)^2 + \frac{(\theta^\epsilon)^2}{4\psi}|\nabla\psi|^2 \\
\le & \ [\sup_x \frac{|\nabla\psi(x)|^2}{2\psi(x)}] \int_{\{\psi>0\}} \frac{\epsilon}{2}|\nabla\varphi^\epsilon|^2 + \frac{1}{2}(\theta^\epsilon)^2 \\
\le & \ \| D^2\psi \|_\infty \, \hat{\mu}^\epsilon(t;\{\psi>0\}).
\end{aligned}$$

Here we used the fact that for any positive C^2 function,

$$\frac{|\nabla\psi(x)|^2}{2\psi(x)} \le ||D^2\psi||_\infty \, .$$

Hence there is a constant $C(\psi)$, independent of ϵ, such that the map

$$t \mapsto \int \psi(x)\hat{\mu}^\epsilon(t;dx) - C(\psi)t$$

is non increasing.

2.3 Subsequence

Using the above monotonicity in a diagonal argument (see [22, Section 5.4] for details), we construct a subsequence, denoted by ϵ again, a family of Radon measures $\hat{\mu}(t,\cdot)$ and $\mu(t,\cdot)$ satisfying

$$\hat{\mu}^\epsilon(t,\cdot) \rightharpoonup \hat{\mu}(t,\cdot), \mu^\epsilon(t,\cdot) \rightharpoonup \mu(t,\cdot) \quad \forall t \geq 0 , \tag{2.6}$$

in the weak* topology of Radon measures. Now for a Borel subset $B \subset [0,\infty) \times \mathcal{R}^d$, define

$$\hat{\mu}(B) = \iint_B \hat{\mu}(t;dx)dt , \quad \mu(B) = \iint_B \mu(t;dx)dt ,$$

and

$$\Gamma = spt\mu \quad , \quad \Gamma(t) = spt\mu(t;\cdot) , \quad t \geq 0.$$

We will show that the t-section, Γ_t, of Γ is essentially equal to $\Gamma(t)$, see Section 7 below.

The energy estimate (2.5) yields that

$$\sup_{\epsilon,t>0} \| \theta^\epsilon(t,\cdot) \|_{L^2(\mathcal{R}^d)} < \infty.$$

Hence there are a subsequence, denoted by ϵ, and an L^2 function θ such that

$$\theta^\epsilon \rightharpoonup \theta \text{ in weak } L^2((0,T) \times \mathcal{R}^d), \tag{2.7}$$

for every $T > 0$. We will show that the above convergence is in fact in the strong topology (see Section 4, below). Moreover, by the arguments of Bronsard and Kohn [6], this sequence can be chosen so that, for every $T > 0$,

$$h(\varphi^\epsilon) \to h(\varphi) = \frac{2}{3}\varphi \ \text{ in } L^1_{loc}, \quad \varphi^\epsilon \to \varphi \quad \text{a.e.} , \tag{2.8}$$

where φ is a function of bounded variation and $|\varphi(t,x)| = 1$, for almost every (t,x).

2.4 Initial data and assumptions

In addition to $(A1), (A2)$ we assume that,

$$\| \nabla z_0^\epsilon \|_\infty \leq 1 , \tag{A3}$$

$$\sup_{0<\epsilon\leq 1} \epsilon \| D^2 z_0^\epsilon \|_\infty < \infty , \tag{A4}$$

$$\sup_{0<\epsilon\leq 1}\ \sup_{\substack{x\in\mathcal{R}^d \\ R>0}} \frac{\mu^\epsilon(0;B_R(x))}{R^{d-1}} < \infty\,, \tag{A5}$$

where $B_R(x)$ is the sphere centered at x with radius R. We also assume that,

$$\sup_{0<\epsilon\leq 1} \{\| \theta_0^\epsilon \|_1 + \| \theta_0^\epsilon \|_\infty + \sqrt{\epsilon}\, [\| \nabla\theta_0^\epsilon \|_\infty]\} < \infty, \tag{A6}$$

$$\sup_{0<\epsilon\leq 1} \{\epsilon^3 \| D^2\varphi_0^\epsilon \|_\infty + \epsilon^2 \| D^2\theta_0^\epsilon \|_\infty\} < \infty\,. \tag{A7}$$

Observe that since for $1 \leq p < \infty$,

$$\| \theta_0^\epsilon \|_p \leq \| \theta_0^\epsilon \|_1 \| \theta_0^\epsilon \|_\infty^{p-1},$$

(A6) implies that

$$\sup_{0<\epsilon\leq 1} \| \theta_0^\epsilon \|_p = K(p) < \infty. \tag{2.9}$$

Finally we assume that there is $\theta_0 \in L^2(\mathcal{R}^d)$ such that

$$\theta_0^\epsilon \to \theta_0 \text{ in } L^2\text{-strong}. \tag{A8}$$

There are functions satisfying (A1)-(A8). Indeed if $\theta_0^\epsilon = \theta_0$ is a smooth, compactly supported function, then θ_0 satisfies (A6), (A7) and (A8) trivially. Suppose that Γ_0 is a bounded, smooth hypersurface in $\mathcal{R}^d$. Let $d(x)$ be the signed distance of x to Γ_0 and let $\hat{d}$ be an appropriate modification of d outside of a tubular neighborhood of Γ_0 such that all derivatives of $\hat{d}$ up to order three are bounded and $2|\hat{d}| \geq |d|$. Then $z^\epsilon = \hat{d}$ satisfies all the above conditions.

Finally we note that the term $\sqrt{\epsilon}$ appearing in (A6) is not essential. Indeed if (A6) holds with ϵ^ν for some $\nu \geq 1/2$, then we can prove the same results with minor changes.

2.5 Main Results

Our main result states that the "limit" of $(\theta^\epsilon, \varphi^\epsilon)$ solves (1.5), (1.6). However the boundary of the set $\{\lim \varphi^\epsilon = 1\}$ is not necessarily smooth and a radial example with one interface shows that the classical solutions of (1.5), (1.6) may not be poitwise bounded, see Remark 3.1 below. Hence a weak formulation of the Mullins-Sekerka problem, (1.5), (1.6), is needed in order to state our main result.

Fix $T \geq 0$. Let $\mu(t, dx)$ be the limit of $\mu^\epsilon(t, dx)$ and $\Gamma(t)$ be the support of $\mu(t,\cdot)$. Whenever $\mu(t;)$ is $(d-1)$-rectifiable (c.f. [22, Section 1.7] or [27]), we say that $\mu(t;)$ has a *generalized mean curvature vector*

$$H(t,\cdot) \in L^1_{loc}(\mathcal{R}^d \to \mathcal{R}^d; \mu(t;dx))$$

if for all smooth, compactly supported vector field $Y(x)$, H satisfies,

$$\int_{\mathcal{R}^d} tr(DY(x)P(t,x))\mu(t,dx) = -\int_{\mathcal{R}^d} Y(x)\cdot H(t,x)\mu(t,dx),$$

where $P(t,x)$ is the projection on the the tangent space $T_x\mu(t,\cdot)$ of $\mu(t;)$ at x. Here the left hand side is equal to the first variation of the varifold $V_{\mu(t,\cdot)}$ corresponding $(d-1)$-varifold $V_{\mu(t,\cdot)}$ (c.f. [22, Section 1.7], [27]).

Our main regularity result is;

Theorem 2.1 (Regularity) *For almost every $t \geq 0, \mu(t,\cdot)$ is $(d-1)$ rectifiable and has a generalized mean curvature vector $H(t,x)$. Moreover for every $T>0$,*

$$|H| \in L^2((0,T)\times\mathcal{R}^d\,;\,d\mu).$$

Definition of rectifiability requires the existence of the tangent plane $T_x\mu(t,\cdot)$. However, a weak formulation of (1.6) requires not only the tangent plane but also the normal vector. Therefore to describe the limit of the phase field equations we have to introduce yet another measure, m^ϵ, that keeps track of the normal direction. At this point we should point out that the mean curvature vector is independent of orientation and therefore in [22], and in [28], $\mu(t,\cdot)$ is enough to describe the asymptotic behavior of the Cahn-Allen equation.

Let S^{d-1} denote the set of all d-dimensional unit vectors. For $(t,x,n)\in[0,\infty)\times\mathcal{R}^d\times S^{d-1}$, define

$$\nu^\epsilon(t,x) = \begin{cases} \dfrac{\nabla\varphi^\epsilon(t,x)}{|\nabla\varphi^\epsilon(t,x)|}, & \text{if } |\nabla\varphi^\epsilon(t,x)| \neq 0, \\ \nu_0\,, & \text{if } \nabla\varphi^\epsilon(t,x) = 0\,, \end{cases}$$

$$dm^\epsilon(t,x,n) = dt\mu^\epsilon(t;dx)\delta_{\{\nu^\epsilon(t,x)\}}(dn),$$

where $\nu_0\in S^{d-1}$ is arbitrary and $\delta_{\{\nu^\epsilon\}}$ is the Dirac measure located at ν^ϵ. Since S^{d-1} is compact, there is a further sequence, denoted by ϵ again, such that dm^ϵ is weak* convergent. By a slicing argument (c.f. [15, Theorem 10, page 14]) we conclude that there exists a probability measure $N(t,x,\cdot)$ on S^{d-1} such that as ϵ tends to zero, we have

$$dm^\epsilon \rightharpoonup dm = dt\mu(t;dx)N(t,x;dn).$$

Theorem 2.2 (Convergence) *There is a normal velocity function*

$$v(t,x,n)\in L^2((0,T)\times\mathcal{R}^d\times S^{d-1}\to\mathcal{R}^1;dm),\;\; T>0$$

satisfying

$$(\theta_t - \Delta\theta)dxdt = \int_{S^{d-1}} v(t,x,n)N(t,x;dn)d\mu(t,x), \tag{2.10}$$

and

$$\int_{S^{d-1}} nv(t,x,n)N(t,x,dn) = -H(t,x) - \theta(t,x)\int_{S^{d-1}} nN(t,x,dn) \qquad (2.11)$$

for $d\mu$-almost every (t,x). Moreover,

$$sptN(t,x,\cdot) \perp T_x\mu(t,\cdot).$$

for $d\mu$-almost every (t,x), i.e., there are measurable functions $\alpha(t,x) \in [0,1]$ and $\nu(t,x) \in S^{d-1}$ satisfying,

$$N(t,x;dn) = \alpha(t,x)\delta_{\{\nu(t,x)\}}(dn) + (1-\alpha(t,x))\delta_{\{-\nu(t,x)\}}(dn),$$

and

$$\nu(t,x) \perp T_x\mu(t,\cdot),$$

for $d\mu$-almost every (t,x).

In (2.10), (2.11) we interpret

$$\int_{S^{d-1}} v(t,x,n)N(t,x;dn), \quad \int_{S^{d-1}} nv(t,x,n)N(t,x;dn), \quad \int_{S^{d-1}} nN(t,x;dn)$$

as, respectively, the normal velocity, the normal velocity vector and the outward unit normal vector of the interface. Then clearly (2.11) is a weak formulation of (1.6). One dimensional considerations suggest that the term $\frac{4}{3}(\chi_\Omega)_t$ appearing in (1.5) is there is formally equal to $Vd\mu$. Hence (2.10) is a weak version of (1.5).

The definition of the generalized mean curvature vector and the orthogonality of the $sptN(t,x,\cdot)$ to $T_x\mu(t,\cdot)$ imply that (2.11) is equivalent to

$$\int [Y \cdot n(v+\theta) + DY : (I - n \otimes n)]\, dm = 0,$$

for any compactly supported, smooth vector field $Y(t,x)$. We also note that (2.10) implies that the distribution $(\theta_t - \Delta\theta)$ is indeed a Radon measure which is absolutely continuous with respect to $\mu(t,dx)dt$. In Section 8, $v(t,x,n)$ will be constructed as the limit of $-z_t^\epsilon(t,x)$.

In the case of the Cahn-Allen equation, radially symmetric examples indicate that $N(t,x,dn)$ may not be a Dirac measure. This corresponds to the "piling-up" of the interfaces. However it is not clear that if $N(0,x,dn)$ is a Dirac measure whether $N(t,x;dn)$ has to be a Dirac measure maybe expect on a set of lower dimension. Stoth [30] proved this result for the limit of radially symmetric Cahn-Hilliard equations.

Our final result is a Brakke [5] type inequality satisfied by the limit inteface. This inequality is a straightforward extension of Brakke's definition of a varifold moving by its mean curvature and it may be useful in proving further regularity of the limit interface.

Theorem 2.3 (Brakke type inequality) *For any compactly supported smooth function $\phi(x)$, and $t > 0$, if*

$$d := \limsup_{s \to t} \frac{1}{s-t} \int \phi(x)(\hat{\mu}(s; dx) - \hat{\mu}(t; dx)) > -\infty,$$

then $\mu(t, \cdot)$ restricted to $\{\phi > 0\}$ is $(d-1)$-rectifiable with a generalized mean curvature vector $\hat{H}(t, \cdot)$. Moreover

$$\begin{aligned} d \le & -\iint \phi(x) \left| \hat{H}(t,x) + \theta(t,x) n \right|^2 \mu(t; dx) N(t, x; dn) \\ & - \int \phi(x) \left| \nabla \theta(t,x) \right|^2 dx + \frac{1}{2} \int \Delta \phi(x)\ \theta^2(t,x) dx \qquad (2.12) \\ & - \iint D\phi(x) \cdot [\hat{H}(t,x) + n\theta(t,x)] \mu(t; dx) N(t, x; dn). \end{aligned}$$

3 Elementary Estimates

In this section we obtain several elementary estimates using the heat kernel representation of the solutions. Let

$$G(\tau, \xi) = (4\pi\tau)^{-\frac{d}{2}} \exp(-\frac{|\xi|^2}{4\tau}),\ (\tau, \xi) \in (0, \infty) \times \mathcal{R}^d.$$

Since $h' = g$, using the heat equation (HE) and integration by parts we have,

$$\theta^\epsilon(t,x) = A^\epsilon(t,x) + B^\epsilon(t,x) \qquad (3.1)$$

where

$$\begin{aligned} A^\epsilon(t,x) &= (G(t,\cdot) * [\theta_0^\epsilon + H^\epsilon(0,\cdot) - H^\epsilon(t,\cdot)])(x) \\ B^\epsilon(t,x) &= \int_0^t (G_\tau(\tau,\cdot) * (H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot)))(x) d\tau\,, \\ H^\epsilon(t,x) &= h(\varphi^\epsilon(t,x))\,, \end{aligned}$$

and * denotes convolution in the x-variable.

For $T > 0$, let $\|\cdot\|_{\infty,T}$ denote the norm in $L^\infty((0,T) \times \mathcal{R}^d)$. All the constants in this and later sections depend on T but we will suppress this dependence in our notation. Also all constants independent of ϵ will be denoted by K. We should warn the reader that this constant may change from one line to the next.

Lemma 3.1 *There is a constant K independent of $0 < \epsilon \leq 1$ satisfying,*

$$\epsilon \parallel \nabla \varphi^\epsilon \parallel_{\infty,T} \leq K , \tag{3.2}$$

$$\parallel \theta^\epsilon \parallel_{\infty,T} \leq K \left[1 + |\ln \epsilon|\right] , \tag{3.3}$$

Proof:
1. Fix $T > 0$ and set

$$m^\epsilon(T) = \epsilon \parallel \nabla \varphi^\epsilon \parallel_{\infty,T}, \quad n^\epsilon(T) = \parallel \theta^\epsilon \parallel_{\infty,T} ,$$

$$f^\epsilon = -\frac{1}{\epsilon^2} W'(\varphi^\epsilon) + \frac{1}{\epsilon} g(\varphi^\epsilon)\theta^\epsilon .$$

Then

$$\parallel f^\epsilon \parallel_{\infty,T} \leq \frac{1}{\epsilon^2} \left[2 + \epsilon n^\epsilon(T)\right] ,$$

and the order parameter equation (OPE) can be rewritten as,

$$\varphi^\epsilon_t - \Delta \varphi^\epsilon = f^\epsilon .$$

2. Fix $(t,x) \in [0,T] \times \mathcal{R}^d$. Then for any $\sigma \in (0,t]$, (OPE) yields

$$\nabla \varphi^\epsilon(t,x) = (\nabla G(\sigma,\cdot) * \varphi^\epsilon(t-\sigma,\cdot))(x) + \int_0^\sigma (\nabla G(\tau,\cdot) * f^\epsilon(t-\tau,\cdot))(x) d\tau .$$

Observe that for any $\tau > 0$,

$$\sqrt{\tau} \parallel \nabla G(\tau,\cdot) \parallel_1 = (\pi)^{-\frac{d}{2}} \int_{\mathcal{R}^d} |y|\, e^{-|y|^2} dy = K .$$

Therefore,

$$\left| \int_0^\sigma [\nabla G(\tau,\cdot) * f^\epsilon(t-\tau,\cdot)](x) dt \right| \leq \frac{2K(2 + \epsilon n^\epsilon(T))}{\epsilon^2} \sqrt{\sigma} ,$$

$$|(\nabla G(\sigma,\cdot) * \varphi^\epsilon(t-\sigma,\cdot))(x)| \leq \frac{K}{\sqrt{\sigma}} .$$

Also if $\sigma = t$,

$$|(\nabla G(\sigma,\cdot) * \varphi^\epsilon(t-\sigma,\cdot))(x)| = |(G(t,\cdot) * \nabla \varphi^\epsilon_0)(x)| \leq \parallel \nabla \varphi^\epsilon_0 \parallel_\infty .$$

3. Now use the above inequalities with $\sigma = \epsilon^2 \wedge t$ to obtain,

$$|\nabla \varphi^\epsilon(t,x)| \leq \frac{K}{\epsilon} [5 + 2\epsilon n^\epsilon(T)], \text{ if } t \geq \epsilon^2,$$

$$|\nabla \varphi^\epsilon(t,x)| \leq \frac{K}{\epsilon} [\epsilon \parallel \nabla \varphi^\epsilon_0 \parallel_\infty + 2(2 + \epsilon n^\epsilon(T))], \text{ if } t \leq \epsilon^2 .$$

Since by (A3) $\epsilon \parallel \nabla\varphi_0^\epsilon \parallel_\infty \leq K$, we conclude that,

$$m^\epsilon(T) \leq K[1 + \epsilon n^\epsilon(T)]. \tag{3.4}$$

4. Let A^ϵ, B^ϵ be as in (3.1). Then,

$$|A^\epsilon(t,x)| \leq \parallel \theta_0^\epsilon \parallel_\infty + 2.$$

For $\sigma \in (0, t]$,

$$\left| \int_\sigma^t [G_\tau(\tau,\cdot) * (H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot))](x) d\tau \right| \leq 2 \int_\sigma^t \parallel G_\tau(\tau,\cdot) \parallel d\tau.$$

Observe that

$$\tau \parallel G_\tau(\tau,\cdot) \parallel_1 \leq (\pi)^{-\frac{d}{2}} \int_{\mathcal{R}^d} (\frac{d}{2} + |y|^2) e^{-|y|^2} dy \leq K.$$

Hence

$$\left| \int_\sigma^t [G_\tau(\tau,\cdot) * (H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot))](x) d\tau \right| \leq K \ln(\frac{t}{\sigma}).$$

Since $\Delta G = G_\tau$, and $\nabla H = g\nabla\varphi^\epsilon$, by integration by parts we obtain,

$$\begin{aligned}
&\left| \int_0^\sigma [G_\tau(\tau,\cdot) * (H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot))](x) d\tau \right| \\
&= \left| \int_0^\sigma \int_{\mathcal{R}^d} \nabla G(\tau, x-y) \cdot [\nabla H(t,y) - \nabla H(t-\tau,y)] dy d\tau \right| \\
&\leq \int_0^\sigma \parallel \nabla G(\tau,\cdot) \parallel_1 (\parallel \nabla\varphi^\epsilon(t-\tau,\cdot) \parallel_\infty + \parallel \nabla\varphi^\epsilon(t,\cdot) \parallel_\infty) d\tau. \\
&\leq \frac{K}{\epsilon} m^\epsilon(T)\sqrt{\sigma}.
\end{aligned}$$

5. Estimates obtained in step 4 and (3.1) yield,

$$|\theta^\epsilon(t,x)| \leq \parallel \theta_0^\epsilon \parallel_\infty + 2 + K \ln(\frac{t}{\sigma}) + \frac{K\sqrt{\sigma}}{\epsilon} m^\epsilon(T).$$

Choose $\sigma = \epsilon^2 \wedge t$ to obtain

$$n^\epsilon(T) \leq \parallel \theta_0^\epsilon \parallel_\infty + 2 + K \ln T + K |\ln \epsilon| + K m^\epsilon(T). \tag{3.5}$$

Combine (3.4), (3.5) and use (A6) to obtain

$$n^\epsilon(T) \leq K_0(1 + |\ln \epsilon| + \epsilon n^\epsilon(T)).$$

Hence (3.2) and (3.3) holds for all $\epsilon > 0$ satisfying

$$2K_0\epsilon \leq 1 \Leftrightarrow \epsilon \leq \epsilon_0 = (1/2K_0).$$

For $1 \geq \epsilon \geq \epsilon_0$, (3.2), (3.3) can be proved easily. ∎

Remark 3.1.

θ^ϵ is not necessarily uniformly bounded in ϵ. Indeed consider the Mullins-Sekerka problem with radial symmetry and one interface. If the radius R_0 of the initial interface is sufficiently small and the initial temperature θ_0 is sufficiently large, then we can show that the radius $R(t)$ of the interface becomes zero at a finite time T. Moreover, $|\theta(R(t),t)|$ behaves like $|\ln(T-t)|$ as t tends to T. Since the phase field equations with radial symmetry known to approximate the Mullins-Sekerka problem [29], this example shows that θ^ϵ is not uniformly bounded in ϵ.

Next we will use the above elementary technique to obtain uniform bounds for $\epsilon^2 \left|D^2\varphi^\epsilon\right|$ and $\epsilon\left|\nabla\theta^\epsilon\right|$.

Lemma 3.2

$$\sup_{0<\epsilon\leq 1} \left\{\epsilon^2 \left[\| D^2\varphi^\epsilon \|_{\infty,T} + \| \varphi_t^\epsilon \|_{\infty,T}\right] + \epsilon \| \nabla\theta^\epsilon \|_{\infty,T}\right\} < \infty. \tag{3.6}$$

Proof:

1. Differentiate the (OPE) to obtain

$$\varphi^\epsilon_{x_j t} - \Delta\varphi^\epsilon_{x_j} = F_j^\epsilon$$

$$F_j^\epsilon = -\frac{1}{\epsilon^2}W''(\varphi^\epsilon)\varphi^\epsilon_{x_j} + \frac{1}{\epsilon}g'(\varphi^\epsilon)\varphi^\epsilon_{x_j}\theta^\epsilon + \frac{1}{\epsilon}g(\varphi^\epsilon)\theta^\epsilon_{x_j}.$$

Using (3.2), (3.3) we conclude that

$$\| F_j^\epsilon \|_{\infty,T} \leq K[\frac{1}{\epsilon^3} + \frac{1}{\epsilon} \| \nabla\theta^\epsilon \|_{\infty,T}],$$

for some constant K. Set

$$\overline{m}^\epsilon(T) = \epsilon^2 \| D^2\varphi^\epsilon \|_{\infty,T}, \quad \overline{n}^\epsilon(T) = \epsilon \| \nabla\theta^\epsilon \|_{\infty,T}.$$

Then we use (A4), (3.2) as in step 1 of Lemma 3.1 to obtain,

$$\overline{m}^\epsilon(T) \leq K[1 + \epsilon\overline{n}^\epsilon(T)]. \tag{3.7}$$

Observe that (3.3), (3.7), and (OPE) yield

$$\epsilon^2 \| \varphi_t^\epsilon \|_{\infty,T} \leq K[1 + \epsilon\overline{n}^\epsilon(T)]. \tag{3.8}$$

2. Let A^ϵ, B^ϵ, be as in (3.1). Then,

$$\nabla A^\epsilon(t,x) = (G(t,\cdot) * \nabla(\theta_0^\epsilon + H^\epsilon(0,\cdot) - H^\epsilon(t,\cdot)))(x),$$

$$\nabla B^\epsilon(t,x) = \int_0^t (\nabla G_\tau(\tau,\cdot) * [H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot)])(x)d\tau.$$

Fix $(t,x) \in [0,T] \times \mathcal{R}^d$. In view of (A6) and (3.2) we have

$$|\nabla A^\epsilon(t,x)| \leq \| \nabla\theta_0^\epsilon \|_\infty + 2 \| \nabla\varphi^\epsilon \|_{\infty,T} \leq \frac{K}{\epsilon}.$$

Also for $\sigma \in (0, t \wedge 1]$,

$$\left|\int_\sigma^t [\nabla G_\tau(\tau,\cdot) * (H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot))](x)d\tau\right|$$

$$\leq 2\int_\sigma^t \| \nabla G_\tau(\tau,\cdot) \|_1 \, d\tau \leq \int_\sigma^t K\tau^{-\frac{3}{2}} d\tau \leq K\left(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sqrt{t}}\right).$$

3. By integration by parts in the τ-variable, we obtain,

$$\left|\int_0^\sigma (\nabla G_\tau(\tau,\cdot) * [H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot)])(x)d\tau\right|$$

$$\leq |\nabla G(\sigma,\cdot) * [H^\epsilon(t,\cdot) - H^\epsilon(t-\sigma,\cdot)])(x)| + |\int_0^\sigma (\nabla G(\tau,\cdot) * H_t^\epsilon(t-\tau,\cdot)(x)d\tau|$$

$$\leq \frac{K}{\sqrt{\sigma}} \|H^\epsilon(t,\cdot) - H^\epsilon(t-\sigma,\cdot)\|_\infty + \int_0^\sigma \|\nabla G(\tau,\cdot)\|_1 \, \|H_t^\epsilon\|_{\infty,T} \, d\tau.$$

$$\leq K\sqrt{\sigma}\, \|H_t^\epsilon\|_{\infty,T}$$

$$\leq K\sqrt{\sigma}\, \|\varphi_t^\epsilon\|_{\infty,T}\cdot$$

3. Combine steps 2 and 3, and choose $\sigma = \epsilon^2 \wedge t$ to obtain,

$$|\nabla\theta^\epsilon(t,x)| \leq \frac{K}{\epsilon}\left(1 + \epsilon^2 \, \|\varphi_t^\epsilon\|_{\infty,T}\right)\cdot$$

As in the last step of the previous lemma, the above estimate together with (3.7) and (3.8) imply (3.6) for sufficiently small $\epsilon \leq \epsilon_0$. But for $\epsilon \geq \epsilon_0$, (3.6) holds trivially. ∎

Assumption (A7) and the arguments of Lemma 3.2 yield,

$$\sup_{0<\epsilon\leq 1} \left\{\epsilon^3 \|D^3\varphi^\epsilon\|_{\infty,T} + \epsilon^2 \|D^2\theta^\epsilon\|_{\infty,T} + \epsilon^2 \|\theta_t^\epsilon\|_{\infty,T}\right\} < \infty. \tag{3.9}$$

Lemma 3.3 *For any* $1 < p < \infty$, $T \geq 0$,

$$\sup_{0<\epsilon\leq 1,\, t\leq T} \|\theta^\epsilon(t,\cdot)\|_{L^p(\mathcal{R}^d)} < \infty. \tag{3.10}$$

Proof:
1. For any $0 < \epsilon \leq 1$, $0 \leq t_0 \leq t_1 \leq T$, we have,

$$\begin{aligned} \|H^\epsilon(t_1,\cdot) - H^\epsilon(t_0,\cdot)\|_1 &\leq \int_{t_0}^{t_1} \int_{\mathcal{R}^d} |h'(\varphi^\epsilon)|\, |\varphi_t^\epsilon|\, dxds \\ &\leq \left(\int_{t_0}^{t_1} \int_{\mathcal{R}^d} \frac{1}{2\epsilon}(h'(\varphi^\epsilon))^2\right)^{1/2} \left(\int_{t_0}^{t_1} \int_{\mathcal{R}^d} \frac{\epsilon}{2}(\varphi_t^\epsilon)^2\right)^{1/2}. \end{aligned}$$

Recall that $(h'(\varphi^\epsilon))^2 = 2W(\varphi^\epsilon)$. Hence the energy estimate (2.5) yields,

$$\|H^\epsilon(t_1,\cdot) - H^\epsilon(t_0,\cdot)\|_1 \leq C_1^* \sqrt{t_1 - t_0}.$$

Since $|H^\epsilon| \leq 1$, we have

$$\|H^\epsilon(t_1,\cdot) - H^\epsilon(t_0,\cdot)\|_p^p \leq K\sqrt{t_1 - t_0}.$$

2. Let A^ϵ, B^ϵ be as in (3.1). Then by (2.9) and the previous step we have,

$$\|A^\epsilon\|_p \leq \|\theta_0^\epsilon\|_p + \|H^\epsilon(t,\cdot) - H^\epsilon(0,\cdot)\|_p \leq K(1 + t^{\frac{1}{2p}}).$$

3. By the first step,

$$\begin{aligned} \|B^\epsilon\|_p &\leq \int_0^t \|G_\tau(\tau,\cdot) * (H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot))\|_p \, d\tau \\ &\leq \int_0^t \|G_\tau(\tau,\cdot)\|_1 \, \|H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot)\|_p \, d\tau \\ &\leq K \int_0^t \tau^{-1+\frac{1}{2p}} \, d\tau. \end{aligned}$$

∎

We close this section by proving the strong convergence of the sequence θ^ϵ.

Proposition 3.4 *θ^ϵ converges to θ strongly in $L^2_{loc}((0,\infty)\times\mathcal{R}^d)$.*

Proof:
1. Let $\overline{\theta}^\epsilon = \theta^\epsilon - \hat{\theta}^\epsilon$ where $\hat{\theta}^\epsilon$ is the unique solution of

$$\hat{\theta}^\epsilon_t - \Delta\hat{\theta}^\epsilon = 0\ ,\ (0,\infty)\times\mathcal{R}^d$$

with initial data $\hat{\theta}^\epsilon(0,x) = \theta^\epsilon_0(x)$. Then (A8) implies that $\hat{\theta}^\epsilon(t,\cdot)$ converges to

$$\hat{\theta}(t,\cdot) = G(t,\cdot) * \theta_0$$

strongly in $L^2(\mathcal{R}^d)$.
2. By integration by parts, we write $\overline{\theta}^\epsilon = \varphi^{\epsilon,1} + \varphi^{\epsilon,2}$ where

$$\begin{aligned}\varphi^{\epsilon,1}(t,\cdot) &= G(t,\cdot) * [H^\epsilon(0,\cdot) - H^\epsilon(t,\cdot)]\\ \varphi^{\epsilon,2}(t,\cdot) &= \int_0^t G_\tau(\tau,\cdot) * [H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot)]d\tau.\end{aligned}$$

Clearly (2.8) implies that $\varphi^{\epsilon,1}(t,\cdot)$ converges to

$$\varphi^1(t,\cdot) = G(t,\cdot) * [H(0,\cdot) - H(t,\cdot)].$$

For $t,\sigma > 0$, set $\delta = \min\{\sigma, t\}$. Then it is easy to show that

$$\int_\delta^t G_\tau(\tau,\cdot) * [H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot)]d\tau$$

converges to

$$\frac{2}{3}\int_\delta^t G_\tau(\tau,\cdot) * [\varphi(t,\cdot) - \varphi(t-\tau,\cdot)]d\tau$$

strongly in $L^2_{loc}(\mathcal{R}^d)$. And by step 1 of the previous lemma,

$$\begin{aligned}&\|\int_0^\delta (G_\tau(\tau,\cdot) * [H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot)]d\tau)\|_2\\ &\qquad \le K\int_0^\delta \frac{1}{\tau}\|H^\epsilon(t,\cdot) - H^\epsilon(t-\tau,\cdot)\|_1^{1/2}d\tau,\\ &\qquad \le K(\delta)^{1/4} \le K\sigma^{1/4}.\end{aligned}$$

A similar argument shows that

$$\|\int_0^\delta G_\tau(\tau,\cdot) * [\varphi(t,\cdot) - \varphi(t-\tau,\cdot)]d\tau\|_2 \le K\sigma^{1/4}.$$

Therefore for all $\sigma, T, R > 0$,

$$\limsup_{\epsilon \to 0} \|\theta^\epsilon - \theta\|_{L^2((0,\infty)\times B_R)} \leq K\sigma^{1/4},$$

where

$$\begin{aligned}\theta(t,x) \quad &= \hat{\theta}(t,x) + \frac{2}{3}(G(t,\cdot) * [\varphi(0,\cdot) - \varphi(t,\cdot)])(x) \\ &+ \frac{2}{3}\int_0^t G_\tau(\tau,\cdot) * [\varphi(t,\cdot) - \varphi(t-\tau,\cdot)]d\tau.\end{aligned}$$

∎

An elementary argument very similar to the proof Proposition 3.4 shows that the map

$$t \mapsto \|\theta^\epsilon(t,\cdot)\|_2$$

is uniformly Hölder continuous in $\epsilon \in (0,1]$. However this fact will not be used in our analysis.

4 Estimate of $|\nabla z^\epsilon|$.

Main result of this section is,

Theorem 4.1 *For $T > 0$, there exists a constant $K^* = K^*(T)$ satisfying*

$$|\nabla z^\epsilon(t,x)|^2 \leq 1 + \sqrt{\epsilon}K^*(1 + |z^\epsilon(t,x)|), \tag{4.1}$$

for all $(t,x) \in [0,T] \times \mathcal{R}^d$, and $0 < \epsilon \leq 1$.

Proof of this estimate will be completed in several steps. Before we start our long analysis, let us briefly explain the main idea. Set

$$w^\epsilon = |\nabla z^\epsilon|^2.$$

Using the equation (ZE), we obtain

$$w_t^\epsilon + \mathcal{L}_t^\epsilon w^\epsilon + R^\epsilon(t,x,w^\epsilon) - 2\nabla\theta^\epsilon \cdot \nabla z^\epsilon \leq 0\,, \tag{4.2}$$

where for $\psi \in C^2(\mathcal{R}^d)$,

$$\begin{aligned}\mathcal{L}_t^\epsilon\psi(x) = &\quad -\Delta\psi(x) + \frac{4\varphi^\epsilon(t,x)}{\epsilon}\nabla z^\epsilon(t,x)\cdot\nabla\psi(x), \\ R^\epsilon(t,x,r) = &\quad \frac{4}{\epsilon^2}q'(\frac{z^\epsilon(t,x)}{\epsilon})r(r-1), \quad r \geq 0.\end{aligned}$$

In [28, Section 8], we obtained pointwise estimates for a differential inequality obtained by setting the last term involving $\nabla\theta^\epsilon$ in (4.2) to zero. Here we start by using the technique developed in [28]. By (3.6) we first crudely estimate that,

$$|2\nabla\theta^\epsilon \cdot \nabla z^\epsilon| \leq 2\|\nabla\theta^\epsilon\|_{\infty,T}\, w^\epsilon \leq \frac{K}{\epsilon} w^\epsilon,$$

on $t \leq T$, $w^\epsilon \geq 1$. Then the proof of Proposition 8.1 in [28] yields that w^ϵ is uniformly bounded in ϵ.

Then our next step is to obtain a uniform bound for $\epsilon\,|z_t^\epsilon|$, see (4.8) below. Using these estimates, we obtain a bound for $|\nabla\theta^\epsilon|$ which is slightly better than (3.6). Finally, we use this new estimate of $|\nabla\theta^\epsilon|$ in (4.2) together with an argument similar to the ones used in [28] to obtain (4.1).

Remark 4.1. In the phase field equations if we choose

$$g = 2W = (q')^2,$$

then w satisfies,

$$w_t^\epsilon + \mathcal{L}_t^\epsilon w^\epsilon + R^\epsilon(t,x,w^\epsilon) - 2q'\nabla\theta^\epsilon \cdot \nabla z^\epsilon - \frac{2}{\epsilon} q''\theta^\epsilon w^\epsilon \leq 0\,,$$

Then the proof of the estimate (4.1) simplifies greatly. Indeed an attendant modification of the proof of Proposition 4.1, below, yields this estimate.

As in Section 3, we fix $T > 0$ and all constants depending only on T will be denoted by K.

Proposition 4.2 *There is $K = K(T)$ satisfying*

$$|\nabla z^\epsilon(t,x)|^2 \leq K(1 + |z^\epsilon(t,x)|),\ (t,x) \in [0,T] \times \mathcal{R}^d. \qquad (4.3)$$

Proof:

1. Fix $T > 0$ and set

$$K_0 = 2 \sup_{0<\epsilon\leq 1} \epsilon\|\nabla\theta^\epsilon\|_{\infty,T},$$

$$\hat{\mathcal{L}}\psi = \mathcal{L}_t^\epsilon\psi - 2\,|\nabla\theta^\epsilon(t,x)|\,\psi.$$

Then

$$w_t^\epsilon + \hat{\mathcal{L}}w^\epsilon + R^\epsilon(t,x,w^\epsilon) \leq 0. \qquad (4.4)$$

In the next several steps we will construct a "supersolution" to (4.4).

2. Let $z_0 > 0$ be the point satisfying,

$$q'(\frac{z_0}{\epsilon}) = \epsilon^{1/4} \Rightarrow z_0 = \epsilon(q')^{-1}(\epsilon^{1/4}).$$

Then z_0 behaves like $\epsilon\,|\ln\epsilon|$ as ϵ tends to zero. Indeed

$$\lim_{\epsilon\to 0}\frac{z_0}{\epsilon|\ln\epsilon|}=\frac{1}{8}.$$

Now define

$$h_\epsilon(r)=\begin{cases}\frac{1}{2}C_\epsilon r^2+1\,, & |r|\le z_0\,,\\ (K_0+1)[|r|-z_0]+h_\epsilon(z_0)\,, & |r|>z_0,\end{cases}$$

where

$$C_\epsilon=\frac{K_0+1}{z_0}.$$

Observe that h_ϵ is continuously differentiable with Lipschitz derivatives. Finally we set

$$W=1+h_\epsilon(z^\epsilon).$$

3. We directly calculate that

$$\begin{aligned} I &= W_t+\hat{\mathcal{L}}W+R^\epsilon(t,x,W)\\ &\ge h'_\epsilon(z^\epsilon)\,[z^\epsilon_t-\Delta z^\epsilon]-h''_\epsilon(z^\epsilon)w^\epsilon-\frac{K_0}{\epsilon}W\\ &\quad+\frac{4}{\epsilon}\varphi^\epsilon h'_\epsilon(z^\epsilon)w^\epsilon+\frac{4}{\epsilon^2}q'(\frac{z^\epsilon}{\epsilon})h_\epsilon(z^\epsilon)W\\ &\ge \frac{2}{\epsilon}\varphi^\epsilon h'_\epsilon(z^\epsilon)(w^\epsilon+1)+\frac{4}{\epsilon^2}q'(\frac{z^\epsilon}{\epsilon})h_\epsilon(z^\epsilon)W\\ &\quad-\frac{K_0}{\epsilon}W-h''_\epsilon(z^\epsilon)w^\epsilon+h'_\epsilon(z^\epsilon)\theta^\epsilon. \end{aligned}$$

Observe that $h_\epsilon\ge 1$, $|h'_\epsilon|\le K_0+1$ and

$$\|h''_\epsilon\|_\infty=C_\epsilon\ ,\ \lim_{\epsilon\to 0}\epsilon C_\epsilon=0. \tag{4.5}$$

Hence

$$I\ge\frac{2}{\epsilon}\varphi^\epsilon h'_\epsilon(w^\epsilon+1)+\frac{4}{\epsilon^2}q'W-\frac{K_0}{\epsilon}W-C_\epsilon w^\epsilon-(K_0+1)\|\theta^\epsilon\|_{\infty,T}.$$

4. Suppose that

$$|z^\epsilon(t,x)|\le z_0.$$

(The opposite case will be discussed in the next step). In this case we have

$$q'(\frac{z^\epsilon}{\epsilon})\ge q'(\frac{z_0}{\epsilon})=\epsilon^{1/4}.$$

Since $\varphi^\epsilon h'_\epsilon \geq 0,\ W \geq 1$, we use (3.3) to obtain,

$$\begin{aligned} I &\geq \frac{4}{\epsilon^2} q' W - \frac{K_0}{\epsilon} W - C_\epsilon w^\epsilon - (K_0+1)\|\theta^\epsilon\|_{\infty,T} \\ &\geq \left(\frac{4}{\epsilon^{7/4}} - \frac{K_0}{\epsilon}\right) W - C_\epsilon w^\epsilon - (K_0+1)\|\theta^\epsilon\|_{\infty,T} \\ &\geq C_\epsilon (W - w^\epsilon), \end{aligned}$$

for sufficiently small $\epsilon > 0$.
5. Suppose that $|z^\epsilon(t,x)| \geq z_0$. Then

$$h'_\epsilon(|z^\epsilon(t,x)|) = K_0 + 1,$$

and

$$|\varphi^\epsilon(x,t)| = q\left(\frac{|z^\epsilon(t,x)|}{\epsilon}\right) \geq q\left(\frac{z_0}{\epsilon}\right) \geq \frac{1}{2},$$

for sufficiently small $\epsilon > 0$. Therefore

$$\varphi^\epsilon(t,x) h'_\epsilon(z^\epsilon(t,x)) = |\varphi^\epsilon(t,x)|\, h'_\epsilon(|z^\epsilon(t,x)|) \geq \frac{1}{2}\,[K_0 + 1].$$

Since $q' \geq 0$, we have

$$\begin{aligned} I &\geq \frac{2\varphi^\epsilon}{\epsilon} h'_\epsilon(z^\epsilon)(w^\epsilon + 1) - \frac{K_0}{\epsilon} W - C_\epsilon w^\epsilon - (K_0+1)\|\theta^\epsilon\|_{\infty,T} \\ &\geq \left(\frac{K_0+1}{\epsilon} - C_\epsilon\right) w^\epsilon - \frac{K_0}{\epsilon} W + (K_0+1)\left[\frac{1}{\epsilon} - \|\theta^\epsilon\|_{\infty,T}\right]. \end{aligned}$$

We now use (3.3) and (4.5) to conclude that $I \geq 0$ on $\{w^\epsilon \geq W\}$.
6. In steps 3, 4 and 5, we proved that for every $T > 0$ there is $\epsilon_0 = \epsilon_0(T) > 0$ satisfying

$$W_t + \hat{\mathcal{L}} W + R^\epsilon(t,x,W) \ \geq\ C_\epsilon (W - w^\epsilon),$$

on $(0,T) \times \mathcal{R}^d\ \cap\ \{w^\epsilon \geq W\}$ for all $0 < \epsilon \leq \epsilon_0(T)$. Also in step 1, we showed that

$$w^\epsilon_t + \hat{\mathcal{L}} w^\epsilon + R^\epsilon(t,x,W) \leq 0, \quad (0,T) \times \mathcal{R}^d.$$

Since $W \geq 1 \geq w^\epsilon(0,x)$, by the maximum principle we conclude that $W \geq w^\epsilon$ on $(0,T) \times \mathcal{R}^d$. See the proof of Proposition 4.2 in [28, section 8] for the application of the maximum principle in a very similar situation.
7. Since

$$h_\epsilon(z) \leq (K_0 + 1)|z| + 1,$$

we have,

$$w^\epsilon \leq W = 1 + h_\epsilon(z^\epsilon) \leq 1 + (K_0 + 1)(|z^\epsilon| + 1).$$

∎

Our next step is a crude estimate of $|z_t^\epsilon|$. We will obtain a better estimate in Lemma 4.4.

Lemma 4.3 *For $0 < \epsilon \leq 1$ we have,*

$$|z_t^\epsilon(t,x)| \leq \frac{K}{\epsilon^2}, \quad (t,x) \in [0,T] \times \mathcal{R}^d. \tag{4.6}$$

Proof: **1.** For $\alpha > 0$, set

$$\Omega = \{(t,x) \in [0,T) \times \mathcal{R}^d \ : \ |\frac{z^\epsilon(t,x)}{\epsilon}| > \alpha\}.$$

By (3.6) we have,

$$|z_t^\epsilon| = \epsilon(q'(\frac{z^\epsilon}{\epsilon}))^{-1} |\varphi_t^\epsilon| \ \leq \ \frac{K}{\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^{-1}.$$

Hence on the complement of Ω,

$$|z_t^\epsilon| \ \leq \ \frac{K}{\epsilon q'(\alpha)}.$$

2. Set $v = z_t^\epsilon$. Differentiate (ZE) to obtain,

$$v_t - \Delta v + \frac{4\varphi^\epsilon}{\epsilon}\nabla z^\epsilon \cdot \nabla v + \frac{2}{\epsilon^2}q'(\frac{z^\epsilon}{\epsilon})(|\nabla z^\epsilon|^2 - 1)v = \theta_t^\epsilon. \tag{4.7}$$

3. For $K_1 > 0$, let

$$V = \frac{K_1}{\epsilon}(1 + \frac{|z^\epsilon|}{\epsilon}).$$

We will show that for appropriately chosen K_1 and α, V is a supersolution of (4.7) in Ω. Indeed in Ω,

$$\begin{aligned}
I &= V_t - \Delta V + \frac{4\varphi^\epsilon}{\epsilon}\nabla z^\epsilon \cdot \nabla V + \frac{2}{\epsilon^2}q'(\frac{z^\epsilon}{\epsilon})(|\nabla z^\epsilon|^2 - 1)V \\
&\geq \frac{K_1 z^\epsilon}{\epsilon^2|z^\epsilon|}[z_t^\epsilon - \Delta z^\epsilon + \frac{4\varphi^\epsilon}{\epsilon}|\nabla z^\epsilon|^2] - \frac{2K_1}{\epsilon^3}q'(\frac{z^\epsilon}{\epsilon})[\frac{|z^\epsilon|}{\epsilon} + 1] \\
&\geq \frac{K_1}{\epsilon^3}\{2\,|\varphi^\epsilon|\,(1 + |\nabla z^\epsilon|^2) - \epsilon\|\theta^\epsilon\|_{\infty,T} - 2\sup_{r\geq\alpha} q'(r)(r+1)\} \\
&\geq \frac{K_1}{\epsilon^3}\{2q(\alpha) - \epsilon\|\theta^\epsilon\|_{\infty,T} - 2\sup_{r\geq\alpha} q'(r)(r+1)\}.
\end{aligned}$$

Since $q'(r)$ is exponentially small for large values of r, (3.3), (3.9) imply that there are K_1 and α such that

$$I \geq \|\theta_t^\epsilon\|_{\infty,T} \quad \text{in } \Omega,$$

for all sufficiently small ϵ. By redefining K_1, if necessary, we may assume that

$$\inf_{\partial\Omega} V = \frac{K_1}{\epsilon}(1+\alpha) \geq \frac{K}{\epsilon q'(\alpha)} = \sup_{\Omega^c} |z_t^\epsilon|.$$

4. We proved that there is $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, V is a supersolution of (4.7) in Ω. Moreover $V \geq v$ on $\partial\Omega$. Therefore by the maximum principle,

$$V \geq v, \quad \text{in } \Omega,\ \epsilon \leq \epsilon_0.$$

Hence by step 1

$$v = z_t^\epsilon \leq \frac{K}{\epsilon}(1 + \frac{|z^\epsilon|}{\epsilon})$$

for all $0 < \epsilon \leq \epsilon_0$. And for $\epsilon_0 \leq \epsilon < 1$, the above estimate is easy to prove. Above arguments also yields the same bound for $-z_t^\epsilon$.

5. Set

$$\hat{\Omega} = \{|z^\epsilon| \geq 1\}, \quad \hat{V} = \frac{K}{\epsilon^2}(1+t).$$

Then on $[0,T] \times \mathcal{R}^d \cap \hat{\Omega}$,

$$\begin{aligned}
&\hat{V}_t - \Delta\hat{V} + \frac{4\varphi^\epsilon}{\epsilon}\nabla z^\epsilon \cdot \nabla\hat{V} + \frac{2}{\epsilon^2}q'(\frac{z^\epsilon}{\epsilon})(|\nabla z^\epsilon|^2 - 1)\hat{V} \\
&\geq \frac{K}{\epsilon^2} - \frac{K(1+T)}{\epsilon^4}q'(\frac{1}{\epsilon}) \\
&\geq \|\theta_t^\epsilon\|_{\infty,T},
\end{aligned}$$

for sufficiently small ϵ. Also by step 4, $\hat{V} \geq |v|$ on $\partial\hat{\Omega}$. Hence (4.6) follows from the maximum principle. ∎

Next we will improve (4.6).

Lemma 4.4 *There is $K = K(T)$ satisfying*

$$|z_t^\epsilon(t,x)| + |D^2 z^\epsilon(t,x)| \leq \frac{K}{\epsilon}(1 + |z^\epsilon(t,x)|),\ (t,x) \in [0,T] \times \mathcal{R}^d. \tag{4.8}$$

Proof: Fix $T > 0$. All the constants in this proof depend on T. Set

$$k^\epsilon = \sup\left\{\frac{\epsilon|z^\epsilon_{x_i x_j}(t,x)|}{1+|z^\epsilon(t,x)|} \;:\; (t,x) \in [0,T]\times\mathcal{R}^d,\ i,j = 1,\ldots,d\right\}.$$

1. In view of (4.3) we have

$$(|z^\epsilon(t,y)|+1) \le e^{K|x-y|}\,(|z^\epsilon(t,x)|+1).$$

Also (4.6) implies that there is K^* satisfying

$$\begin{aligned}(|z^\epsilon(t-\tau,y)|+1) &\le (1+\frac{K^*\tau}{\epsilon^2})\,(|z^\epsilon(t,y)|+1)\\ &\le (1+\frac{K^*\tau}{\epsilon^2})e^{K^*|x-y|}\,(|z^\epsilon(t,x)|+1). \qquad (4.9)\end{aligned}$$

2. Fix $(t_0,x_0) \in [0,T]\times\mathcal{R}^d$. For any $h \in (0, t_0 \wedge 1]$, (ZE) yields,

$$z^\epsilon_{x_i x_j}(t_0,x_0) = a + b + c\,,$$

where

$$\begin{aligned}a &= (G_{x_i}(h,\cdot) * z^\epsilon_{x_j}(t_0-h,\cdot)(x_0)\\ b &= \int_0^h (G_{x_i}(\tau,\cdot) * \theta^\epsilon_{x_j}(t_0-\tau,\cdot))(x_0)d\tau,\\ c &= \int_0^h (G_{x_i}(\tau,\cdot) * F^\epsilon_{x_j}(t_0-\tau,\cdot))(x_0)d\tau,\\ F^\epsilon &= \frac{2\varphi^\epsilon}{\epsilon}\left(1-|\nabla z^\epsilon|^2\right).\end{aligned}$$

3. If $h = t_0$, by (A4) we have

$$|a| \le \|G(h,\cdot)\|_1\|D^2 z^\epsilon_0\|_\infty \le \frac{K}{\epsilon}.$$

When $h < t_0$, (4.3) and (4.9) imply that

$$\begin{aligned}|a| &\le \int_{\mathcal{R}^d} K\,|\nabla G(h,x_0-y)|\,[1+|z^\epsilon(t_0-h,y)|]^{1/2}\,dy.\\ &\le K\left(1+\frac{K^*h}{\epsilon^2}\right)^{1/2}[1+|z^\epsilon(t_0,x_0)|]^{1/2}\int_{\mathcal{R}^d} e^{\frac{1}{2}K^*|w|}\,|\nabla G(h,w)|\,dw\\ &\le \frac{K}{\sqrt{h}}\left(1+\frac{K^*h}{\epsilon^2}\right)^{1/2}(1+|z^\epsilon(t_0,x_0)|)^{1/2}.\end{aligned}$$

4. By (3.6) we have

$$|b| \leq \frac{K}{\epsilon}\sqrt{h}.$$

5. Differentiate F^ϵ to obtain,

$$\nabla F^\epsilon = \frac{2}{\epsilon^2}q'(\frac{z^\epsilon}{\epsilon})\nabla z^\epsilon(1-|\nabla z^\epsilon|^2) - \frac{4\varphi^\epsilon}{\epsilon}D^2 z^\epsilon \nabla z^\epsilon.$$

Definition of k^ϵ, (4.3) and (4.9) yield,

$$\begin{aligned} |\, F^\epsilon_{x_j}(t_0-\tau,y)\,| &\leq \frac{K}{\epsilon^2}\left[\sup_r q'(r)(1+\epsilon r)^{3/2} + (1+|z^\epsilon(t_0-\tau,y)|)^{3/2}\, k^\epsilon\right] \\ &\leq \frac{K}{\epsilon^2}\left[1+k^\epsilon(1+\frac{K^*\tau}{\epsilon^2})^{3/2}\,(1+|z^\epsilon(t_0,x_0)|)^{3/2}\exp(\frac{3}{2}K^*|x_0-y|)\right]. \end{aligned}$$

Therefore

$$|c| \leq C^*\frac{\sqrt{h}}{\epsilon^2}[1+k^\epsilon(1+\frac{K^*h}{\epsilon^2})^{3/2}\,(1+|z^\epsilon(t_0,x_0)|)^{3/2}],$$

for some C^*. Without loss of generality assume that $C^* \geq 1$.

6. Choose

$$h = \min\{t_0, \epsilon^2\left[4(1+K^*)^3(1+|z^\epsilon(t_0,x_0)|)(C^*)^2\right]^{-1}\}.$$

Since $h \leq \epsilon^2$, we have $(1+\frac{K^*h}{\epsilon^2}) \leq (1+K^*)$ and therefore

$$|c| \;\leq\; \frac{1}{2\epsilon}(1+\,|\,z^\epsilon(t_0,x_0)\,|)k^\epsilon + \frac{C^*}{\epsilon}\ ,$$

and by step 3,

$$|a| \leq \frac{K}{\epsilon}(1+K^*)^2C^*(1+|z^\epsilon(t_0,x_0)|).$$

Therefore,

$$|\,z^\epsilon_{x_ix_j}(t_0,x_0)\,| \leq \frac{1}{\epsilon}(1+\,|\,z^\epsilon(t_0,x_0)\,|)[K+\frac{1}{2}k^\epsilon].$$

Now (4.8) follows from the above estimate and (ZE). ∎

We continue by improving the $|\nabla\theta^\epsilon|$ estimate.

Lemma 4.5 *For every $(t,x) \in [0,T]\times\mathcal{R}^d$, we have,*

$$|\nabla\theta^\epsilon(t,x)| \leq \frac{K}{\sqrt{\epsilon[\epsilon+(|z^\epsilon(t,x)|\wedge 1)]}}, \tag{4.10}$$

for some constant $K = K(T)$.

Proof: Fix $(t_0, x_0) \in [0,T] \times \mathcal{R}^d$ and set

$$p^\epsilon = \frac{|z^\epsilon(t_0,x_0)| \wedge 1}{\epsilon}.$$

If $p^\epsilon \leq 1$, (4.10) at (t_0, x_0) follows from (3.6). So we may assume that $p^\epsilon \geq 1$.

1. For $\epsilon, \lambda > 0$, set,

$$O_{\epsilon,\lambda} = [t_0 - \epsilon^2 \lambda p^\epsilon, t_0 + \epsilon^2 \lambda p^\epsilon] \times B^\epsilon \ , \ \ B^\epsilon = \{|x - x_0| \leq \epsilon \lambda p^\epsilon\}.$$

We claim that there exists $\lambda = \lambda(T) > 0$ satisfying,

$$|z^\epsilon(t,x)| \geq \frac{\epsilon}{2} p^\epsilon \ , \ \ \forall (t,x) \in O_{\epsilon,\lambda}.$$

Use (4.3) and (4.8) to construct a constant $K = K(T)$ satisfying,

$$|z^\epsilon(s,y)| + 1 \leq (|z^\epsilon(t,x)| + 1) \exp K(\frac{|t-s|}{\epsilon} + |x-y|),$$

for all $s, t \leq T$. Now suppose that for some $(\tau, y) \in \mathcal{R}^{d+1}$, we have

$$z^\epsilon(t_0 + \epsilon^2 \tau, x_0 + \epsilon y) = \frac{\epsilon}{2} p^\epsilon.$$

We use the previous estimate with $(s,y) = (t_0, x_0)$ and $(t,x) = (t_0 + \epsilon^2 \tau, x_0 + \epsilon y)$ to obtain,

$$1 + \epsilon p^\epsilon \leq 1 + |z^\epsilon(t_0,x_0)| \leq (1 + \frac{\epsilon}{2} p^\epsilon)\, e^{\epsilon K(|\tau| + |y|)}.$$

Since $\epsilon p^\epsilon \leq 1$,

$$\epsilon K(|\tau| + |y|) \geq \ln\left(\frac{1 + \epsilon p^\epsilon}{1 + \frac{\epsilon}{2} p^\epsilon}\right) \geq \ln\left(1 + \frac{\epsilon}{4} p^\epsilon\right) \geq \frac{\epsilon}{8} p^\epsilon.$$

Hence for $\lambda = 1/8K$,

$$|z^\epsilon(t,x)| \geq \frac{\epsilon}{2} p^\epsilon \ , \ \ \forall (t,x) \in O_{\epsilon,\lambda}.$$

2. As in the proof of Lemma 3.2, for any $0 < \sigma \leq t_0$,

$$\nabla \theta^\epsilon(t_0, x_0) = a + b + c \, ,$$

where

$$\begin{aligned}
a &= (\nabla G(t_0, \cdot) * [\theta_0^\epsilon + H^\epsilon(0, \cdot) - H^\epsilon(t_0, \cdot)])(x_0) \\
b &= \int_\sigma^{t_0} (\nabla G_\tau(\tau, \cdot) * [H^\epsilon(t_0, \cdot) - H^\epsilon(t_0 - \tau, \cdot)])(x_0) d\tau, \\
c &= \int_0^\sigma (\nabla G_\tau(\tau, \cdot) * [H^\epsilon(t_0, \cdot) - H^\epsilon(t_0 - \tau, \cdot)])(x_0) d\tau.
\end{aligned}$$

We choose

$$\sigma = \min\{t_0, \epsilon^2 \lambda p^\epsilon\}.$$

3. Assumption (A6) implies that

$$|(\nabla G(t_0,\cdot) * \theta_0^\epsilon)(x_0)| = |(G(t_0,\cdot) * \nabla\theta_0^\epsilon)(x_0)| \leq \frac{K}{\sqrt{\epsilon}}.$$

Also when $t_0 \geq \epsilon^2 \lambda p^\epsilon$ we have

$$|(\nabla G(t_0,\cdot) * [H^\epsilon(0,\cdot) - H^\epsilon(t_0,\cdot)])(x_0)| \leq \frac{K}{\sqrt{t_0}} \leq \frac{K}{\epsilon\sqrt{\lambda p^\epsilon}}.$$

However if $t_0 \leq \epsilon^2 \lambda p^\epsilon$,

$$\{0\} \times B^\epsilon \subset O_{\epsilon,\lambda}$$

and by (3.2) and step 1 we have

$$\begin{aligned} |\nabla H^\epsilon(0,y)| &= |g(\varphi^\epsilon(0,y))\nabla\varphi^\epsilon(0,y)| \leq \frac{K}{\epsilon} q'(\frac{z^\epsilon(0,y)}{\epsilon}) \\ &\leq \frac{K}{\epsilon} q'(\frac{p^\epsilon}{2}), \quad \forall y \in B^\epsilon. \end{aligned}$$

Hence,

$$\begin{aligned} &|\nabla G(t_0,\cdot) * H^\epsilon(0,\cdot))(x_0)| = |(G(t_0,\cdot) * \nabla H^\epsilon(0,\cdot))(x_0)| \\ &\leq \int_{B^\epsilon} G(t_0, x_0 - y) q'(\frac{p^\epsilon}{\epsilon})\frac{K}{\epsilon} dy + \int_{\mathcal{R}^d - B^\epsilon} G(t_0, x_0 - y)\frac{K}{\epsilon} dy \\ &\leq \frac{K}{\epsilon}[q'(\frac{p^\epsilon}{\epsilon}) + \int G(1,w)\chi_{\{\sqrt{4t_0}|w| \geq \epsilon\lambda p^\epsilon\}}\, dw] \\ &\leq \frac{K}{\epsilon}[q'(\frac{p^\epsilon}{\epsilon}) + G(1, \frac{\epsilon\lambda p^\epsilon}{\sqrt{4t_0}})]. \end{aligned}$$

Since $t_0 \leq \epsilon^2 \lambda p^\epsilon$ and $p^\epsilon \geq 1$, we have,

$$|(\nabla G(t_0,\cdot) * H^\epsilon(0,\cdot))(x_0)| \leq \frac{K}{\epsilon\sqrt{p^\epsilon}}.$$

Indeed, we can estimate the above quantity by a function decaying faster than the square root, but this sharper estimate will not improve the final estimate.

Next, we estimate $|(\nabla G(t_0,\cdot)) * H^\epsilon(t_0,\cdot))(x_0)|$ exactly the same way to obtain,

$$|a| \leq \frac{K}{\epsilon\sqrt{p^\epsilon}}.$$

4. Since $\|\nabla G_\tau(\tau,\cdot)\|_1 \leq K\tau^{-3/2}$,

$$|b| \leq K\left(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sqrt{t_0}}\right) \leq \frac{K}{\epsilon\sqrt{p^\epsilon}}\,.$$

5. By integration by parts in the t-variable, we obtain,

$$\begin{aligned} |c| \;\; &\leq \;\; \left|\int_0^\sigma (\nabla G(\tau,\cdot) * g(\varphi^\epsilon(t_0-\tau,\cdot))\varphi_t^\epsilon(t_0-\tau,\cdot))(x_0)d\tau\right| \\ &\quad + |(\nabla G(\sigma,\cdot) * [H^\epsilon(t_0,\cdot) - H^\epsilon(t_0-\sigma,\cdot)])(x_0))| \end{aligned}$$

Since $\sigma \leq \epsilon^2\lambda p^\epsilon$,

$$[t_0-\sigma, t_0] \times B^\epsilon \subset O_{\epsilon,\lambda}.$$

Therefore for any $y \in B^\epsilon, \tau \in [0,\sigma]$ by (3.6) we have,

$$|g^\epsilon(\varphi^\epsilon(t_0-\tau,y))\varphi_t^\epsilon(t_0-\tau,y)| \leq |\varphi_t^\epsilon| q'(\frac{p^\epsilon}{2}) \leq \frac{K}{\epsilon^2} q'(\frac{p^\epsilon}{2}).$$

Proceed as in step 3 to obtain,

$$\begin{aligned} &\left|\int_0^\sigma (\nabla G(\tau,\cdot) * g(\varphi^\epsilon(t_0-\tau,\cdot))\varphi_t^\epsilon(t_0-\tau,\cdot))(x_0)d\tau\right| \\ &\quad \leq \int_0^\sigma \int_{B^\epsilon} |\nabla G(\tau, x_0-y)| \frac{K}{\epsilon^2} q'(\frac{p^\epsilon}{2}) d\tau + \int_0^\sigma \int_{\mathcal{R}^d - B^\epsilon} |\nabla G(\tau, x_0-y)| \frac{K}{\epsilon^2} d\tau \\ &\quad \leq \frac{K\sqrt{\sigma}}{\epsilon^2}\left[q'(\frac{p^\epsilon}{2}) + |\nabla G(1, \frac{\epsilon\lambda p^\epsilon}{\sqrt{4\sigma}})|\right] \\ &\quad \leq \frac{K}{\epsilon\sqrt{p^\epsilon}}. \end{aligned}$$

Also if $\sigma = \epsilon^2\lambda p^\epsilon$,

$$|(\nabla G(\sigma,\cdot) * [H^\epsilon(t_0,\cdot) - H^\epsilon(t_0-\sigma,\cdot)])\,(x_0)| \leq K\|\nabla G(\sigma,\cdot)\|_1 \leq \frac{K}{\sqrt{\sigma}} \leq \frac{K}{\epsilon\sqrt{p^\epsilon}}.$$

Finally if $\sigma = t_0$, then by step 3 we have,

$$\begin{aligned} &|(\nabla G(\sigma,\cdot) * [H^\epsilon(t_0,\cdot) - H^\epsilon(t_0-\sigma,\cdot)])\,(x_0)| \\ &\qquad = |(G(t_0,\cdot) * [\nabla H^\epsilon(t_0,\cdot) - \nabla H^\epsilon(0,\cdot)])\,(x_0)| \leq \frac{K}{\epsilon\sqrt{p^\epsilon}}\,. \end{aligned}$$

6. Combine steps 3, 4 and 5 to conclude that

$$|\nabla\theta^\epsilon(t_0,x_0)| \leq \frac{K}{\epsilon\sqrt{p^\epsilon}}\,.$$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1 This proof is very similar to the proof of Proposition 4.2.

1. Let z_0 be as in Proposition 4.3,i.e.,

$$q'(\frac{z_0}{\epsilon}) = \epsilon^{1/4}.$$

Set

$$K_\epsilon = (z_0)^{-3/2}.$$

Since $q'(r)$ decays exponentially, we have

$$\lim_{\epsilon\to 0} \frac{z_0}{\epsilon\,|\ln\epsilon|} = \frac{1}{8}, \quad \lim_{\epsilon\to 0} (\epsilon\ |\ln\epsilon|)^{3/2}\, K_\epsilon < \infty. \tag{4.11}$$

2. For a real number r, set

$$f_\epsilon(r) = \begin{cases} \frac{1}{2}K_\epsilon r^2 + 1, & |r| \le z_0, \\ 2[\sqrt{r} - \sqrt{z_0}] + f_\epsilon(z_0), & |r| \in [z_0, 1], \\ |r| - 1 + f_\epsilon(1), & |r| \ge 1. \end{cases}$$

Observe that f_ϵ is continuously differentiable with Lipschitz continuous derivatives.

3. For $K^* > 1$ define,

$$W = 1 + \sqrt{\epsilon}K^* f_\epsilon(z^\epsilon).$$

In the next three steps, we will show that for K^* large enough, W is a "super-solution" of (4.4).

Let $\hat{\mathcal{L}}, R^\epsilon$ and w^ϵ be as in Proposition 4.2. Then,

$$\begin{aligned} I &= W_t + \hat{\mathcal{L}}W + R^\epsilon(t,x,W) \\ &= \sqrt{\epsilon}K^* f'_\epsilon [z^\epsilon_t - \Delta z^\epsilon + \frac{4\varphi^\epsilon}{\epsilon} w^\epsilon] - \sqrt{\epsilon}K^* f''_\epsilon w^\epsilon \\ &\quad + \frac{4}{\epsilon^2} q'\,(\frac{z^\epsilon}{\epsilon})\sqrt{\epsilon}K^* f_\epsilon W \ - 2\,|\nabla\theta^\epsilon(t,x)|\, W \\ &\ge \sqrt{\epsilon}K^* \left\{ f'_\epsilon [\frac{2\varphi^\epsilon}{\epsilon}(w^\epsilon + 1) + \theta^\epsilon] - f''_\epsilon w^\epsilon + \frac{4}{\epsilon^2} q'(\frac{z^\epsilon}{\epsilon}) W \right\} - 2\,|\nabla\theta^\epsilon|\, W. \end{aligned}$$

4. We split the estimate of I into three cases:

$$(a)\quad |z^\epsilon| \le z_0\ ,\quad (b)\quad |z^\epsilon| \in [z_0, 1]\ ,\quad (c)\, |z^\epsilon| \ge 1.$$

We start with case a. Since $z_0 = \epsilon(q')^{-1}(\epsilon^{1/4})$,

$$q'(\frac{z^\epsilon}{\epsilon}) \ge q'(\frac{z_0}{\epsilon}) = \epsilon^{1/4}.$$

By (4.11) we also have

$$\sqrt{\epsilon}\,|f_\epsilon'(z^\epsilon)| = \sqrt{\epsilon}K_\epsilon\,|z^\epsilon| \le \sqrt{\epsilon}z_0 K_\epsilon \le \frac{K}{\sqrt{|\ln\epsilon|}}\ ,$$

for some constant K. Since $f_\epsilon'\varphi^\epsilon \ge 0$,

$$I\ \ge\ K^*[-\frac{K}{\sqrt{|\ln\epsilon|}}\|\theta^\epsilon\|_{\infty,T} - \sqrt{\epsilon}K_\epsilon w^\epsilon + 4\epsilon^{-5/4}W] - 2\|\nabla\theta^\epsilon\|_{\infty,T}W.$$

Using (3.3), (3.6), (4.11), and the fact that $W \ge 1$, we construct $\epsilon_0 = \epsilon_0(T) > 0$ such that,

$$I \ge \sqrt{\epsilon}K^*K_\epsilon(W - w^\epsilon),\quad \epsilon \le \epsilon_0,\ t \le T,$$

for any $K^* \ge 1$.

5. Suppose that $|z^\epsilon| \ge 1$. Then for sufficiently small ϵ,

$$f_\epsilon'\varphi^\epsilon = |\varphi^\epsilon| \ge \frac{1}{2}.$$

Moreover, $f_\epsilon''(z^\epsilon) = 0$ and by (4.10),

$$|\nabla\theta^\epsilon(t,x)| \le \frac{\hat{K}}{\sqrt{\epsilon}}.$$

Since $q' \ge 0$, we have:

$$I \ge \frac{K^*}{\sqrt{\epsilon}}(w^\epsilon + 1) - \sqrt{\epsilon}K^*\|\theta^\epsilon\|_{\infty,T} - \frac{2\hat{K}}{\sqrt{\epsilon}}W.$$

So if $K^* \ge 2\hat{K}$, (3.3) implies that $I \ge 0$ on $\{w^\epsilon \ge W\}$ for all sufficiently small ϵ.

6. Finally we consider the case $|z^\epsilon| \in [z_0, 1]$. In this case, for sufficiently small $\epsilon > 0$, we have,

$$f_\epsilon'(z^\epsilon)\varphi^\epsilon = f_\epsilon'(|z^\epsilon|)\,|\varphi^\epsilon| \ge \frac{1}{2}f_\epsilon'(|z^\epsilon|) = \frac{1}{2\sqrt{|z^\epsilon|}}.$$

Moreover by the construction of f_ϵ,

$$\sqrt{\epsilon}f_\epsilon'(|z^\epsilon|) \le 1,\quad f_\epsilon''(z^\epsilon) \le 0,\ \Rightarrow -f_\epsilon''w^\epsilon \ge 0.$$

Since $\epsilon \leq z_0 \leq |z^\epsilon| \leq 1$, by (4.10) we have,

$$|\nabla \theta^\epsilon(t,x)| \leq \frac{\hat{K}}{\sqrt{\epsilon |z^\epsilon|}}.$$

We now have,

$$I \geq \frac{K^*}{\sqrt{\epsilon|z^\epsilon|}}(w^\epsilon + 1) - K^*\|\theta^\epsilon\|_{\infty,T} - \frac{2\hat{K}}{\sqrt{\epsilon\,|z^\epsilon|}}W$$

Hence on $\{w^\epsilon \geq W\}$, $I \geq 0$ provided that $K^* \geq 2\hat{K}$ and ϵ is sufficiently small.
7. Combining steps 3, 4 and 5 we conclude that for $K^* \geq 2\hat{K}$,

$$I \geq \sqrt{\epsilon} K^* K_\epsilon (W - w^\epsilon) \;\; \text{on } \{w^\epsilon \geq W\}$$

for $\epsilon \in (0, \epsilon_0]$. Since by (A3) $W(0,x) \geq 1 \geq |\nabla z_0^\epsilon|^2$, the maximum principle implies that $W \geq w^\epsilon$ for $\epsilon \leq \epsilon_0$, (see [28, Section 8] for the details of this applications.) Also

$$f_\epsilon(r) \leq |r| + f_\epsilon(1) \leq |r| + 4.$$

This proves (4.1) for all $\epsilon \leq \epsilon_0$. For $\epsilon \geq \epsilon_0$, (4.1) follows from (4.3). ∎

The following lemma will be useful in the next section.

Lemma 4.6 *Suppose that there is a bounded open set $O \subset (0,\infty) \times \mathcal{R}^d$ satisfying,*

$$\beta = \liminf_{\epsilon \to 0} \inf_{(s,y) \in \bar{O}} |\varphi^\epsilon(s,y)| > 0.$$

Then for every $(s,y) \in O$,

$$\liminf_{(s',y') \to (s,y)\; \epsilon \to 0} |z^\epsilon(s',y')| \geq \inf\{|\hat{y} - y| : (s,\hat{y}) \notin O\}$$

Proof:
1. Since $\bar{O}$ is compact, there is $\epsilon_0 > 0$ satisfying

$$|\varphi^\epsilon(s,y)| \geq \beta/2\,, \quad \forall (s,y) \in \bar{O}\,, \quad \epsilon \leq \epsilon_0\,.$$

Since φ^ϵ is continuous and $h(\varphi^\epsilon)$ is convergent in L^1_{loc} we have either

$$\varphi^\epsilon(s,y) \geq \beta/2\,, \quad \forall (s,y) \in \bar{O},\ \epsilon \leq \epsilon_0, \tag{4.12}$$

or

$$\varphi^\epsilon(s,y) \leq -\beta/2\,, \quad \forall (s,y) \in \bar{O},\ \epsilon \leq \epsilon_0, \tag{4.13}$$

2. Multiply (ZE) (of Section 2.1) by ϵ to obtain,

$$2\varphi^\epsilon(|\nabla z^\epsilon|^2 - 1) = \epsilon(-z_t^\epsilon + \Delta z^\epsilon) + \epsilon\theta^\epsilon.$$

In view of (3.3),

$$\lim_{\epsilon\to 0} \epsilon\|\theta^\epsilon\|_{\infty,T} = 0.$$

Set

$$z^*(t,x) = \limsup_{\epsilon\to 0,(s,y)\to(t,x)} z^\epsilon(s,y),$$

$$z_*(t,x) = \liminf_{\epsilon\to 0,(s,y)\to(t,x)} z^\epsilon(s,y).$$

Now, pass to the limit in the above equation to obtain,

$$|Dz_*| - 1 \geq 0 \quad \text{in } O \quad \text{if (4.12) holds,} \tag{4.14}$$

$$|Dz^*| - 1 \leq 0 \quad \text{in } O \quad \text{if (4.13) holds,}$$

The above inequalities are to be understood in the viscosity sense [13]. The details of this limit argument is given in [28, Lemma 4.1].

For $(s,y) \in \bar{O}$ set

$$d(s,y) = \inf\{|y'-y| \ : \ (s,y') \notin O\} \ \text{ if (4.12) holds,}$$

$$d(s,y) = -\inf\{|y'-y| \ : \ (s,y') \in O\} \ \text{ if (4.13) holds.}$$

Then $d(s,y)$ solves (4.14) in the viscosity sense and by the comparison results for the Eikonal equation [13] we obtain

$$z_*(s,y) \geq d(s,y) \quad \text{if (4.12) holds,}$$

$$z^*(s,y) \leq d(s,y) \quad \text{if (4.13) holds.}$$

∎

5 Monotonicity formula

In this section we obtain an extension of the monotonicity formula of Ilmanen [22]. Ilmanen proved his formula for solutions of the Allen-Cahn equation (i.e. (OPE) without the θ^ϵ term). Ilmanen's formula itself is an extension of the Huisken's formula for smooth manifolds moving by their mean curvature [21].

For $x, x_0 \in \mathcal{R}^d$, $0 \leq t < t_0$, let

$$\begin{aligned} \rho(t,x;t_0,x_0) &= (4\pi(t_0-t))^{1/2}\, G(t_0-\tau, x_0-x) \\ &= (4\pi(t_0-t))^{-\frac{(d-1)}{2}} \exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right). \end{aligned}$$

Then,

$$\nabla_x \rho = -\frac{(x-x_0)}{2(t_0-t)}\rho\,,$$

$$\rho_t = \left[\frac{d-1}{2(t_0-t)} - \frac{|x-x_0|^2}{4(t_0-t)^2}\right]\rho,$$

$$D_x^2\rho = \left[-\frac{1}{2(t_0-t)}I + \frac{(x-x_0)\otimes(x-x_0)}{4(t_0-t)^2}\right]\rho,$$

where I is the identity matrix and $\otimes$ is the tensor product. For $t \geq 0$ and a Borel set $A \subset \mathcal{R}^d$, let $\mu^\epsilon(t;A), \hat{\mu}^\epsilon(t;A)$ be as in Section 2.2 and define,

$$\xi^\epsilon(t;A) = \int_A \left(\frac{\epsilon}{2}|\nabla\varphi^\epsilon|^2 - \frac{1}{\epsilon}W(\varphi^\epsilon)\right)dx,$$

$$\alpha^\epsilon(t;t_0,x_0) = \int_{\mathcal{R}^d} \rho(t,x;t_0,x_0)\hat{\mu}^\epsilon(t;dx).$$

Theorem 5.1 *There is a constant C_d, depending only on the dimension, such that*

$$\frac{d}{dt}\alpha^\epsilon(t;t_0,x_0) \leq \frac{1}{2(t_0-t)}\int_{\mathcal{R}^d}\rho(t,x;t_0,x_0)\xi^\epsilon(t;dx) + \frac{C_d}{\sqrt{t_0-t}}. \tag{5.1}$$

Proof: Fix (t_0,x_0). We suppress the dependence on (t_0,x_0) in our notation.
1. Using (OPE) and (HE) we calculate,

$$\begin{aligned}
\frac{d}{dt}\alpha^\epsilon(t) &= \int \rho_t\hat{\mu}^\epsilon + \int \rho\left(\epsilon\nabla\varphi^\epsilon\cdot\nabla\varphi_t^\epsilon + \frac{1}{\epsilon}W'(\varphi^\epsilon)\varphi_t^\epsilon + \theta^\epsilon\theta_t^\epsilon\right)\\
&= \int \rho_t\hat{\mu}^\epsilon - \int \epsilon\nabla\rho\cdot\nabla\varphi^\epsilon\,\varphi_t^\epsilon\\
&\quad + \int\left[\epsilon\varphi_t^\epsilon\left(-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon)\right) + \theta^\epsilon\left(\Delta\theta^\epsilon - g(\varphi^\epsilon)\varphi_t^\epsilon\right)\right]\rho\\
&= \int \rho_t\hat{\mu}^\epsilon - \int \epsilon\nabla\rho\cdot\nabla\varphi^\epsilon\varphi_t^\epsilon - \int \epsilon(\varphi_t^\epsilon)^2\rho + \int \theta^\epsilon\Delta\theta^\epsilon\rho\\
&= \int \rho_t\hat{\mu}^\epsilon + \epsilon\int\left[\nabla\rho\cdot\nabla\varphi^\epsilon\varphi_t^\epsilon + \frac{(\nabla\rho\cdot\nabla\varphi^\epsilon)^2}{\rho}\right]\\
&\quad -\epsilon\int\left(\varphi_t^\epsilon + \frac{\nabla\rho\cdot\nabla\varphi^\epsilon}{\rho}\right)^2\rho - \int|\nabla\theta^\epsilon|^2\rho + \int\Delta\rho\frac{1}{2}(\theta^\epsilon)^2.
\end{aligned}$$

Since

$$\rho_t + \Delta\rho = -\frac{1}{2(t_0 - t)}\rho ,$$

$$\begin{aligned}\frac{d}{dt}\alpha^\epsilon(t) &= -\epsilon\int\left(\varphi_t^\epsilon + \frac{\nabla\rho\cdot\nabla\varphi^\epsilon}{\rho}\right)^2\rho \\ &\quad + \int\left[\rho_t\mu^\epsilon + \nabla\rho\cdot\nabla\varphi^\epsilon\left(\epsilon\Delta\varphi^\epsilon - \frac{1}{\epsilon}W'(\varphi^\epsilon)\right) + \epsilon\frac{(\nabla\rho\cdot\nabla\varphi^\epsilon)^2}{\rho}\right] \\ &\quad - \int\rho\left[|\nabla\theta^\epsilon|^2 + \frac{1}{4(t_0-t)}(\theta^\epsilon)^2\right] + \int\nabla\rho\cdot\nabla\varphi^\epsilon g(\varphi^\epsilon)\theta^\epsilon.\end{aligned}$$

2. Let $\nu = \nu^\epsilon$ be as in Section 2.5, i.e., when $|\nabla\varphi^\epsilon| \neq 0$,

$$\nu = \frac{\nabla\varphi^\epsilon}{|\nabla\varphi^\epsilon|},$$

and set

$$T = \epsilon(\nu\otimes\nu - \frac{1}{2}I)\,|\nabla\varphi^\epsilon|^2 - \frac{1}{\epsilon}W(\varphi^\epsilon)I.$$

Then,

$$T = \epsilon\nabla\varphi^\epsilon\otimes\nabla\varphi^\epsilon - \left(\frac{\epsilon}{2}|\nabla\varphi^\epsilon|^2 + \frac{1}{\epsilon}W(\varphi^\epsilon)\right)I,$$

and

$$\sum_{i=1}^d \frac{\partial}{\partial x_i}T_{ij} = \varphi_{x_j}^\epsilon\ \ (\epsilon\Delta\varphi^\epsilon - \frac{1}{\epsilon}W'(\varphi^\epsilon)).$$

Since $\xi^\epsilon + \mu^\epsilon = \epsilon\,|\nabla\varphi^\epsilon|^2\,dx$, we have

$$T = (\nu\otimes\nu)\,\xi^\epsilon - (I - \nu\otimes\nu)\mu^\epsilon.$$

Let k be the second term appearing in the expression at the end of step 1, i.e.,

$$k = \int\left[\rho_t\mu^\epsilon + \nabla\rho\cdot\nabla\varphi^\epsilon(\epsilon\Delta\varphi^\epsilon - \frac{1}{\epsilon}W'(\varphi^\epsilon)) + \epsilon\frac{(\nabla\rho\cdot\nabla\varphi^\epsilon)^2}{\rho}\right].$$

Integration by parts and the identity $\xi^\epsilon + \mu^\epsilon = \epsilon\,|\nabla\varphi^\epsilon|^2\,dx$, yield

$$\begin{aligned}k &= \int\rho_t\mu^\epsilon - D^2\rho : T + \frac{(\nabla\rho\cdot\nu)^2}{\rho}(\xi^\epsilon + \mu^\epsilon) \\ &= \int\left[\rho_t + D^2\rho : (I - \nu\otimes\nu) + \frac{(\nabla\rho\cdot\nu)^2}{\rho}\right]\mu^\epsilon \\ &\quad + \int\left[\frac{(\nabla\rho\cdot\nu)^2}{\rho} - D^2\rho : \nu\otimes\nu\right]\xi^\epsilon ,\end{aligned}$$

where for two symmetric matrices M, N , $M : N =\text{trace}\, MN$. Explicit formulae for the derivatives of ρ imply that for any unit vector ν we have,

$$\rho_t + D^2\rho : (I - \nu\otimes\nu) + \frac{(\nabla\rho\cdot\nu)^2}{\rho} = 0,$$
$$\frac{(\nabla\rho\cdot\nu)^2}{\rho} - D^2\rho : \nu\otimes\nu = \frac{\rho}{2(t_0 - t)},$$

Hence

$$k = \int \frac{\rho}{2(t_0 - t)}\xi^\epsilon(t; dx).$$

3. Recall that $H^\epsilon = h(\varphi^\epsilon)$ and $\nabla H^\epsilon = \nabla\varphi^\epsilon g(\varphi^\epsilon)$. By integration by parts we obtain

$$\int \nabla\rho\cdot\nabla\varphi^\epsilon g(\varphi^\epsilon)\theta^\epsilon = -\int [\Delta\rho H^\epsilon\theta^\epsilon + H^\epsilon\nabla\rho\cdot\nabla\theta^\epsilon]$$

By steps 1 and 2 we have,

$$\frac{d}{dt}\alpha^\epsilon(t) \le \frac{1}{2(t_0 - t)}\int \rho\xi^\epsilon(t; dx) + I + J\ ,$$

where

$$I = -\int \left[\rho\,|\nabla\theta^\epsilon|^2 + H^\epsilon\nabla\rho\cdot\nabla\theta^\epsilon\right]\ ,$$
$$J = -\int \left[\frac{\rho}{4(t_0 - t)}(\theta^\epsilon)^2 + \Delta\rho H^\epsilon\theta^\epsilon\right].$$

4 Since $|\varphi^\epsilon| < 1$, $|H^\epsilon| \le 2/3 < 1$. Hence

$$\begin{aligned} I &= -\int \rho\left|\nabla\theta^\epsilon + H^\epsilon\frac{\nabla\rho}{2\rho}\right|^2 + \frac{1}{4}\int \frac{|\nabla\rho|^2}{\rho}\,|H^\epsilon|^2 \\ &\le \frac{1}{4}\int \frac{|\nabla\rho|^2}{\rho}dx \\ &= \frac{1}{4}\int \frac{|x - x_0|^2}{4(t_0 - t)^2}\rho(t, x; t_0, x_0)dx \\ &= \frac{1}{2\sqrt{t_0 - t}}\int_{\mathcal{R}^d}(\pi)^{-\frac{(d-1)}{2}}\,|y|^2\,e^{-|y|^2}dy. \end{aligned}$$

5. To estimate J, we observe that

$$\begin{aligned} |\Delta\rho H^\epsilon\theta^\epsilon| &\leq \left[\frac{d-1}{2(t_0-t)}+\frac{|x-x_0|^2}{4(t_0-t)^2}\right]|H^\epsilon|\ |\theta^\epsilon|\rho \\ &= \frac{\rho}{2(t_0-t)}\left[d-1+\frac{|x-x_0|^2}{2(t_0-t)}\right]|H^\epsilon|\ |\theta^\epsilon|. \end{aligned}$$

Since

$$2|H^\epsilon|\ |\theta^\epsilon| \leq \left[d-1+\frac{|x_0-x|^2}{2(t_0-t)}\right]^{-1}(\theta^\epsilon)^2+\left[d-1+\frac{|x_0-x|^2}{2(t_0-t)}\right]|H^\epsilon|^2 \ ,$$

we have

$$|\Delta\rho H^\epsilon\theta^\epsilon| \leq \frac{(\theta^\epsilon)^2\rho}{4(t_0-t)}+C(t,x)\rho\,,$$

where

$$C(t,x)=\frac{1}{4(t_0-t)}\left[d-1+\frac{|x_0-x|^2}{2(t_0-t)}\right]^2.$$

Hence

$$\begin{aligned} J &\leq \int C(t,x)\rho dx \\ &= \frac{1}{2\sqrt{t_0-t}}\int_{\mathcal{R}^d}(\pi)^{-\frac{d-1}{2}}(d-1+2|y|^2)e^{-|y|^2}dy. \end{aligned}$$

6. Combining the previous steps, we obtain (5.1) with

$$C_d=\frac{1}{2}(\pi)^{\frac{1-d}{2}}\int_{\mathcal{R}^d}\left[|y|^2+(d-1+2|y|^2)^2\right]e^{-|y|^2}dy\,.$$

∎

The monotonicity formula together with the gradient estimate (4.1) yield the following,

Corollary 5.2 *For any $T>0$, there exists a constant $K=K(T)$ such that for any $x_0\in\mathcal{R}^d$, and $0\leq t\leq r<t_0\leq T$ we have,*

$$\alpha^\epsilon(r;t_0,x_0)\leq\alpha^\epsilon(t;t_0,x_0)\left(\frac{t_0-t}{t_0-r}\right)^{K\sqrt{\epsilon}}+K\int_t^r\left(\frac{t_0-\tau}{t_0-r}\right)^{K\sqrt{\epsilon}}\frac{d\tau}{\sqrt{t_0-\tau}}. \quad (5.2)$$

Moreover as ϵ tends to zero we obtain,

$$\alpha(r;t_0,x_0)\ \leq\ \alpha(t;t_0;x_0)+C_d\left[\sqrt{t_0-t}-\sqrt{t_0-r}\right]. \quad (5.3)$$

Proof: Since

$$\xi^\epsilon(t;dx) = \frac{1}{2\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2 \left(|\nabla z^\epsilon|^2 - 1\right),$$

(4.1) and (5.1) yield,

$$\frac{d}{dt}\alpha^\epsilon(t) \le \frac{1}{2(t_0-t)}\int \rho \frac{1}{2\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2 K^* \sqrt{\epsilon}(1+|z^\epsilon|) + \frac{C_d}{\sqrt{t_0-t}}.$$

Observe that

$$\frac{1}{2\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2 dx \le \frac{1}{2\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2[1+|\nabla z^\epsilon|^2]dx = \mu^\epsilon(t;dx)$$

and

$$(q'(\frac{z^\epsilon}{\epsilon}))^2\,|z^\epsilon|^2 \;\le\; \epsilon^2(\sup_{r\ge 0} q'(r)r)^2 \le 4\epsilon^2.$$

Hence

$$\begin{aligned}\frac{1}{2\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2\,(1+|z^\epsilon|)\,dx &\le \frac{1}{2\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2\,(\frac{3}{2}+\frac{1}{2}|z^\epsilon|^2)dx\\ &\le \frac{3}{2}\mu^\epsilon(t;dx) + \epsilon dx\,,\end{aligned}$$

and consequently

$$\begin{aligned}\frac{d}{dt}\alpha^\epsilon(t) &\le \frac{K\sqrt{\epsilon}}{(t_0-t)}\int \rho\,(\mu^\epsilon + \epsilon dx) + \frac{C_d}{\sqrt{t_0-t}}\\ &\le \frac{K\sqrt{\epsilon}}{(t_0-t)}\alpha^\epsilon(t) + \frac{K\epsilon\sqrt{\epsilon}}{t_0-t}\int \rho dx + \frac{C_d}{\sqrt{t_0-t}}\\ &\le \frac{K\sqrt{\epsilon}}{(t_0-t)}\alpha^\epsilon(t) + \frac{K}{\sqrt{t_0-t}}.\end{aligned}$$

Now an application of Gronwall's inequality yields (5.2). ∎

6 Clearing-out

In this section, we follow the proof of [28, Theorem 5.1] to prove an extension of the cleaning out lemma proved in [22, 28].

Theorem 6.1 *For every $T>0$, there are positive constants $\eta, t^* > 0$, depending on T, such that if*

$$\int \rho(t,x;t_0,x_0)\mu(t;dx) \le \eta \tag{6.1}$$

for some t, t_0, x_0 satisfying

$$(t_0 - t^*) \leq t < t_0 \leq T, \tag{6.2}$$

then there exists a neighborhood O of (t_0, x_0) such that

$$\liminf_{(s',y')\to(s,y)\ \epsilon\to 0} |z^\epsilon(s', y')| > 0, \quad \forall (s,y) \in O. \tag{6.3}$$

In particular,

$$(t_0, x_0) \notin \overline{\bigcup_{t\geq 0} \{t\} \times \text{spt}\mu(t,\cdot)}. \tag{6.4}$$

Proof: Suppose that (6.1), (6.2) hold for some η, t^* that will be chosen later in this proof.
1. Hölder's inequality yields,

$$\begin{aligned} \int \rho(t,x;t_0,x_0)(\theta^\epsilon(t,x))^2 dx &\leq \|\rho(t,\cdot;t_0,x_0)\|_{p'} \|(\theta^\epsilon(t,\cdot))^2\|_p \\ &= (4\pi(t_0-t))^{\frac{1}{2}} \|G(t_0-t,\cdot)\|_{p'} \|\theta^\epsilon(t,\cdot))^2\|_p, \end{aligned}$$

for any $1 \leq p \leq \infty$, where p' is the conjugate of p, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Since

$$\|G(\tau,\cdot)\|_{p'} \leq K(p)\tau^{-\frac{d}{2p}},$$

$$\int \rho(t,x;t_0,x_0)(\theta^\epsilon(t,x))^2 dx \leq \hat{K}(p)(t_0-t)^{\frac{1}{2}(1-\frac{d}{p})} \|(\theta^\epsilon(t,\cdot))^2\|_p.$$

Choose $p = d + \delta$, use (3.10) to obtain

$$\int \rho(t,x;t_0,x_0)(\theta^\epsilon(t,x))^2 dx \leq K^*(t_0-t)^\gamma, \ \ 0 < \epsilon \leq 1,$$

for some constants K^* and $\gamma > 0$.
2. Continuity of ρ and the convergence of μ^ϵ to μ, implies that there are a constant $\epsilon_0 > 0$ and a neighborhood U of (t_0, x_0) such that for all $\epsilon \leq \epsilon_0$ and $(s,y) \in U$,

$$t < s - \epsilon^2,$$

and

$$\int \rho(t,x;s,y)\mu^\epsilon(t;dx) \leq 2\eta.$$

Step 1 yields,

$$\begin{aligned} \alpha^\epsilon(t;s,y) &= \int \rho(t,x;s,y)[\mu^\epsilon(t;dx) + \frac{1}{2}(\theta^\epsilon(t,x))^2 dx] \\ &\leq 2\eta + \tfrac{1}{2}K^*(s-t)^\gamma, \quad (s,y) \in U, \ \ \epsilon \leq \epsilon_0. \end{aligned}$$

Here $\hat{\epsilon}_0$ may depend on η, t and U.
3. Use (5.2) with $(t_0, x_0) = (s, y)$ and $r = s - \epsilon^2$ to obtain,

$$\begin{aligned}
\alpha^\epsilon(s-\epsilon^2; s, y) &\le \left(\frac{s-t}{\epsilon^2}\right)^{K\sqrt{\epsilon}} \alpha^\epsilon(t; s, y) + K \int_t^{s-\epsilon^2} \left(\frac{s-\tau}{\epsilon^2}\right)^{K\sqrt{\epsilon}} \frac{d\tau}{\sqrt{s-\tau}} \\
&\le \left(\frac{s-t}{\epsilon^2}\right)^{K\sqrt{\epsilon}} [\alpha^\epsilon(t; s, y) + K \int_t^{s-\epsilon^2} \frac{d\tau}{\sqrt{s-\tau}} d\tau] \\
&\le \left(\frac{s-t}{\epsilon^2}\right)^{K\sqrt{\epsilon}} [\alpha^\epsilon(t; s, y) + 2K\sqrt{s-t}].
\end{aligned}$$

Since $\epsilon^{-2K\sqrt{\epsilon}}$ converges to one as ϵ approaches to zero, there is $0 < \hat{\epsilon}_0 \le \epsilon_0$ satisfying

$$\left(\frac{s-t}{\epsilon^2}\right)^{K\sqrt{\epsilon}} \le 2, \quad \epsilon \le \hat{\epsilon}_0, \quad (s, y) \in U.$$

Then by step 2 we have,

$$\alpha^\epsilon(s-\epsilon^2; s, y) \le 4\eta + K^*(s-t)^\gamma + 4K\sqrt{s-t},$$

for all $(s, y) \in U$ and $\epsilon \le \hat{\epsilon}_0$. Set

$$\hat{U} = U \cap (t_0 - t^*, t_0 + t^*) \times \mathcal{R}^d.$$

Recall that for any $(s, y) \in \hat{U}$, we have $t < s - \epsilon^2 < s$. Also for any $(s, y) \in \hat{U}$, and t, t_0 satisfying (6.2) we have $(s - t) \le 2t^*$. Now choose $t^* = t^*(\eta)$ so that

$$K^*(s-t)^\gamma + 4K\sqrt{s-t} \le K^*(2t^*)^\gamma + 4K\sqrt{2t^*} \le \eta.$$

Therefore,

$$\alpha(s-\epsilon^2, s, y) \le 5\eta, \quad (s, y) \in \hat{U}, \ \epsilon \le \hat{\epsilon}_0.$$

Recall that the above estimate is obtained under the assumption that (6.1) holds with t, t_0 satisfying (6.2) with $t^* = t^*(\eta)$.
4. Let $B_\epsilon(y)$ be the sphere centered at y with radius ϵ. For any $x \in B_\epsilon(y)$,

$$\begin{aligned}
\rho(s-\epsilon^2, x; s, y) &= (4\pi\epsilon^2)^{-\frac{d-1}{2}} \exp(-\frac{|x-y|^2}{4\epsilon^2}) \\
&\ge \left[(4\pi)^{-\frac{d-1}{2}} e^{-\frac{1}{4}}\right] \epsilon^{-(d-1)} \\
&= (K_* \epsilon^{d-1})^{-1},
\end{aligned}$$

for some constant K_*. Therefore,

$$\hat{\mu}^\epsilon(s-\epsilon^2;B_\epsilon(y)) \;\leq\; \left[\min_{x\in B_\epsilon(y)} \rho(s-\epsilon^2,x;s,y)\right]^{-1} \alpha^\epsilon(s-\epsilon^2;s,y) \tag{6.5}$$
$$\leq\; 5K_*\eta\epsilon^{d-1} \quad \forall(s,y)\in\hat{U},\ \epsilon\leq\hat{\epsilon}_0.$$

5. Define

$$\beta = \liminf_{\epsilon\to 0} \inf_{(s,y)\in\hat{U}} |\varphi^\epsilon(s,y)|.$$

Let c be any number sufficiently close to one, say $\frac{7}{8}$. In this step we will show that for a carefully chosen η, we have $\beta \geq c = \frac{7}{8}$.

Suppose that $\beta < \frac{7}{8}$. Then there are $\epsilon_n \to 0$ and $(s_n,y_n)\in\hat{U}$ satisfying

$$\left|\varphi^{\epsilon_n}(s_n-\epsilon_n^2,y_n)\right| < \frac{7}{8} \Rightarrow \left|z^{\epsilon_n}(s_n-\epsilon_n^2,y_n)\right| < \epsilon_n q^{-1}\left(\frac{7}{8}\right).$$

Using (4.1) we construct K_0, n_0, independent of η, such that for all $n\geq n_0$ we have

$$|z^{\epsilon_n}(s_n-\epsilon_n^2,x)| < \epsilon_n[q^{-1}(\frac{7}{8})+K_0], \quad \forall x\in B_{\epsilon_n}(y_n).$$

Therefore $W(\varphi^{\epsilon_n}(s_n-\epsilon_n^2,x)) > W(q(q^{-1}(\frac{7}{8})+K_0))$ for all $x\in B_{\epsilon_n}(y_n)$, and $n\geq n_0$. Hence for $n\geq n_0$, we have,

$$\mu^{\epsilon_n}(s_n-\epsilon_n^2;B_{\epsilon_n}(y_n)) \;\geq\; \int_{B_{\epsilon_n}(y_n)} \frac{1}{\epsilon_n} W(\varphi^{\epsilon_n}(\epsilon_n-\epsilon_n^2,x))dx$$
$$>\; w_d W(q(q^{-1}(\tfrac{7}{8})+K_0))(\epsilon_n)^{d-1},$$

where w_d is the volume of the d dimensional unit sphere. Now choose

$$\eta = \frac{w_d}{5K_*} W(q(q^{-1}(\frac{7}{8})+K_0)), \tag{6.6}$$

where K_* is the constant appearing in (6.5). With this choice of η, (6.6) contradicts (6.5). Hence $\beta\geq 7/8$.

So we have proved the following. Let η and t^* be as above, and suppose that (6.1) holds for some t, t_0 satisfying (6.2). Then there exists a neighborhood $\hat{U}$ of (t_0,x_0) such that

$$\beta = \liminf_{\epsilon\to 0} \inf_{(s,y)\in\hat{U}} |\varphi^\epsilon(s,y)| \geq \frac{7}{8}.$$

Now Lemma 4.6 implies that (6.3) holds on any open set O satisfying $\bar{O}\subset\hat{U}$. Let O be such an open set. Then by Lemma 4.6,

$$\liminf_{\epsilon\to 0} \inf_{(s,y)\in\bar{O}} |z^\epsilon(s,y)| > 0,$$

and (4.1) yields

$$\begin{aligned}\mu^\epsilon(O) &= \iint_{\bar{O}} \tfrac{1}{2\epsilon}(q'(\tfrac{z^\epsilon}{\epsilon}))^2(|\nabla z^\epsilon|^2+1)dxdt \\ &\le \iint_{\bar{O}} \tfrac{1}{2\epsilon}(q'(\tfrac{z^\epsilon}{\epsilon}))^2(2+K\sqrt{\epsilon}(1+|z^\epsilon|))dxdt.\end{aligned}$$

Now it is easy to show that $\mu^\epsilon(\bar{O})$ converges to zero as ϵ tends to zero. Hence (6.4) holds. ∎

7 Dimension of Γ and Equipartition of Energy

Let Γ and $\Gamma(t)$ be as in Section 2.3. Then it is immediate that

$$\Gamma \subset \overline{\bigcup_{t\ge 0} \{t\} \times \Gamma(t)}.$$

Suppose that $(t_0, x_0) \not\in \Gamma$. Then there is a neighborhood U of (t_0, x_0) such that $U \cap \Gamma = \emptyset$. Therefore

$$\lim_{t\uparrow t_0} \int \rho(t,x;t_0,x_0)\mu(t;dx) = 0,$$

and by Theorem 6.1, (t_0, x_0) satisfy (6.4). Hence,

$$\Gamma = \overline{\bigcup_{t\ge 0} \{t\} \times \Gamma(t)}.$$

Let Γ_t be the t-section of Γ. In this section we will first estimate the Hausdorff dimension of Γ_t (cf. [16]). Then we will show that ξ^ϵ, defined in Section 5, converges to zero. Hence proving the equipartition of energy. Our arguments closely follow Sections 6, 7 and 8 in [22].

The following follows from [32, Theorem 5.12.4].

Theorem 7.1 *Let φ be a smooth function in $BV(\mathcal{R}^d)$ and μ be a positive Borel measure satisfying,*

$$M(\mu) = \sup_{x\in\mathcal{R}^d, R>0} \frac{\mu(B_R(x))}{R^{d-1}} < \infty.$$

Then there is a constant K_d, depending only on the dimension d but not on μ and φ, such that

$$\left|\int \varphi(x)\mu(dx)\right| \le K_d M(\mu)\|\nabla\varphi\|_1.$$

We continue with an estimate of the dimension of the interface.

Proposition 7.2 *For every $T > 0$ there is $K(T) > 0$ such that*

$$\mu^\epsilon(r; B_R(x)) \leq K(T) R^{d-1}, \tag{7.1}$$

$$\mathcal{H}^{d-1}(\Gamma_r \cap B_R(x)) \leq K(T) R^{d-1}, \tag{7.2}$$

for all $0 < \epsilon \leq 1,\ \ R > 0,\ \ 0 \leq r \leq T$.

Proof:
1. Theorem 7.1, (A5) and (A6) imply that

$$\begin{aligned}
\alpha^\epsilon(0; t_0, x_0) &= \int \rho(0, x; t_0, x_0) \left[\mu^\epsilon(0; dx) + \frac{1}{2} (\theta_0^\epsilon(x))^2\, dx\right] \\
&\leq K[\|\nabla_x \rho(0, \cdot; t_0, x_0)\|_1 + \|\rho(0, \cdot; t_0, x_0)\|_1] \\
&\leq K(\sqrt{t_0} + 1),
\end{aligned}$$

for some constant K, independent of ϵ.
2. Energy estimate (2.5) yields

$$\mu^\epsilon(r; B_R(x)) \leq \hat{\mu}^\epsilon(r; R^d) \leq C_1^* \leq R^{d-1},$$

if $R \geq (C_1^*)^{\frac{1}{d-1}} = R_0$. Hence (7.1) holds for all $r \geq 0$ and $R \geq R_0$ with constant $K(T) = 1$.
3. Fix $0 \leq r \leq T, \epsilon \leq R \leq R_0$, and $x_0 \in \mathcal{R}^d$. Then for $t_0 > r$,

$$\begin{aligned}
\mu^\epsilon(r; B_R(x_0)) &\leq \hat{\mu}^\epsilon(r; B_R(x_0)) \\
&\leq [\inf_{x \in B_R(x_0)} \rho(r, x; t_0, x_0)]^{-1} \alpha^\epsilon(r; t_0, x_0) \\
&= (4\pi(t_0 - r))^{\frac{d-1}{2}} \exp(\frac{R^2}{4(t_0 - r)}) \alpha^\epsilon(r; t_0, x_0).
\end{aligned} \tag{7.3}$$

Choose $t_0 = r + R^2$. Then $t_0 \leq T + R_0^2 = T_*$. By step 1, and (5.2) we obtain,

$$\begin{aligned}
\alpha^\epsilon(r; t_0, x_0) &\leq \alpha^\epsilon(0; t_0, x_0) \left(\frac{t_0}{R^2}\right)^{K\sqrt{\epsilon}} + K \int_0^r \left(\frac{t_0 - \tau}{R^2}\right)^{K\sqrt{\epsilon}} \frac{d\tau}{\sqrt{t_0 - \tau}} \\
&\leq K \left(\frac{t_0}{R^2}\right)^{K\sqrt{\epsilon}} (\sqrt{t_0} + 1).
\end{aligned}$$

Since $R \geq \epsilon$, there is $K = K(T)$ satisfying

$$\alpha^\epsilon(r; t_0, x_0) \leq K, \quad 0 < \epsilon \leq 1, \quad r \leq T,$$

with $t_0 = r + R^2$. Then (7.3) implies,

$$\mu^\epsilon(r; B_R(x_0)) \leq (4\pi)^{\frac{d-1}{2}} e^{\frac{1}{4}} \; K \; R^{d-1}, \;\; \epsilon \leq R \leq R_0.$$

Hence (7.1) holds for all $R \geq \epsilon$.

4. In this step we study the case $0 < R \leq \epsilon$. (3.2) yields, that for $0 \leq r \leq T$,

$$\begin{aligned} \mu^\epsilon(r; B_R(x_0)) &= \int_{B_R(x_0)} \frac{\epsilon}{2} |\nabla \varphi^\epsilon|^2 + \frac{1}{\epsilon} W(\varphi^\epsilon) \\ &\leq \frac{K}{\epsilon} |B_R(x_0)| = \frac{K}{\epsilon} R^d. \end{aligned}$$

Since $R \leq \epsilon$, $R^d \epsilon^{-1} \leq R^{d-1}$. This completes the proof of (7.1) for all R.

5. (7.2) follows from Theorem 6.1 and the proof of [22, 6.3]. ∎

In the remainder of this section, we will prove that ξ^ϵ converges to zero. Our proof is a direct modification of Sections 7 and 8 in [22].

Let η be as in Theorem 6.1. Define

$$Z^- = \left\{ (t,x) \in \Gamma \cap [0,T] \times \mathcal{R}^d \; : \limsup_{s \downarrow t} \int \rho(t,x;s,y) \mu(s;dy) < \eta \right\}$$

Then Section 7 in [22] implies that there is $\delta > 0$ satisfying,

$$\mathcal{H}^{d-2+\delta}(Z_t^-) = 0 \quad \text{for a.e. } t \in [0,T]. \tag{7.4}$$

Let ξ^ϵ be as in Section 5. For a Borel set $A \subset [0,T] \times \mathcal{R}^d$ define

$$\xi^\epsilon(A) = \int_A \xi^\epsilon(t; dx) dt.$$

Since $|\xi^\epsilon| = \mu^\epsilon$, by passing to a further subsequence we assume that ξ^ϵ converges to a Borel measure ξ in the weak* topology of Radon measures.

Proposition 7.3 $\xi = 0$.

Proof:

1. For $\epsilon > 0$. Let

$$\begin{aligned} \nu^\epsilon(A) &= \int_A \frac{1}{2\epsilon} (q'(\frac{z^\epsilon}{\epsilon}))^2 (|\nabla z^\epsilon|^2 - 1)^+ dx, \\ \lambda^\epsilon(A) &= \int_A \frac{1}{2\epsilon} (q'(\frac{z^\epsilon}{\epsilon}))^2 (|\nabla z^\epsilon|^2 - 1)^- dx, \end{aligned}$$

where for a real number b, $(b)^+ = \max\{b, 0\}$, $b^- = \max\{-b, 0\}$. Then

$$\xi^\epsilon = \nu^\epsilon - \lambda^\epsilon.$$

2. (4.1) and the proof of Corollary 5.2 imply,

$$\nu^\epsilon(A) \le K\sqrt{\epsilon}\,[\mu^\epsilon(A) + \epsilon\,|A|]\,, \quad 0 < \epsilon \le 1, \tag{7.5}$$

for any $A \subset [0,T] \times \mathcal{R}^d$. Hence ν^ϵ converges to zero, and λ^ϵ converges to $-\xi$.
3. Fix $(s,y) \in [0,\infty) \times \mathcal{R}^d$, and $0 < \sigma \le s$. Integrate (5.1) on $[0, s-\sigma]$. Using (7.5) and the exponential decay of ρ we let ϵ go to zero to obtain,

$$\begin{aligned}\alpha(s-\sigma; s, y) - \alpha(0; s, y) \quad &\le \quad -\int_0^{s-\sigma}\int_{\mathcal{R}^d} \frac{1}{2(s-t)}\rho(t,x;s,y)\lambda(t;dx)dt \\ &\qquad +2C_d\left(\sqrt{s} - \sqrt{\sigma}\right).\end{aligned}$$

Above inequality and step 1 of Proposition 7.2 yield,

$$\int_0^{s-\sigma}\int_{\mathcal{R}^d} \frac{1}{2(s-t)}\rho(t,x;s,y)\lambda(t;dx)dt \le K(\sqrt{s}+1).$$

Fix $T > 0$ and integrate the above inequality against $\mu(s;dy)ds$ and then use (2.5) to obtain,

$$\begin{aligned}\int_0^{T+1}\int_{\mathcal{R}^d}\int_0^{s-\sigma}\int_{\mathcal{R}^d} &\frac{1}{2(s-t)}\rho(t,x;s,y)\lambda(t;dx)dt\mu(s;dx)ds \\ &\le \int_0^{T+1}\int_{\mathcal{R}^d} K(\sqrt{s}+1)\mu(s;dy)ds \le \hat{C}(T),\end{aligned}$$

for some constant $\hat{C}(T)$ depending on T.
4. Fubini's theorem and the monotone convergence theorem enable us to send σ to zero to obtain,

$$\int_0^{T+1}\int_{\mathcal{R}^d}\int_t^{T+1}\int_{\mathcal{R}^d} \frac{1}{2(s-t)}\rho(t,x;s,y)\mu(s;dy)ds\lambda(t;dx)dt \le \hat{C}(T).$$

Hence

$$\int_t^{t+1} \frac{1}{2(s-t)}\int_{\mathcal{R}^d} \rho(t,x;s,y)\mu(s;dy)ds \le C(x,t) < \infty \tag{7.6}$$

for λ almost every $(t,x) \in [0,T] \times \mathcal{R}^d$.
5. Fix (t,x) such that (7.6) holds. For $s \in (t, t+1]$ define,

$$\beta = \ln(s-t)$$

$$h(s)=\int_{\mathcal{R}^d}\rho(t,x;s,y)\mu(s;dy).$$

Then (7.6) implies that

$$\int_{-\infty}^{0} h(t+e^{\beta})d\beta<\infty. \tag{7.7}$$

We wish to prove that

$$\lim_{s\downarrow t} h(s)=0.$$

Clearly (7.7) implies that $h(t+e^{\beta})$ converges to zero on a subsequence. We will now use the monotonicity of h to prove the convergence on the whole sequence.

6. Following [22], for $\gamma\in(0,1]$ we choose a decreasing sequence $\beta_i\to-\infty$ such that

$$|\beta_{i+1}-\beta_i|\leq\gamma,\quad h(t+e^{\beta_i})\leq\gamma.$$

Then for any $\beta\in[\beta_i,\beta_{i-1})$,

$$\begin{aligned} h(t+e^{\beta}) &= \int\rho(t,x;t+e^{\beta},y)\mu(t+e^{\beta};dy)\\ &= \int\rho(t+e^{\beta};t+2e^{\beta},x)\mu(t+e^{\beta},dy)\\ &= \alpha(t+e^{\beta};t+2e^{\beta},x).\end{aligned}$$

Use (5.3) to obtain,

$$\begin{aligned} h(t+e^{\beta}) &\leq \alpha(t+e^{\beta_i};t+2e^{\beta},x)+C_d\left[\sqrt{2e^{\beta}-e^{\beta_i}}-\sqrt{e^{\beta}}\right]\\ &\leq \alpha(t+e^{\beta_i};t+2e^{\beta},x)+C_d\sqrt{2e^{\beta}}.\end{aligned} \tag{7.8}$$

Also the previous identity with $\beta=\beta_i$ yields,

$$\gamma\geq h(t+e^{\beta_i})=\alpha(t+e^{\beta_i};t+2e^{\beta_i},x). \tag{7.9}$$

7. We claim that for any $\delta>0$ there is $\gamma(\delta,T)>0$ satisfying

$$\alpha(t_0;t_0+R_1,x)\leq(1+\delta)\alpha(t_0;t_0+R_0,x)+\delta, \tag{7.10}$$

for all $0\leq t_0\leq T+1$, $x\in\mathcal{R}^d$ and $0\leq R_0\leq R_1\leq(\gamma(\delta)+1)R_0$. This result follows from (7.1) and it is stated in [22, Lemma 3.4(iv)]. We postpone the elementary proof of (7.10) to the next step and complete the proof of the proposition.

We use (7.10) with,

$$t_0=t+e^{\beta_i}\ \ R_1=2e^{\beta}-e^{\beta_i},\quad R_0=e^{\beta_i}.$$

Then

$$\frac{R_1}{R_0} = \sqrt{2e^{\beta-\beta_i}-1} \leq \sqrt{2[e^{\beta-\beta_i}-1]+1} \leq 1 + K\gamma$$

for some constant K. So if $K\gamma \leq \gamma(\delta)$, (7.8), (7.9) yield,

$$\begin{aligned}
h(t+e^{\beta}) &\leq \alpha(t+e^{\beta_i}; t+2e^{\beta}, x) + C_d\sqrt{2e^{\beta}} \\
&\leq (1+\delta)\alpha(t+e^{\beta_i}; t+2e^{\beta_i}, x) + \delta + C_d\sqrt{2e^{\beta}} \\
&= (1+\delta)h(t+e^{\beta_i}) + \delta + C_d\sqrt{2e^{\beta}} \\
&\leq (1+\delta)\gamma + \delta + C_d\sqrt{2e^{\beta}}.
\end{aligned}$$

Above holds for all $\delta > 0$ and $0 < \gamma \leq \gamma_0(\delta)$. Now pass to the limit $i \to \infty$, $\gamma \to 0$ and then $\delta \to 0$, to obtain,

$$\lim_{s \downarrow t} h(s) = 0,$$

for every (t, x) satisfying (7.6). Hence above holds for λ-almost every (t, x). On the other hand, (7.4) and (7.1) imply that

$$\limsup_{s \downarrow t} h(s) \geq \eta > 0$$

for μ-almost every (t, x). Since $\lambda = -\xi << \mu$, we conclude that $\lambda = -\xi = 0$.

8. In this step we will prove (7.10). Recall that

$$\alpha(t_0; t_0+\tau, x_0) = \int \left(\frac{1}{4\pi\tau}\right)^{\frac{d-1}{2}} e^{-\frac{|x_0-y|^2}{4\tau}} \mu(t_0; dy).$$

Without loss of generality we take $x_0 = 0$. Set $\mu(dy) = \mu(t_0; dy)$,

$$f(\tau) = \int \left(\frac{1}{4\pi\tau}\right)^{\frac{d-1}{2}} e^{-\frac{|y|^2}{4\tau}} \mu(dy).$$

For any $0 < \alpha < 1$,

$$f(\frac{\tau}{1-\alpha}) \leq \int \left(\frac{1}{4\pi\tau}\right)^{\frac{d-1}{2}} e^{-\frac{|y|^2}{4\tau}(1-\alpha)} \mu(dy).$$

For $\delta > 0$

$$\begin{aligned}
I &= f(\frac{\tau}{1-\alpha}) - (1+\delta)f(\tau) \\
&\leq \int \left(\frac{1}{4\pi\tau}\right)^{\frac{d-1}{2}} e^{-\frac{|y|^2}{4\tau}} \left(e^{\alpha\frac{|y|^2}{4\tau}} - (1+\delta)\right) \\
&\leq \int_{|y|\geq A} \left(\frac{1}{4\pi\tau}\right)^{\frac{d-1}{2}} e^{-\frac{|y|^2}{4\tau}} \mu(dy),
\end{aligned}$$

where

$$\Lambda = \sqrt{\frac{4\tau}{\alpha}\ln(1+\delta)}.$$

Since by (7.1), $\mu(\{|y| \leq R\}) \leq KR^{d-1}$, and the integrand is radially symmetric, by integration by parts we obtain,

$$I \leq \int_{\Lambda}^{\infty} \left(\frac{1}{(4\pi\tau)}\right)^{\frac{d-1}{2}} \frac{R}{2\tau} e^{-\frac{R^2}{4\tau}} KR^{d-1} dR.$$

By a change of variables,

$$I \leq K \int_{\sqrt{\tau}\Lambda}^{\infty} |\xi|^d e^{-|\xi|^2} d\xi \leq K \exp(-\frac{2\ln(1+\delta)}{\alpha}) \leq \delta,$$

for sufficiently small δ. ∎

8 Passage to the limit.

For $(t,x) \in [0,\infty) \times \mathcal{R}^d$ and $0 < \epsilon \leq 1$ recall that

$$\nu^\epsilon(t,x) = \begin{cases} \dfrac{\nabla\varphi^\epsilon(t,x)}{|\nabla\varphi^\epsilon(t,x)|}, & \text{if } \; |\nabla\varphi^\epsilon(t,x)| \neq 0 \\ \nu_0 & \text{if } \; |\nabla\varphi^\epsilon(t,x)| = 0, \end{cases}$$

where $\nu_0 \in S^{d-1}$ is any unit vector. Let $A \subset [0,\infty) \times \mathcal{R}^d \times S^{d-1}$ be a Borel subset. Set

$$m^\epsilon(A) = \int_A dt\, \mu^\epsilon(t,dx)\delta_{\{\nu^\epsilon(t,x)\}}(dn),$$

$$\tilde{m}^\epsilon(A) = -\int_A z_t^\epsilon(t,x) dm^\epsilon.$$

We start with the following lemma.

Lemma 8.1 *For any $T > 0$, $\alpha \geq 0$ there are constant $K_\alpha = K(T,\alpha), K = K(T)$ such that for any Borel set $B \subset \mathcal{R}^d$, we have*

$$\sup_{0<\epsilon\leq 1} \int_0^T \int_B (1 + |z^\epsilon(t,x)|)^\alpha \mu^\epsilon(t;dx)\, dt \leq K_\alpha(1 + |B|), \qquad (8.1)$$

$$\int_0^T \int_B (z_t^\epsilon(t,x))^2 \mu^\epsilon(t;dx) dt \leq K\left(1 + \sqrt{\epsilon}|B|\right). \qquad (8.2)$$

Proof: Set

$$\Omega = \{(t,x) \in [0,T] \times B \; ; \; |z^\epsilon(t,x)| \leq 1\}.$$

Energy estimate (2.5) yields,

$$\int_0^T \int_B (1 + |z^\epsilon(t,x)|)^\alpha \, \mu^\epsilon(t;dx)dt$$

$$\leq 2^\alpha \int_\Omega \mu^\epsilon(t;dx)dt + \int_{\Omega^c} (1 + |z^\epsilon|)^\alpha \, \mu^\epsilon(t;dx)dt$$

$$\leq (2^\alpha + 1)\, C_1^* T + \int_{\Omega^c} [(1 + |z^\epsilon|)^\alpha - 1]\, \mu^\epsilon(t;dx)dt,$$

where Ω^c denote the complement of Ω. Observe that for any $r \geq 1$,

$$(1+r)^\alpha - 1 \leq \alpha r \;\max\left\{1, (1+r)^{\alpha-1}\right\} \leq \alpha r (1+r)^\alpha.$$

We use the above inequality and (4.3) to obtain,

$$[(1 + |z^\epsilon|)^\alpha - 1]\, \mu^\epsilon(t;dx) = [(1 + |z^\epsilon|)^\alpha - 1]\, \frac{1}{2\epsilon}\, (q'(\frac{z^\epsilon}{\epsilon}))^2 (|\nabla z^\epsilon|^2 + 1)$$

$$\leq K\alpha \frac{|z^\epsilon|}{\epsilon}\, (1 + |z^\epsilon|)^{\alpha+1} (q'(\frac{z^\epsilon}{\epsilon}))^2$$

$$\leq K\alpha \sup_{0<\epsilon\leq 1} \sup_{r \geq 1} \frac{r}{\epsilon}\, (1+r)^{\alpha+1} (q'(\frac{r}{\epsilon}))^2$$

$$= K\alpha \sup_{0<\epsilon\leq 1} \sup_{\bar{r}>0} \bar{r}\, (1+\epsilon\bar{r})^{\alpha+1} (q'(\bar{r}))^2$$

$$= C^*(\alpha) < \infty.$$

This proves (8.1). To prove (8.2), first recall that

$$\int_0^T \int_{\mathcal{R}^d} \epsilon\, (\varphi_i^\epsilon)^2 \, dx\, dt = \int_0^T \int_{\mathcal{R}^d} \frac{1}{\epsilon}\, (z_i^\epsilon)^2\, (q')^2 dx\, dt$$

where q' is evaluated at (z^ϵ/ϵ). Hence by (2.5) and (4.1) we have

$$\int_0^T \int_B (z_i^\epsilon)^2 \mu^\epsilon(t;dx)dt = \int_0^t \int_B \frac{1}{2\epsilon} (z_i^\epsilon)^2 (q')^2 \left(1 + |\nabla z^\epsilon|^2\right) dx\, dt$$

$$\leq \int_0^T \int_{\mathcal{R}^d} \frac{1}{\epsilon} (z_i^\epsilon)^2 (q')^2 dx\, dt + \int_0^T \int_B \frac{1}{2\epsilon} (z_i^\epsilon)^2 (q')^2 \left(|\nabla z^\epsilon|^2 - 1\right) dx\, dt$$

$$\leq C_1^* + K\sqrt{\epsilon} \left[\int_0^T \int_B \frac{1}{2\epsilon} (z_i^\epsilon)^2 (q')^2 (1 + |z^\epsilon|)\right] dx\, dt$$

$$\leq C_1^* + K\sqrt{\epsilon} \left[\int_0^T \int_{\mathcal{R}^d} \frac{\epsilon}{2} (\varphi_i^\epsilon)^2 dx\, dt + \int_0^T \int_B \frac{1}{2\epsilon} (z_i^\epsilon)^2 (q')^2 |z^\epsilon| dx\, dt\right].$$

By (4.6) and (2.5) we have

$$\int_0^T \int_B \frac{1}{2\epsilon} (z_i^\epsilon)^2 (q')^2 |z^\epsilon| dx\, dt$$

$$\leq \int_0^T \int_B \frac{1}{2\epsilon} (z_i^\epsilon)^2 (q')^2 [1 + |z^\epsilon| \chi_{|z^\epsilon| \geq 1}] dx\, dt$$

$$\leq \int_0^T \int_{\mathcal{R}^d} \epsilon (\varphi_i^\epsilon)^2 + \int_0^T \int_B \frac{K}{\epsilon^5} (q')^2 |z^\epsilon| \chi_{|z^\epsilon| \geq 1} dx\, dt$$

$$\leq C_1^* + \int_0^T \int_B \frac{K}{\epsilon^5} \sup_{r \geq 1} r\, q'(\frac{r}{\epsilon}) dx\, dt$$

$$\leq K[1 + |B|T].$$

∎

Next we choose a subsequence, denoted by ϵ, such that m^ϵ and $\bar{m}^\epsilon$ converge to m and $\bar{m}$ in weak*, respectively. By a slicing argument (c.f. [15, Theorem 10, page 14]), we conclude that

$$dm(t, x, n) = dt\, \mu(t; dx) N(t, x; dn),$$

for some probability measure $N(t, x, \cdot)$ on S^{d-1}. Moreover (8.2) implies that

$$d\bar{m} = v(t, x, n) dm$$

for some $v \in L^2(dm)$.

Proof of Theorem A.

Following [22, Section 9.3], for $t > 0$ let $V^\epsilon(t;\cdot)$ be the unique varifold (c.f. [27]) satisfying

$$\|V^\epsilon(t;\cdot)\| = \mu^\epsilon(t;\cdot),$$

and $(V^\epsilon(t;\cdot))^{(x)}$ is supported at $(\nu^\epsilon(t,x))^\perp$, i.e.,

$$V^\epsilon(t; dx\, dS) = \delta_{\{(\nu^\epsilon)^\perp\}}(dS)\mu^\epsilon(t;dx).$$

1. It follows from (2.5) that,

$$\sup_{0<\epsilon\leq 1} \int_0^T \int \left[\epsilon\,(\varphi_t^\epsilon)^2 + |\nabla\theta^\epsilon|^2\right] dx\, dt < \infty. \tag{8.3}$$

In this step, we will show that

$$\sup_{0<\epsilon\leq 1} \int_0^T \int \epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2} W'(\varphi^\epsilon)\right]^2 dx\, dt < \infty. \tag{8.4}$$

Since $g^2 = 2W$, (OPE) yields that

$$\begin{aligned} \epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2} W'(\varphi^\epsilon)\right]^2 &= \epsilon \left[-\varphi_t^\epsilon + \frac{1}{\epsilon} g(\varphi^\epsilon)\theta^\epsilon\right]^2 \\ &\leq 2\epsilon\,(\varphi_t^\epsilon)^2 + \frac{2}{\epsilon} W(\varphi^\epsilon)\,(\theta^\epsilon)^2 . \end{aligned}$$

Theorem 7.1 and (7.1) yield,

$$\frac{1}{2\epsilon}\int W(\varphi^\epsilon)\,(\theta^\epsilon)^2\, dx \leq \int (\theta^\epsilon)^2\, d\mu^\epsilon \leq K \|\nabla\,(\theta^\epsilon)^2\,\|_1 \leq K \int (\theta^\epsilon)^2 + (\nabla\theta^\epsilon)^2 .$$

Hence,

$$\begin{aligned} &\int_0^T \int \epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2} W'(\varphi^\epsilon)\right]^2 dx\, dt \\ &\leq \int_0^T \int \left[\epsilon\,(\varphi_t^\epsilon)^2 + K\,|\nabla\theta^\epsilon|^2 + K\,(\theta^\epsilon)^2\right] dx\, dt. \end{aligned}$$

Now (8.4) follows from (8.3) and (2.5).

2. For any smooth vector field Y the definition of V^ϵ and the definition of the first variation (c.f. [27]), imply

$$\begin{aligned} \delta V^\epsilon(t;\cdot)(Y) &= \int DY : S\; V^\epsilon(t; dx \times dS) \\ &= \int DY : (I - \nu^\epsilon \otimes \nu^\epsilon)\mu^\epsilon(t;dx). \end{aligned}$$

Let T be as in step 2 of Theorem 5.1. Recall that

$$(I - \nu^\epsilon \otimes \nu^\epsilon)\mu^\epsilon = (\nu^\epsilon \otimes \nu^\epsilon)\xi^\epsilon - T.$$

We now proceed as in Theorem 5.1 to obtain,

$$\begin{aligned} \delta V^\epsilon(t;\cdot)(Y) &= -\int \epsilon Y \cdot \nabla\varphi^\epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon)\right] dx \\ &\quad + \int DY : \nu^\epsilon \otimes \nu^\epsilon \, d\xi^\epsilon. \end{aligned}$$

Hence

$$\begin{aligned} |\delta V^\epsilon(t;\cdot)(Y)| &\le \left(\int \epsilon |Y|^2 |\nabla\varphi^\epsilon|^2 \, dx\right)^{\frac{1}{2}} \left(\int \epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W(\varphi^\epsilon)\right]^2\right)^{\frac{1}{2}} \\ &\quad + \int |DY| \, d|\xi^\epsilon|. \end{aligned}$$

3. By Fatou's lemma and (8.4) we have

$$\int_0^T \liminf_{\epsilon\to 0} \int \epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon)\right]^2 dx\, dt < \infty.$$

Hence for almost every $t \ge 0$ there is a constant $K(t) < \infty$ satisfying

$$\liminf_{\epsilon\to 0} \int \epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon)\right]^2 dx \le K(t).$$

Step 2 together with Proposition 7.3 yield,

$$\liminf_{\epsilon\to 0} |\delta V^\epsilon(t;\cdot)(Y)| \le \hat{K}(t) \left(\int |Y|^2 \mu(t;dx)\right)^{\frac{1}{2}} \le \hat{K}(t)\|Y\|_\infty$$

for some constant $\hat{K}(t) < \infty$ for almost every $t \ge 0$.
4. Fix $t \ge 0$ such that $\hat{K}(t) < \infty$. Choose a subsequence ϵ_n such that $V^{\epsilon_n}(t;\cdot)$ converges to a varifold $\tilde{V}$ and

$$\lim |\delta V^{\epsilon_n}(t;\cdot)(Y)| \le \hat{K}(t)\|Y\|_\infty.$$

Then

$$\left|\delta\tilde{V}(Y)\right| = \lim |\delta V^{\epsilon_n}(t;\cdot)(Y)| \le \hat{K}(t)\|Y\|_\infty. \tag{8.5}$$

Also observe that

$$\|\tilde{V}\| = \lim \|V^{\epsilon_n}(t;\cdot)\| = \lim \mu^{\epsilon_n}(t;\cdot) = \mu(t,\cdot).$$

Therefore by Allard's Rectifiability Theorem [1, 5.5(2)], (7.2) and (8.5), we conclude that $\mu(t,\cdot)$ is $(d-1)$-rectifiable. Moreover since a rectifiable varifold is uniquely determined by $\|\tilde{V}\|$, we conclude that $\tilde{V}$ is equal to the varifold $V_{\mu(t,\cdot)}$ defined in Section 2.5.

5. In the previous step we proved that for almost every $t \geq 0$, $\delta V_{\mu(t,\cdot)}$ is $(d-1)$-rectifiable and

$$|\delta V_{\mu(t,\cdot)}(Y)| \leq \liminf_{\epsilon \to 0} \left(\int \epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2} W'(\varphi^\epsilon) \right]^2 dx \right)^{\frac{1}{2}} \left(\int |Y|^2 \mu(t; dx) \right)^{\frac{1}{2}}.$$

Hence for almost every $t \geq 0, \mu(t,\cdot)$ has a generalized mean curvature vector $\mathrm{H}(t,x)$ and

$$\int |H(t,x)|^2 \mu(t;dx) \leq \liminf_{\epsilon \to 0} \int \epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2} W'(\varphi^\epsilon) \right]^2 dx.$$

Step 2 implies that $H \in L^2\left((0,T) \times \mathcal{R}^d; dt \times \mu(t;dx)\right)$, and for later use we note that,

$$\iint Y \cdot H(t,x)\, dt\, \mu(t;dx) = -\lim_{\epsilon \to 0} \iint \epsilon Y \cdot \nabla\varphi^\epsilon \left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2} W'(\varphi^\epsilon) \right] dx\, dt, \tag{8.6}$$

for any compactly supported smooth vector field Y. ∎

Proof of Theorem B:

1. Let $\Psi(t,x)$ be a smooth compactly supported function. Then the action of the distribution $\theta_t - \Delta\theta$ on Ψ is given by,

$$I(\Psi) = \iint (\theta_t - \Delta\theta)\, \Psi\, dx\, dt = \lim_{\epsilon \to 0} \iint (\theta_t^\epsilon - \Delta\theta^\epsilon)\, \Psi\, dx\, dt.$$

Then by (HE) we have,

$$\begin{aligned} I(\Psi) &= -\lim_{\epsilon \to 0} \iint g(\varphi^\epsilon)\varphi_t^\epsilon\, \Psi\, dx\, dt \\ &= -\lim_{\epsilon \to 0} \iint \frac{1}{\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2 z_t^\epsilon\, \Psi\, dx\, dt \\ &= \lim_{\epsilon \to 0} \iint \Psi d\bar{m}^\epsilon - \lim_{\epsilon \to 0} \iint \Psi\, z_t^\epsilon d\xi^\epsilon \\ &= \iint v(t,x,n)\Psi(t,x) dm - \lim_{\epsilon \to 0} \iint \Psi z_t^\epsilon d\xi^\epsilon. \end{aligned}$$

We claim that the second term in the last expresion is equal to zero. Indeed, since $|\xi^\epsilon| \le \mu^\epsilon$, by Cauchy-Schwartz we obtain,

$$\left|\iint \Psi z_i^\epsilon d\xi^\epsilon\right| \le \left(\iint |\Psi|^2 d|\xi^\epsilon|\right)^{\frac{1}{2}} \left(\iint_{spt\Psi} |z_i^\epsilon|^2 d\mu^\epsilon\right)^{\frac{1}{2}}.$$

By Proposition 7.3 and (8.2) we conclude thatthe above term converges to zero as ϵ tends to zero. Therefore (2.10) holds.

2. Let Y be a compactly supported smooth vector field. The definition of $v(t,x,n)$ yields that

$$\begin{aligned} L(Y) &= \iiint v(t,x,n)n\cdot Y(t,x)dm = \iiint n\cdot Y(t,x)d\bar{m} \\ &= -\lim_{\epsilon\to 0}\iint z_i^\epsilon \nu^\epsilon\cdot Y\,\mu^\epsilon(t;dx)dt \\ &= -\lim_{\epsilon\to 0}\iint z_i^\epsilon\nu^\epsilon\cdot Y\frac{1}{\epsilon}(q'(\tfrac{z^\epsilon}{\epsilon}))^2 dx\,dt + \lim_{\epsilon\to 0}\iint z_i^\epsilon\nu^\epsilon\cdot Y\,d\xi^\epsilon\,dt. \end{aligned}$$

As in step 1, we can show that the second term in the above expression is zero. Next we use the identities,

$$z_i^\epsilon q'(\frac{z^\epsilon}{\epsilon}) = \epsilon\varphi_i^\epsilon, \quad \nu^\epsilon q'(\frac{z^\epsilon}{\epsilon}) = \frac{\epsilon\nabla\varphi^\epsilon}{|\nabla z^\epsilon|}, \quad g(\varphi^\epsilon) = \sqrt{2W(\varphi^\epsilon)} = q',$$

together with (OPE) and (8.6) to obtain,

$$\begin{aligned} L(Y) &= -\lim_{\epsilon\to 0}\iint \epsilon\varphi_i^\epsilon\nabla\varphi^\epsilon\cdot Y\frac{1}{|\nabla z^\epsilon|}dxdt \\ &= -\lim_{\epsilon\to 0}\iint q'(\frac{z^\epsilon}{\epsilon})\nabla\varphi^\epsilon\cdot Y\frac{1}{|\nabla z^\epsilon|}\theta^\epsilon dxdt \\ &\quad + \lim_{\epsilon\to 0}\iint \epsilon Y\cdot\nabla\varphi^\epsilon\left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon)\right]\frac{1}{|\nabla z^\epsilon|}dxdt \\ &= -\lim_{\epsilon\to 0}\iint \frac{1}{\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2\nu^\epsilon\cdot Y\theta^\epsilon dxdt \\ &\quad + \lim_{\epsilon\to 0}\iint \epsilon Y\cdot\nabla\varphi^\epsilon\left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon)\right]dxdt \\ &\quad + \lim_{\epsilon\to 0}\iint \epsilon Y\cdot\nabla\varphi^\epsilon\left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon)\right]\left(\frac{1-|\nabla z^\epsilon|}{|\nabla z^\epsilon|}\right)dxdt \\ &= \lim_{\epsilon\to 0} I^\epsilon - \iint H\cdot Y dt\mu(t;dx) + \lim_{\epsilon\to 0}E^\epsilon. \end{aligned} \tag{8.7}$$

3. In this step we show that E^ϵ converges to zero. By (8.4) we have,

$$|E^\epsilon| \le \|Y\|_\infty \left(\iint_{sptY} \epsilon\left[-\Delta\varphi^\epsilon + \frac{1}{\epsilon^2}W'(\varphi^\epsilon)\right]^2\right)^{\frac{1}{2}} \left(\iint_{sptY} \epsilon|\nabla\varphi^\epsilon|^2 \left(\frac{1-|\nabla z^\epsilon|}{|\nabla z^\epsilon|}\right)^2\right)^{\frac{1}{2}}.$$

$$\le K(Y)\left(\iint_{sptY} \frac{1}{\epsilon}(q'(\frac{z^\epsilon}{\epsilon}))^2(1-|\nabla z^\epsilon|)^2\right)^{\frac{1}{2}}.$$

By (4.1) we have,

$$\begin{aligned}(1-|\nabla z^\epsilon|)^2 &= (1-|\nabla z^\epsilon|)^2\chi_{\{|\nabla z^\epsilon|\ge 1\}} + (1-|\nabla z^\epsilon|)^2\chi_{\{|\nabla z^\epsilon|<1\}} \\ &\le K\epsilon(1+|z^\epsilon|)^2 + \left|1-|\nabla z^\epsilon|^2\right| .\end{aligned}$$

Hence

$$E^\epsilon \le K(Y)\left(\iint_{sptY} K\epsilon(1+|z^\epsilon|)^2\,d\mu^\epsilon + d|\xi^\epsilon|\right)^{\frac{1}{2}}.$$

Now by Proposition 7.3 and (8.1) we conclude that the limit of the above term is zero.

4. In the next two steps we will show that $\theta \in L^1_{loc}(d\mu)$ and that the limit of I^ϵ is equal to

$$I(\theta) := -\iiint \theta(t,x)\,n\cdot Y(t,x)dm(t,x,n).$$

In view of (2.5), there exists a sequence of smooth functions θ_k satisfying,

$$\lim_{k\to\infty} \|\theta_k-\theta\|_{2,T} = 0, \quad \sup_k \|\nabla\theta_k\|_{2,T} < \infty,$$

for every $T>0$, where $\|\cdot\|_{p,T}$ denote the $L^p((0,T)\times\mathcal{R}^d)$ norm. Fix $\lambda>0, T>0$. Then by Theorem 7.1, (7.1) and (2.5), we have

$$\begin{aligned}\int_0^T\int_{\mathcal{R}^d} |\theta_k-\theta_\ell|\,dt\mu(t;dx) &\le \int_0^T\int_{\mathcal{R}^d}[\frac{\lambda}{2}|\theta_k-\theta_\ell|^2 + \frac{1}{2\lambda}]dt\mu(t;dx) \\ &\le K[\frac{\lambda}{2}\|\nabla(|\theta_k-\theta_\ell|^2)\|_{1,T} + \frac{T}{\lambda}] \\ &\le K[\lambda\|\theta_k-\theta_\ell\|_{2,T}\|\nabla(\theta_k-\theta_\ell)\|_{2,T} + \frac{T}{\lambda}].\end{aligned}$$

Hence $\theta\in L^1_{loc}(d\mu)$ and

$$\lim_{k,\ell\to\infty}\left|\iiint(\theta_k-\theta_\ell)\,n\cdot Y\,dm\right| = 0.$$

Moreover $I(\theta)$ is well-defined and

$$I(\theta) = -\lim_{k\to\infty} \iiint \theta_k n \cdot Y dm.$$

5. Let $I(\theta)$, θ_k be as in Step 4. Then,

$$\begin{aligned} I(\theta) - I^\epsilon &= \iint \frac{1}{\epsilon}(q')^2 \nu^\epsilon \cdot Y\,(\theta^\epsilon - \theta_k)\,dx\,dt \\ &+ \iiint (\frac{1}{\epsilon}(q')^2\nu^\epsilon \cdot Y\,dx\,dt - n\cdot Y\,dm^\epsilon)\theta_k \\ &+ \iiint n\cdot Y\,\theta_k(dm^\epsilon - dm) \\ &+ \iiint n\cdot Y(\theta_k - \theta)dm. \end{aligned}$$

Since

$$[\frac{1}{\epsilon}(q')^2\nu^\epsilon \cdot Y\,dx\,dt - n\cdot Y\,dm^\epsilon] = \nu^\epsilon \cdot Y\,d\xi^\epsilon,$$

Proposition 7.3 and the convergence of m^ϵ to m yields,

$$\begin{aligned} \limsup_{\epsilon\to 0} |I^\epsilon - I(\theta)| &\le \limsup_{\epsilon\to 0} \left| \iint \frac{1}{\epsilon}(q')^2 \nu^\epsilon \cdot Y\,(\theta^\epsilon - \theta_k)\,dx\,dt \right| \\ &+ \left| \iiint n\cdot Y\,(\theta_k - \theta)\,dm \right|. \end{aligned}$$

Recall that θ^ϵ converges to θ strongly in L^2_{loc} (c.f. Proposition 3.4). So we proceed as in Step 4 to obtain,

$$\begin{aligned} &\limsup_{\epsilon\to 0} \left| \iint \frac{1}{\epsilon}(q')^2 \nu^\epsilon \cdot Y\,(\theta^\epsilon - \theta_k)\,dx\,dt \right| \\ &\le \|Y\|_\infty \limsup_{\epsilon\to 0} \iint_{sptY} |\theta^\epsilon - \theta_k|\,dt\,\mu^\epsilon(t;dx) \\ &\le K\|Y\|_\infty \left[\lambda\|\theta - \theta_k\|_{2,T} + \frac{T}{\lambda}\right], \end{aligned}$$

for any $\lambda > 0$. Now let first k and then λ go to infinity to show that I^ϵ converges to $I(\theta)$.

6. Combining the previous steps we conclude that

$$\iiint vn\cdot Y\,dm = -\iiint (\theta n + H)\cdot Y\,dm,$$

for any smooth vector field Y. ∎

Theorem C now follows from an attendant modification of Section 9 in [22]. Main tool in this proof is the following identity which is derived as in Step 1 of Theorem 5.1. Let ϕ be a compactly supported smooth function. Then,

$$\begin{aligned}\frac{d}{dt}\int \phi(x)\hat{\mu}^\epsilon(t;dx) &= -\int \phi\left[\epsilon\,(\varphi_t^\epsilon)^2 + |\nabla\theta^\epsilon|^2\right]dx + \frac{1}{2}\int \Delta\phi\,(\theta^\epsilon)^2\,dx \\ &\quad -\epsilon\int \nabla\phi\cdot\nabla\varphi^\epsilon\varphi_t^\epsilon dx.\end{aligned} \tag{8.8}$$

Since by (8.6)

$$\iint Y\cdot(H+n\theta)dm = \lim \iint \epsilon Y\cdot\nabla\varphi^\epsilon\varphi_t^\epsilon dxdt,$$

for any smooth vector field Y, (2.12) now formally follows from (8.6), (8.8).

References

[1] W. Allard, On the first variation of a varifold, Ann. of Math., 95/2 (1972), 417-491.

[2] N.D. Alikakos, P.W. Bates and X. Chen, Convergence of the Cahn-Hilliard equation to the Hele-Shaw model, (1993), preprint.

[3] F. Almgren and L. Wang, Mathematical existence of crystal growth with Gibbs-thompson curvature effects, (1993), preprint.

[4] G. Barles, H.M. Soner, and P.E. Souganidis, Front propagation and phase field theory, SIAM. J. Cont. Opt., March 1993, issue dedicated to W.H. Fleming.

[5] K.A. Brakke, The motion of a surface by its mean curvature, Princeton University Press, Princeton, NJ, (1978).

[6] L. Bronsard and R. Kohn, Motion by mean curvature as the singular limit of Ginzburg-Landau model, Jour. Diff. Equations, 90 (1991), 211-237.

[7] G. Caginalp, Surface tension and supercooling in solidification theory, Lecture Notes in Physics, 216 (1984), 216-226.

[8] G. Caginalp, An analysis of a phase field model of a free boundary, Arch. Rat. Mech. An., 92 (1986), 205-245.

[9] G. Caginalp, Stefan and Hele Shaw type models as asymptotic limits of the phase-field equations, Physical Review A, 39/11 (1989), 5887-5896.

[10] G. Caginalp and X. Chen, Phase field equations in the singular limit of sharp interface equations, (1992), preprint.

[11] X. Chen, Spectrums for the Allen-Cahn, Cahn-Hilliard and phase-field equations for generic interfaces, (1993), preprint.

[12] X. Chen and F. Reitich, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling, J. Math. Anal. Appl., 164/2 (1992), 350-362.

[13] M.G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial diffrential equations, Bull. AMS, 27/1 (1992), 1-67.

[14] J.B. Collins and H. Levine, Diffuse interface model of diffusion-limited crystal growth, Phys. Rev. B., 31 (1985), 6119-6122.

[15] L.C. Evans, Weak Convergence Methods for Nonlinear Partial Differential Equations, CBMS Regional Series in Mathematics 74, (1990).

[16] L.C. Evans and R. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Ann Harbor, (1992).

[17] L.C. Evans, H.M. Soner and P.E. Souganidis, Phase transitions and generalized motion by mean curvature, Comm. Pure Appl. Math., 45 (1992), 1097-1123.

[18] G. Fix, Phase field methods for free boundary problems, in Free boundary problems : theory and applications, edited by B. Fasano and M. Primicerio, Pitman, London (1983), 580-589.

[19] E. Fried and M. Gurtin, Continuum phase transitions with an order parameter; accretion and heat conduction, (1992), preprint.

[20] M.E. Gurtin, Thermodynamics of Evolving Phase Boundaries in the Plane, Oxford University Press, (1993).

[21] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Differential Geometry, 31 (1990), 285-299.

[22] T. Ilmanen, Convergence of the Allen-Cahn equation to the Brakke's motion by mean curvature, J. Differential Geometry, 38 (1993), 417-461.

[23] J.S. Langer, unpublished notes (1978).

[24] S. Luckhaus, Solutions of the two phase Stefan problem with the Gibbs-Thompson law for the melting temperature, European J. Appl. Math., 1 (1990), 101-111.

[25] W.W. Mullins and R.F. Sekerka, Stability of a planar interface during solidification of a dilute binary alloy, J. Appl. Phys., 35 (1964), 444-451.

[26] O. Penrose and P. Fife, Theormodynamically consistent models for the kinetics of phase transitions, Physica D, 43 (1990), North-Holland, 44-62.

[27] L. Simon, Lectures on Geometric Measue Theory, Centre for Mathematical Analysis, Australian National University, (1984).

[28] H.M. Soner, Ginzburg-Landau equation and motion by mean curvature,I: convergence, II: development of the initial interface, Jour. Geometric Analysis, (1993),to appear.

[29] B. Stoth, The Stefan problem with the Gibbs-Thompson law as singular limit of phase-field equations in the radial case, European J. Appl. Math., (1992), to appear.

[30] B. Stoth, Convergence of the Cahn-Hilliard equation to the Mullins-Sekerka problem in spherical symmetry, preprint.

[31] S.L. Wang, R.F. Sekerka, A.A. Wheeler, B.T. Murray, S.R. Coriell, R.J. Braun, and G.B. McFadden, Thermodynamically-consistent phase-field models for solidification, Physica D (1993), to appear.

[32] W. Ziemer, Weakly Differentiable Functions, Springer-Verlag, New York, (1989).