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Irene Fonseca
Carnegie Mellon University

Markos A. Katsoulakis
University of Wisconsin-Madison

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Irene Fonseca
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Markos A. Katsoulakis
Center for Mathematical Sciences
University of Wisconsin-Madison
Madison, WI 53706

1 Introduction

In this paper we show that *viscosity solutions* to *curvature evolution* equations may be obtained as limits of minimizers for Γ-limits of inhomogeneous, anisotropic singular perturbations for certain nonconvex variational problems. We consider the energy

$$I(u) := \int_\Omega W(u(x))dx$$

where Ω is an open, bounded, strongly Lipschitz domain of $\mathbb{R}^N$, $u : \Omega \to \mathbb{R}^n$, and W supports two phases, i.e. W has two isolated (global) minimum points a and b. The minimization of the energy $E(.)$ subject to fixed volume fraction θ, $0 < \theta < 1$, admits infinitely many solutions which are piecewise constant measurable functions of the form $u = \chi_A a + (1 - \chi_A)b$, with meas$(A) = \theta$ meas(Ω). In search of a selection criterion for resolving this non-uniqueness, we fix an initial phase-a configuration, A_0, and we introduce the family of perturbed problems

$$I_\epsilon^h(u) := \int_\Omega W(u(x))dx + \int_\Omega \epsilon^2\Lambda^2(x, \nabla u(x))dx + \int_\Omega \epsilon f_h(x, u(x); A_0)dx$$

where $\epsilon, h > 0$, $\partial^* A_0$ is the reduced boundary of A_0 (see Section 2) and $\Lambda(x,.)$ has linear growth. A particularly interesting example of the last contribution to the total energy is given by the density

$$f_h(x, u; A_0) := |u - \chi_{A_0}(x)a - (1 - \chi_{A_0}(x))b|^p g\left(\frac{d(x, \partial^* A_0)}{h}\right) \qquad (1.1)$$

where $d(x,\partial^* A_0)$ denotes the signed distance from x to $\partial^* A_0$.

In order to study the behavior of minimizing sequences, we rescale the energy to obtain

$$E_\epsilon^h(u;A_0) := \int_\Omega \frac{1}{\epsilon} W(u(x))dx + \epsilon \int_\Omega \Lambda^2(x,\nabla u(x))dx + \int_\Omega f_h(x,u(x);A_0)dx.$$

Using the results by [BF] with $f_h = 0$ (see also [Mo] for the isotropic, scalar case, [Bo] and [OSt] for the anisotropic scalar case and [Ba], [FT], [KoSt] for the isotropic, vectorial case), we identify the $\Gamma(L^1)$ -limit of $\{E_\epsilon^h(.;A)\}_{\epsilon>0}$, as $\epsilon \to 0$,

$$J_h(u;A_0) = \begin{cases} \int_{\partial_* A\cap\Omega} H(x,\nu(x))dH^{N-1}(x) + \int_\Omega f_h(x,u(x);A_0)dx & \text{if } u = \chi_A a + (1-\chi_A)b \\ & \text{and } Per(A) < +\infty \\ +\infty & \text{otherwise,} \end{cases}$$

where H is given by

$$H(x_0,\nu) := \inf\left\{ \int_{Q_\nu} W(\xi(y) + \Lambda^\infty(x_0,\nabla\xi(y))dy \,|\, \xi \in \mathcal{A}(\nu)\right\},$$

$$\mathcal{A}(\nu) := \Big\{\xi \in H^1(Q_\nu;\mathrm{IR}^n) \,|\, \xi(y) = a \text{ if } y\cdot\nu = -\tfrac{1}{2},\ \xi(y) = b \text{ if } y\cdot\nu = \tfrac{1}{2} \text{ and } \xi \text{ is periodic with period one in the directions of } \nu_1,\dots,\nu_{N-1}\Big\},$$

$\{\nu_1,\dots,\nu_{N-1},\nu\}$ forms an orthonormal basis of IR^N and S_ν is the strip

$$S_\nu := \left\{ y \in \mathrm{IR}^N \,|\, |y\cdot\nu| < \frac{1}{2}\right\}.$$

In the above $\Lambda^\infty(x,.)$ stands for the *recession function* of $k(x,.)$, given by

$$\Lambda^\infty(x,y) := \limsup_{t\to+\infty} \frac{\Lambda(x,ty)}{t}$$

and H^{N-1} is the $N-1$ dimensional Hausdorff measure. Denoting by

$$u_{j+1}^h := a\chi_{A_{j+1}^h} + (1-\chi_{A_{j+1}^h})b$$

the minimizer of $J_h(.;A_j^h)$ obtained as L^1 limit of a sequence $\{v_{\epsilon,h}^j\}$ of minimizers for $\{E_\epsilon^h(.,A_j^h)\}_{\epsilon>0}$, then $\{u_{j+1}^h\}$ generates a (generalized) minimizing movement $u(t) := a\chi_{A_t} + (1 - \chi_{A_t})b$ in the sense of De Giorgi [DG2] and we prove that, in accordance to De Giorgi's conjecture, the boundary of A_t moves with normal velocity

$$V = \Phi\,(K_H + a)\,, \tag{1.2}$$

where Φ is an increasing real function, a is a function depending on the derivatives of the inhomogeneous, anisotropic surface energy density H and K_H is the (anisotropic) curvature of

$$\Gamma_t := \partial A_t,$$

associated to H.

Our results hold globally in time, *past possible singularities* that may arise in the evolution $\{\Gamma_t\}_{t>0}$ even for smooth initial data Γ_0. We conclude that the notion of minimizing movements may indeed unify problems in the calculus of variations and in particular it can relate singular perturbations and their Γ-limits to generalized curvature flows. A prototypical example, capturing many important features of this class of equations, is the *motion by mean curvature*, i.e. where

$$V = \kappa = \sum_{i=1}^{N-1} \kappa_i,$$

V is the normal velocity of the interface Γ_t and $\kappa_1, ..., \kappa_{n-1}$ are its principal curvatures.

As noted earlier, surface evolutions depending on their curvature tensor can start out smooth and yet at a later time they may develop singularities, change topological type and exhibit various other pathologies. A great deal of work has been done recently in order to interpret the curvature evolution of surfaces *past singularities*. Here we will be using a combination of the so-called *level-set* and *distance-function* approaches. For a detailed description of all approaches, their relationship, as well as their consequences, we refer to the papers cited below and references therein.

The *level set* formulation represents the evolving surface as the level set of an auxiliary function solving an appropriate nonlinear partial differential equation. This approach has been extensively developed by Evans and Spruck in [ES1] in the case of motion by mean curvature and, independently, by Chen, Giga and Goto (see [CGG]), who considered evolutions of the type of (1.2). In both works, the analysis is based on the theory of *viscosity* *solutions* to fully nonlinear second order parabolic equations, which were introduced by Crandall and Lions in [CL] and Lions in [L]. For a detailed overview of the theory of viscosity solutions as well as a complete list of references (until at least 1991) we refer the reader to the *User's Guide* [CIL] by Crandall, Ishii and Lions.

The *distance-function* approach, introduced by Soner in [S] and later extended to very general situations by Barles, Soner and Souganidis (see [BSS]), is a more intrinsic one. It describes the motion in terms of the properties of the distance function to the evolving surface. For the precise relation between the two approaches, as well as their consequences, we refer to [BSS].

The generalized mean curvature evolution $\{\Gamma_t\}_{t\geq 0}$, defined in the viscosity sense, starting with a given closed surface $\Gamma_0 \subset \mathbb{R}^N$ exists for all $t \geq 0$. The sets obtained from the level set approach are uniquely defined as opposed to the ones obtained by the distance function, unless there is *no interface fattening.* The last issue is rather intriguing and we return to it later in the paper. Finally, the generalized mean curvature evolution agrees with the classical differential-geometric flow, as long as the latter exists.

These notions of generalized motion have been proven in several occasions to be the *right* way to extend the classical motion past singularities. One of the most definitive results in this direction was obtained by Evans, Soner and Souganidis (see [ESS]), who proved that the generalized mean curvature motion governs the asymptotic behavior of solutions to the following semilinear reaction-diffusion equation (*Allen-Cahn model*)

$$u_t^\lambda - \Delta u^\lambda + \frac{1}{\lambda^2} W'(u^\lambda) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty),$$

in the limit as $\lambda \to 0^+$, where W is a double well potential with wells of equal depth. In a statistical mechanics framework Katsoulakis and Souganidis studied the asymptotic behavior of Ising particle systems with dynamics of *Glauber-Kawasaki* type ([KS1]) and simple *Glauber with long range interactions* ([KS2]), obtaining again after appropriate rescaling of the stochastic system, generalized mean curvature evolutions, past possible singularities.

The work on this paper had just started before the conference on "Surface tension and movement by mean curvature " was held in Trento in July, 1992. There, discussing with DeGiorgi [DGF] we realized that it could be extended to a more general framework. During that same conference [ATW] announced their results on the time evolution of A_t, starting directly from the Γ-limit $J_h(.; A_0)$, assuming that H is homogeneous and f_h is given by a multiple of (1.1) with $g \equiv 1$ and proved that such evolutions agree with *classical smooth* flows starting with *smooth* initial data. Our methods differ from those of [ATW] except in what concerns the regularity properties of the reduced boundaries $\partial^* A^h_{j+1}$ (see Theorem 3.5). The proof of this regularity result presented here is that of [ATW], and by completeness, as well as for convenience for those readers less familiar with concepts from geometric measure theory, it is included in the Appendix.

The paper is organized as follows: in Section 2 we recall some results on functions of bounded variation and sets of finite perimeter as well as the integral representation for the Γ-limit of $\{E^h_\epsilon\}_{\epsilon>0}$ in the case where $f_h = 0$. Furthermore, we introduce the notion of viscosity solutions and discuss some of their fundamental properties. In Section 3 we determine the Γ-limit J_h of $\{E^h_\epsilon\}_{\epsilon>0}$ in the general case, we construct a minimizing movement associated to minimizers for J_h and in Theorem 3.6 we deduce the Euler-Lagrange equation satisfied by each of these minimizers, namely

$$f_h(x,b;A^h_j) - f_h(x,a;A^h_j) = K_H(x,\nu(x)) + \sum_{i=1}^N \frac{\partial^2 H}{\partial x_i \partial y_i}(x,\nu(x))$$

for a.e. $x \in \Omega$ and where $\nu(x)$ is the normal to $\partial^* A^h_{j+1}$ at x. This result relies heavily on the apriori knowledge of smoothness properties of the interfaces, Theorem 3.5. In Section 4 we prove that a minimizing movement is contained in the unique viscosity solution $\{\Gamma_t\}_{t>0}$ of (1.2); if such a solution has no interior then the minimizing movement coincides with the viscosity solution.

2 Preliminaries

In what follows $\Omega \subset \mathbb{R}^N$ is an open, bounded, strongly Lipschitz domain, $n, N \geq 1$, $\{e_1, \ldots, e_N\}$ is the standard orthonormal basis of $\mathbb{R}^N$ and $M^{n\times N}$ is the vector space of all $n \times N$ real matrices. If $A \in M^{n\times N}$ then $||A|| := \sqrt{tr(A^T A)}$.

Given $\nu \in S^{N-1} := \{x \in \mathbb{R}^N \mid |x| = 1\}$, we denote by Q_ν an open unit cube centered at the origin with two of its faces normal to ν, i.e. if $\{\nu_1, \ldots, \nu_{N-1}, \nu\}$ is an orthonormal basis of $\mathbb{R}^N$ then

$$Q_\nu := \left\{x \in \mathbb{R}^N \mid |\, x\cdot\nu_i \,|< \frac{1}{2},\ |\, x\cdot\nu \,|< \frac{1}{2},\ i = 1,\ldots,N-1\right\}.$$

Definition 2.1 ([DG1]) *Let $J_\epsilon, J_0 : L^1(\Omega;\mathbb{R}^n) \to \mathbf{R}$ be a family of functionals. We say that J_0 is the $\Gamma(L^1(\Omega))$ limit of J_ϵ if* *i) given any $u \in L^1(\Omega; R^n)$ and any sequence $\{u_\epsilon\}$ such that $u_\epsilon \to u$ in $L^1(\Omega; \mathbb{R}^n)$, then*

$$J_0(u) \leq \liminf_{\epsilon \to 0^+} J_\epsilon(u_\epsilon);$$

ii) for every $u \in L^1(\Omega; \mathbb{R}^n)$ there exists a sequence $\{u_\epsilon\}$ such that $u_\epsilon \to u$ in $L^1(\Omega; \mathbb{R}^n)$ and $J_0(u) = \lim_{\epsilon \to 0^+} J_\epsilon(u_\epsilon)$.

We recall briefly some properties of functions of bounded variation and sets of finite perimeter. For more details, we refer the reader to Evans and Gariepy [EG], Federer [F], Giusti [Giu] and Ziemer [Z].

Definition 2.2 *A function $u \in L^1(\Omega; \mathbb{R}^n)$ is said to be of bounded variation, $u \in BV(\Omega; \mathbb{R}^n)$, if for all $i \in \{1, \ldots, n\}, j \in \{1, \ldots, N\}$, there exists a finite measure μ_{ij} such that*

$$\int_\Omega u_i(x) \frac{\partial \varphi}{\partial x_j}(x) dx = - \int_\Omega \varphi(x) d\mu_{ij}(x)$$

for every $\varphi \in C_c^1(\Omega)$. The distributional derivative Du is the matrix-valued measure μ with entries μ_{ij}.

Definition 2.3 *A set $E \subset \Omega$ is said to be of finite perimeter in Ω (of finite perimeter, when $\Omega = \mathbb{R}^N$) if $\chi_E \in BV(\Omega)$, where χ_E denotes the characteristic function of E. The perimeter of E in Ω is defined by*

$$\mathrm{Per}_\Omega(E) := \sup \left\{ \int_E \mathrm{div}\ \varphi(x) dx \mid \varphi \in C_c^1(\Omega; \mathbb{R}^N), \|\varphi\|_\infty \leq 1 \right\}.$$

Theorem 2.4 [Isoperimetric Inequality]
If E is a bounded set of finite perimeter in $\mathbb{R}^N$ then

$$\mathcal{L}_N(E) \leq C(N) \mathrm{Per}_{\mathbb{R}^N}(E)^{\frac{N}{N-1}}$$

where $\mathcal{L}_N$ stands for the N-dimensional Lebesgue measure in $\mathbb{R}^N$.

Definition 2.5 *If E is a set of finite perimeter, we say that x_0 belongs to the reduced boundary of $E, x_0 \in \partial^* E$, if there exists $\nu_E(x_0) \in S^{N-1}$ such that*

i) $\mathrm{Per}_{B(x_0,\epsilon)}(E) > 0$ for all $\epsilon > 0$;

ii) $\lim_{\epsilon \to 0^+} \frac{1}{\mathrm{meas} B(x_0,\epsilon)} \int_{B(x_0,\epsilon)} D\chi_E = -\nu_E(x_0)$, and $\nu_E(x_0)$ is called the measure theoretic unit outer normal to E at x_0.

Theorem 2.6 [Structure theorem for sets of finite perimeter]
If E has locally finite perimeter in $\mathbb{R}^N$, then

i) $\partial^* E = \bigcup_{k=1}^{\infty} F_k \cup G$ *where* $H^{N-1}(G) = 0$ *and* F_k *is a compact subset of a* C^1*−hypersurface ;*

ii) $\mathrm{Per}_A(E) = H^{N-1}(\partial^* E \cap A)$ *for every Borel set* $A \subset \mathbb{R}^N$.

Theorem 2.7 [Generalized Gauss-Green Theorem]
If $E \subset \mathbb{R}^N$ has locally finite perimeter then

$$\int_E \mathrm{div}\ \varphi(x)dx = \int_{\partial^* E} \varphi \cdot \nu_E dH^{N-1}(x)$$

for all $\varphi \in C^1_c(\mathbb{R}^N; \mathbb{R}^N)$.

Definition 2.8 *Let $E \subset \mathbb{R}^N$ be a set of finite perimeter, $x_0 \in \mathbb{R}^N$. The $(N-1)$-upper density of $\partial^* E$ at x_0 is defined as*

$$\theta^{*N-1}(\partial^* E, x_0) := \limsup_{\epsilon \to 0^+} \frac{H^{N-1}(\partial^* E \cap B(x_0, \epsilon))}{\alpha(N-1)\epsilon^{N-1}}$$

where $\alpha(N-1) := H^{N-1}(B_{\mathbb{R}^{N-1}}(0,1))$.

It is possible to prove that if $x_0 \in \partial^* E$ then

$$\theta^{*N-1}(\partial^* E, x_0) \ = 1.$$

Lemma 2.9 *If E is a set of finite perimeter then*

$$\mathcal{L}_N(E) \le \frac{\mathrm{diam}\ E}{N} \mathrm{Per}(E).$$

Proof Fix $x_0 \in E$. By the Gauss-Green Theorem (see Theorem 2.7)

$$\begin{aligned} \mathcal{L}_N(E) &= \textstyle\int_E \frac{\mathrm{div}(x-x_0)}{N} dx \\ &= \tfrac{1}{N} \textstyle\int_{\partial^* E} (x - x_0) \cdot \nu_E(x) dH^{N-1}(x) \\ &\le \tfrac{1}{N} (\mathrm{diam}\ E) \textstyle\int_{\partial^* E} dH^{N-1}(x) \\ &= \tfrac{1}{N} (\mathrm{diam}\ E) \mathrm{Per}(E), \end{aligned}$$

where we have used Theorem 2.6 (ii). ∎

Let E be a set of finite perimeter, $x_0 \in \mathbb{R}^N$ and let

$$m(r) := H^{N-1}(\partial^* E \cap B(x_0, r)).$$

If E was smooth, e.g. if E was a polyhedral set, then

$$m'(r) = H^{N-2}(L_r)$$

where L_r is the boundary curve of $\partial^* E \cap B(x_0, r)$. Suppose that $x_0 \in \text{ext}E$ and let $x_0^* = \text{proj}_{\partial^* E} x_0$. If we consider the cone

$$C_\theta = x_1 \times (\partial^* E \cap B(x_0, r)) =$$

$$:= \{\lambda x_1 + (1-\lambda)x \mid x \in \partial^* E \cap B(x_0, r),\ \lambda \in [0,1]\}$$

where $x_1 = \theta x_0 + (1-\theta)x_0^*$, $\theta \in (0,1)$, we see that due to the isoperimetric inequality in $\mathbb{R}^{N-1}$ (see Theorem 2.4)

$$\lim_{\theta \to 0^+} H^{N-1}(\partial^* C_\theta \setminus (\partial^* E \cap B(x_0, r))) \le C(N-1)[m'(r)]^{\frac{N-1}{N-2}}$$

and so, there exists $\theta \in (0,1)$ such that

$$H^{N-1}(\partial^* C_\theta \setminus (\partial^* E \cap B(x_0, r)) \le 2C(N-1)[m'(r)]^{\frac{N-1}{N-2}}. \tag{2.1}$$

Note that

$$C_\theta = (x_0 \times (\partial^* E \cap B(x_0, r)) \setminus (x_0 \times \partial_L^* C_\theta)$$

where $\partial_L^* C_\theta$ stands for the lateral surface of C_θ. In this case, $\partial^*(E \cup C_\theta) = \partial^* E \setminus (\partial^* E \cap B(x_0, r)) \cup X$ where $X = \partial_L^* C_\theta = \partial^* C_\theta \setminus (\partial^* E \cap B(x_0, r))$ is the lateral surface of a cone satisfying, by (2.1),

$$H^{N-1}(X) \le C'(N-1)[m'(r)]^{\frac{N-1}{N-2}}. \tag{2.2}$$

If $x_0 \in \text{int } E$ then the cone C_θ will be interior to the polyhedron and $\partial^*(E \setminus C_\theta) = \partial^* E \setminus (\partial^* E \cap B(x_0, r)) \cup X$. If $x_0 \in \partial^* E$, let $x_1 := x_0 - r\nu_E(x_0)$ and it is easy to verify that

$$\partial^*(E \setminus C_\theta) = \partial^* E \setminus (\partial^* E \cap B(x_0, r)) \cup X$$

and that (2.2) is satisfied. We conclude that for every $x_0 \in \mathbb{R}^N$ there exists a set Q (where $Q = E \cup C_\theta$ if $x_0 \in \text{ext}E$ and $Q = E \backslash C_\theta$ if $x_0 \in \bar{E}$) such that $E\Delta Q \subset B(x_0, r)$,

$$\partial^*(E\Delta Q) \subset (\partial^* E \cap B(x_0, r)) \cup X,$$

$$H^{N-1}(X) \leq C'(N-1)[m'(r)]^{\frac{N-1}{N-2}},$$

$$\partial^* Q \subset X \cup (\partial^* E \backslash B(x_0, r)).$$

Here, and in what follows, we use the notation

$$A\Delta B := (A \setminus B) \cup (B \setminus A).$$

The analogue of this property for arbitrary sets of finite perimeter may be stated as follows.

Lemma 2.10 *There exists a constant $C = C(N-1)$ such that if E is a set of finite perimeter, $x_0 \in \mathbb{R}^N$, $m(r) := H^{N-1}(\partial^* E \cap B(x_0, r))$ and $m(r) > 0$ for all $r > 0$, then for $\mathcal{L}_1$ a.e. $r > 0, H^{N-1}(\partial^* E \cap \partial B(x_0, r)) = 0$ and there exists a set of finite perimeter Q such that*

$$Q\Delta E \subset B(x_0, r),$$

$$\partial^*(E\Delta Q) \subset (\partial^* E \cap B(x_0, r)) \cup X,$$

$$H^{N-1}(X) \leq C(N-1)[m'(r)]^{\frac{N-1}{N-2}}$$

and

$$\partial^* Q \subset (\partial^* E \backslash B(x_0, r)) \cup X.$$

The proof of this result relies heavily on geometric measure theory, and so we only sketch the proof following [ATW].

Using slicing properties of integral currents by Lipschitz functions (here we use the slicing of $\partial^* E$ by $\rho(x) := |x - x_0|$), for $\mathcal{L}_1$a.e. $r > 0$, we have that $\partial(\partial^* E \cap B(x_0, r))$ is an integral $N-2$ current (see [F], 4.3) with mass

$$M(\partial(\partial^* E \cap B(x_0, r)) \leq C_1(N) m'(r).$$

As $\partial[\partial(\partial^* E \cap B(x_0, r))] = 0$, there exists a $N-1$ integral current X with $\partial X = \partial(\partial^* E \cap B(x_0, r))$, supp $X \subset$ convex hull $\partial(\partial^* E \cap B(x_0, r))$ such that

$$\begin{aligned} H^{N-1}(X) &\leq C_2(N)[M(\partial(\partial^* E \cap B(x_0, r)))]^{\frac{N-1}{N-2}} \\ &\leq C_3(N)[m'(r)]^{\frac{N-1}{N-2}} \end{aligned} \quad (2.3)$$

where we used a result by Federer (see [F] 4.2.10 and also [A2]) ensuring the existence of X satisfying the isoperimetric inequality (2.3). It suffices to consider the cone

$$C := x_0 \times (\partial^* E \cap B(x_0, r)) \setminus (x_0 \times X).$$

The following necessary condition for lower semicontinuity can be found in Reshetnyak [R] (see also Goffman and Serrin [GS] and [Fo]).

Theorem 2.11 *Let $f \in C(\Omega \times S^{N-1})$ be a nonnegative function and extend $f(x, \cdot)$ as a positively homogeneous of degree one function. If*

$$\int_{\partial^* E \cap \Omega} f(x, \nu_E(x)) dH^{N-1}(x) \leq \liminf_{n \to +\infty} \int_{\partial^* E_n \cap \Omega} f(x, \nu_{E_n}(x)) dH^{N-1}(x)$$

whenever E_n, E are sets of finite perimeter such that $\mathcal{L}_N(E_n \Delta E) \underset{n \to +\infty}{\rightarrow} 0$, then $f(x, \cdot)$ is convex for every $x \in \Omega$.

Let $W : \mathbb{R}^n \to [0, +\infty)$ and $\Lambda : \Omega \times M^{n \times N} \to [0, +\infty)$ be continuous functions satisfying the following hypotheses:

(H1) $W(u) = 0$ if and only if $u \in \{a, b\}$;

(H2) there exists a constant $C > 0$ such that

$$\frac{1}{C}|u|^p - C \leq W(u) \leq C(1 + |u|^p)$$

for some $p \geq 2$, and for all $u \in \mathbb{R}^n$;

(H3)

$$\frac{1}{C}||A|| - C \leq \Lambda(x, A) \leq C(1 + ||A||)$$

for all $x \in \Omega, A \in M^{n \times N}$.

Let $\Lambda^\infty : \Omega \times M^{n \times N} \to [0, +\infty)$ be the *recession function* of Λ, i.e.

$$\Lambda^\infty(x, A) := \limsup_{t \to +\infty} \frac{\Lambda(x, tA)}{t}.$$

In addition, we assume that

(H4) there exist $0 < m < 2, C', L > 0$ such that

$$\left|(\Lambda^\infty)^2(x,A) - \frac{\Lambda^2(x,tA)}{t^2}\right| \le C' \frac{\|A\|^{2-m}}{t^m}$$

for all $(x,A) \in \Omega \times M^{n\times N}$ and for all $t > 0$ such that $t\|A\| > L$;

(H5) for all $x_0 \in \Omega$ and for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$| \Lambda^2(x_0, A) - \Lambda^2(x,A) | \le \epsilon C(1 + \|A\|^2)$$

whenever $|x - x_0| < \delta$;

(H6) for every $h > 0$, $f_h : \Omega \times \mathbb{R}^n \times \{A \subset \mathbb{R}^N \mid A \text{ is measurable}\} \to \mathbb{R}$ is a bounded, Carathéodory function (i.e., for any A, $f(\cdot,\cdot,A)$ is measurable in the first variable and continuous in the second one) such that

$$| f_h(x,u;A) | \le g_h(\mathcal{L}_N(A))(1 + |u|^p)$$

for all $u \in \mathbb{R}^n$ and some $g_h \in L^\infty_{loc}(\mathbb{R})$;

(H7) $f_h(x,a;A) = 0$ if $x \in A$, $f_h(x,b;A) = 0$ if $x \notin A$.

We consider the family of anisotropic singular perturbations

$$E^h_\epsilon(u,A) := \int_\Omega \left[\frac{1}{\epsilon}W(u(x)) + \epsilon\Lambda^2(x,\nabla u(x))\right] dx + \int_\Omega f_h(x,u(x);A)dx$$

for $\epsilon, h > 0$. Heuristically, we want to show that as $h, \epsilon \to 0^+$, minimizers of E^h_ϵ will converge in some sense to a piecewise constant u with two phases, a and b, and the interface separating the two phases moves by mean curvature, i.e.

$$u_t = |\nabla u| \; \Phi(K_H)$$

where K_H is the mean curvature of the interface associated to a possibly anisotropic surface energy H and Φ an increasing real function, related to f (see Sections 3 and 4 below).

An example of functions f_h satisfying (H6) and (H7) for which this property is satisfied is given by (see Section 4)

$$f_h(x,u;A) := | u - a\chi_A(x) - b(1 - \chi_A(x)) |^p \, g\left(\frac{d(x,\partial^* A)}{h}\right)$$

and d is the signed distance function (see (2.13)).

Let $\nu \in S^{N-1}$ and define

$$\begin{aligned}\mathcal{A}(\nu): \ = \{&\xi \in H^1(\mathbb{R}^n \cap \{| x \cdot \nu |< \tfrac{1}{2}\}; \mathbb{R}^n) \mid \xi(y) = a \text{ if } y \cdot \nu = -\tfrac{1}{2}, \\ &\xi(y) = b \text{ if } y \cdot \nu = \tfrac{1}{2} \text{ and } \xi \text{ is periodic with period one} \\ &\text{in the remaining directions } \nu_1, \ldots, \nu_{N-1}\},\end{aligned}$$

where the boundary values of ξ are understood in the sense of traces. Recall that a function ξ is periodic with period one in the direction of ν_i if

$$\xi(y) = \xi(y + k\nu_i)$$

for all $k \in \mathbf{Z}, y \in \mathbb{R}^n \cap \{| x \cdot \nu |< \frac{1}{2}\}$.

We define the inhomogeneous, anisotropic surface energy density

$$\begin{aligned}H(x_0, \nu_0) \ := \inf\Big\{&\textstyle\int_{Q_{\nu_0}} M\, W(\xi(y)) \\ &+ \tfrac{1}{M} (\Lambda^\infty)^2 (x_0, \nabla\xi(y)) dy \mid M > 0, \xi \in \mathcal{A}(\nu_0)\Big\}.\end{aligned}$$

In [BF] it was shown that

$$I_\epsilon \overset{\Gamma(L^1(\Omega))}{\underset{\epsilon \to 0^+}{\longrightarrow}} J_0 \tag{2.4}$$

where

$$I_\epsilon(u) := \int_\Omega \frac{1}{\epsilon} W(u) + \epsilon \Lambda^2(x, \nabla u(x)) dx, \ u \in H^1(\Omega; \mathbb{R}^n)$$

and

$$J_0(u) := \begin{cases} \int_{\partial^* A \cap \Omega} H(x, \nu(x)) dH^{N-1}(x) & \text{if } u(x) = a\chi_A(x) + b(1 - \chi_A(x), \\ & Per_\Omega(A) < +\infty \\ +\infty\,, & \text{otherwise.} \end{cases}$$

As remarked by Gurtin [Gu2], the assumption that the two potential wells of W have equal depth involves no loss of generality, as we can always add an affine function of u to the functional E^h_ϵ.

In the isotropic, scalar-valued case, i.e. $u : \Omega \to \mathbb{R}$ and $\Lambda = \|\cdot\|$, the $\Gamma(L^1(\Omega))$–limit of I, J_0, was studied by Gurtin [Gu1], [Gu2] and Modica [Mo]. This result was generalized by Owen and Sternberg [OSt] and by Bouchitté [Bo] to anisotropic functions Λ with linear growth such that Λ^2 is convex. The isotropic vector-valued case, $u : \Omega \to \mathbb{R}^n (n > 1)$ and $\Lambda = \|\cdot\|$, was studied by Kohn and Sternberg [KoSt], Sternberg [St] and by [FT].

We remark that, as Ω is a strongly Lipschitz domain, $\mathrm{Per}_\Omega(A) < +\infty$ if and only if $\mathrm{Per}(A) < +\infty$. Also

Proposition 2.12 *i) $H(x,\nu)$ is continuous for every $\nu \in S^{N-1}$;*

ii) $H(x,\cdot)$ is convex, positively homogeneous of degree one for every $x \in \Omega$;

iii) there exist $\alpha, \beta > 0$ such that

$$\alpha \le H(x,\nu) \le \beta$$

for every $(x,\nu) \in \Omega \times S^{N-1}$.

Proof

i) The proof can be found in [BF], Proposition 2.8.

ii) Due to Theorem 2.11, it suffices to prove that if E_k, E have finite perimeter, $\mathcal{L}_N(E_k \Delta E) \underset{k\to+\infty}{\longrightarrow} 0$, then

$$\int_{\partial^* E\cap\Omega} H(x,\nu_E(x))dH^{N-1}(x) \le \liminf_{k\to+\infty} \int_{\partial^* E_k\cap\Omega} H(x,\nu_{E_k}(x))dH^{N-1}(x).$$

Indeed, by part ii) of Definition 2.1 and by (2.4), for every $k \in \mathbf{N}$ there exists $u_k \in H^1(\Omega;\mathbb{R}^n)$ such that

$$\|u_k - a\chi_{Ek} - b(1-\chi_{E_k})\|_{L^1} < \frac{1}{k}$$

and

$$\left| I_{\epsilon_k}(u_k) - \int_{\partial^* E_k\cap\Omega} H(x,\nu_{E_k}(x))dH^{N-1}(x)\right| < \frac{1}{k},$$

for some $\epsilon_k \underset{k\to+\infty}{\longrightarrow} 0^+$. Then $u_k \to a\chi_E + b(1-\chi_E)$ in L^1 and by Definition 2.1 (i) and (2.4)

$$\begin{aligned}\textstyle\int_{\partial^* E\cap\Omega} H(x,\nu_E)dH^{N-1} &\le \liminf_{k\to+\infty} I_{\epsilon_k}(u_k)\\ &= \liminf_{k\to+\infty} \textstyle\int_{\partial^* E\cap\Omega} H(x,\nu_{E_k}(x))dH^{N-1}(x).\end{aligned}$$

iii) It is easy to see that (H3) yields

$$\frac{1}{C}\|A\| \leq \Lambda^{\infty}(x,A) \leq C\|A\|.$$

Hence, given $\xi_0 \in \mathcal{A}(\nu)$ defined as $\xi_0(x) := (x\cdot\nu)b + (1-(x\cdot\nu)a) + \frac{b-a}{2}$,

$$\begin{aligned} H(x,\nu) &\leq \int_{Q_\nu} W(\xi_0(y)) + (\Lambda^\infty)^2(x,\nabla\xi(y))dy \\ &\leq \int_{Q_\nu} W(\xi_0(y)) + C^2 \parallel \nabla\xi_0(y) \parallel^2 dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[W(tb+(1-t)a) + \tfrac{b-a}{2} + C^2 \mid b-a \mid^2\right] dt \\ &:= \beta. \end{aligned}$$

On the other hand,

$$\begin{aligned} H(x,\nu) = &\ \inf_{\xi\in\mathcal{A}(\nu),M>0} \int_{Q_\nu} M\, W(\xi(y)) + \frac{1}{M}(\Lambda^\infty)^2(x,\nabla\xi(y))dy \\ \geq &\ 2 \inf_{\xi\in\mathcal{A}(\nu)} \int_{Q_\nu} \sqrt{W(\xi(y))}\,\frac{1}{C} \parallel\nabla\xi(y)\parallel dy \\ = &\ \tfrac{2}{C} \inf_{\xi\in\mathcal{A}(e_N)} \int_{Q_{e_N}} \sqrt{W(\xi(y))}\ \ \|\nabla\xi(y)\|\, dy \end{aligned}$$

because if $\eta \in \mathcal{A}(e_N)$ then $\xi(y) := \eta(Ry) \in \mathcal{A}(\nu)$ where $\nu = R^T e_N, R$ is a rotation, $Q_\nu = R^T Q_{e_N}$ and

$$\begin{aligned} \int_{Q_\nu} \sqrt{W(\xi(y))}\, \|\nabla\xi(y) \parallel dy &= \int_{R^T Q_{e_N}} \sqrt{W(\eta(Ry))}\ \parallel \nabla\eta(Ry)R \parallel dy \\ &= \int_{Q_{e_N}} W(\eta(x)\ \ \||\ \nabla\eta(x) \parallel dx. \end{aligned}$$

By results in [FT],

$$\begin{aligned} \alpha := &\ \inf_{\xi\in\mathcal{A}(e_N)} \int_{Q_{e_N}} \sqrt{W\xi(y))}\ \parallel\nabla\xi(y)\parallel dy \\ = &\ \inf\Big\{ \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{W(g(t))}\ |g'(t)|\, dt \mid g \text{ is piecewise } C^1, \\ &\ g\left(-\tfrac{1}{2}\right) = a, g\left(\tfrac{1}{2}\right) = b\Big\} \end{aligned}$$

which is the geodesic distance between a and b with respect to the Riemannian (degenerate) metric associated to $\sqrt{W}$. Thus $\alpha > 0$. ∎

Now we recall the *level-set* definition of generalized curvature evolution with normal velocity

$$V = \Phi\left(K_H + a(x,\nu)\right) \tag{2.5}$$

where Φ is increasing, $H(x,\cdot)$ is as before, homogeneous of degree one, and convex, K_H is given by (3.12) in Section 3, ν is the unit normal to the interface and also $\nabla^2 H \in C(\mathbb{R}^N - \{0\})$. In addition

$$a(x,y) := \sum_{i=1}^{N} \frac{\partial^2 H(x,y)}{\partial x_i \partial y_i}. \tag{2.6}$$

Given a closed set $\Gamma_0 \subset \mathbb{R}^N$, $n \geq 2$, choose $h \in UC(\mathbb{R}^N)$, where $UC(\Omega)$ denotes the space of uniformly continuous functions defined on Ω, satisfying

$$\Gamma_0 = \{r \in \mathbb{R}^N : h(r) = 0\} \tag{2.7}$$

and consider the curvature evolution PDE associated to H and Φ

$$u_t - |\nabla u| \Phi\left(\frac{1}{|\nabla u|}\left\{\mathrm{tr}\left[\nabla^2 H\left(x, \frac{\nabla u}{|\nabla u|}\right)\nabla^2 u\right] + a\left(x, \frac{\nabla u}{|\nabla u|}\right)\right\}\right) = 0 \tag{2.8}$$

$$\text{in} \quad \mathbb{R}^N \times (0,\infty)$$

$$u(x,0) = h(x) \qquad \text{in} \quad \mathbb{R}^N. \tag{2.9}$$

In the particular case of *motion by mean curvature*, $\Phi(r) = r$ and $H(x,y) = |y|$, the above equation reduces to

$$u_t - \mathrm{tr}\left[\left(I - \frac{Du \otimes Du}{|Du|^2}\right) D^2 u\right] = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty). \tag{2.10}$$

We consider the closed sets

$$\Gamma_t := \{r \in \mathbb{R}^n : u(r,t) = 0\}, \qquad t \geq 0 \tag{2.11}$$

and expect that $\{\Gamma_t\}_{t\geq 0}$ is the evolution by the curvature rule (2.5) starting from Γ_0 (see [AG], [Gu3], [Gu4]). As we mentioned in the introduction, such evolving surfaces may start out smooth and yet develop singularities at some later time. However we can define weak solutions of (2.8) (hence surfaces Γ_t) using the notion of viscosity solutions:

Definition 2.13 *i) A function $u : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is a viscosity subsolution of (2.8)-(2.9) if for all smooth functions $\phi : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ and all local maximum points (x_0, t_0) of $u^* - \phi$,*

$$\phi_t(x_0,t_0) -$$

$$|\nabla\phi(x_0,t_0)|\Phi\left(\frac{1}{|\nabla\phi(x_0,t_0)|}\left\{\mathrm{tr}\left[\nabla^2 H\left(x_0,\frac{\nabla\phi(x_0,t_0)}{|\nabla\phi(x_0,t_0)|}\right)\nabla^2\phi(x_0,t_0)\right]\right.\right.$$

$$\left.\left.+a\left(x_0,\frac{\nabla\phi(x_0,t_0)}{|\nabla\phi(x_0,t_0)|}\right)\right\}\right) \leq 0 \quad \textit{if} \quad \nabla\phi(x_0,t_0) \neq 0,$$

$$\phi_t(x_0,t_0) \leq 0 \qquad \textit{otherwise}$$

and

$$u(\cdot,0) \leq h.$$

ii) A function $u : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ *is a viscosity supersolution of (2.8)-(2.9) if for all smooth functions* $\phi : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ *and all local minimum points* (x_0,t_0) *of* $u_* - \phi$,

$$\phi_t(x_0,t_0)-$$

$$|\nabla\phi(x_0,t_0)|\Phi\left(\frac{1}{|\nabla\phi(x_0,t_0)|}\left\{\mathrm{tr}\left[\nabla^2 H\left(x_0,\frac{\nabla\phi(x_0,t_0)}{|\nabla\phi(x_0,t_0)|}\right)\nabla^2\phi(x_0,t_0)\right]\right.\right.$$

$$\left.\left.+a\left(x_0,\frac{\nabla\phi(x_0,t_0)}{|\nabla\phi(x_0,t_0)|}\right)\right\}\right) \geq 0 \quad \textit{if} \quad \nabla\phi(x_0,t_0) \neq 0,$$

$$\phi_t(x_0,t_0) \geq 0 \qquad \textit{otherwise}$$

and

$$u(\cdot,0) \geq h.$$

iii) If a function u is both a viscosity sub- and supersolution then we say that u is a viscosity solution of (2.8)-(2.9).

In the latter definition we used the notation u^* and u_* to denote the upper- (resp. lower-) semicontinuous envelopes of a function u, defined as

$$u^*(z) := \lim_{r\to 0} \quad \sup\{u(y) : |z-y| < r\}$$

and

$$u_*(z) := \lim_{r\to 0} \quad \inf\{u(y) : |z-y| < r\}.$$

It was proved in [ES1] for (2.10) and in [CGG] for a more general equation (see also [BSS] for a mild relaxation of the assumptions of [CGG] and [ES1] and [IS] for noncompact hypersurfaces) that the initial value problem (2.8)-(2.9) admits a unique solution $u \in UC(\mathbb{R}^n \times [0,\infty)\ )$ (see [BSS], [CGG], [ES1], [IS] for the relevant proofs, comments etc.). In particular the following theorems hold.

Theorem 2.14 [Comparison Principle]
Let u and v be, respectively, viscosity sub- and supersolutions of 2.8 in $\mathbb{R}^N \times [0,T]$ *for some $T > 0$. If $u^*(0,x) \leq v_*(0,x)$ then $u^* \leq v_*$ in* $\mathbb{R}^N \times [0,T]$.

This result, together with Perron's method (see [CIL]), yields:

Theorem 2.15 *There exists a unique continuous viscosity solution of (2.8)-(2.9) .*

The solution of (2.8) describes a *geometric* evolution of level sets, therefore the evolution should be invariant under any arbitrary relabelling of the initial level set. Indeed, the following theorem, which will be used subsequently, can be proved along the lines of [ES1, Theorem 2.8].

Theorem 2.16 *Assume that u is the viscosity solution of (2.8) and let Ψ : $\mathbb{R} \to \mathbb{R}$ be a nondecreasing function. Then*

$$v :\equiv \Psi(u)$$

is the viscosity solution of (2.8) with $v(0,x) = \Psi(h(x))$.

We conclude with the *distance-function* definition of generalized mean curvature evolution. Let Γ, P and $R \subset \mathbb{R}^N \times [0,\infty)$ be such that

$$\Gamma \cup P \cup R = \mathbb{R}^N \times [0,\infty) \quad \text{and} \quad \Gamma \cap \mathrm{P} = \Gamma \cap \mathrm{R} = \emptyset, \tag{2.12}$$

define Γ_t to be the t-section of Γ and let $d(x,t)$ be the *signed-distance* between x and Γ_t, i.e.

$$D(x,t) = \begin{cases} \mathrm{dist}(x,\Gamma_t) & (x,t) \in P, \\ -\mathrm{dist}(x,\Gamma_t) & (x,t) \in R, \\ 0 & (x,t) \in \Gamma, \end{cases} \tag{2.13}$$

with the understanding that $dist(x,\Gamma_t) = \infty$ for all x if $\Gamma_t = \emptyset$,and where $\mathrm{dist}(x,A)$ denotes the usual nonnegative distance from x to the set A.

We say that $\{\Gamma_t\}_{t\geq 0}$ is the (distance function) *generalized motion by mean curvature starting from* Γ_0 if $(D \vee 0)^*$ is a supersolution and $(D \wedge 0)_*$ is a subsolution of (2.8) (see [S] and [BSS]). Here, and in what follows, we use the notation

$$a \vee b := \max\{a,b\}, \qquad a \wedge b := \min\{a,b\}.$$

In the case where Γ_t is defined by (2.11), it turns out that the two definitions given above are equivalent if and only if there is no interface fattening, i.e. if and only if the set

$$\cup_{t>0}(\Gamma_t \times \{t\})$$

has *empty interior* in $\mathbb{R}^N \times [0,\infty)$ (see [BSS], [S], etc.). Here, as it is standard in the literature, we say that $\mathrm{int}\Gamma_t = \emptyset$, where $\Gamma_t = \{x| \quad u(x,t) = 0\}$, if given $u(x,t) = 0$ there exist sequences $x_n \to x$, $y_n \to x$, such that

$$u(x_n,t) > 0, \; u(y_n,t) < 0 \quad \text{for all } n \in \mathbf{N}.$$

If there is interior, then there is no uniqueness in the distance function approach but rather a minimal and maximal front (see [S] for the details). Whether interior develops or not is a rather intriguing issue; we refer to [BSS] for a detailed discussion and a general sufficient condition yielding no fattening.

3 Γ-limit and generalized minimizing movement for E^h_ϵ

Let $A \subset \Omega$ be a set of finite perimeter, $\epsilon, h > 0$, and set

$$E^h_\epsilon(u;A) := \int_\Omega \left[\frac{1}{\epsilon}W(u(x)) + \epsilon\Lambda^2(x,\nabla u(x))\right] dx + \int_\Omega f_h(x,u(x);A)dx$$

where $u \in H^1(\Omega;\mathbb{R}^n)$.

Theorem 3.1 *The $\Gamma(L^1(\Omega))$-limit of E^h_ϵ as $\epsilon \to 0^+$ is*

$$J_h(u;A) := \begin{cases} \int_{\partial^* L\cap\Omega} H(x,\nu(x))dH^{N-1}(x) + \int_\Omega f_h(x,u(x);A)dx & \text{if } u = a\chi_L + b(1-\chi_L), \\ & \mathrm{Per}(L) < +\infty \\ +\infty, & \text{otherwise.} \end{cases}$$

Before proving this theorem we show that

Lemma 3.2 *Let $h > 0$ be fixed. If $v_\epsilon \in H^1(\Omega;\mathbb{R}^n)$ is such that $v_\epsilon \to u$ in $L^1(\Omega;\mathbb{R}^n)$ and if*

$$\limsup_{\epsilon\to 0^+} E^h_\epsilon(v_\epsilon;A) < +\infty \tag{3.1}$$

then

$$u(x) = a\chi_L(x) + b(1 - \chi_L(x))$$

for some set of finite perimeter $L \subset \Omega$, *and*

$$v_\epsilon \to u \text{ in } L^p(\Omega; \mathbb{R}^n).$$

Proof For a subsequence, and by (H2) and (H6), we have

$$\lim_{\epsilon \to 0^+} \int_\Omega \frac{1}{\epsilon} W(v_\epsilon) dx < +\infty$$

and so by Fatou's lemma

$$\int_\Omega W(u(x))dx \leq \limsup_{\epsilon \to 0^+} \int_\Omega W(v_\epsilon(x))dx = 0 \,, \tag{3.2}$$

i.e.

$$u(x) = a\chi_L(x) + b(1 - \chi_L(x))$$

for some measurable set L. As shown in [BF], Proposition 3.1, Step 1, if $\text{Per}(L) = +\infty$ then

$$\lim_{\epsilon \to 0^+} I_\epsilon(v_\epsilon) = +\infty$$

contradicting (3.1), thus $\text{Per}(L) < +\infty$. Moreover, by (H2)

$$\begin{aligned} |v_\epsilon - u|^p \leq & \; C[|v_\epsilon|^p + |u|^p] \\ \leq & \; C'[W(v_\epsilon) + 1] \end{aligned}$$

and so, extracting a subsequence for which $v_\epsilon(x) \to u(x)$ a.e. $x \in E$, by Fatou's Lemma we obtain

$$\begin{aligned} C'\mathcal{L}_N(\Omega) = & \; \int_\Omega \liminf_{\epsilon \to 0^+} \{C'[W(v_\epsilon(x)) + 1] - |v_\epsilon(x) - u(x)|^p\} \, dx \\ \leq & \; \liminf_{\epsilon \to 0^+} \int_\Omega \{C'[W(v_\epsilon(x)) + 1] - |v_\epsilon(x) - u(x)|^p\} \, dx \\ = & \; C'\mathcal{L}_N(\Omega) - \limsup_{\epsilon \to 0^+} \int_\Omega |v_\epsilon(x) - u(x)|^p dx \end{aligned}$$

where we have used (3.2). Hence

$$\| v_\epsilon - u \|^p_{L^p(\Omega;\mathbb{R}^n)} \underset{\epsilon \to 0^+}{\to} 0.$$

∎

Proof of Theorem 3.1. Let $u \in L^1(\Omega; \mathbb{R}^n)$, $u_\epsilon \in H^1(\Omega; \mathbb{R}^n), u_\epsilon \to u$ in $L^1(\Omega; \mathbb{R}^n)$ with

$$\liminf_{\epsilon \to 0^+} E_\epsilon^h(u_\epsilon; A) < +\infty.$$

By Lemma 3.2

$$u(x) = a\chi_L(x) + b(1 - \chi_L(x)) \quad \text{a.e. } x \in \Omega$$

with $\mathrm{Per}(L) < +\infty$ and

$$\| u_\epsilon - u \|_{L^p} \underset{\epsilon \to 0^+}{\to} 0. \tag{3.3}$$

By [BF] (see (2.4))

$$\liminf_{\epsilon \to 0^+} I_\epsilon(u_\epsilon) \geq J_0(u)$$

and by Fatou's Lemma, (3.3) and (H6)

$$\lim_{\epsilon \to 0^+} \int_\Omega f_h(x, u_\epsilon(x); A)dx = \int_\Omega f_h(x, u; A)dx$$

and we conclude that

$$\begin{aligned} J_h(u; A) = & \ J_0(u) + \textstyle\int_\Omega f_h(x, u; A)dx \\ \leq & \ \liminf_{\epsilon \to 0^+} \{I_\epsilon(u_\epsilon) + \int_\Omega f_h(x, u; A)dx\} \\ = & \ \liminf_{\epsilon \to 0^+} E_\epsilon^h(u_\epsilon). \end{aligned}$$

Conversely, choose $v_\epsilon \in H^1(\Omega; R^n)$ such that $v_\epsilon - u$ in L^1 and $I_\epsilon(v_\epsilon) \underset{\epsilon \to 0^+}{\to}$ $J_0(u)$, $J_0(u) < +\infty$. By Lemma 3.2

$$v_\epsilon \underset{\epsilon \to 0^+}{\to} u \text{ in } L^p$$

and so by Fatou's Lemma, (3.3) and (H6)

$$\int_\Omega f_h(x, v_\epsilon(x); A)dx \underset{\epsilon \to 0^+}{\to} \int_\Omega f_h(x, u(x); A)dx$$

and we deduce that

$$\begin{aligned} E_\epsilon^h(v_\epsilon; A) & = I_\epsilon(v_\epsilon) + \textstyle\int_\Omega f_h(x, v_\epsilon(x); A)dx \\ & \underset{\epsilon \to 0^+}{\to} J_0(u) + \textstyle\int_\Omega f_h(x, u; A)dx \\ & = J_h(u; A). \end{aligned}$$

∎

We recall De Giorgi's notions of *minimizing movement* and *generalized minimizing movement*.

Definition 3.3 [DG2] *Let S be a topological space, $\mathcal{F} : (1,+\infty) \times \mathbf{Z} \times S \times S \to \mathbb{R} \cup \{+\infty\}$. We say that u is a minimizing movement associated to $\mathcal{F}$ and S, $u \in MM(\mathcal{F}, S)$, if there exists $w : (1,+\infty) \times \mathbf{Z} \to S$ such that for all $t \in \mathbb{R}$*

$$u(t) = \lim_{\lambda \to +\infty} w(\lambda, [\lambda t])$$

and if $\lambda > 1$, $k \in \mathbf{Z}$

$$\min_{s \in S} \mathcal{F}(\lambda, k, s, w(\lambda, k)) = \mathcal{F}(\lambda, k, w(\lambda, k+1), w(\lambda, k)).$$

Definition 3.4 [DG2] *Let S be a topological space, $\mathcal{F} : (1,+\infty) \times \mathbf{Z} \times S \times S \to \mathbb{R} \cup \{+\infty\}$. We say that u is a generalized minimizing movement associated to $\mathcal{F}$ and S, $u \in GMM(\mathcal{F}, S)$, if there exist $w : (1,+\infty) \times \mathbf{Z} \to S$ and a sequence $\{\lambda_i\}_{i \in \mathbf{N}}$ such that $\lim_{i \to +\infty} \lambda_i = +\infty$ and for all $t \in \mathbb{R}$*

$$u(t) = \lim_{i \to +\infty} w(\lambda_i, [\lambda_i t])$$

and if $\lambda > 1$, $k \in \mathbf{Z}$

$$\min_{s \in S} \mathcal{F}(\lambda, k, s, w(\lambda, k)) = \mathcal{F}(\lambda, k, w(\lambda, k+1), w(\lambda, k)).$$

We use the notation $[x] := \max\{z \in \mathbf{Z} \mid z \leq x\}$. In our context, associating the set of finite perimeter L to the function of bounded variation $u(x) := a\chi_L(x) + b(1 - \chi_L(x))$, we consider $S := \{A \subset \Omega \mid \mathrm{Per}(A) < +\infty\}$ and we set

$$\mathcal{F}(\lambda, k, L, A) := \begin{cases} \mathcal{L}_N(L \Delta A) & \text{if } k \leq 0 \\ \int_{\partial^* L \cap \Omega} H(x, \nu(x)) dH^{N-1}(x) & \\ \quad + \int_\Omega f_{\frac{1}{\lambda}}(x, a\chi_L + b(1 - \chi_L); A) dx & \text{if } k > 0. \end{cases}$$

In the case where

$$f_h(x, u; A) := | u - a\chi_A(x) - b(1 - \chi_A(x)) |^p \; g\left(\frac{d(x, \partial^* A)}{h}\right)$$

then for $k > 0$

$$\begin{aligned} \mathcal{F}(\lambda, k, L, A) &= J_{\frac{1}{\lambda}}(a\chi_L + (1 - \chi_L)b; A) \\ &= \int_{\partial^* L \cap \Omega} H(x, \nu(x)) dH^{N-1}(x) \\ &+ \int_\Omega | a\chi_L(x) + b(1 - \chi_L(x))) - a\chi_A(x) - b(1 - \chi_A(x)) |^p \times \end{aligned}$$

$$\times \quad g(\lambda \text{dist}\,(x,\partial^* A))dH^{N-1}(x)$$
$$= \int_{\partial^* L\cap\Omega} H(x,\nu(x))dH^{N-1}(x)$$
$$+ \int_{A\Delta L} \mid b-a\mid^p \, g(\lambda\ \text{dist}(x,\partial^* A))dH^{N-1}(x). \tag{3.4}$$

This was exactly the functional considered in [ATW], with $g(t) := t$ and $H = H(\nu)$.

Fix $A_0 \subset \Omega, \text{Per}(A_0) < +\infty$. Using the Direct Method of the Calculus of Variations, it is easy to see that $E^h_\epsilon(\cdot;A_0)$ admits a minimizer $v^{(0)}_{\epsilon,h}$. It is a well-known fact that minimizers converge to minimizers of the Γ-limit.

Proposition 3.5 $\{v^{(0)}_{\epsilon,h}\}_{\epsilon>0}$ *admits a subsequence* $\{v^{(0)}_{\eta,h}\}_{\eta>0}$ *such that*

$$v^{(0)}_{\eta,h} \underset{\eta\to 0^+}{\to} u^{(0)}_h \ \textit{in}\ L^1(\Omega;\mathbb{R}^n),$$

$$u^{(0)}_h = a\chi_{A^h_1}(x) + b(1-\chi_{A^h_1}(x)),\ \text{Per}(A^h_1) < +\infty$$

and $u^{(0)}_h$ *minimizes* $J_h(\cdot;A_0)$. *In addition, setting*

$$u_0(x) := a\chi_{A_0}(x) + b(1-\chi_{A_0}(x)),$$

we have

$$J_h(u^{(0)}_h;A_0) \le J_0(u_0).$$

Proof By Theorem 5.2 in [BF], as $I_\epsilon(v^{(0)}_{\epsilon,h})$ remains uniformly bounded in ϵ there exists a subsequence $\{v^{(0)}_{\eta,h}\}_{\eta>0}$ such that

$$v^{(0)}_{\eta,h} \underset{h\to 0^+}{\to} u^{(0)}_h \text{ in } L^1(\Omega;\mathbb{R}^n).$$

By Lemma 3.2

$$u^{(0)}_h = a\chi_{A^h_1} + b(1-\chi_{A^h_1}),\ \ \text{Per}(A^h_1) < +\infty.$$

Due to Theorem 3.1, given $u = a\chi_L + b(1-\chi_L), \text{Per}(L) < +\infty$, let $v_\epsilon \in H^1(\Omega;\mathbf{R}^n), v_\epsilon \to u$ in L^1 and $E^h_\epsilon(u_\epsilon;A_0) \to J_h(u;A_0)$. Then

$$\begin{aligned} J_h(u^{(0)}_h;A_0) \le &\ \liminf_{\eta\to 0^+} E^h_\eta(v^{(0)}_{\eta,h};A_0) \\ \le &\ \liminf_{\eta\to 0^+} E^h_\eta(v_\eta;A_0) \\ = &\ J_h(u;A_0) \end{aligned}$$

hence $u_h^{(0)}$ minimizes $J_h(\cdot; A_0)$. Finally,

$$\begin{aligned} J_h(u_h^{(0)}; A_0) &\leq J_h(u_0; A_0) \\ &= J_0(u_0) \end{aligned}$$

since, due to (H6), $\int_\Omega f(x, u_0(x); A_0)dx = 0$. ∎

Construction of a Generalized Minimizing Movement:

We consider a minimizer $v_{\epsilon,h}^{(1)}$ for $E_\epsilon^h(\cdot; A_1^h)$ and, as before, for some subsequence

$$v_{\epsilon,h}^{(1)} \underset{\epsilon \to 0^+}{\to} u_h^{(1)} = a\chi_{A_2^h} + b(1 - \chi_{A_2^h}),\ \mathrm{Per}(A_2^h) < +\infty\ ,\ u_h^{(1)} \text{ minimizes } J_h(\cdot; A_1^h),$$

and

$$\begin{aligned} J_h(u_h^{(1)}; A_1^h) &\leq J_h(u_h^{(0)}; A_1^h) \\ &= J_0(u_h^{(0)}) \\ &\leq J_h(u_h^{(0)}; A_0) \\ &\leq J_0(u_0). \end{aligned}$$

Recursively, we construct a sequence of minimizers $\{v_{\epsilon,h}^{(j)}\}$ for $E_\epsilon^h(\cdot; A_j^h)$ such that (for some subsequence)

$$v_{\epsilon,h}^{(j)} \underset{\epsilon \to 0^+}{\to} u_h^{(j)} \text{ in } L^1(\Omega; \mathbb{R}^n),$$

$$u_h^{(j)}(x) = a\chi_{A_{j+1}^h} + b(1 - \chi_{A_{j+1}^h}), \mathrm{Per}(A_{j+1}^h) < +\infty,$$

$$u_h^{(j)} \text{ minimizes } J_h(\cdot; A_j^h),$$

$$\begin{aligned} J_h(u_h^{(j)}; A_j^h) &\leq J_h(u_h^{(j-1)}, A_j^h) \\ &= J_0(u_h^{(j-1)}) \\ &\leq J_h(u_h^{(j-1)}; A_{j-1}^h) \\ &\leq \ldots \leq J_0(u_0). \end{aligned} \tag{3.5}$$

By Proposition 2.12 iii), $H(x, \nu(x)) \geq \alpha > 0$ and so by (3.5) and as $\|u_h^{(j)}\|_\infty \leq \max\{|a|, |b|\}$,

$$\sup_{j,h} \|u_h^{(j)}\|_{BV(\Omega;\mathbb{R}^n)} < +\infty. \tag{3.6}$$

With $\mathcal{F}(\lambda,k,L,A)$ defined as in (3.4), we set

$$w(\lambda;[t\lambda]) := u_{\frac{1}{\lambda}}^{([t\lambda])}, \quad \text{i.e.} \quad w\left(\frac{1}{h},j\right) := u_h^{(j)}.$$

By (3.6), and after extracting a diagonal subsequence, we find a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to+\infty}\lambda_i = +\infty$ and for every $t>0$ in a countable dense subset of $(0,+\infty)$

$$w(\lambda_i;[t\lambda]) \underset{i\to+\infty}{\to} U(t) \text{ in } L^1(\Omega;\mathbb{R}^n), \mathrm{U(t)} \in \mathrm{BV}(\Omega;\mathbb{R}^n).$$

In addition, as $u_h^{(j)} \in \{a,b\}$a.e., then

$$U(t) = a\chi_{A_t} + b(1-\chi_{A_t}),\ \mathrm{Per}(A_t) < +\infty.$$

Hölder regularity results obtained in [ATW] guarantee that the convergence holds for all $t>0$.

Note that $u \in GMM(F,S)$, i.e. u is a generalized minimizing movement associated to $\mathcal{F}$ (see Definition 3.4) since

$$\begin{aligned}
\min_{\mathrm{Per}(L)<+\infty} \mathcal{F}(\lambda,k,L,w(\lambda,k)) &= \min_{\mathrm{Per}(L)<+\infty} \mathcal{F}(\lambda,k,L,u_{\frac{1}{\lambda}}^{(k)}) \\
&= \min_{\mathrm{Per}(L)<+\infty} \mathcal{F}(\lambda,k,L,A_{k+1}^{\frac{1}{\lambda}}) \\
&= \mathcal{F}(\lambda,k,A_{k+2}^{\frac{1}{\lambda}},A_{k+1}^{\frac{1}{\lambda}}) \\
&= \mathcal{F}(\lambda,k,w(\lambda,k+1),\ w(\lambda,k)).
\end{aligned}$$

De Giorgi [DG2] conjectured that $\partial^* A_t$ moves along its mean curvature (see also [ATW]). In order to justify this conjecture, we begin by determining the Euler-Lagrange equation satisfied by the minimizers A_{j+1}^h of

$$\begin{aligned}
L\in S \mapsto J_h(L;A_j^h) := &\int_{\partial^* L\cap\Omega} H(x,\nu(x))dH^{N-1}(x) \\
+ &\int_\Omega f_h(x,a\chi_L(x)+b(1-\chi_L(x));A_j^h)dx \\
= &\int_{\partial^* L\cap\Omega} H(x,\nu(x))dH^{N-1}(x) \\
+ &\int_{L\setminus A_j^h} f_h(x,a;A_j^h)dx \\
+ &\int_{A_j^h\setminus L} f_h(x,b;A_j^h)dx.
\end{aligned}$$

For simplicity of notation, in the sequel we consider a set of finite perimeter A_0 and a functional

$$L \mapsto J(L) := \int_{\partial^* L \cap \Omega} H(x,\nu(x))dH^{N-1}(x) + \int_{L\setminus A_0} f(x,a)dx + \int_{A_0\setminus L} f(x,b)dx$$

where f satisfies (H6) and (H7), $\mathrm{Per}(L) < +\infty$ and

(H8) $H \in C^3(\Omega \times (\mathbb{R}^n\setminus\{0\}))$, $H(x,\cdot)$ is convex, positively homogeneous of degree one, $H \geq 0$;

(H9) there exists $\alpha, \beta > 0$ such that

$$\alpha \leq H(x,\nu) \leq \beta$$

for every $(x,\nu) \in \Omega \times S^{N-1}$.

Note that the surface density H that we obtained in the Γ-limit of the functionals E^h_ϵ satisfies (H8) and (H9) (see Proposition 2.12) when $\Lambda = \Lambda(\nu)$ (and so $H = H(\nu)$).
Also, as $H(x,\cdot)$ is homogeneous of degree one

$$H(x,ty) = tH(x,y) \text{ for all } t > 0 \tag{3.7}$$

hence, differentiating this equation with respect to y, we have

$$\frac{\partial H}{\partial y}(x,ty) = \frac{\partial H}{\partial y}(x,y), \tag{3.8}$$

i.e. $\frac{\partial H}{\partial y}(x,\cdot)$ is homogeneous at degree zero. If we differentiate (3.7) with respect to t at $t = 1$ we get

$$\frac{\partial H}{\partial y}(x,y)\cdot y = H(x,y) \tag{3.9}$$

thus

$$\frac{\partial}{\partial y_j}\left[\frac{\partial H}{\partial y_k}(x,y)y_k\right] = \frac{\partial H}{\partial y_j}(x,y)\,,$$

i.e.

$$\frac{\partial^2 H}{\partial y_j \partial y_k}(x,y)y_k = 0 \text{ for every } j = 1,\dots N. \tag{3.10}$$

We introduce the concept of mean curvature of a surface $\Gamma \subset \Omega$ with respect to H. Suppose that Γ is a C^2 surface with a local parameterization $\Gamma \equiv$

$\{(x', F(x')) \mid x' \in D\}$, where D is a domain of $\mathbb{R}^{N-1}$ and $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Then the *normal vector* to Γ is given by

$$\nu := \frac{< -\nabla F, 1 >}{\sqrt{1+ \mid \nabla F \mid^2}}$$

and the *mean curvature* is defined as

$$\begin{aligned} K := \operatorname{div}(\nu) &= \frac{-1}{\sqrt{1+ \mid \nabla F \mid^2}} \operatorname{tr}\left[\left(\mathbf{I} - \frac{\nabla F \otimes \nabla F}{1+ \mid \nabla F \mid^2}\right) D^2 F\right] \\ &= \frac{-1}{\sqrt{1+ \mid \nabla F \mid^2}} \left[\Delta F - \tfrac{1}{(1+|\nabla F|^2)} \frac{\partial F}{\partial x_\alpha} \frac{\partial F}{\partial x_\beta} \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}\right] \quad (3.11) \end{aligned}$$

where we have used the summation convention for repeated indices, greek indices range from 1 to $N-1$ and $i, j \in \{1, \ldots, N\}$.

Now if H satisfies (H8) and (H9), then the *mean curvature of* Γ *with respect to* H is given by

$$\begin{aligned} K_H &:= -\frac{\partial^2 H}{\partial y_\alpha \partial y_\beta}((x', F(x')), < -\nabla F(x'), 1 >) \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}(x') \\ &= D_y^2 H \cdot D^2 G \qquad (3.12) \end{aligned}$$

where $\Gamma \equiv \{G = 0\}$, and $G(x) = x_N - F(x')$. Of course, when $H(x, y) = \mid y \mid$ then (3.12) reduces to (3.11).

The uniform ellipticity of equation (3.12) will be used to obtain regularity properties for F which, in turn, will be helpful to deduce the Euler-Lagrauge equations satisfied by a minimizer A of $J(\cdot)$, with $\operatorname{Per}(A) < +\infty$. This suggests the introduction of an additional hypothesis, not necessary if $H(x, \cdot)$ is strictly convex:

(H10) H is *elliptic,* i.e. for every $R > 0$ there exists $\lambda_R > 0$ such that

$$\sum_{\substack{i,j=1,\cdots,N \\ i,j \neq k}} \frac{\partial^2 H}{\partial y_i \partial y_j}(x, y) \xi_i \xi_j \geq \lambda_R \sum_{\substack{i=1,\cdots,N \\ i \neq k}} \xi_i^2$$

for all $x \in \Omega, k \in \{1, \ldots, N\}$, $\xi \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$ such that $y_k = 1, ||y|| \leq R$.

As it turns out (see [ATW], Theorem 3.8 and (3.4)) there is a regularity result asserting that

Theorem 3.6 *Let H satisfy (H8), (H9), (H10) and let $f(x,u) = f(x,u;A_0)$ verify (H6) and (H7). There is an open set $U \subset \mathbb{R}^n$ such that $H^{N-1}(\partial^* A \backslash U) = 0$ and $\overline{\partial^* A} \cap U$ is a $C^1(N-1)$ dimensional submanifold of $\mathbb{R}^n$. If $N = 3$ then $\partial^* A$ is a compact C^1 2-dimensional manifold of $\mathbb{R}^3$ without boundary.*

As the proof of this theorem presented in [ATW] uses heavily notions of geometric measure theory, we found it convenient for the reader less familiar with those concepts to rewrite it analytically and include it in the appendix.

Theorem 3.7 *Let H satisfy (H8), (H9) and (H10) and let $f(x,u) = f(x,u;A_0)$ satisfy (H6), (H7), $f(\cdot,b)-f(\cdot,a)$ Lipschitz in a neighborhood of H^{N-1}a.e. $x \in \partial^* A$. Then for H^{N-1} a.e. $x_0 \in \partial^* A$ there exists $\delta > 0$ such that*

$$f(x,b) - f(x,a) = K_H + \sum_{i=1}^{N} \frac{\partial^2 H}{\partial x_i \partial y_i} \tag{3.13}$$

for every $x \in \partial^* A \cap B(x_0,\delta)$.

Remark 3.8 Let $\psi : \mathbb{R} \to \mathbb{R}$ be an increasing, locally Lipschitz function, and set

$$g(t) := \frac{-\text{sign }t}{|\, b-a \,|^p}\psi(t), \quad f(x,u) :=|\, u - a\chi_{A_0}(x) - b(1-\chi_{A_0}(x)) \,|^p \, g\left(\frac{d(x,\partial^* A_0)}{h}\right)$$

for some $h > 0$, where the *signed distance to $\partial^* A_0$* is defined as

$$d(x,\partial^* A_0) = \begin{cases} \text{dist}(x,\partial^* A_0), & \text{if } x \notin A_0 \\ \\ -\text{dist}(x,\partial^* A_0), & \text{if } x \in A_0. \end{cases}$$

By Proposition 2.12 ii) H is convex. Moreover,

$$\begin{aligned} f(x,b) - f(x,a) = & \; [\chi_{A_0}(x) \,|\, b-a \,|^p - (1-\chi_{A_0}(x)) \,|\, b-a \,|^p] \, g\left(\tfrac{d(x,\partial^* A_0)}{h}\right) \\ = & \; - \,|\, b-a \,|^p \text{ sign } d(x,\partial^* A) g\left(\tfrac{d(x,\partial^* A)}{h}\right) \\ = & \; \psi\left(\tfrac{d(x,\partial^* A_0)}{h}\right) \end{aligned}$$

which, by Theorem 3.5 and if we assume that H is elliptic, is locally Lipschitz for H^{N-1}a.e. $x \in \partial^* A$. Hence, by Theorem 3.6 and (3.13) we conclude that

$$\psi\left(\frac{d(x, \partial^* A_0)}{h}\right) = K_H + a(x, \nu)$$

where

$$a(x, \nu) = \sum_{i=1}^{N} \frac{\partial^2 H}{\partial x_i \partial y_i},$$

and so

$$\frac{d(x, \partial^* A_0)}{h} = \Phi\left(K_H + a(x, \nu)\right)$$

with

$$\Phi := \psi^{-1}.$$

Proof of Theorem 3.7. By Theorem 3.6, for H^{N-1}a.e. $x_0 \in \partial^* A$ we may find $\delta, \delta' > 0$ and a C^1 function $F : B'(0, \delta) \subset \mathbb{R}^{N-1} \to \mathbb{R}$ such that, for some rotation R,

$$R(A - x_0) \cap [B'(0, r) \times (-\delta, \delta')] = \{(x', x_N) \in B'(0, \delta) \times \mathbb{R} \mid -\delta' < x_N < F(x')\}.$$

Without loss of generality, we may assume that $R = \mathbf{I}, x_0 = 0$. Let $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$, $\operatorname{supp}\varphi \subset B'(0, \delta) \times (-\delta', \delta')$ and choose $\epsilon > 0$ small enough so that

$$w_\epsilon(x) := x + \epsilon\varphi(x)$$

is a diffeomorphism. Then by (H6)

$$\begin{aligned}
0 = \; & \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Big\{ \int_{\partial^* w_\epsilon(A) \cap \Omega} H(y, \nu_\epsilon(y)) dH^{N-1}(y) + \int_{w_\epsilon(A) \setminus A_0} f(y, a) dy \\
& + \int_{A_0 \setminus w_\epsilon(A)} f(y, b) dy \Big\} \\
= \; & \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left\{ \int_{\partial^* w_\epsilon(A) \cap \Omega} H\left(y, \frac{[Dw_\epsilon(w_\epsilon^{-1}(y))]^{-T} \nu_A(w_\epsilon^{-1}(y))}{\mid [Dw_\epsilon(w_\epsilon^{-1}(y))]^{-T} \nu_A(w_\epsilon^{-1}(y)) \mid}\right) dH^{N-1}(y) \right. \\
& \left. + \int_{w_\epsilon(A)} f(y, a) dy + \int_\Omega f(y, b) dy - \int_{w_\epsilon(A)} f(y, b) dy \right\} \\
= \; & \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left\{ \int_{\partial^* A \cap \Omega} H\left(w_\epsilon(x), \frac{[Dw_\epsilon(x))]^{-T} \nu(x)}{\mid [Dw_\epsilon(x)]^{-T} \nu(x) \mid}\right) \cdot \right. \\
& . \mid \operatorname{adj} Dw_\epsilon(x) \nu(x) \mid dH^{N-1}(x) \\
& \left. + \int_A [f(w_\epsilon(x), a) - f(w_\epsilon(x), b)] \det \nabla w_\epsilon(x) dx \right\}.
\end{aligned}$$

Now

$$Dw_\epsilon^{-T}(x)\nu(x) = \frac{\text{adj}Dw_\epsilon(x)\nu(x)}{\det Dw_\epsilon(x)}$$

and

$$\frac{d}{d\epsilon}\,|_{\epsilon=0}\text{ adj }Dw_\epsilon(x) = -D\varphi(x)^T + \text{div}\varphi(x)\mathbf{I}.$$

Hence, using the homogeneity of $H(x,\cdot)$ and (3.8)-(3.12),

$$\begin{aligned}
0 = & \tfrac{d}{d\epsilon}\,|_{\epsilon=0}\Big\{\textstyle\int_{\partial^* A\cap\Omega} H(w_\epsilon(x),\text{adj }Dw_\epsilon(x))dH^{N-1}(x)\\
& + \textstyle\int_A [f(w_\epsilon(x),a) - f(w_\epsilon(x),b)]\det\nabla w_\epsilon(x)]dx\Big\}\\
= & \textstyle\int_{\partial^* A\cap\Omega}\frac{\partial H}{\partial x}(x,\nu(x))\varphi(x)dx\\
& + \textstyle\int_{\partial^* A\cap\Omega}\frac{\partial H}{\partial y}(x,\nu(x))\cdot\Big(\text{div}\varphi(x)\mathbf{I} - D\varphi^T(x)\Big)\nu(x)dH^{N-1}(x)\\
& + \textstyle\int_A\Big\{\Big[\frac{\partial f}{\partial x}(x,a) - \frac{\partial f}{\partial x}(x,b)\Big]\cdot\varphi(x) + [f(x,a) - f(x,b)]\text{div }\varphi(x)\Big\}dx
\end{aligned}$$

and as

$$\begin{aligned}
& \textstyle\int_A\Big\{\Big[\frac{\partial f}{\partial x}(x,a) - \frac{\partial f}{\partial x}(x,b)\Big]\cdot\varphi(x) + [f(x,a) - f(x,b)]\text{ div }\varphi(x)\Big\}dx\\
= & \textstyle\int_A \text{div}[\varphi(x)(f(x,a) - f(x,b))]dx,
\end{aligned}$$

by the Gauss-Green formula (see Theorem 2.7) we obtain

$$\begin{aligned}
& \textstyle\int_{\partial^* A\cap\Omega}[f(x,b) - b(x,a)]\varphi(x)\cdot\nu(x)dH^{N-1}(x)\\
& \quad = \textstyle\int_{\partial^* a\cap\Omega}\frac{\partial H}{\partial x}(x,\nu(x))\cdot\varphi(x)dx\\
& + \textstyle\int_{\partial^* A\cap\Omega}\frac{\partial H}{\partial y}(x,\nu(x))\cdot\Big(\text{div }\varphi(x)\mathbf{I} - D\varphi^T(x)\Big)\cdot\nu(x)dH^{N-1}(x). \quad (3.14)
\end{aligned}$$

For convenience we assume that $F\in C^2(B'(0,\delta))$, although we will drop this hypothesis at the end of the proof. By (3.8) we have

$$\frac{\partial H}{\partial y}((x',F(x')),\frac{<-\nabla F(x'),1>}{\sqrt{1+\|\nabla F(x')\|^2}}) = \frac{\partial H}{\partial y}((x',F(x')),<-\nabla F(x')),1>).$$

Writing the last integral in (3.14) in local coordinates and using (3.8)-(3.12), we obtain

$$\begin{aligned}
\int_{\partial^* A\cap\Omega}[f(x,b) - f(x,a)](\varphi\cdot\nu)dH^{N-1} = & \int_{\partial^* A\cap\Omega}\frac{\partial}{\partial x_\alpha}\left(\frac{\partial H}{\partial y_\alpha}\right)(\varphi\cdot\nu)dH^{N-1}\\
& + \int_{\partial^* A\cap\Omega}\frac{\partial^2 H}{\partial x_N\partial y}\cdot\nu\sqrt{1+\|\nabla F\|^2}(\varphi\cdot\nu)dH^{N-1}
\end{aligned}$$

and if $F \in C^1(B'(0,\delta)$ then

$$f(x,b)-f(x,a) = \frac{\partial}{\partial x_\alpha}\left[\frac{\partial H}{\partial y_\alpha}((x',F(x')),< -\nabla F(x'),1 >)\right] - \frac{\partial^2 H}{\partial x_N \partial y_\alpha}((x',F(x')),< -\nabla F(x'),1 >)\frac{\partial F}{\partial x_\alpha} + \frac{\partial^2 H}{\partial x_N \partial y_N}((x',F(x')),< -\nabla F(x'),1 >) \quad (3.15)$$

in $\mathcal{D}'(B'(\sigma,\delta))$. As F is locally Lipschitz then $\|< -\nabla F,1 >\|$ remains bounded and so, due to the ellipticity hypothesis (H10), we deduce that (3.15) is an elliptic equation for ∇F. The hypothesis (H10) and classical regularity results (see [LU]) imply that $F \in C^{1,k}$ and, as in [ATW], differentiating (3.15) once more with respect to $x_\theta, \theta = 1,\ldots,N-1$, we obtain that $\frac{\partial F}{\partial x_\theta}$ is a solution of a second order linear elliptic partial differential equation with Hölder continuous coefficients, of the form

$$-D_\alpha a_\alpha\left(x,\frac{\partial F}{\partial \theta},\nabla\left(\frac{\partial F}{\partial \theta}\right)\right) + bx,\frac{\partial F}{\partial \theta},\nabla\left(\frac{\partial F}{\partial \theta}\right)\Big) = 0.$$

From Theorem 1.2, page 219 in [Gi], we conclude that $\frac{\partial F}{\partial \theta} \in C^{1,2}$ and so

$$\begin{aligned} f(x,b)-f(x,a) = {} & \frac{\partial^2 H}{\partial y_\alpha \partial x_\alpha} + \frac{\partial^2 H}{\partial y_\alpha \partial x_N}\frac{\partial F}{\partial x_\alpha} - \frac{\partial^2 H}{\partial y_\alpha \partial y_\beta}\frac{\partial^2 F}{\partial x_\alpha \partial x_\beta} \\ & - \frac{\partial^2 H}{\partial x_N \partial y_\alpha}\frac{\partial F}{\partial x_\alpha} + \frac{\partial^2 H}{\partial x_N \partial y_N} \\ = {} & K_H + \sum_{i=1}^{N} \frac{\partial^2 H}{\partial x_i \partial y_i}. \end{aligned}$$

∎

4 Minimizing flows are viscosity solutions

In this section we prove that a minimizing flow is a viscosity solution for the corresponding curvature-type equation. Notice that this result is past the possible singularities appearing in the solutions of such equations.

We recall the construction of the generalized minimizing movement, as described in the previous section. Let Ω be any bounded set such that

$A_0 \subset\subset \Omega$. Given a time step h, we write $[0,+\infty) = \cup_{i=0}^{\infty}[t_i^h, t_{i+1}^h)$ where $t_0^h = 0$, $t_{i+1}^h = t_i^h + h$. Then

$$u_{h_j}^{([t/h_j])} \to u = a\chi_{A_t} + b(1-\chi_{A_t}) \text{ in } L^1(\Omega;\mathbb{R}^n) \,\text{as} j \to +\infty$$

for some sequence $h_j \to 0^+$, i.e.

$$\chi_{A_t^{h_j}} \underset{j\to+\infty}{\to} \chi_{A_t} \qquad \text{in } L^1(\Omega;\mathbb{R})$$

with

$$A_t^h := A_{[t/h]}^h = A_i^h \quad \text{if } t \in [t_i^h, t_{i+1}^h).$$

We set $A_0^h = A_0$, and as before the signed distance to $\partial^* A_t^h$ is given by

$$d(x,\partial^* A_t^h) = \begin{cases} -\text{dist}(x,\partial^* A_t^h) & \text{if } x \in A_t^h \\ \text{dist}(x,\partial^* A_t^h) & \text{if } x \notin A_t^h \end{cases}$$

where, by Theorem 3.6, $\partial^* A_t^h$ is a compact C^1 $(N-1)$-dimensional manifold.

We define

$$D_0(x) := -d(x,\partial^* A_0), \quad D^h(x,t) := -d(x,\partial^* A_t^h).$$

In the case where $H(x,y) = |y|$ and g is given as in Remark 3.8, the Euler Lagrange equation derived in Theorem 3.7 becomes

$$\begin{aligned} &\psi\left(\frac{-D^h(x,t-h)}{h}\right) \\ &= \frac{1}{|\nabla D^h(x,t)|}\text{tr}\left\{\left[\mathrm{I} - \frac{\nabla D^h(x,t)\otimes\nabla D^h(x,t))}{|\nabla D^h(x,t)|^2}\right]\nabla^2 D^h(x,t)\right\} \\ &= \Delta D^h(x,t) \end{aligned} \tag{4.1}$$

for all $x \in \partial^* A_t^h$ where the manifold is twice continuously differentiable. Furthermore ψ is a locally Lipschitz, increasing function; note also that in the second equality we have used the fact that $|\nabla D^h| = 1$.

Setting

$$D_\infty(x,t) := -d(x,\partial^* A_t),$$

heuristically we expect (4.1) to yield D_∞ as a solution of (2.10) and (2.9) with $h(x,0) := D_0(x)$ (at least when $\psi(r) = r$), or for general H and ψ,

$$\begin{cases} u_t - |\nabla u|\Phi\left(\frac{1}{|\nabla u|}\left\{\text{tr}\left[\nabla^2 H(x,\frac{\nabla u}{|\nabla u|})\nabla^2 u\right] + a\left(x,\frac{\nabla u}{|\nabla u|}\right)\right\}\right) = 0 \\ \qquad\qquad \text{in} \quad \mathbb{R}^n \times (0,\infty) & (4.2) \\ u(x,0) = D_0(x) \qquad \text{in} \quad \mathbb{R}^n & (4.3) \end{cases}$$

where, as before,

$$a(x,y) = \sum_{i=1}^{N} \frac{\partial^2 H(x,y)}{\partial x_i \partial y_i}, \ \Phi = \psi^{-1} \qquad \text{is increasing}$$

and where $t^* \in (0,+\infty]$ is the *extinction time* for the level sets $\Gamma_t = \{x \in \mathbb{R}^n \mid u(x,t) = 0\}$, i.e.

$$t^* := \sup\{t \mid t > 0 \quad \text{and} \quad \Gamma_t \neq \emptyset\}.$$

Here $u : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ is uniformly continuous and it is the unique viscosity solution of (4.2), (4.3) (see Theorem 2.15).

Following the method introduced in [BP], we define the *lower semicontinuous envelope for* $\{D^h\}_{h>0}$,

$$D_*(x,t) := \liminf_{\substack{(y,s)\to(x,t) \\ h\to 0}} D^h(y,s)$$

and the *upper semicontinuous envelope*

$$D^*(x,t) := \limsup_{\substack{(y,s)\to(x,t) \\ h\to 0}} D^h(y,s).$$

It is clear that

$$D^* \geq D_* \quad \text{in} \quad \mathbb{R}^n \times [0,+\infty).$$

Let

$$T := \sup\{t > 0 \ \mid \ | D^*(x,t) |, | D_*(x,t) | < +\infty \text{ in } \mathbb{R}^n \times [0,t)\}.$$

Now we state our main theorem.

Theorem 4.1 *Let u be the unique viscosity solution of (4.2), (4.3). Then*

$$\{u < 0\} \subset \{D^* < 0\}, \qquad \{u > 0\} \subset \{D_* > 0\}. \tag{4.4}$$

If, in addition, int $\Gamma_t = \emptyset$ *for all* $t \in [0,t^*)$, *then*

$$D^*(x,t) = D(x,t) \quad \text{in } \{u < 0\}, \tag{4.5}$$

$$D^*(x,t) = [D(x,t)]^* \quad \text{in } \{u > 0\}, \tag{4.6}$$

$$D_*(x,t) = D(x,t) \quad \text{in } \{u > 0\} \tag{4.7}$$

and

$$D_*(x,t) = [D(x,t)]_* \quad in\ \{u < 0\} \tag{4.8}$$

where $[D]^*$ *(resp.* $[D]_*$*) is the upper semicontinuous envelope of* D *(resp. lower semicontinuous envelope) and*

$$D(x,t) := \begin{cases} \text{dist } (x,\Gamma_t) & \text{if } u(x,t) > 0 \\ -\text{dist } (x,\Gamma_t) & \text{if } u(x,t) < 0 \\ 0 & \text{if } u(x,t) = 0. \end{cases}$$

Remark 4.2 Notice that the viscosity solution is always contained inside the generalized minimizing movement. Moreover, in the no interior case, at the points where the signed distance function D is continuous,

$$D^*(x,t) = D_*(x,t) = D(x,t),$$

the convergence of the approximating motion to the minimizing movement is not only in L_1 but also locally uniform and finally the generalized minimizing movement turns out to be a minimizing movement which coincides with the viscosity solution.

The proof of Theorem 4.1 is based on a series of lemmas that we present below.

Lemma 4.3 *The function* $D_* \vee 0$ *(resp.* $D^* \wedge 0$*) is a supersolution (resp. subsolution) of (4.2) in* $\mathbb{R}^n \times (0,t^*]$ *and*

$$(D_* \vee 0)(x,0) \geq (D_0 \vee 0)(x,0) \, in \, \mathbb{R}^n$$

(resp. $(D^* \wedge 0)(x,0) \leq (D_0 \wedge 0)(x,0)$ *in* $\mathbb{R}^n$*).*

Lemma 4.4 *The function* D_**(resp.* D^**) is a supersolution (resp. subsolution) of (4.2) in* $\{D_* > 0\} \cap (\mathbb{R}^n \times (0,t^*))$ *(resp.* $\{D^* < 0\} \cap (\mathbb{R}^n \times (0,t^*))$*). In addition*

$$|\nabla D_*| - 1 = 0 \ \text{ in } \ \{D_* > 0\} \cap (\mathbb{R}^n \times (0,t^*)) \tag{4.9}$$

$$(\textit{resp.} 1 - |\nabla D^*| = 0 \text{ in } \{D^* < 0\} \cap (\mathbb{R}^n \times (0,t^*)))$$

in the viscosity sense.

Proof of Theorem 4.1. We only present the proof in the case where $H(x,y) = |y|$, the general statement being a straightforward adaptation of our arguments. By Theorem 2.16, $\varphi(u) = u \vee 0$ is a solution of (4.2) and so, by the comparison principle (see Theorem 2.14) and Lemma 4.3

$$u(x,t) \vee 0 \le D_*(x,t) \vee 0 \qquad \text{in } \mathbb{R}^n \times [0,t^*).$$

In particular

$$(\mathbb{R}^n \times [0,t^*)) \cap \{u > 0\} \subset \{D_* > 0\} \cap (\mathbb{R}^N \times [0,t^*)).$$

Similarly

$$(\mathbb{R}^n \times [0,t^*)) \cap \{u < 0\} \subset \{D^* < 0\} \cap (\mathbb{R}^N \times [0,t^*)).$$

If int $\Gamma_t = \emptyset$ then

$$\{u > 0\} = \{D_* > 0\} \qquad \text{and} \qquad \{u < 0\} = \{D^* < 0\}.$$

Indeed, suppose that $D_*(x_0,t) > 0, u(x_0,t) \le 0$. If $u(x_0,t) < 0$ then $D^*(x_0,t) < 0$ and this implies that $D^*(x_0,t) < 0$, contradicting our assumption. On the other hand, if $u(x_0,t) = 0$ and as $\text{int}\Gamma_t = \emptyset$ then we may find a sequence $x_n \to x_0$ such that $u(x_n,t) < 0$. Thus $D_*(x_n,t) \le D^*(x_n,t) < 0$ and due to the lower semicontinuity property we conclude that

$$D_*(x_0,t) \le \liminf_{n\to+\infty}(x_n,t) \le 0,$$

again in contradiction with the assumption.

Since $D(\cdot,t)$ is a distance function, it satisfies

$$|\nabla D| - 1 = 0 \text{ in } \{D > 0\} \cap (\mathbb{R}^n \times [0,t^*)). \tag{4.10}$$

$$1 - |\nabla D| = 0 \text{ in } \{D < 0\} \cap (\mathbb{R}^n \times [0,t^*)). \tag{4.11}$$

By the uniqueness (in the viscosity sense) property of the above equations and by (4.9) we conclude that

$$D_*(x,t) = D(x,t) \qquad \text{in } \{D > 0\} \cap (\mathbb{R}^n \times [0,t^*])$$

and, in a similar way,

$$D^*(x,t) = D(x,t) \qquad \text{in } \{D < 0\} \cap (\mathbb{R}^n \times [0,t^*]).$$

Relations (4.6) and (4.8) are proved at the end of this section. ∎

The proofs of Lemmas 4.3 and 4.4 use the preliminary results below, Lemmas 4.5 and 4.7.

Lemma 4.5 *The function D_* (resp. D^*) is a supersolution (resp. subsolution) of (4.2) in $\{D_* > 0\} \cap (\mathbb{R}^n \times (0,T])$ (resp. $\{D^* < 0\} \cap (\mathbb{R}^n \times (0,T])$). In addition*

$$| \nabla D_* | - 1 = 0 \ \textit{in} \quad \{D_* > 0\} \cap (\mathbf{R}^N \times (0,T)) \tag{4.12}$$

and

$$1 - | \nabla D^* | = 0 \ \textit{in} \ \{D^* < 0\} \cap (\mathbb{R}^n \times (0,T))$$

in the viscosity sense.

The proof of Lemma 4.5 is based on the fact that strict local minima of D_* are approximated by local minima of D^h, precisely

Lemma 4.6 *Let $U \subset \mathbb{R}^k$ be an open set, $\psi_n : U \to \mathbb{R}$ continuous functions, and let $\psi_*(x) := \liminf_{\substack{n\to+\infty \\ x_n \to x}} \psi_n(x_n)$. Let x_0 be a strict local minimum for ψ_n. Then there exist a subsequence $n_k \to +\infty$ and points of local minimum for ψ_{n_k}, x_{n_k}, such that $x_{n_k} \to x_0$.*

This result is standard in the literature on viscosity methods. For completeness we include its proof in the Appendix.

Proof of Lemma 4.5. To prove that D_* is a supersolution of (4.2), we consider a smooth function φ, we assume that $(x_0, t_0) \in \{D_* > 0\} \cap (\mathbb{R}^n \times (0,T))$ is a strict minimum for $D_* - \varphi$ and we prove that

$$\varphi_t(x_0,t_0) - | \nabla\varphi(x_0,t_0) | \times$$

$$\times \Phi\left(\frac{1}{| \nabla\varphi(x_0,t_0) |} \operatorname{tr}\left\{\left[1 - \frac{\nabla\varphi(x_0,t_0) \otimes \nabla\varphi(x_0,t_0)}{| \nabla\varphi(x_0,t_0) |^2}\right] \nabla^2\varphi(x_0,t_0)\right\}\right) \geq 0. \tag{4.13}$$

By Lemma 4.6, there are points of minimum $(x_h, t_h) \in \{D^h > 0\}$ for $D^h - \varphi$ (where we assume that a subsequence $h_k \to 0^+$ has been extracted), such that $(x_h, t_h) \underset{h\to 0}{\to} (x_0, t_0)$. We fix a time step h and we assume that $t_h \in [t_i^h, t_{i+1}^k)$ for some $i = i(h) \in \mathbf{N}$. Then

$$D^h(x_h, t_h) = \operatorname{dist}(x, \partial^* A_i^h) = | x_h - z_h |$$

for some $z_h \in \partial^* A^h_{t_h} = \partial^* A^h_t$. According to Theorem 3.10 in [ATW], there is a neighborhood of z_h where $\partial^* A^h_t$ is twice continuously differentiable. By (4.1) we have

$$\frac{-D^h(z_h, t_h - h)}{h} =$$

$$= \Phi\left(\frac{1}{|\nabla D^h(z_h,t_h)|} \operatorname{tr}\left\{\left[1 - \frac{\nabla D^h(z_h,t_h)\otimes \nabla D^h(z_h,t_h)}{|\nabla D^h(z_h,t_h)|^2}\right]\nabla^2 D^h(z_h,t_h)\right\}\right). \tag{4.14}$$

We define

$$\xi(x,t) := \varphi(x + x_h - z_h, t) - \varphi(x_h, t_h),\ \xi(z_h,t_h) = 0,$$

and we claim that

$$\{\xi > 0\} \subset \{D^h > 0\} \text{ and } D^h(x,t) - \xi(x,t) \geq D^h(z_h,t_h) - \xi(z_h,t_h) = 0. \tag{4.15}$$

Indeed, if there exists (x,t) such that $\xi(x,t) > 0$ and $D^h(x,t) \leq 0$, since (x_h,t_h) minimizes $D^h - \varphi$ we obtain

$$\begin{aligned} D^h(x + x_h - z_h, t) \geq\ & \varphi(x + x_h - z_h, t) + D^h(x_h,t_h) - \varphi(x_h,t_h) \\ =\ & \xi(x,t) + D^h(x_h,t_h) \end{aligned}$$

and since $\xi(x,t) > 0$,

$$D^h(x + x_h - z_h, t) \geq D^h(x_h,t_h) = | x_h - z_h | .$$

On the other hand, as $D^h(x,t) \leq 0$ we conclude that

$$| x_h - z_h | < D^h(x + x_h - z_h, t) - D^h(x,t) \leq | x_h - z_h |$$

yielding a contradiction. Also,

$$\begin{aligned} D^h(x,t) \geq\ & D^h(x + x_h - z_h, t) - | x_h - z_h | \\ =\ & D^h(x + x_h - z_h, t) - D^h(x_h,t_h) \\ \geq\ & \varphi(x + x_h - z_h, t) - \varphi(x_h,t_h) \\ =\ & \xi(x,t) \end{aligned}$$

and this proves (4.13). Now, as in Theorem 2.2 in [ESS], we may construct an increasing function $\Psi : \mathbb{R} \to \mathbb{R}$ such that

$$\Psi(\xi(x,t)) \le D^h(x,t), \quad \Psi(0) = 0 \tag{4.16}$$

in a (uniform in h) neighborhood of (z_h, t_h). Furthermore, since $|\nabla D^h| = 1$, by (4.16) and as $\xi(z_h,t_h) = 0 = D^h(z_h,t_h)$, we have that $D^h - \Psi \circ \xi$ has a local minimum at (z_h, t_h) and so

$$\Psi'(0) = 1, \tag{4.17}$$

where we used the fact that $D\xi(z_h,t_h) = D\varphi(x_h,t_h)$ and $| D\varphi(x_h,t_h) | = 1$. By (4.14), (4.16) we obtain

$$\frac{-\Psi(\xi(z_h, t_h - h))}{h} \ge$$

$$\ge \Phi\left(\frac{1}{| \nabla D^h(z_h,t_h) |} \operatorname{tr}\left\{\left[1 - \frac{\nabla D^h(z_h,t_h) \otimes \nabla D^h(z_h,t_h)}{| \nabla D^h(z_h,t_h) |^2}\right] \nabla^2 D^h(z_h,t_h)\right\}\right)$$

and by (4.14), (4.17), letting $h \to 0$ and using the fact that

$$\nabla D^h(z_h,t_h) = \nabla\xi^h(z_h,t_h) = \nabla\varphi(x_h,t_h)$$

with $| \nabla\varphi(x_h,t_h) | = 1$, we conclude (4.12):

$$\varphi_t(x_0,t_0) \ge$$

$$\ge | \nabla\varphi(x_0,t_0) | \, \Phi\left((\frac{1}{| \nabla\varphi(x_0,t_0) |} \operatorname{tr}\left\{\left[1 - \frac{\nabla\varphi(x_0,t_0) \otimes \nabla\varphi(x_0,t_0)}{| \nabla\varphi(x_0,t_0) |^2}\right] \times \right.\right.$$

$$\left.\left. \times \, \nabla^2\varphi(x_0,t_0)\right\}\right).$$

The dual statements for D^* follow along similar arguments. ∎

Lemma 4.7 *The function $D_* \vee 0$ (resp. $D^* \wedge 0$) is a supersolution (resp. subsolution) of (4.2) on $\mathbb{R}^n \times (0,T]$.*

Proof *Step 1.*Given $\delta > 0$ we consider a smooth function $\varphi_\delta : \mathbb{R} \to \mathbb{R}_+$ such that

$$\varphi_\delta = 0 \text{ on } (-\infty, \delta],\ \varphi'_\delta \geq 0 \text{ in } \mathbb{R},\ \lim_{\delta\to 0^+} \varphi_\delta(s) = s \vee 0.$$

Define

$$v_\delta(x,t) := \varphi_\delta(D_*(x,t)).$$

Let ψ be a smooth function and let $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$ be a strict minimum for $v_\delta - \psi$.

If $D_*(x_0,t_0) > 0$, then $D_* \vee 0 = D_*$ near (x_0,t_0) and so, by Lemma 4.6 and Theorem 2.16, we have that v_δ is a supersolution. We conclude that (4.14) is satisfied by ψ at (x_0,t_0).

If $D_*(x_0,t_0) \leq 0$, due to the Lipschitz continuity of v_δ in x we have $v_\delta = 0$ in a small ball centered at x_0. Thus

$$\nabla\psi(x_0,t_0) = 0, \quad \nabla^2\psi(x_0,t_0) \leq 0$$

and to obtain (4.14) it suffices to prove that

$$\psi_t(x_0,t_0) \geq 0.$$

Set

$$v_\delta^h(y,s) := \varphi_\delta(D^h(y,s)).$$

We have

$$v_\delta(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ h\to 0}} v_\delta^h(y,s)$$

and so, by Lemma 4.6, $v_\delta^h - \psi$ has a local minimum at (x_h,t_h) , where $(x_h,t_h) \underset{h\to 0}{\to} (x_0,t_0)$. In addition, for each $h > 0$ there exists $i = i(h)$ such that $t_h \in [t_i^h, t_{i+1}^h)$ and

$$v_\delta^h(x_h,t) = v_\delta^h(x_h,t_h) \text{ for all } t \in [t_i^h, t_{i+1}^h)$$

and so

$$\begin{aligned} 0 = \ & v_\delta^h(x_h,t) - v_\delta^h(x_h,t_h) \\ \geq \ & \psi(x_h,t) - \psi(x_h,t_h) \ \text{ for } t > t_h,\ t \in [t_i^h, t_{i+1}^h) \end{aligned}$$

from which we conclude that

$$\psi_t(x_0,t_0) \geq 0.$$

Step 2. To prove that $D_* \vee 0$ is a supersolution of (4.2), we consider a smooth function ψ and we assume that $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$ is a strict minimum for $D_* \vee 0 - \psi$.
By Lemma 4.5 we may find a sequence of strict local minima $(x_\delta, t_\delta) \in \mathbb{R}^n \times (0,T)$ for $v_\delta - \Phi$ such that $(x_\delta, t_\delta) \underset{\delta \to 0^+}{\longrightarrow} (x_0, t_0)$.

Applying Step 1 to (x_δ, t_δ), we conclude that equation (4.11) is satisfied by ψ at (x_δ, t_δ) and due to the continuity of ψ, we obtain equation (4.11) for ψ at (x_0, t_0).

Step 3. Finally, we turn our attention to the endpoint $\{t = T\}$. The statement reduces to a standard observation in the theory of parabolic equations, provided $D_* \vee 0$ is bounded. In our context, we can always reduce to this situation by considering a bounded, continuous, strictly increasing function Ψ and using Theorem 2.16 ensuring that $\Psi(D_* \vee 0)$ is also a supersolution. We conclude by applying Ψ^{-1} and, again,Theorem 2.16. ∎

From Lemmas 4.5 and 4.7 we conclude that Lemmas 4.3 and 4.4 will follow, provided

$$\begin{aligned} D_*(x,0) \vee 0 \quad &\geq \quad D_0(x) \vee 0 \qquad \text{in } \mathbf{R}^N , \\ D^*(x,0) \wedge 0 \quad &\leq \quad D_0(x) \wedge 0 \qquad \text{in } \mathbb{R}^n \end{aligned} \tag{4.18}$$

and

$$t^* \leq T. \tag{4.19}$$

The estimates (4.18) and (4.19) are an easy consequence of the following result, proven in [ATW] in a more general form (see Theorem 5.4 [ATW]).

Lemma 4.8 *Let $R_0, h_0 > 0, t_0 \geq 0$ and let $R_t := [R_0^2 - 2(N-1)(t-t_0)]^{1/2}$. The following hold true:*

i) if for all $0 < h < h_0$, $B(x_0, R_0) \subset A_{t_0}^h$ then

$$B(x_0, R_t) \subset A_t^h \, ;$$

ii) if for all $0 < h < h_0$, $A_{t_0}^h \subset B(x_0, R_0)$ then

$$A_t^h \subset B(x_0, R_t) \, ;$$

iii) if for all $0 < h < h_0$, $A_{t_0}^h \subset [B(x_0, R_0)]^C$ then

$$A_t^h \subset [B(x_0, R_t)]^C$$

for all $t_0 \le t \le t_0 + \frac{R_0^2}{2(N-1)}$.

We prove that

$$D^*(x,0) = D_0(x) \text{ in } \mathbb{R}^n. \tag{4.20}$$

Since by definition $D^*(x,0) \ge D_0(x)$, it suffices to show that $D^*(x,0) \le D_0(x)$.

Define the set

$$\mathcal{A}_0^\epsilon := \{x : \text{dist}(x, A_0) \le \epsilon\}$$

and consider the signed distance $\mathcal{D}_0^\epsilon$ from the boundary of $\mathcal{A}_0^\epsilon$. For all $x \in [\mathcal{A}_0^\epsilon]^C$,

$$A_0 = A_0^h \subset [B(x,\epsilon)]^C$$

and so by Lemma 4.8 iii) we have

$$A_t^h \subset [B(x, \epsilon_t)]^C$$

for $0 \le t \le \frac{\epsilon^2}{2(N-1)}$, where $\epsilon_t := [\epsilon^2 - 2(N-1)t]^{\frac{1}{2}}$. In particular

$$A_t^h \subset [B(x, \frac{\epsilon}{\sqrt{2}})]^C$$

for all $x \in [\mathcal{A}_0^\epsilon]^C$, $0 \le t \le \frac{\epsilon^2}{4(N-1)}$. Consequently

$$A_t^h \subset \cap_{x \in [\mathcal{A}_0^\epsilon]^C} [B(x, \frac{\epsilon}{\sqrt{2}})]^C =: E^\epsilon,$$

where $A_0 \subset E^\epsilon \subset \mathcal{A}_0^\epsilon$. It follows that

$$D^h(x,t) \le \mathcal{D}_0^\epsilon(x) \qquad \textit{for} \qquad 0 \le t \le \frac{\epsilon^2}{4(N-1)}$$

which yields $D^*(x,0) \le D_0(x)$.

Finally, we turn to the proof of (4.19). Suppose that $T < t^*$. By (4.20), Lemma 4.5 and the comparison principle for viscosity solutions we have

$$\{u < 0\} \subset \{D^* < 0\} \text{ and } \{u > 0\} \subset \{D_* > 0\} \tag{4.21}$$

for $t \le T$. By definition of t^* there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0, T) \ne 0$; suppose that $u(x_0,T) > 0$. By (4.4)

$$D_*(x_0, T) > \gamma > 0$$

and

$$D^h(y,T) > \frac{\gamma}{2} \text{ for all } \ y \in B(x_0, R_0)\ ,\ 0 < h \leq h_0.$$

Hence $B(x_0, R_0) \subset A_T^h$

$$B(x_0, R_t) \subset A_{T+t}^h \tag{4.22}$$

for $0 \leq t \leq \frac{R_0^2}{2(N-1)}$. Thus

$$D^h(x, T+t) \geq D(x,\ B(x_0, R_t)) \tag{4.23}$$

for all $x \in \mathbb{R}^n, 0 < \mathrm{h} \leq \mathrm{h}_0$, from which we deduce that

$$D_*(x,t'), \quad D^*(x,t') > -\infty \tag{4.24}$$

for all $x \in \mathbb{R}^n$, $0 \leq t' \leq T + \frac{R_0^2}{2(N-1)}$.

On the other hand, choose M large enough so that $T < \frac{M^2}{2(N-1)}$ and $x_0 \in \mathbb{R}^n$ is such that

$$A_0^h \subset B(x_0, M), \text{ for } 0 < h << 1.$$

By Lemma 4.8 ii) we have

$$A_T^h \subset B(x_0, M_T), \quad 0 < h < 1$$

and once again, Lemma 4.8 iii) yields

$$A_{T+t}^h \subset B(x_0, (M_T)_t) \subset B(x_0, 2M_T)$$

for all $0 \leq t \leq \frac{M_T^2}{2(N-1)}$. Thus

$$D^h(x, T+t) \leq D(x, B(x_0, 2M_T))$$

for all $x \in \mathbb{R}^n$, yielding

$$D_*(x,t'), \quad D^*(x,t') < +\infty$$

for $0 \leq t' \leq T + \frac{M_T^2}{2(N-1)}$. Therefore D_*, D^* remain finite beyond T and this is a contradiction.

Proof of (4.6) and (4.8).

As usual we prove only one of the statements, (4.8). Let (x_0, t_0) be such that $D_*(x_0, t_0) = -c < 0$ and $u(x_0, t_0) < 0$. Due to (4.5) we have that

$$D_*(x_0, t_0) \leq [D^*]_*(x_0, t_0) = [D]_*(x_0, t_0),$$

so it suffices to show that $[D]_*(x_0, t_0) \leq D_*(x_0, t_0)$.

We claim that

$$\text{dist}(x_0, \{x : D_*(x, t_0) > 0\}) \geq c.$$

Suppose that there exists x_1 such that $D_*(x_1, t_0) > 0$ and $|x - x_1| < c - \delta$, for some $\delta > 0$. As in (4.22) and (4.23) we have that $B(x_1, R_s) \subset A_s^h$, for $0 < h < h_0, t_0 < s < t_0 + \epsilon(x_1)$ and for some $R_s > 0$.
If we consider $y \in B(x_0, \frac{\delta}{2})$ then

$$\begin{aligned} \text{dist}(y, \partial^* A_s^h) &\leq |y - x_0| + |x_0 - x_1| \\ &\leq \tfrac{\delta}{2} + c - \delta = c - \tfrac{\delta}{2}, \end{aligned}$$

from which we deduce that

$$D^h(y, s) \geq -c + \frac{\delta}{2} \quad \text{for all} \quad (y, s) \in B(x_0, \frac{\delta}{2}) \times [t_0, t_0 + \epsilon(x_1)) \quad \text{and} \quad 0 < h \leq h_0.$$

This implies that

$$D_*(x_0, t_0) > c$$

which contradicts the hypothesis and the claim is proved.

As $\text{int}\Gamma_s = \emptyset$, given a point y near x_0 and $\delta > 0$ there exists x_1 such that $u(x_1, s) > 0$ and

$$\text{dist}(y, \Gamma_s) \geq |y - x_1| - \delta.$$

By (4.7) we have $D_* = D$ in $\{u > 0\}$ and so $D_*(x_1, s) > 0$, from which we conclude that

$$\text{dist}(y, \{x : D_*(x, s) > 0\}) \leq \text{dist}(y, \Gamma_s) + \delta$$

and letting $\delta \to 0^+$ we obtain

$$\text{dist}(y, \{x : D_*(x, s) > 0\}) \leq \text{dist}(y, \Gamma_s).$$

Also, given $\delta > 0$ and x_2 such that $D_*(x_2, s) > 0$ and $\text{dist}(y, \{x : D_*(x, s) > 0\}) \geq |y - x_2| - \delta$. By (4.4) we have $u(x_2, s) \geq 0$ or else $D^*(x_2, s) < 0$ which contradicts $D_*(x_2, s) > 0$. Since y is near x_0 and $u(x_0, s) < 0$, we deduce that

$$\text{dist}(y, \Gamma_s) \leq |y - x_2| \leq \text{dist}(y, D_*(x_2, s)) + \delta.$$

Consequently

$$\limsup_{\substack{y\to x_0, s\downarrow t_0\\ h\to 0}} \operatorname{dist}(y,\Gamma_s) = \limsup_{\substack{y\to x_0, s\downarrow t_0\\ h\to 0}} \operatorname{dist}(y,\{x : D_*(\cdot,s) > 0\}) \geq c$$

and we conclude that

$$\begin{aligned}[D]_*(x_0,t_0) &= \liminf_{y\to x_0, s\downarrow t_0} D(y,s)\\ &= \liminf_{\substack{y\to x_0, s\downarrow t_0\\ h\to 0}} [-\operatorname{dist}(y,\Gamma_s)]\\ &\leq -c,\end{aligned}$$

and so

$$[D]_*(x_0,t_0) \leq -c = D_*$$

which completes the proof. ∎

Appendix

Here we prove Theorem 3.5 and we follow exactly the proof given in [AMT], although we try to avoid the use of geometric measure theory, hopefully rendering it more accessible to analysts.

We recall the statement. Let A_0 be a set of finite perimeter and let

$$J(L) := \int_{\partial^* L\cap\Omega} H(x,\nu(x))dH^{N-1}(x) + \int_{L\setminus A_0} f(x,a)dx + \int_{A_0\setminus L} f(x,b)dx$$

where $H : \Omega \times \mathbf{R}^N \to [0,+\infty)$ satisfies (H8) - (H10) (see Proposition 2.12), $f = f(x,u;A_0)$ verifies (H6),(H7)and $Per(L) < +\infty$.

Theorem 3.5 *If A minimizes $J(\cdot)$ then there is an open set $U \subset \mathbb{R}^n$ such that $H^{N-1}(\partial^* A\setminus U) = 0$ and $\overline{\partial^* A} \cap U$ is an open $C^1(N-1)$ dimensional submanifold of $\mathbf{R}^N$. If $N = 3$ then $\partial^* A$ is a compact C^1 2−dimensional manifold of $\mathbf{R}^3$ without boundary.*

Note that by the Structure Theorem for sets of finite perimeter (see Theorem 2.6)

$$\partial^* A = \bigcup_{i=1}^{\infty} K_i \cup G\,,$$

$H_{N-1}(G) = 0$ and K_i are compact subsets of C^1 manifolds. However, there is no apriori guarantee that $\bigcup_{i=1}^{\infty} K_i$ is an open set.

Due to Theorem III.3 of [A1], Theorem 3.5 will hold if we prove that there exist $C, \delta > 0$ such that if $0 < r < \delta$ then

(A)

$$\int_{(\partial^* A\cap W)\cap\Omega} H(x,\nu_A(x))dH^{N-1}(x) \leq$$

$$\leq (1+rc) \int_{\varphi(\partial^* A\cap W)\cap\Omega} \underset{\varphi(\partial^* A)}{H(x,\nu(x))}\, dH^{N-1}(x),$$

(B) $H^{N-1}(\overline{\partial^* A} \bigtriangleup \partial A) = 0$, whenever $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a Lipschitz mapping, $W := \{x \in \mathbb{R}^n | \varphi(x) \neq x\} \subset\subset \Omega$, and diam $(W \cup \varphi(W)) = r$.

Proof of (A). Let $\varphi : \mathbb{R}^N \to \mathbb{R}^n$ be a Lipschitz mapping, let $W = \{x \in \mathbb{R}^N | \varphi(x) \neq x\}$ be a bounded set, diam $(W \cup \varphi(W)) = r$ and we compare the energy at A with the energy at $\varphi(A)$,

$$\int_{\partial^* A\cap\Omega} H(x,\nu_A)dH^{N-1}(x) + \int_{A\setminus A_0} f(x,a)dx + \int_{A_0\setminus A} f(x,b)dx \leq$$

$$\leq \int_{\varphi(\partial^* A)\cap\Omega} H(x,\nu_{\varphi(A)})dH^{N-1} + \int_{\varphi(A)\setminus A_0} f(x,a)dx + \int_{A_0\setminus A} f(x,b)dx.$$

Thus, by (H6)

$$\int_{\partial^* A\cap\Omega} H(x,\nu_A(x))dH^{N-1}(x) \leq \int_{\varphi(\partial^* A)\cap\Omega} H(x,\nu_{\varphi(A)})dH^{N-1}(x)$$

$$+ \int_{\varphi(A)\setminus A} f(x,a)dx + \int_{A\setminus A_0} f(x,a)dx - \int_{(\varphi(A)\setminus A)\cap A_0} f(x,a)dx$$

$$- \int_{(A\setminus A_0)\setminus\varphi(A)} f(x,a)dx + \int_{A_0\setminus A} f(x,b)dx + \int_{A\setminus\varphi(A)} f(x,b)dx$$

$$- \int_{(A_0\setminus A)\cap\varphi(A)} f(x,b)dx - \int_{(A\setminus\varphi(A))\setminus A_0} f(x,b)dx$$

$$- \int_{A\setminus A_0} f(x,a)dx - \int_{A_0\setminus A} f(x,b)dx$$

$$\leq \int_{\varphi(\partial^* A)\cap\Omega} \underset{\varphi(A)}{H(x,\nu(x))}\, dH^{N-1}(x) +$$

$$+6\mathcal{L}_N(\varphi(A) \bigtriangleup A). \operatorname*{esssup}_{x\in\Omega} \{| f(x,a)| + | f(x,b) |\}. \tag{A.1}$$

However,

$$\begin{aligned}
\partial^*\varphi(A) &= \partial^*[\varphi(A\backslash W)\cup\varphi(A\cap W)]\backslash(A\cap\partial^* W),\\
\partial^*\varphi(A\cap W) &= \varphi(\partial^* A\cap W)\cup(A\cap\partial^* W),\\
\partial^*\varphi(A\backslash W) &= \partial^*(A\backslash W)=(\partial^* A\backslash W)\cup(A\cap\partial^* W)
\end{aligned}$$

and we conclude that

$$\partial^*\varphi(A)=\varphi(\partial^* A\cap W)\cup(\partial^* A\backslash W)$$

and so (A.1) reduces to

$$\begin{aligned}
\int_{(\partial^* A\cap W)\cap\Omega} H(x,\nu_A(x))dH^{N-1}(x) \;\leq\; & \int_{\varphi(\partial^* A\cap W)\cap\Omega} H(x,\nu_{\varphi(A)}(x))dH^{N-1}(x)\\
& +6C_0\mathcal{L}_N(\varphi(A)\bigtriangleup A)
\end{aligned} \qquad (A.2)$$

where

$$C_0:=\operatorname*{esssup}_{x\in\Omega}\{|\,f(x,a)|+|\,f(x,b)\,|\}.$$

As $\varphi(x)=x$ if $x\notin W$,

$$\partial^*(\varphi(A)\bigtriangleup A)=\partial^* A\cap W\cup\varphi(\partial^* A\cap W) \qquad (A.3)$$

and

$$\varphi(A)\bigtriangleup A\subset W\cup\varphi(W). \qquad (A.4)$$

Indeed, if $y\notin W$, $y\notin\varphi(W)$ and if $y\in\varphi(A)\bigtriangleup A$, then either $y\in A\backslash\varphi(A)$ or $y\in\varphi(A)\backslash A$. In the first case we obtain

$$y\in A \text{ and } \varphi(y)\neq y \text{ (or else } y\in\varphi(A))$$

and this implies $y\in W$, contracting the assumption. If $y\in\varphi(A)\backslash A$ then $y=\varphi(x)$ for some $x\in A$ and, as $y\notin W$, we have

$$\varphi(y)=y=\varphi(\varphi(x)).$$

Hence $\varphi(\varphi(x))\neq x$, otherwise $y=x\in A$, thus $\varphi(x)\in W$. We conclude that $y=\varphi(\varphi(x))\in\varphi(W)$.

Now, by (H9), (A.2), (A.3), (A.4) and Lemma 2.9 we deduce that

$$\int_{\partial^* A\cap W)\cap\Omega} H(x,\nu_A(x))dH^{N-1}(x) \le \int_{\varphi(\partial^* A\cap W)\cap\Omega} H(x,\nu_{\varphi(A)}(x))dH^{N-1}(x)$$
$$+ \tfrac{6}{N}C_0 \text{ diam } (W\cup\varphi(W))\,[H^{N-1}(\partial^* A\cap W) + H^{N-1}(\varphi(\partial^* A\cap W))]$$
$$\le \int_{\varphi(\partial^* A\cap W)\cap\Omega} H(x,\nu_{\varphi(A)}(x))dH^{N-1}(x) + \tfrac{6}{N}\tfrac{C_0}{\alpha} \text{ diam } (W\cup\varphi(W\cup\varphi(W)).$$
$$\cdot\left[\int_{\varphi(\partial^* A\cap W)\cap\Omega} H(x,\nu_A(x))dH^{N-1}(x) + \int_{\varphi(\partial^* A\cap W)\cap\Omega} H(x_1\nu_{\varphi(A)}(x))dH^{N-1}(x)\right]$$

where we have used the fact that $W \subset\subset \Omega$ and so $\partial^* A\cap W \subset\subset \Omega$, $\varphi(\partial^* A\cap W) \subset\subset \Omega$. Finally,

$$\int_{(\partial^* A\cap W)\cap\Omega} H(x,\nu_A(x))dH^{N-1}(x)$$

$$\le \frac{1+\frac{6}{N}\frac{C_0}{\alpha}r}{1-\frac{6}{N}\frac{C_0}{\alpha}r} \int_{\varphi(\partial^* A\cap W)\cap\Omega} H(x,\nu_A(x))dH^{N-1}(x)$$

and setting $C := \dfrac{18}{N}\dfrac{C_0}{\alpha}$ it follows that

$$\frac{1+r\frac{6}{N}\frac{C_0}{\alpha}r}{1-\frac{6}{N}\frac{C_0}{\alpha}r} \le 1+rC$$

whenever $0 < r \le \frac{1}{C}$. ∎

Proof of (B). To prove that $H^{N-1}(\overline{\partial^* A}\,\triangle\, A) = 0$ it suffices to show that the $N-1$ density ratios at points of $\overline{\partial^* A}$ (see Definition 2.8) are uniformly bounded from below (see [F], [ATW] 3.1.3), i.e. there exist continuous functions $a, b : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that if $x_0 \in \overline{\partial^* A}$, $\theta^{*N-1}(\partial^* A, x_0) > 0$ then

$$\frac{m(r)}{r^{N-1}} \ge a(p)$$

for every $0 < r < b(p)$, where

$$m(r) := H^{N-1}(\partial^* A\cap B(x_0,r)).$$

As $\theta^{*^{N-1}}(\partial^* A, x_0) > 0$ and $m(r)$ is an increasing function, we have

$$m(r) > 0 \text{ for all } r > 0.$$

By Lemma 2.10 for a.e. $r > 0$ we have $H^{N-1}(\partial^* A \cap \partial B(x_0, r)) = 0$ and there exists a set Q of finite perimeter such that

$$A \triangle Q \subset B(x_0, r), \partial^*(A \triangle Q) \subset (\partial^* A \cap B(x_0, r)) \cup X,$$
$$H^{N-1}(X) \leq (N-1)[m'(r)]^{\frac{N-1}{N-2}}$$

and

$$\partial^* Q \subset (\partial^* A \backslash B(x_0, r)) \cup X.$$

Then,

$$\begin{aligned} & \int_{\partial^* A \cap \Omega} H(x, \nu_A(x)) dH^{N-1}(x) + \int_{A \backslash A_0} f(x,a)dx + \int_{A_0 \backslash A} f(x,b)dx \\ \leq & \int_{\partial^* Q \cap \Omega} H(x, \nu_Q(x)) dH^{N-1}(x) + \int_{Q \backslash A_0} f(x,a)dx + \int_{A_0 \backslash Q} f(x,b)dx \end{aligned}$$

and, as $H \geq 0$, we have

$$\begin{aligned} \int_{\partial^* A \cap B(x_0, r)} & \leq \int_X H(x, \nu_Q(x)) dH^{N-1}(x) + \int_{A \Delta Q} |f(x,a)| dx \\ & + \int_{A \Delta Q} |f(x,b)| dx \\ & \leq \int_X H(x, \nu_Q(x)) dH^{N-1}(x) + \mathcal{L}_N(A \triangle Q) C_0 \end{aligned}$$

where $C_0 := \operatorname{esssup}_{x \in \Omega} \{| f(x,a |) + | f(x,b) |\}$. By (H9) and Lemma 2.9

$$\alpha m(r) \leq \beta C(N-1)[m'(r)]^{\frac{N-1}{N-2}}$$

$$+ \frac{C_0}{N} \operatorname{diam}(A \triangle Q) \{H^{N-1}(\partial^* A \cap B(x_0, r)) + H^{N-1}(X)\}$$

$$\leq \beta C(N-1)[m'(r)]^{\frac{N-1}{N-2}} + \frac{C_0}{N} r [m(r) + C(N-1) m'(r)^{\frac{N-1}{N-2}}]$$

and so

$$\begin{aligned} m(r) & \leq m'(r)^{\frac{N-1}{N-2}} \frac{C(N-1)}{\alpha} \left(\beta + r\frac{C_0}{N}\right) (1 - \tfrac{C_0 r}{N\alpha})^{-1} \\ & = m'(r)^{\frac{N-1}{N-2}} \frac{C(N-1)}{\alpha} \beta \left(1 + \frac{rC_0}{N\beta}\right) \left(1 - \frac{C_0 r}{N\alpha}\right)^{-1}. \end{aligned}$$

Let $r_0 := \frac{N\alpha}{3C_0} = b(x_0)$. If $r \le r_0$ then

$$m(r) \le m'(r)^{\frac{N-1}{N-2}} C(N-1)\left(\frac{\beta}{\alpha} + r\frac{C_0}{N\alpha}\right)\frac{3}{2}$$

and $\frac{3r}{2}\frac{C_0}{N_\alpha} \le \frac{1}{2}$, thus

$$\begin{aligned} m(r) &\le m'(r)^{\frac{N-1}{N-2}} C(N-1)\left(\tfrac{3}{2}\tfrac{\beta}{\alpha} + \tfrac{1}{2}\right) \\ &= m'(r)^{\frac{N-1}{N-2}} \hat{C}^{\frac{N-1}{N-2}} \end{aligned}$$

where

$$\hat{C}^{\frac{N-1}{N-2}} := C(N-1)\left(\frac{3}{2}\frac{\beta}{\alpha} + \frac{1}{2}\right).$$

We conclude that

$$[(N-1)m(r)^{\frac{1}{N-1}}]' \ge \frac{1}{\hat{C}}.$$

As m is increasing

$$(N-1)m(r)^{\frac{1}{N-1}} \ge \frac{r}{\hat{C}}$$

and so

$$m(r) \ge \left[\frac{1}{(N-1)\hat{C}}\right]^{N-1} r^{N-1},$$

i.e.

$$\frac{m(r)}{r^{N-1}} \ge \left[\frac{1}{(N-1)\hat{C}}\right]^{N-1} =: a(x_0).$$

∎

For completeness, we include the proof of Lemma 4.6. We recall its statement.

Lemma 4.6 *Let $U \subset \mathbb{R}^k$ be an open set, $\psi_n : U \to \mathbb{R}$ continuous functions, and let $\psi_*(x) := \liminf_{\substack{n\to+\infty \\ x_n \to x}} \psi_n(x_n)$. Let x_0 be a strict local minimum for ψ_n. Then there exist a subsequence $n_k \to +\infty$ and points of local minimum for ψ_{n_k}, x_{n_k}, such that $x_{n_k} \to x_0$.*

Proof Let $r > 0$ be such that

$$\psi_*(x) > \psi_*(x_0) \text{ for all } x \in \bar{B}(x_0, r) \backslash \{x_0\}. \qquad (A.5)$$

Claim 1. There exists $\alpha > 0$ such that

$$\psi_*(x) \geq \psi_*(x_0) + \alpha \text{ for every } x \in \partial B(x_0, r).$$

Indeed, if we can find $x_\delta \in \partial B(x_0, r)$ such that

$$\psi_*(x_\delta) \leq \psi_*(x_0) + \delta,$$

we may assume that $x_\delta \rightarrow x_\infty \in \partial B(x_0, r)$ and, due to the lower semicontinuity of ψ_*, we obtain

$$\psi_*(x_\infty) \leq \psi_*(x_0)$$

contradicting (A.5).

Now we choose, by definition of ψ^*, a sequence $x_n \rightarrow x_0$ such that $\psi_n(x_n) \rightarrow \psi_*(x_0)$ and $| x_n - x_0 |< r$. We write

$$\{x_n\}_{n=1}^{\infty} = A \cup B$$

where

$$A := \{x_n \mid \exists y_n \in \bar{B}(x_0, r) \text{suchthat } y_n \text{ is a point of local minimum for} \varphi_n$$

$$\text{and } \varphi_n(y_n) \leq \varphi_n(x_n) \},$$

$$B := \{x_n\}_{n=1}^{\infty} \backslash A.$$

Claim 2. If $x_n \in B$ then there exists $z_n \in \partial B(x_0, r)$ such that $\psi_n(z_n) \leq \psi_n(x_n)$.

If claim 2 was not satisfied, there would exist $x_n \in B$ such that $\psi_n(z) > \psi_n(x_n)$ for every $z \in \partial B(x_0, r)$. On the other hand, as $x_n \in B$ there must exist $\xi \in B(x_0, r)$ such that

$$\psi_n(\xi) < \psi_n(x_n)$$

or else x_n would be a local minimum for φ_n, contradicting $x_n \in B$. We conclude that

$$\phi \neq \{z \in B(x_0, r) \mid \psi_n(z) < \psi_n(x_n)\} \subset K \subset\subset B(x_0, r)$$

where K is a compact set. Therefore

$$\psi_n \mid_K \text{ attains a local minimum at} z_n \in K$$

such that $\psi_n(z_n) \leq \psi_n(x_n)$, contradicting $x_n \in B$. Hence the claims holds true.

Claim 3 $\#B < +\infty$.

Suppose that B has infinitely many elements. Then it is possible to construct a subsequence $z_m \in \partial B(x_0, r)$ such that, by Claim 2 and by (A.5)

$$\psi_*(x_0) \leq \psi_m(z_m) \leq \psi_m(x_m).$$

Letting $z_m \to z_\infty \in \partial B(x_0, r)$, by the lower semicontinuity of ψ_* and as $\psi_n(x_n) \to \psi_*(x_0)$, we deduce that

$$\psi_*(x_0) = \psi(z_\infty)$$

contradicting (A.5). This proves Claim 3.

In view of Claim 3 we are able to extract a subsequence of points of A, x_{n_k}, with corresponding local minima $y_{n_k} \in \bar{B}(x_0, r)$ such that

$$\psi_{n_k}(y_{n_k}) \leq \psi_{n_k}(x_{n_k}).$$

Again, assuming that $y_{n_k} \to y_\infty \in \bar{B}(x_0, r)$, we conclude that

$$\begin{aligned} \psi_*(y_\infty) \leq & \lim \psi_{n_k}(x_{n_k}) \\ = & \psi_*(x_0) \end{aligned}$$

and, by (A.5), this yields $y_\infty = x_0$. ∎

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References

[A1] Almgren, F., Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, *Mem. Amer. Math. Soc.* **165** (1976), 199 + vii pp.

[A2] Almgren, F., Optimal isoperimetric inequalities, *Indiana U. Math. J.* **35** (1986), 451–547.

[ATW] Almgren, F., J. Taylor and L. Wang, Curvature driven flows: a variational approach. Preprint of the Geometry Center, University of Minnesota.

[AG] Angenent, S. and M. E. Gurtin, Multiphase thermomechanics with interfacial structure. 2. Evolution of an isothermal interface, *Arch. Rat. Mech. Anal.* **108** (1989), 323-391.

[Ba] Baldo, S., Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids, *Ann. Inst. H. Poincaré - Analyse Non-linéaire* **7** (1990), 37–65.

[BP] Barles, G. and B. Perthame, Discontinuous solutions of deterministic optimal stopping-time problems, *Model. Math. et Anal. Num.* **21** (1987), 557–579.

[BSS] Barles, G., H. M. Soner and P. E. Souganidis, Front Propagation and Phase Field Theory, *SIAM J. Cont. Opt.* **31** (1993), 439–469.

[BF] Barroso, A.C. and I. Fonseca, Anisotropic singular perturbations. To appear in *Proc. Royal Soc. Edin.*

[Bo] Bouchitté, G., Singular perturbations of variational problems arising from a two-phase transition model, *Appl. Math. Optim.* **21** (1990), 289–314.

[CGG] Chen, Y.-G. Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Diff. Geom.* **33** (1991), 749–786.

[CIL] Crandall, M.G., I. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bul. AMS* **27** (1992), 1–67.

[CL] Crandall, M.G. and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. AMS* **277** (1983), 1–42.

[DG1] De Giorgi, E., Sulla convergenza di alcune succession d'integrali del tipo dell'area, *Rend. Matematica* **8** (1975), 277–294.

[DG2] De Giorgi, E., New problems on minimizing movements. Preprint.

[DGF] De Giorgi, E. and I. Fonseca, Informal talk during the Conference "*Surface tension and movement by mean curvature*", Trento, July 1992.

[EG] Evans, L.C. and R.F. Gariepy, *Lecture Notes on Measure Theory and Fine Properties of Functions.* CRC Press, 1992.

[ESS] Evans, L.C., Soner, H.M. and P. E. Souganidis, Phase transitions and generalized motion by mean curvature, *Comm. Pure and Appl. Math.* **XLV** (1992), 1097–1123.

[ES1] Evans, L.C. and J. Spruck, Motion of level sets by mean curvature I, *J. Diff. Geom.* **33** (1991), 635–681.

[F] Federer, H., *Geometric Measure Theory.* Springer, 1969.

[Fo] Fonseca, I., Lower semicontinuity of surface energies, *Proc. Soc. Edin.* **120A** (1992), 99-115.

[FT] Fonseca, I. and L. Tartar, The gradient theory of phase transitions for systems with two potential wells, *Proc. Royal Soc. Edin.* **111A** (1989), 89–102.

[Gi] Giaquinta, M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Princeton Univ. Press, 1983.

[Giu] Giusti, E., *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, 1984.

[GS] Goffman, C., and J. Serrin, Sublinear functions of measures and variational integrals, *Duke Math. J.* **31** (1964), 159–178.

[Gu1] Gurtin, M. E., Some remarks concerning phase transitions in $\mathbf{R}^N$. Dept. Math. Carnegie Mellon Univ. 1983.

[Gu2] Gurtin, M. E., Some results and conjecturee in the gradient theory of phase transitions. *Measurability and Incompletely Posed Problems.* Antman, S., J. L. Ericksen, D. Kinderlehrer, I. Müller, eds., Springer, 1987, 135-146.

[Gu3] Gurtin, M. E., Multiphase thermomechanics with interfacial structure, 1. Heat conduction and the capillary balance law, *Arch. Rat. Mech. Anal.* **104** (1988), 195–222.

[Gu4] Gurtin, M. E., *Thermomechanics of Evolving Phase Boundaries in the Plane.* Clarendon Press, Oxford, 1993.

[IS] Ishii, H. and P. E. Souganidis, Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor. Preprint.

[KS1] Katsoulakis, M.A. and P. E. Souganidis, Interacting particle systems and generalized mean curvature evolution, Arch. Rat. Mech. Anal., to appear.

[KS2] Katsoulakis, M.A. and P. E. Souganidis, Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range interactions and Glauber dynamics, Comm. Math. Phys., to appear.

[KoS] Kohn, R. and P. Sternberg, Local minimezers and singular perturbations, *Proc. Royal Soc. Edin.* **111A** (1989), 69.

[LU] Ladyzhenskaya, O. A. and N. N. Ural'tseva, *Linear and Quasilinear Equations.* Academic Press, New York, London, 1968.

[L] Lions, P.L., Optimal control of diffusion processes. Part 2: Viscosity solutions and uniqueness, *Comm. P.D.E.* **8** (1983), 1229–1276.

[Mo] Modica, L., Gradient theory of phase transitions and minimal interface criterion, *Arch. Rat. Mech. Anal.* **98** (1987), 123–142.

[OSt] Owen, N.C., and P. Sternberg, Nonconvex variational problems with anisotropic perturbations, *Nonlinear Analysis, Theory, Methods and Applications* vol 16, **78** (1991), 705–719.

[R] Reshetnyak, Y. G., Weak Convergence of completely additive vector functions on a set, *Siberian Math. J.* **9** (1968), 1039-1045.

[S] Soner, H.M., Motion of a set by the curvature of its boundary, *J. Diff. Eq.* **101** (1993), 313–372.

[St] Sternberg, P., The effect of a singular perturbation on nonconvex variational problems, *Arch. Rat. Mech. Anal* **101** (1988), 209-260.

[Z] Ziemer, W., *Weakly Differentiable Functions.* Springer, 1989.