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Relaxation of Multiple Integrals in Subcritical Sobolev Spaces

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1 Introduction

In this paper we study lower semicontinuity and relaxation of some integrals of the calculus of variations of which a prototype example is given by

$$I(u) := \int_\Omega \{f(\nabla u(x)) + g(\det \nabla u(x))\}\, dx\ , \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is an open, bounded domain,

$$\frac{1}{C_1}|\xi|^p - C_2 \leq f(\xi) \leq C_1(1+|\xi|^p)\ , \frac{1}{C_1}|t| - C_2 \leq g(t) \leq C_1(1+|t|)$$

for some $C_1 > 0, C_2 \geq 0, N-1 < p < N$ and for all $\xi \in M^{N\times N}, t \in \mathbf{R}$.

Integrands of the type (1.1) are considered in nonlinear elasticity and the condition $p < N$ plays a fundamental role (see [B]) as it allows discontinuous deformations.

It is well known that if $W : M^{n\times N} \to \mathbf{R}$ is a quasiconvex function (see Section 2) satisfying

$$0 \leq W(\xi) \leq C(1+|\xi|^q) \tag{1.2}$$

then

$$\liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx \geq \int_\Omega W(\nabla u(x))dx$$

whenever $u_k, u \in W^{1,q}(\Omega; \mathbf{R}^n)$, $u_k \rightharpoonup u$ in $W^{1,q}(\Omega; \mathbf{R}^n)$ (see [AF], [Mar1]).

The lower semicontinuity problem for *polyconvex* integrands below the critical exponent q was firstly stated by Marcellini [Mar1]. In particular, if we restrict to our prototype example (1.1), Marcellini [Mar1] proved the lower semicontinuity for $p > \frac{N^2}{N+1}$. This result was extended to the case $p > N-1$ by Dacorogna and Marcellini [DM]; Gangbo [G] incorporated in this setting a dependence on x and u as well. Later, Dal Maso and Sbordone [DMS], Fusco and Hutchinson [FH], and Celada and Dal Maso [CDM] extended the lower semicontinuity result to the limiting case $p = N-1$.

Counterexamples to lower semicontinuity have been provided by Malý [Mal1] for $p < N-1$ and by Gangbo [G].

In the *quasiconvex* case (1.2), lower semicontinuity below the critical exponent was first studied by [Mar2] for $p > q\frac{N}{N+1}$, under some structure conditions on the quasiconvex integrand. Malý [Mal2] considered the case where $p > q\frac{N-1}{N}$, $p \geq q-1$, and he analyzed the lower semicontinuity on smooth mappings.

Here we study a class of integrands larger than the ones included in (1.1), and not necessarily polyconvex. Considering the functional (1.1) with $W(\xi) := f(\xi) + g(\det\xi)$, we define the relaxed energy for every $u \in W^{1,p}(\Omega; \mathbf{R}^N)$ as

$$\mathcal{F}_p(u) := \inf_{\{u_k\}} \left\{ \begin{array}{c} \liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx| \ u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbf{R}^N), \\ u_k \in W^{1,N}(\Omega; \mathbf{R}^N) \end{array} \right\}$$

and we prove that, for every $p > N-1$,

$$\mathcal{F}_p(u) \geq \int_\Omega QW(\nabla u(x))dx$$

and equality holds provided $u \in W^{1,N}(\Omega; \mathbf{R}^N)$.

The novelty of this result is that, in the relaxation process, the integrand QW is at best quasiconvex and the convergence takes place in $W^{1,p}(\Omega; \mathbf{R}^N)$ where p is smaller than the critical exponent. Hence the results obtained earlier for $p \geq q$ or for polyconvex integrands cannot be applied here.

Also this is, as far as we know, the first relaxation result for subcritical Sobolev spaces as opposed to the lower semicontinuity results obtained in the previous works mentioned above. We emphasize the fact that when relaxing an energy functional the energy density is, in general, represented by a quasiconvex function which is not polyconvex.

This paper is organized as follows:

In Section 1 we prove the general theorem of lower semicontinuity when $q-1 < p < q$, for quasiconvex integrands with q growth satisfying a suitable structure condition (see Theroem 2.1). In Corollary 2.3 we obtain a relaxation result and the integral representation for $\mathcal{F}_p(u)$ when the integrand is not quasiconvex. In Section 3 we show that Theorem 2.1 can be applied to integrands of the type (1.1) (see Theorem 3.1) and in Theorem 4.1, Section 4, under the same hypothesis on the integrands of (1.1), together with polyconvexity, we prove that

$$\mathcal{F}_p(u) > \int_\Omega f(\nabla u) + g(\det \nabla u)dx$$

for a certain class of deformations $u \notin W^{1,N}(\Omega;\mathbf{R}^N)$. This shows that the condition $u \in W^{1,q}(\Omega;\mathbf{R}^N)$ imposed in (2.17) cannot be relaxed. Precisely, we prove that for some radial deformations $u \notin W^{1,N}(\Omega;\mathbf{R}^N)$, singular at the origin, $\mathcal{F}_p(u)$ is equal to the sum of the integral in the right hand side of (1.1) plus a Dirac mass supported at the origin.

2 The General Case

Let $\Omega \subset \mathbf{R}^N$ be an open, bounded domain, $N, n \in \mathbf{N}$; let $W : M^{n\times N} \to [0,+\infty)$ be a Borel function such that

$$(H1) \qquad 0 \leq W(\xi) \leq C(1+|\xi|^q) \ \text{ for all } \ \xi \in M^{n\times N}$$

and there exists $\lambda \in [0,+\infty)$ such that

$$(H2)_\lambda \qquad W(AB) \leq C(1+W(B))$$

for all $B \in M^{n\times N}$, $A = \theta\mathbf{1}+(-\lambda-\theta)z\otimes z$, for all $\theta \in [0,1], z \in S^{n-1}\cup\{\vec{0}\}$ and, if $\lambda = 0$, then $W(\theta(\mathbf{1}-z\otimes z)B) \leq C$ if $|z| = 1, \theta \in [0,1]$, for some $C > 0$, $q > 1$.

Let $p \geq 1$, $p > q-1$, and for $u \in W^{1,p}(\Omega;\mathbf{R}^n)$ consider the relaxed energy

$$\mathcal{F}_p(u) := \inf_{\{u_k\}} \left\{ \begin{array}{c} \liminf\limits_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx | u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbf{R}^n), \\ u_k \in W^{1,q}(\Omega;\mathbf{R}^n) \end{array} \right\}$$

We recall that W is said to be ***quasiconvex*** (see [D],[Mo]) if

$$W(A) \leq \frac{1}{\text{meas}(D)} \int_D W(A+\nabla\varphi(x))dx$$

for all $A \in M^{n\times N}$, $D \subset \mathbf{R}^N$ open, bounded, Lipschitz domain, $\varphi \in W_0^{1,\infty}(D;\mathbf{R}^n)$. Also, the ***quasiconvex envelope*** of W is defined as

$$QW(A) := \inf\left\{\frac{1}{\text{meas}(Q)}\int_Q W(A+\nabla\varphi(x))dx | \varphi \in W_0^{1,\infty}(Q;\mathbf{R}^n)\right\}$$

where $Q = \left(\frac{-1}{2},\frac{1}{2}\right)^N$.

Theorem 2.1 *Let $W : M^{n\times N} \to [0,+\infty)$ be a quasiconvex function satisfying (H1) and (H2)$_\lambda$, for some $\lambda \geq 0$. Let $u \in W^{1,p}(\Omega;\mathbf{R}^n), u_k \in W^{1,q}(\Omega;\mathbf{R}^n)$ such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbf{R}^n)$. Then*

$$\liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx \geq \int_\Omega W(\nabla u(x))dx.$$

The proof of this result is based on a method introduced by Marcellini [Mar2] and also on a truncation argument used by Celada and Dal Maso [CDM]. The truncating functions are of the form, for $\lambda \in (0,+\infty)$,

$$\varphi_\lambda(t) := \begin{cases} t\,, & 0 \leq t \leq 1 \\ -\lambda t + \lambda + 1, & 1 < t < \Lambda \\ 0, & t \geq \Lambda \end{cases}$$

where $\Lambda := 1 + \frac{1}{\lambda}$; while, for $\lambda = 0$,

$$\varphi_0(t) := \begin{cases} t, & 0 \leq t \leq 1 \\ 1, & t > 1. \end{cases}$$

For $\rho > 0$ set

$$\Psi_{\lambda,\rho}(y) := \begin{cases} \rho\varphi_\lambda\left(\frac{|y|}{\rho}\right)\frac{y}{|y|}\,, & y \neq 0 \\ 0, & y = 0. \end{cases} \tag{2.1}$$

If $\lambda > 0, \rho < |y| < \Lambda\rho$, then

$$\nabla\Psi_{\lambda,\rho}(y) := \theta\mathbf{1} + (-\lambda - \theta)z\otimes z\,,\ |z| = 1,\ \theta \in [0,1], \theta := \frac{\rho}{|y|}\varphi_\lambda\left(\frac{|y|}{\rho}\right), z = \frac{y}{|y|}\,,$$

$$\begin{aligned} \nabla\Psi_{\lambda,\rho}(y) &= 0 \quad \text{if} \quad |y| > \Lambda\rho \\ \nabla\Psi_{\lambda,\rho}(y) &= \mathbf{1} \quad \text{if} \quad |y| < \rho. \end{aligned} \tag{2.2}$$

Also if $\lambda = 0$

$$\begin{aligned} \nabla\Psi_{0,\rho}(y) &= \mathbf{1} \quad \text{if} \quad |z| < \rho \\ \nabla\Psi_{0,\rho}(y) &= \theta(\mathbf{1} - z\otimes z) \quad \text{if} \quad |y| > \rho. \end{aligned} \tag{2.3}$$

Thus we conclude that $\Psi_{\lambda,\rho} \in W^{1,\infty}(\mathbf{R}^n;\mathbf{R}^n)$,

$$||\Psi_{\lambda,\rho}||_{1,\infty} \leq C(\lambda). \tag{2.4}$$

As in [Mar2] we will make use of the following lemma (see [Mar1], [Mar2] Lemma 2.2).

Lemma 2.2 *Let $W : M^{n\times N} \to \mathbf{R}$ be a quasiconvex function satisfying the growth condition (H1). If $q-1 \leq p \leq q$ then there exists a constant $C_1 > 0$ such that*

$$\int_\Omega |W(\xi(x)) - W(\eta(x))|\, dx \leq$$

$$\leq C_1 \| 1 + |\xi| + |\eta| \|_{L^p}^{q-1} \|\xi - \eta\|_{L^r}$$

whenever $\xi, \eta \in L^r(\Omega; M^{n\times N})$, and where $r := \frac{p}{p-q+1}$.

Proof .It was proven in [Mar 1] that the quasiconvexity of W together with $(H1)$ yield

$$|f(\xi) - f(\eta)| \leq C_1(1 + |\xi| + |\eta|)^{q-1} |\xi - \eta|$$

for all $\xi, \eta \in M^{n\times N}$ and some $C_1 > 0$.

The result now follows using Hölder's inequality with exponents $\frac{p}{q-1}$ and r.∎

Proof of Theorem 2.1. The case $p \geq q$ is well known and it holds without assuming $(H2)_\lambda$ (see [AF], [Mar 1]), and so we restrict our attention to $q - 1 < p < q$. We divide the proof into four steps.

Step 1. Let $F_0 \in M^{n\times N}$ be a fixed matrix, $u_k \in W^{1,q}(\Omega; \mathbf{R}^n), u_k \rightharpoonup F_0 x$ in $W^{1,p}(\Omega; \mathbf{R}^n)$. Assume in addition that

$$u_k \rightharpoonup F_0 x \text{ in } L^r(\Omega; \mathbf{R}^n). \tag{2.5}$$

Without loss of generality we may assume that

$$\liminf_{k\to+\infty} \int_\Omega f(\nabla u_k(x))dx = \lim_{k\to+\infty} \int_\Omega f(\nabla u_k(x))dx < +\infty.$$

Here we follow an argument similar to that of [Mar 2] Lemma 2.3 (see also [DG]).

Let Ω_0 be a fixed open set compactly contained in Ω, let $R := \frac{\text{dist}(\bar\Omega_0, \partial\Omega)}{2}$, let $M \in \mathbf{N}$ be fixed. For $i \in \{1, \ldots, M\}$ we define

$$\Omega_i := \{x \in \Omega |\ \text{dist}\,(x, \Omega_0) < \frac{i}{M} R\}.$$

Then $\Omega = \cup_{i=1}^M \Omega_i$, $\Omega_i \supset \Omega_{i-1}$, and given $i \in \{1, \ldots, M\}$ we consider cut-off functions $\varphi_i \in C_0^1(\Omega_i)$ such that

$$0 \leq \varphi_i \leq 1,\ \varphi_i(x) = \begin{cases} 1, & x \in \Omega_{i-1} \\ 0, & x \in \Omega \setminus \Omega_i \end{cases}, \qquad \|\nabla\varphi_i\|_\infty \leq \frac{M+1}{R}.$$

We set

$$u_k^i := \varphi^i u_k + (1 - \varphi^i) F_0 x.$$

As W is quasiconvex, $u_k^i \in W_0^{1,q}(\Omega;\mathbf{R}^n)$ and W satisfies (H1) we have (see [BM])

$$\begin{aligned}\int_\Omega W(F_0)dx &= W(F_0)\text{meas}(\Omega) \leq \int_\Omega W(\nabla u_k^i(x))dx \\ &= \int_{\Omega\setminus\Omega_i} W(F_0)dx + \int_{\Omega_i\setminus\Omega_{i-1}} W(\nabla u_k^i(x))dx + \int_{\Omega_{i-1}} W(\nabla u_k(x))dx\end{aligned}$$

hence

$$\int_{\Omega_{i-1}} W(\nabla u_k(x))dx - W(F_0)\text{meas}(\Omega_i) \geq$$

$$\geq -\int_{\Omega_i\setminus\Omega_{i-1}} W(\nabla u_k^i(x))dx.$$

Summing this inequality with respect to $i = 1, 2, \ldots, M$ and dividing by M yields

$$\int_\Omega W(\nabla u_k(x))dx - W(F_0)\frac{\sum_{i=1}^M \text{meas}(\Omega_i)}{M} \geq -\frac{1}{M}\sum_{i=1}^M \int_{\Omega_i\setminus\Omega_{i-1}} W(\nabla u_k^i(x))dx. \tag{2.6}$$

On the other hand

$$\nabla u_k^i = \varphi^i \nabla u_k + (1-\varphi^i)F_0 + (u_k - F_0 x)\otimes\nabla\varphi^i$$

and using Lemma 2.2 we obtain

$$\begin{aligned}\int_{\Omega_i\setminus\Omega_{i-1}} W(\nabla u_k^i(x))dx &= \int_{\Omega_i\setminus\Omega_{i-1}} (W(\nabla u_k^i(x)) - W(\varphi^i(x)\nabla u_k(x)))dx \\ &\quad + \int_{\Omega_i\setminus\Omega_{i-1}} W(\varphi^i(x)\nabla u_k(x))dx =: I_1 + I_2 \quad (2.7)\end{aligned}$$

where

$$\begin{aligned}I_1 &\leq C_1 \| 1 + |\nabla u_k^i| + |\varphi^i \nabla u_k| \|_{L^p(\Omega_i\setminus\Omega_{i-1})}^{q-1} \|\nabla u_k^i - \varphi^i\nabla u_k\|_{L^r(\Omega_{i-1})} \\ &\leq C_1 \| 1 + 2|\nabla u_k| + |F_0| + \|\nabla\varphi^i\|_\infty |u_k - F_0 x| \|_{L^p(\Omega_i\setminus\Omega_i - 1)}^{q-1} \cdot \\ &\quad \cdot (\| 1 + |F_0| + \|\nabla\varphi^i\|_\infty |u_k - F_0 x| \|_{L^r(\Omega_i\setminus\Omega_{i-1})}) \\ &\leq C_2 \left(\text{meas}(\Omega_i\setminus\Omega_{i-1}) + \left(\frac{M+1}{R}\right)^{q-1} \|u_k - F_0 x\|_{L^p(\Omega_i\setminus\Omega_{i-1})}^{q-1}\right) \\ &\quad \cdot \left(\text{meas}(\Omega_i\setminus\Omega_{i-1}) + \frac{M+1}{R}\|u_k - F_0 x\|_{L^r(\Omega_i\setminus\Omega_{i-1})}\right) \\ &\leq C_3 \left(\text{meas}(\Omega_i\setminus\Omega_{i-1}) + \left(\frac{M+1}{R}\right)^{q-1} \|u_k - F_0 x\|_{L^r(\Omega_i|\Omega_{i-1})}^{q-1}\right)^2 \quad (2.8)\end{aligned}$$

On the other hand, using $(\mathrm{H2})_\lambda$ with $z=0$ we have

$$I_2 \le \int_{\Omega_i\setminus\Omega_{i-1}} C(1+W(\nabla u_k(x))dx$$

which, together with (2.6), (2.7), (2.8), yields

$$\int_\Omega W(\nabla u_k(x))dx - W(F_0)\sum_{i=1}^{M}\frac{\mathrm{meas}(\Omega_i)}{M}$$

$$\ge -\tfrac{C_4}{M}\left(1+C(M,R)\|u_k-F_0x\|_{L^r(\Omega)}^{2(q-1)}\right) - \tfrac{1}{M}C\left(\mathrm{meas}(\Omega)+\int_\Omega W(\nabla u_k(x))dx\right).$$

Taking the limit as $k\to+\infty$ followed by the limit as $M\to+\infty$ we conclude that

$$\liminf_{k\to+\infty}\int_\Omega W(\nabla u_k(x))dx - W(F_0)\mathrm{meas}(\Omega)\ge 0\ ,$$

where we have used the fact that

$$\int_\Omega W(\nabla u_k(x))dx$$

remains bounded.

Step 2. Under the same hypotheses of Step 1, we remove the additional assumption (2.5).

i) $\lambda>0$. Let $m\in\mathbf{N}$ and choose an increasing sequence $\rho_m, \rho_m \underset{m\to+\infty}{\longrightarrow} +\infty, \rho_m\Lambda > \|F_0x\|_{L^\infty(\Omega)}$, where $\Lambda = 1+\frac{1}{\lambda}$,

$$\frac{1}{\rho_m}\sup_k\int_\Omega |u_k(x)|dx < \frac{1}{m}. \tag{2.9}$$

Let $C_0>0$ be such that

$$\sup_k\int_\Omega f(\nabla u_k(x))dx \le C_0.$$

Since

$$C_0 \ge \int_\Omega W(\nabla u_k(x))dx \ge \sum_{i=1}^{m}\int_{\{\rho_m\Lambda^{i-1}<|u_k|<\rho_m\Lambda^i\}} W(\nabla u_k(x))dx$$

then we must be able to find $i(m)\in\{0,\dots,m-1\}$ and a sequence

$$\{u_k^{(m)}\}_{k=1}^\infty \subset \{u_k\}_{k=1}^\infty$$

such that

$$\int_{\{\rho_m\Lambda^{i(m)}<|u_k|<\rho_m\Lambda^{i(m)+1}\}} W(\nabla u_k^{(m)}(x))dx \leq \frac{C_0}{m}. \tag{2.10}$$

Set

$$\begin{aligned} \beta(m) : &= \rho_m \Lambda^{i(m)} , \\ U_m : &= \Psi_{\lambda,\beta_m}(u) , \ u(x) = F_0 x , \\ U_{m,k} : &= \Psi_{\lambda,\beta_m}(u_k) \end{aligned}$$

(see (2.1)-(2.4)). Then $U_{m,k} \underset{k\to+\infty}{\rightharpoonup} U_m$ in $W^{1,p}(\Omega;\mathbf{R}^n), U_{m,k} \in W^{1,q}(\Omega;\mathbf{R}^n)$ and $\sup_k \|U_{m,k}\|_{L^\infty} \leq \beta(m)$, thus $U_{m,k} \underset{k\to+\infty}{\rightarrow} U_m$ strongly in $L^r(\Omega;\mathbf{R}^n)$. Moreover, as $\varphi_m\Lambda > \|F_0 x\|_{L^\infty(\Omega;\mathbf{R}^n)}$, then

$$U_m(x) = F_0 x$$

and by Step 1 we have

$$\begin{aligned} \int_\Omega W(F_0)dx &= W(F_0)\text{meas}(\Omega) = \int_\Omega W(\nabla U_m(x))dx \\ &\leq \liminf_{k\to+\infty} \int_\Omega W(\nabla U_{m,k}(x))dx \\ &\leq \liminf_{k\to+\infty} \left\{ \int_{\{|u_k|<\beta_m\}} W(\nabla u_k(x))dx + \int_{\{|u_k|>\Lambda\beta_m\}} W(0)dx + \right. \\ &\quad \left. + \int_{\{\beta_m<|u_k|<\Lambda\beta_m\}} W(\nabla\Psi_{\lambda,\beta_m}(u_k)\nabla u_k)dx \right\}. \end{aligned}$$

Using $(\mathrm{H2})_\lambda$, (2.2) and (2.10) we have

$$\begin{aligned} W(F_0)\text{meas}(\Omega) &\leq \liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx \\ &\quad + \limsup_{k\to+\infty} \left\{ W(0)\tfrac{1}{\Lambda\beta_m} \int_\Omega |u_k(x)|\,dx + \frac{CC_0}{m} \right. \\ &\quad \left. + C\text{meas}\{\beta_m < |u_k| < \Lambda\beta_m\} \right\} \\ &\leq \liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx + \frac{W(0)+C}{m} + \frac{CC_0}{m}. \end{aligned}$$

Letting $m \to +\infty$ we conclude that

$$W(F_0)\text{meas}(\Omega) \leq \liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx.$$

ii) $\lambda = 0$. As in part i) , choose β_m an increasing sequence, $\beta_m \underset{m\to+\infty}{\longrightarrow} +\infty$, such that

$$\frac{1}{\beta_m}\sup_k \int_\Omega |u_k(x)|\,dx < \frac{1}{m},$$

$\beta_m > \|F_0 x\|_{L^\infty(\Omega;\mathbf{R}^n)}$. Then $U_{m,k} \rightharpoonup U_m = F_0 x$ in $W^{1,p}(\Omega;\mathbf{R}^n)$, strongly in L^r, and by Step 1

$$\begin{aligned}
\int_\Omega W(F_0)dx &= W(F_0)\text{meas}(\Omega)\\
&\le \liminf_{k\to+\infty}\int_\Omega W(\nabla U_{m,k}(x))dx\\
&\le \liminf_{k\to+\infty}\left\{\int_{\{|u_k|<\beta_m\}} W(\nabla U_k(x))dx\right.\\
&\quad \left. + \int_{\{|u_k|>\beta_m\}} W(\theta(1 - z\otimes z)\nabla u_k(x))dx\right\}
\end{aligned}$$

where $\theta := \frac{\beta_m}{|u_k|}$ and $z = \frac{u_k}{|u_k|}$, and where we have used (2.3).

By $(\text{H2})_0$, $\operatorname*{ess\,sup}_{x\in\Omega} W(\theta(1 - z\otimes z)\nabla u_k(x)) \le C$, and so

$$\begin{aligned}
W(F_0)\cdot\text{meas}(\Omega) &\le \liminf_{k\to+\infty}\left\{\int_\Omega W(\nabla u_k(x))dx\right.\\
&\quad \left. + C\text{meas}\{|u_k| > \beta_m\}\right\}\\
&\le \liminf_{k\to+\infty}\int_\Omega W(\nabla u_k(x))dx + \frac{C}{m}.
\end{aligned}$$

It suffices to let $m \to +\infty$.

Step 3. Let $u \in W^{1,p}(\Omega;\mathbf{R}^n), u_n \in W^{1,q}(\Omega;\mathbf{R}^n), u_n \rightharpoonup u$ in $W^{1,p}$.

In addition, assume that

$$W(\xi) \ge k\,|\xi|^p \tag{2.11}$$

for some $k > 0$ and for all $\xi \in M^{n\times N}$. Without loss of generality we may suppose that

$$\liminf_{k\to+\infty}\int_\Omega W(\nabla u_k(x))dx = \lim_{k\to+\infty}\int_\Omega W(\nabla u_k(x))dx < +\infty.$$

Then there exists a subsequence (for simplicity, we do not change the notation) and a finite, non-negative, Radon measure μ such that

$$W(\nabla u_k)\mathcal{L}^N \underset{k\to+\infty}{\overset{*}{\rightharpoonup}} \mu$$

weakly $-*$ in the sense of measures, where $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbf{R}^N$. Using the Radon-Nikodym Decomposition Theorem (see [EG]), we may write

$$\mu = \mu_a \mathcal{L}^N + \mu_s$$

where $\mu_a \in L^1(\Omega; [0, +\infty))$ and μ_s is a finite, nonnegative Radon measure, singular with respect to $\mathcal{L}^N$.

We claim that

$$\mu_a(x_0) \geq W(\nabla u(x_0)) \text{ for } \mathcal{L}^N a.e. x_0 \in \Omega. \tag{2.12}$$

Assume that the claim holds. Choosing $\varphi^{(m)} \in C_0^\infty(\Omega; [0,1])$, with $\lim_{m\to+\infty} \varphi^{(m)}(x) = 1$ for all $x \in \Omega, \varphi^{(m)} \leq \varphi^{(m+1)}$, then

$$\begin{aligned}
\liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx &\geq \liminf_{k\to+\infty} \int_\Omega \varphi^{(m)}(x) W(\nabla u_k(x))dx \\
&= \int_\Omega \varphi^{(m)}(x) d\mu(x) \\
&\geq \int_\Omega \varphi^{(m)}(x) \mu_a(x) dx \\
&\geq \int_\Omega \varphi^{(m)}(x) W(\nabla u(x)) dx
\end{aligned}$$

and using Lebesgue's Monotone Convergence Theorem, letting $m \to +\infty$, we conclude that

$$\int_\Omega W(\nabla u(x))dx \leq \liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx.$$

Now we prove the claim (2.12).

Let $x_0 \in \Omega$ be a Lebesgue point for u such that

$$\lim_{\epsilon\to0^+} \frac{1}{\epsilon} \fint_{Q(x_0,\epsilon)} |u(y) - u(x_0) - \nabla u(x_0)(y - x_0)|\, dx = 0\,, \tag{2.13}$$

$$\mu_a(x_0) = \lim_{\epsilon\to0^+} \frac{\mu(Q(x_0,\epsilon))}{\epsilon^N} \text{ exists and is finite}\,, \tag{2.14}$$

where

$$Q(x_0, \epsilon) = x_0 + \epsilon \left(\frac{-1}{2}, \frac{1}{2}\right)^N, \; Q := \left(\frac{-1}{2}, \frac{1}{2}\right)^N.$$

It is known that the set of points $x_0 \in \Omega$ for which (2.13) or (2.14) fail has $\mathcal{L}^N$-measure zero (see [EG]).

Then, selecting a sequence $\epsilon \to 0^+$ for which $\mu(\partial Q(x_0,\epsilon)) = 0$, we have

$$\begin{aligned}\mu_a(x_0) &= \lim_{\epsilon\to 0^+} \tfrac{\mu(Q(x_0,\epsilon))}{\epsilon^N} \\ &= \lim_{\epsilon\to 0^+}\lim_{k\to+\infty} \tfrac{1}{\epsilon^N}\int_{Q(x_0,\epsilon)} W(\nabla u_k(x))dx \\ &= \lim_{\epsilon\to 0^+}\lim_{k\to+\infty} \int_Q W(\nabla u_{k,\epsilon}(x))dx\end{aligned}$$

where

$$u_{k,\epsilon}(x) := \frac{u_k(x_0+\epsilon x) - u(x_0)}{\epsilon}.$$

Then $u_{k,\epsilon} \in W^{1,q}(Q;\mathbf{R}^n)$ and by (2.14) and (2.11)

$$\{\nabla u_{k,\epsilon}\} \text{ is bounded in } L^p(Q;M^{n\times N}) \tag{2.15}$$

In addition, as $u_k \to u$ strongly in L^1, by (2.13)

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\lim_{k\to+\infty} \|u_{k,\epsilon} - \nabla u(x_0)x\|_{L^1(Q)} \\ &= \lim_{\epsilon\to 0^+}\lim_{k\to+\infty} \int_Q \left|\frac{u_k(x_0+\epsilon x) - u(x_0) - \nabla u(x_0)(x_0+\epsilon x - x_0)}{\epsilon}\right| dx \\ &= \lim_{\epsilon\to 0^+} \tfrac{1}{\epsilon}\int_Q |u(x_0+\epsilon x) - u(x_0) - \nabla u(x_0)(x_0+\epsilon x - x_0)|\, dx \\ &= \lim_{\epsilon\to 0^+} \tfrac{1}{\epsilon} ⨍_{Q(x_0,\epsilon)} |u(y) - u(x_0) - \nabla u(x_0)(y-x_0)|\, dy.\end{aligned}$$

In view of (2.15) we may take a diagonal subsequence v_j of $u_{k,\epsilon}$ and we obtain $v_j \in W^{1,q}(Q;\mathbf{R}^n)$, $v_j \to \nabla u(x_0)x$ in $L^1(Q;\mathbf{R}^n)$, $\left\{\int_Q |\nabla v_j|^p\right\}$ is bounded. Therefore $v_j \rightharpoonup \nabla u(x_0)x$ in $W^{1,p}(Q;\mathbf{R}^n)$ and by Step 2 we conclude that

$$\begin{aligned}\mu_a(x_0) &= \lim_{j\to+\infty} \int_Q W(\nabla v_j(x))dx \\ &\geq W(\nabla u(x_0)).\end{aligned}$$

Step 4. Under the same conditions of Step 3, we remove the assumption (2.11). Let $k > 0$ be such that

$$\sup_k \int_\Omega |\nabla u_k(x)|^p\, dx \leq \tilde{K} < +\infty.$$

Let $u \in W^{1,p}(\Omega;\mathbf{R}^n)$ and apply Step 3 to

$$W_\epsilon(\xi) := W(\xi) + \epsilon\,|\xi|^p .$$

Indeed, as $p < q$, it is clear that W_ϵ still satisfies (H1) and, given $A = \theta 1 - (\lambda + \theta) z \otimes z$, then $\|A\| \leq C(\lambda)$ and so

$$\begin{aligned} W_\epsilon(AB) = W(AB) + \epsilon\,|AB|^p &\leq C(1 + W(B)) + \epsilon C(\lambda)\,|B|^p \\ &\leq (C + C(\lambda))[1 + W(B) + \epsilon\,|B|^p] \\ &= (C + C(\lambda))(1 + W_\epsilon(B)), \end{aligned}$$

for all $B \in M^{n\times N}$. Hence Step 3 yields

$$\begin{aligned} \int_\Omega W(\nabla u(x))dx &= \lim_{\epsilon\to 0^+} \int_\Omega W(\nabla u(x)) + \epsilon\,|\nabla u(x)|^p\,dx \\ &= \lim_{\epsilon\to 0^+} \int_\Omega W_\epsilon(\nabla u(x))dx \\ &\leq \liminf_{\epsilon\to 0^+} \liminf_{k\to+\infty} \int_\Omega W_\epsilon(\nabla u_k(x))dx \\ &\leq \liminf_{\epsilon\to 0^+} \liminf_{k\to+\infty} \left\{ \int_\Omega W(\nabla u_k(x))dx + \epsilon \tilde{K} \right\} \\ &= \liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx. \end{aligned}$$

∎

Corollary 2.3 *Let $W : M^{n\times N} \to [0,+\infty)$ be a Borel measurable function satisfying (H1), (H2)$_\lambda$ for some $\lambda \in [0,+\infty)$. Let $p > q-1$, $u \in W^{1,p}(\Omega;\mathbf{R}^n)$, $u_k \in W^{1,q}(\Omega;\mathbf{R}^n)$, $u_k \rightharpoonup u$ in $W^{1,p}$. Then*

$$\liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx \geq \int_\Omega QW(\nabla u(x))dx$$

and

$$\mathcal{F}_p(u) \geq \int_\Omega QW(\nabla u(x))dx. \tag{2.16}$$

Moreover, if $u \in W^{1,q}(\Omega;\mathbf{R}^n)$, then

$$\mathcal{F}_p(u) = \int_\Omega QW(\nabla u(x))dx. \tag{2.17}$$

Corollary 2.3 uses the invariance of quasiconvexification under the structure condition (H2)$_\lambda$, namely

Proposition 2.4 *Let $W : M^{n\times N} \to [0,+\infty)$ be a Borel measurable function satisfying (H1) and (H2)$_\lambda$, for some $\lambda \in [0,+\infty)$. Then QW also satisfies (H1) and (H2)$_\lambda$.*

Proof .It is clear that

$$0 \leq QW(\xi) \leq W(\xi) \leq C(1+|\xi|^p)$$

for all $\xi \in M^{n\times N}$, hence (H1) holds. Given $B \in M^{n\times N}, A = \theta \mathbf{1}-(\lambda+\theta)z\otimes z, \theta \in [0,1], z \in S^{n-1} \cup \{\vec{0}\}$, let $\varphi_k \in W_0^{1,\infty}(Q;\mathbf{R}^n), Q := (\frac{-1}{2},\frac{1}{2})^N$, be such that

$$QW(B) = \lim_{k\to+\infty} \int_Q W(B+\nabla\varphi_k(x))dx.$$

Then

$$\begin{aligned} QW(AB) &\leq \liminf_{k\to+\infty} \int_Q W(AB+\nabla(A\varphi_k)(x))dx \\ &= \liminf_{k\to+\infty} \int_Q W(A(B+\nabla\varphi_k(x)))dx \\ &\leq \liminf_{k\to+\infty} \int_Q C(1+W(B+\nabla\varphi_k(x)))dx \\ &= C(1+QW(B)). \end{aligned}$$

∎

Proof of Corollary 2.3. The result is well known for $p \geq q$ (see [AF], [Mar1]), hence we restrict our attention to the case $q-1 < p < q$.

By Proposition 2.4 and Theorem 2.1 we have

$$\begin{aligned} \liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx &\geq \liminf_{k\to+\infty} \int_\Omega QW(\nabla u_k(x))dx \\ &\geq \int_\Omega QW(\nabla u(x))dx. \end{aligned}$$

and we conclude (2.16).

Suppose now that $u \in W^{1,q}(\Omega;\mathbf{R}^n)$. Then

$$\mathcal{F}_q(u) \geq \mathcal{F}_p(u) \tag{2.18}$$

because $p < q$ and, in view of the well-known relaxation results in $W^{1,q}$ (see [AF]) we have

$$\mathcal{F}_q(u) = \int_\Omega QW(\nabla u(x))dx$$

which, together with (2.16) and (2.18), yields

$$\mathcal{F}_p(u) = \int_\Omega QW(\nabla u(x))dx.$$

∎

Remark 2.5 *i)* If $p \geq q$ then we do not need to assume $(\mathrm{H2})_\lambda$ in Theorem 2.1 and Corollary 2.3 (see [AF], [Mar 1]).

ii) If $q\frac{N}{N+1} < p$ then Corollary 2.3 was obtained by Marcellini (see [Mar 2])) under the weakened hypothesis replacing $(\mathrm{H2})_\lambda$,

$$(H2)' \qquad\qquad W(tB) \leq C(1 + W(B))$$

for all $B \in M^{n\times N}, t \in [0,1]$. Note that (H2)$'$ is used in Step 1 of the proof of Theorem 2.1, while the full strength of $(\mathrm{H2})_\lambda$ was only required to carry out the truncation argument in Step 2, where it is shown that we can modify the original sequence $\{u_k\}$, into a new sequence $\{v_k\}$, where

$$v_k \to u \text{ in } L^r \text{strongly, } r = \frac{p}{p-q+1}.$$

However, if $p > q\frac{N}{N+1}$ then the Soblev exponent $\frac{Np}{N-p} > \frac{p}{p-q+1}$ and so $W^{1,p} \to L^r$ (compact embedding). Therefore, in this case we do not need to construct $\{v_k\}$ as the original sequence $\{u_k\}$ converges strongly to u in L^r.

iii) If $q - 1 < p \leq q\frac{N}{N+1}$ and if $u_k \to u$ in L^r strongly (this happens e.g. if $\sup_k \|u_k\|_\infty < +\infty$) then, under the conditions of Theorem 2.1 with $(\mathrm{H2})_\lambda$ replaced by (H2)$'$, we have

$$\liminf_{k\to+\infty} \int_\Omega W(\nabla u_k(x))dx \geq \int_\Omega W(\nabla u(x))dx.$$

We justify this fact using exactly the same argument of ii).

iv) The result of Theorem 2.1 may fail if $p < q - 1$.
To illustrate this fact, we use an example due to Malý (see [Mal1]). Let

$$W(\xi) := |\det \xi|,$$

W is polyconvex, W satisfies (H1) with $q = N$ and $(\mathrm{H2})_\lambda$ with $\lambda = 0$. Indeed, if $A = \theta(1 - z \otimes z), z \in S^{N-1} \cup \{\vec{0}\}$, then $\det A = \theta^N$ if $z = 0$, in which case

$$\begin{aligned} W(AB) &= |\det(AB)| = |\det A|\,|\det B| \\ &= \theta^N |\det B| \leq |\det B| \end{aligned}$$

and if $|z| = 1$ then $\det A = 0$, in which case

$$W(AB) = W(0) \text{ forall } B \in M^{N\times N}.$$

Let $u(x) := x, \Omega = (0,1)^N$. Malý [Mal1] showed that there exists a sequence of diffemorphisms $u_k \in C^1(\bar{\Omega}; \mathbb{R}^N), u_k \to u$ in L^s strong, for all

$s < +\infty, u_s \rightharpoonup u$ in $W^{1,N-1-\epsilon}$ for all $\epsilon > 0$, and

$$\liminf_{k \to +\infty} \int_\Omega W(\nabla u_k(x))dx = \liminf_{k \to +\infty} \int_\Omega |\det \nabla u_k(x)|\, dx$$
$$= 0 < \int_\Omega |\det \mathbf{1}| = \text{meas}(\Omega).$$

v) In Section 4 we will exhibit an example of an energy density satisfying the conditions of Corollary 2.3, and a function $u \in W^{1,q-\epsilon}(\Omega; \mathbf{R}^n)$, for all $\epsilon > 0, u \notin W^{1,q}(\Omega; \mathbf{R}^n)$, such that

$$\mathcal{F}_p(u) > \int_\Omega QW(\nabla u(x))dx$$

whenever $q - 1 < p < q$. This shows that the integral representation (2.17) for the relaxed energy $\mathcal{F}_p(u)$ does not depend on the Sobolev space in which weak convergence takes place, but rather on the regularity of u.

3 A Special class of Energy Functionals

In this section we apply the results of Section 2 to functionals of the type (1.1),

$$I(u) := \int_\Omega \{f(\nabla u(x)) + g(\det \nabla u(x))\}\ dx,$$

where $\Omega \subset \mathbf{R}^N$ is an open, bounded domain. We set

$$W(\xi) := f(\xi) + g(\det \xi).$$

Theorem 3.1 *Let $f : M^{N \times N} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$ be Borel measurable functions such that*

$$\frac{1}{C_1}|\xi|^p - C_2 \le f(\xi) \le C_1(1 + |\xi|^p) \tag{3.1}$$

and

$$\frac{1}{C_1}|t| - C_2 \le g(t) \le C_1(1 + |t|) \tag{3.2}$$

for some $C_1 > 0, C_2 \ge 0$, $N - 1 < p < N$, and for all $\xi \in M^{N\times N}, t \in \mathbf{R}$. Let $u \in W^{1,p}(\Omega; \mathbf{R}^N), u_k \in W^{1,N}(\Omega; \mathbf{R}^N)$, $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbf{R}^N)$. Then

$$\liminf_{k \to +\infty} \int_\Omega \{f(\nabla u_k(x)) + g(\det \nabla u_k(x))\}\, dx \ge \int_\Omega QW(\nabla u(x))dx$$

If, in addition, $u \in W^{1,N}(\Omega; \mathbf{R}^N)$ then

$$\mathcal{F}_N(u) = \inf_{\{v_k\}} \left\{ \liminf_{k \to +\infty} \int_\Omega \{f(\nabla v_k(x)) + g(\det \nabla v_k(x))\}\, dx\ : v_k \in W^{1,N}(\Omega; \mathbf{R}^N), v_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbf{R}^N) \right\} = \int_\Omega QW(\nabla u(x))dx.$$

Proof .In view of Corollary 2.3, it suffices to prove that W satisfies (H1) with $q = N$ and $(\mathrm{H2})_\lambda$ for some $\lambda \geq 0$. In fact, we may assume that $W \geq 0$, i.e. $C_2 = 0$, by adding the constant C_2, and we have

$$\begin{aligned} W(\xi) &= f(\xi) + g(\det \xi) \\ &\leq C_1(1 + |\xi|^p) + C_1(1 + |\det \xi|) \\ &\leq K(1 + |\xi|^N) \end{aligned}$$

for some constant $K > 0$. Also, setting $\lambda = 1$ we have

$$\begin{aligned} A &= \theta 1 - (1+\theta) z \otimes z \\ &= \theta \left(1 - \tfrac{1+\theta}{\theta} z \otimes z\right) \end{aligned}$$

and so $\det A = \theta^N \left(1 - \frac{1+\theta}{\theta} |z|^2\right)$. If $z = 0$ then

$$\begin{aligned} W(AB) &= W(\theta B) = f(\theta B) + g(\theta^N \det B) \\ &\leq C_1(1 + |\theta B|^p) + C_1(1 + |\theta^N \det B|) \\ &\leq C_1(1 + |B|^p + |\det B|) \\ &\leq C_1(1 + C_1 f(B) + C_1 g(\det B)) \\ &\leq \tilde{C}_1\,(1 + f(B) + g(\det B)) \\ &= \tilde{C}_1\,(1 + W(B)) \end{aligned}$$

where we have used (3.1) and (3.2). Also, if $|z| = 1$ then $\det A = \theta^N \left(1 - \frac{1+\theta}{\theta}\right) = -\theta^{N-1}$, $\|A\| \leq C_3$ for all $\theta \in [0,1], |z| = 1$, and so, by (3.1) and (3.2),

$$\begin{aligned} W(AB) &\leq C_1(1 + |AB|^p) + C_1(1 + |\theta^{N-1} \det B|) \\ &\leq C_4(1 + |B|^p + |\det B|) \\ &\leq C_5(1 + f(B) + g\,|\det B|) \\ &\leq C_5((1 + W(B)). \end{aligned}$$

and this concludes $(\mathrm{H2})_\lambda, \lambda = 1$.

∎

In the particular case where $f = 0$, i.e.

$$W(\xi) = g(\det \xi) ,$$

we do not need to assume (3.2), precisely:

Theorem 3.2 *Let $g : \mathbf{R} :\longrightarrow [0, +\infty)$ be a Borel measurable function, let $p > N - 1, u_k \in W^{1,N}(\Omega; \mathbf{R}^N), u \in W^{1,p}(\Omega; \mathbf{R}^N)$. Then*

$$\liminf_{k \to +\infty} \int_\Omega g(\det \nabla u_k(x))dx \geq \int_\Omega g^{**}(\det \nabla u(x))dx. \tag{3.3}$$

If, in addition, $u \in W^{1,N}(\Omega; \mathbf{R}^N)$ then

$$\begin{aligned} \mathcal{F}_p(u) &= \inf_{\{v_k\}} \left\{ \liminf_{k \to +\infty} \int_\Omega g(\det \nabla v_k(x))dx | v_k \in W^{1,N}(\Omega; \mathbf{R}^N), \right. \\ & \left. v_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbf{R}^N) \right\} \\ &= \int_\Omega g^{**}(\det \nabla u(x))dx. \end{aligned} \tag{3.4}$$

Remark 3.3 The lower semicontinuity result (3.3) was obtained in [DM], [G], [DMS], [FH] in the case where $u \in W^{1,N}(\Omega; \mathbf{R}^N)$.

Recently, Celada and Dal Maso [CDM] extended Theorem 3.2 to $p = N - 1, u \in W^{1,N-1}(\Omega; \mathbf{R}^N)$, and the method of their proof relies heavily on a result of Giaquinta, Modica and Souček [GMS] concerning the weak-* convergence (in the sense of measures) of minors, and this result, in turn, is obtained via methods of geometric measure theory.

Proof of Theorem 3.2. Let $p > N - 1, u_k \in W^{1,N}(\Omega; \mathbf{R}^N), u \in W^{1,p}(\Omega; \mathbf{R}^N)$. To establish (3.3) it suffices to prove that

$$\liminf_{k \to +\infty} \int_\Omega g^{**}(\det \nabla u_k(x))dx \geq \int_\Omega g^{**}(\det \nabla u(x))dx.$$

Consider an increasing sequence of piecewise affine, convex function g_m such that $\sup_m g_m = \lim_{m \to +\infty} g_m = g^{**}$ and

$$0 \leq g_m(t) \leq C_m(1 + |t|) , \; C_m > 1.$$

Then, setting

$$W_m(\xi) := g_m(\det \xi),$$

we have (H1) with $q = N$ and we prove $(\mathrm{H2})_\lambda$ for $\lambda = 0$.

If $|z| = 1, \theta \in [0,1]$ then

$$\begin{aligned} W_m(\theta(1 - z \otimes z)B) &= g_m(\theta^N \det(1 - z \otimes z) \det B) \\ &= g_m(0) \leq C_m \end{aligned}$$

for all $B \in M^{N \times N}$ and if $z = 0$ then by the convexity of g_m

$$\begin{aligned} W_m(\theta B) &= g_m(\theta^N \det B) \\ &\leq (1 - \theta^N) g_m(0) + \theta^N g_m(\det B) \\ &\leq C_m + g_m(\det B) \\ &\leq C_m(1 + W_m(B)). \end{aligned}$$

Thus, by Theorem 2.1

$$\begin{aligned} \int_\Omega g_m(\det \nabla u(x))dx &\leq \liminf_{k \to +\infty} \int_\Omega g_m(\det \nabla u_k(x))dx \\ &\leq \liminf_{k \to +\infty} \int_\Omega g(\det \nabla u_k(x))dx \end{aligned}$$

and letting $m \to +\infty$, using Lebesgue's Monotone Convergence Theorem, we conclude that

$$\int_\Omega g^{**}(\det \nabla u(x))dx \leq \liminf_{k \to +\infty} \int_\Omega g(\det \nabla u_k(x))dx.$$

Finally, if $u \in W^{1,N}(\Omega; \mathbf{R}^N)$ then we may find $v_k \in W^{1,\infty}(\Omega; \mathbf{R}^N), v_k \to u$ in $W^{1,N}$ and

$$\begin{aligned} \int_\Omega g^{**}(\det \nabla u) &= \lim_{k \to +\infty} \int_\Omega g(\det \nabla v_k(x))dx \\ &\geq \mathcal{F}_p(u) \end{aligned}$$

which, together with (3.3), yields

$$\mathcal{F}_p(u) = \int_\Omega g^{**}(\det \nabla u)dx.$$

∎

4 Existence of Relaxed Energies with Singular Part

In this section we show that the integral representation (2.17) may fail if $u \notin W^{1,q}(\Omega; \mathbf{R}^n)$.

Precisely, we prove that for a large class of polyconvex functionals of the type (1.1) satisfying (H1), $(\mathrm{H2})_\lambda$ we can find function $u \in W^{1,N-\epsilon}(\Omega;\mathbf{R}^N)$, for all $\epsilon > 0$, such that

$$\mathcal{F}_p(u) = \int_\Omega W(\nabla u(x))dx + \mu_s(\Omega)$$

$\mu_s > 0, \mu_s = \lambda\delta_0$ for some $\lambda > 0, \Omega = B(0,1)$.

A related example was treated by Acerbi and Dal Maso (see Theorem 4.1, [ADM]) by means of geometric measure theory techniques.

Theorem 4.1 *Let $f : M^{n\times N} \to \mathbf{R}$ be a quasiconvex function, $g : \mathbf{R} \to \mathbf{R}$ a convex function such that*

$$\begin{aligned} \tfrac{1}{C}|\xi|^p - C &\le f(\xi) \le C(1+|\xi|^p)\,, \\ \tfrac{1}{C}|t| - C &\le g(t) \le C(1+|t|) \end{aligned}$$

for some $C > 0, N-1 < p < N$, and for all $t \in \mathbf{R}, \xi \in \mathbf{M}^{n\times N}$. Let $B := B(0,1) \subset \mathbf{R}^N$, $W(\xi) := f(\xi)+g(\det\xi), u(x) := v(|x|)\frac{x}{|x|}, v \in W^{1,\infty}(\mathbf{R};\mathbf{R}), v(0) > 0$. Then $u \in W^{1,N-\epsilon}(B;\mathbf{R}^N)$ for all $\epsilon > 0$ and

$$\mathcal{F}_p(u) = \int_B W(\nabla u(x))dx + \mathrm{meas}(B)g^\infty(v(0)^N)\,,$$

where the **recession function** *g^∞ is defined as*

$$g^\infty(s) := \lim_{t\to+\infty} \frac{g(ts)}{t}.$$

This class of radial functions were studied in this context in [Mar 2]. We recall that the recession function is a convex, positively homogeneous of degree one function. Thus for every $s > 0$ we have

$$g^\infty(s) = sg^\infty(1)$$

and so

$$g^\infty(v(0)^N) = v(0)^N g^\infty(1).$$

Proof .It was shown in Section 3 that these integrands satisfy (H1), $(\mathrm{H2})_\lambda$, with $\lambda = 1$. It is clear that

$$\nabla u(x) = \frac{v(r)}{r}\left(1 + \left(\frac{v'(r)r}{v(r)} - 1\right)\frac{x}{r}\otimes\frac{x}{r}\right)$$

where $r := |x|$, hence

$$\begin{aligned} \det\nabla u(x) &= \left(\tfrac{v(r)}{r}\right)^N \cdot \left(1 + v'(r)\tfrac{r}{v(r)} - 1\right) \\ &= v'(r)\left(\tfrac{v(r)}{r}\right)^{N-1}. \end{aligned}$$

Step 1. Let $u_k \in W^{1,N}(B;\mathbf{R}^N)$, $u_k \rightharpoonup u$ in $W^{1,p}(B;\mathbf{R}^N)$. Using the truncation method of Step 2 in the proof of Theorem 2.1, for fixed $m \in \mathbf{N}$ we may find $\rho_m > 1$ such that (see (2.1) - (2.4))

$$\begin{aligned} U_m(x) &:= \Psi_{\lambda,\beta_m}(u(x)) = u(x), \\ U_{m,k}(x) &:= \Psi_{\lambda,\beta_m}(u_k(x)), \end{aligned}$$

$U_{m,k}(x) \underset{k\to+\infty}{\rightharpoonup} U_m$ in $W^{1,p}(B;\mathbf{R}^N)$ and strongly in $L^r(B;\mathbf{R}^N)$, $U_{m,k} \in W^{1,N}(B;\mathbf{R}^N)$ and

$$\liminf_{k\to+\infty} \int_B W(\nabla U_{m,k}(x))dx \leq \liminf_{k\to+\infty} \int_B W(\nabla u_k(x))dx + \frac{1}{m}. \tag{4.1}$$

Fix $m \in \mathbf{N}$ and $\epsilon > 0$. As $u \in C^1(B(0,\epsilon)\backslash\overline{B(0,\epsilon\backslash 2)};\mathbf{R}^N)$, then $\|\nabla u\| \in L^r(B(0,\epsilon)\backslash\overline{B(0,\epsilon/2)})$ and since $U_{m,k} \to u$ in L^r, as in Step 1 of the proof of Theorem 2.1 using the slicing technique we may change $U_{m,k}$ in $B(0,\epsilon)$ near $\partial B(0,\epsilon)$ into a new sequence $w_k^{(m)} \in W^{1,N}(B(0,\epsilon);\mathbf{R}^N)$ such that

$$\begin{aligned} w_k^{(m)} &\rightharpoonup u \quad \text{in} \quad W^{1,p}(B(0,\epsilon);\mathbf{R}^N), \\ w_k^{(m)}(x) &= v(\epsilon)\tfrac{x}{\epsilon} \text{ if } |x| = \epsilon, \\ w_k^{(m)}(x) &= U_{m,k}(x) \text{ if } |x| < \epsilon - \tfrac{1}{k} \end{aligned}$$

and

$$\liminf_{k\to+\infty} \int_{B(0,\epsilon)} g(\det \nabla w_k^{(m)}(x))dx \leq \liminf_{k\to+\infty} \int_{B(0,\epsilon)} g(\det U_{m,k}(x))dx + \epsilon. \tag{4.2}$$

Then

$$\begin{aligned} \liminf_{k\to+\infty} \int_B W(\nabla u_k(x))dx &\geq \liminf_{k\to+\infty} \int_B f(\nabla u_k(x))dx \\ &\quad + \liminf_{k\to+\infty} \int_B g(\det \nabla u_k(x))dx \\ &\geq \int_B f(\nabla u(x))dx \\ &\quad + \liminf_{k\to+\infty} \int_B g(\det \nabla u_k(x))dx, \end{aligned} \tag{4.3}$$

where we have used the fact that f is quasiconvex, it has p-growth and $u \in W^{1,p}(B;\mathbf{R}^N)$ since $p < N$ (see [AF], [Mar 1]). On the other hand, by (4.1), (4.2),

$$
\begin{aligned}
\liminf_{k\to+\infty}\int_B g(\det\nabla u_k(x))dx &= \liminf_{k\to+\infty}\int_{B_\epsilon} g(\det\nabla u_k(x))dx \\
&\quad + \liminf_{k\to+\infty}\int_{B\setminus B_\epsilon} g(\det\nabla u_k(x))dx \\
&\geq \liminf_{k\to+\infty}\int_{B_\epsilon} g(\det\nabla w_k^{(m)}(x))dx - \frac{1}{m} - \epsilon \\
&\quad + \int_{B\setminus B_\epsilon} g(\det\nabla u(x))dx,
\end{aligned}
$$

where we have used Theorem 3.2. Since $w_k^{(m)} \in W^{1,N}(\Omega;\mathbf{R}^N)$ and $w_k^{(m)}(x) = \frac{v(\epsilon)}{\epsilon}x$ on $\partial B(0,\epsilon)$, by the quasiconvexity of $\xi \mapsto g(\det\xi)$ we have

$$
\int_{B_\epsilon} g(\det\nabla w_k^{(m)}(x))dx \geq \text{meas}(B_\epsilon)g\left(\left(\frac{v(\epsilon)}{\epsilon}\right)^N\right)
$$

and by (4.3) we conclude that

$$
\begin{aligned}
\liminf_{k\to+\infty}\int_B W(\nabla u_k(x))dx &\geq \int_B f(\nabla u(x))dx + \int_{B\setminus B_\epsilon} g(\det\nabla u(x))dx \\
&\quad +\text{meas}(B_\epsilon)g\left(\left(\frac{v(\epsilon)}{\epsilon}\right)^N\right) - \frac{1}{m} - \epsilon
\end{aligned}
$$

and letting $\epsilon \to 0^+$

$$
\begin{aligned}
\mathcal{F}_p(u) &\geq \int_B f(\nabla u(x))dx + \int_B g(\det\nabla u(x))dx \\
&\quad + \lim_{\epsilon\to 0^+}\text{meas}(B)(v(\epsilon))^N\left(\tfrac{\epsilon}{v(\epsilon)}\right)^N g\left(\left(\tfrac{v(\epsilon)}{\epsilon}\right)^N\right) - \tfrac{1}{m} \\
&= \int_B W(\nabla u(x))dx + \text{meas}(B)v(0)^N g^\infty(1) - \frac{1}{m} \\
&= \int_B W(\nabla u(x))dx + \text{meas}(B)g^\infty((v(0)^N) - \frac{1}{m}
\end{aligned}
$$

It suffices to let $m \to +\infty$.
Step 2. Consider

$$
v_\epsilon(t) := \begin{cases} v(t)\,, & t > \epsilon \\ \frac{t}{\epsilon}v(\epsilon)\,, & t < \epsilon. \end{cases}
$$

Then $u_\epsilon(x) := v_\epsilon(|x|)\frac{x}{|x|} \in W^{1,N}(B;\mathbf{R}^N)$ and $u_\epsilon \to u$ in $W^{1,p}(B;\mathbf{R}^N)$ strongly. Indeed,

$$\begin{aligned}
\|v_\epsilon - u\|^p_{L^p(B)} &= \int_{B_\epsilon} \left|\frac{|x|}{\epsilon}v(\epsilon)\frac{x}{|x|} - v(|x|)\frac{x}{|x|}\right|^p dx \\
&= C\int_0^\epsilon \frac{r^{N-1}}{\epsilon^p}\,|rv(\epsilon) - \epsilon v(r)|^p\,dr \\
&\leq C'\frac{1}{N}\frac{\epsilon^N}{\epsilon^p} \underset{\epsilon\to 0+}{\to} 0\,,
\end{aligned}$$

$$\begin{aligned}
\|\nabla u_\epsilon - \nabla u\|^p_{L^p(B)} &= \int_{B_\epsilon}\left|\frac{v_\epsilon(r)}{r}\mathbf{1} + \left(v'_\epsilon(r) - \frac{v_\epsilon(r)}{r}\right)\frac{x}{r}\otimes\frac{x}{r}\right. \\
&\quad \left. -\frac{v(r)}{r}\mathbf{1} - \left(v'(r) - \frac{v(r)}{r}\right)\frac{x}{r}\otimes\frac{x}{r}\right|^p dx \\
&\leq C\int_0^\epsilon r^{N-1-p}\left(|v_\epsilon(r) - v(r)|^p + |v'_\epsilon(r) - v'(r)^p|\right)dx \\
&\leq C'\frac{1}{N-p}\epsilon^{N-p} \underset{\epsilon\to 0+}{\to} 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{F}_p(u) &\leq \liminf_{\epsilon\to 0}\int_B f(\nabla u_\epsilon(x))dx + \int_B g(\det\nabla u_\epsilon(x))dx \\
&= \int_B f(\nabla u(x))dx + \liminf_{\epsilon\to 0}\int_B g(\det\nabla u_\epsilon(x))dx \\
&\leq \int_B f(\nabla u(x))dx + \limsup_{\epsilon\to 0+}\int_{B_\epsilon} g\left(v'_\epsilon(r)\left(\frac{v_\epsilon(r)}{r}\right)^{N-1}\right)dx \\
&\quad + \limsup_{\epsilon\to 0+}\int_{B\setminus B_\epsilon} g(\det\nabla u(x))dx \\
&= \int_B W(\nabla u(x))dx + \limsup_{\epsilon\to 0+} w_N\int_0^\epsilon r^{N-1}g\left(\left(\frac{v(\epsilon)}{\epsilon}\right)^N\right)dx \\
&= \int_B W(\nabla u(x))dx + \lim_{\epsilon\to 0+}\frac{w_N}{N}\epsilon^N g\left(\left(\frac{v(\epsilon)}{\epsilon}\right)^N\right) \\
&= \int_B W(\nabla u(x))dx + \lim_{\epsilon\to 0+}\mathrm{meas}(B)(v(\epsilon))^N\left(\frac{\epsilon}{v(\epsilon)}\right)^N g\left(\left(\frac{v(\epsilon)}{\epsilon}\right)^N\right) \\
&= \int_B W(\nabla u(x))dx + \mathrm{meas}(B)(v(0))^N g^\infty(1) \\
&= \int_B W(\nabla u(x))dx + \mathrm{meas}(B)g^\infty((v(0))^N).
\end{aligned}$$

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