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A NECESSARY AND SUFFICIENT CONDITION THAT A NON-DEGENERATE LINEAR CONSTRAINT SET BE EMPTY OR CONTAIN A REDUNDANT CONSTRAINT

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A Necessary and Sufficient Condition that a Non-Degenerate Linear Constraint Set be Empty or Contain a Redundant Constraint

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1. The Statement of the Condition.

Suppose that the linear constraint set

$$\sum_{j=1}^{n} \mathcal{A}_{ij} x_j + \mathcal{A}_{i,n+1} \geq 0, \; i = 1, ..., m \tag{1.1}$$

is in canonical form. That is,

$$\mathcal{A}_{ij} = \delta_{ij}, \; i = 1, ..., n+1, \;\; j = 1, ..., n. \tag{1.2}$$

The author proved in [1] that the polyhedron formed by the coordinate constraints $x_j \geq 0$, $j = 1, ..., n$ and the k^{th} , $k > n$ is empty when $\mathcal{A}_{kj} < 0$ for all $j = 1, ..., n+1$ and contains a redundant constraint when the set $\{\mathcal{A}_{k1}, ..., \mathcal{A}_{kn}, \mathcal{A}_{k,n+1}\}$ consists only or non-negative elements or $\mathcal{A}_{k,n+1} < 0$ and $\mathcal{A}_{kj} > 0$ for exactly one $j \leq n$.

In this note we shall use the above stated condition to obtain the result indicated by the title by obtaining explicit formulas for the coefficients in the constraints when the constraints

$$k_1, ..., k_r, \;\; k_i \geq r+1, \; r \leq n \tag{1.3}$$

have been interchanged with the constraints

$$\ell_1, ..., \ell_r, \; \ell_i \leq n, \tag{1.4}$$

in the order k_i, ℓ_i, $i = 1, ..., r$ and the constraint set is returned to canonical form at each step. In order to state the formulas, we denote by

$$f_\sigma\,(i_1, ..., i_\sigma : j_1, ..., j_\sigma) \tag{1.5}$$

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the minor determinant of $\mathcal{A}_{ij}$, $i = m+1, ..., m$, $j = 1, ..., n$ indexed by the rows $i_1, ..., i_n$ and the columns $j_1, ..., j_r$. Then, with $\mathcal{A}^0_{ij}$ representing the original matrix and $\mathcal{A}^r_{ij}$ the matrix after the constraints indexed by $k_1, ..., k_r$ have replaced those indexed by $\ell, ..., \ell_r$,

$$\begin{aligned} \mathcal{K}_r &= \{k_1, ..., k_r\}, \mathcal{K}'_r = [1, m] - \mathcal{K}_r, \ \mathcal{L}_r = \{\ell_1, ..., \ell_r\}, \ \mathcal{L}'_r = [1, n+1] - \mathcal{L}_r, \\ D_r &= f_r(k_1, ..., k_i : \ell_1, ..., \ell_r) \end{aligned} \tag{1.6}$$

we have the formulas for $i \in \mathcal{K}'_r$,

$$\mathcal{A}^r_{ij} = f_{r+1}(k_1, ..., k_r, \ i : \ell_1, ..., \ell_r, j) / D_r, \ \ j \in \mathcal{L}'_r, \tag{1.7}$$

$$\mathcal{A}^r_{i,\ell_j} = (-1)^{r-j} f_r(k_1, ..., k_{j-1}, k_{j+1}, ..., k_r, i : \ell_1, ..., \ell_r) / D_r, \ \ \ell_j \in \mathcal{L}_r, \tag{1.8}$$

and for $k_i \in \mathcal{K}_r$,

$$\mathcal{A}^r_{k,j} = (-1)^{r+1-i} f_r(k_1, ..., k_r : \ell_1, ..., \ell_{i-1}, \ell_{i+1}, ..., \ell_r, j) / D_r, \ \ j \in \mathcal{L}'_r, \tag{1.9}$$

$$\mathcal{A}^r_{k_i,\ell_j} = (-1)^{i+j} f_{r-1}(k_1, ..., k_{j-1}, k_{j+1}, ..., k_r : \ell_1, ...\ell_{i-1}, \ell_{i+1}, ..., \ell_r) / D_r, \ \ \ell_j \in \mathcal{L}_r. \tag{1.10}$$

Before stating the condition for redundant constraints or empty sets, we shall prove the following theorem.

Theorem 1.1. *The formulas (1.7) - (1.10) are invariant under permutation of $k_1, ..., k_r$ or $\ell_1, ..., \ell_r$ in the sense the sign of either (1.7) (1.8) or (1.9), (1.10) for fixed i and $j = 1, ..., n+1$ are invariant. This makes it possible to state the condition for empty sets or redundant constraints using only the pair (1.7), (1.8) in the order $r = 1, 2, ..., n$.*

Proof. First let us note that we may assume that the k's and l's are in increasing order. This follows from the fact that when $k_1, ..., k_n$ are permutations of the same set, then $k_1, ..., k_{j-1}, k_{j+1}, ...k_r, j = 1, ..., n$ are merely written down in a different order.

To prove this by induction, let $\sigma == (k_1, ..., k_r)$ and $\sigma_j = (k_1, ..., k_{j-1}, k_{j+1}, ..., k_r)$ and suppose that the largest element y of σ is indexed by ℓ. Then after interchanging the y with the last elements of σ and $\sigma_j, j \neq \ell$, the sign of the ratio σ_j/σ is retained when $j < \ell$, changes when $j > \ell$ and is multiplied by $(-1)^{r-\ell}$ when $j = \ell$. Hence by moving the ℓ^{th} ratio to the end of the list and decreasing the order of those. indexed by $k, \ell+1 \leq k \leq r$, we obtain a valid induction proof. Similarly for the l's. □

Theorem 1.2. *In applying the empty set or redundant constraint test, it is sufficient to scan (1.7), (1.8) for all permutation $(k_1, ..., k_r)$ and $(\ell_1, ..., \ell_r)$ in increasing order of r.*

Proof. In proceeding from r to $r+1$ we interchange the constraints indexed by k_{r+1} and ℓ_{r+1}. A simple computation shows that in an $(n+1)$ constraint set in canonical form, an interchange of the $(n+1)^{\text{st}}$ constraint with a basic constraint can't change the sign test indicating an empty set or redundant constraint. But, by Theorem 1.1, we may assume that any h_i and ℓ_i were interchanged.

2. The Recursion Formula.

Assuming that we have computed the matrix $\mathcal{A}^r_{ij}$, the matrix $\mathcal{A}^{r+1}_{ij}$ is obtained by interchanging the constraints indexed by k_{r+1}, ℓ_{r+1} and updating the matrix as in [1]. The result is

$$\mathcal{A}^{r+1}_{k_{r+1},\ell_{r+1}} = 1/\mathcal{A}^r_{k_{r+1},\ell_{r+1}} \tag{2.1}$$

$$\mathcal{A}^{r+1}_{k_{r+1},j} = -\mathcal{A}^r_{k_{r+1},j}/\mathcal{A}^r_{k_{r+1},\ell_{r+1}}, \quad j \neq \ell_{r+1} \tag{2.2}$$

and for $i \neq k_{r+1}$,

$$\mathcal{A}^{r+1}_{i,\ell_{r+1}} = \mathcal{A}^r_{i,\ell_{r+1}} / \mathcal{A}^r_{k_{r+1},\ell_{r+1}}, \tag{2.3}$$

$$\mathcal{A}^{r+1}_{ij} = \mathcal{A}^r_{ij} - \frac{\mathcal{A}^r_{i,\ell_{r+1}} \mathcal{A}^r_{k_{r+1},j}}{\mathcal{A}^r_{k_{r+1},\ell_{r+1}}}, \quad j \neq \ell_{r+1}. \tag{2.4}$$

Note, in particular, that (2.4) is the ratio of a 2×2 minor and a 1×1 minor and when $r = 0$, it agrees with (1.10). Also, when $r = 0$, (2.2), (2.3) agree with (1.8) (1.9). In order to make (2.1) agree with (1.6) we make the convention $f_0 = 1$. Before proving the general result, we shall develop some lemmas on determinants.

3. Some Lemmas on Determinants.

Let us use the usual convention that $\mathcal{B}_{ij}$ is the co-factor of b_{ij}. Then our first and main lemma is:

Lemma 3.1. *Let $\mathcal{B} = (b_{ij})$ be a $k \times k$ matrix and let $\mathcal{C}$ be the $(k-1) \times (k-1)$ matrix*

$$\mathcal{C} = \left(b_{ij} - \frac{b_{i,k} b_{kj}}{b_{kk}}\right), \quad 1 \leq i, \ j \leq k-1. \tag{3.1}$$

Then

$$\det \mathcal{C} = \det \mathcal{B}/b_{kk}. \tag{3.2}$$

Proof. Define:

$$\varphi(\epsilon) = \det\left(b_{ij} - \epsilon\, b_{i,k} b_{kj}\right), \quad 1 \le i, j \le k-1. \tag{3.3}$$

Now we use the fact that the derivative of a determinant is the sum of the determinants obtained by differentiating one row of the matrix. When we differentiate the i^{th} row of $\mathcal{C}$, the new i^{th} row is

$$-\, b_{ik}\left(b_{k1}, b_{k2}, \ldots, b_{k,k-1}\right). \tag{3.4}$$

□

If we interchange this row with each of those indexed by $i+1, \ldots, k-1$, we have the matrix obtained by deleting the i^{th} row from the first k columns of $\mathcal{B}$. Hence, when we take the determinant, we obtain

$$b_{ik}\mathcal{B}_{ik}. \tag{3.5}$$

It follows that

$$\varphi'(0) = \sum_{i=1}^{k-1} b_{ik}\mathcal{B}_{ik}. \tag{3.6}$$

When we differentiate twice we obtain a sum of determinants of matrices having two rows equal. Hence $\varphi''(\epsilon) \equiv 0$ so $\varphi^{(j)}(\epsilon) = 0$ for $j \ge 2$.

Since $\varphi(0) = \mathcal{B}_{kk}$ we then have

$$\varphi(\epsilon) = \mathcal{B}_{kk} + \epsilon \sum_{i=1}^{k-1} b_{ik}\mathcal{B}_{ik}. \tag{3.7}$$

Putting $\epsilon = 1/\mathcal{B}_{kk}$ gives (3.2).

The recursion formula (2.4) with $i > k_{r+1}$, $j > \ell_{r+1}$ can be rewritten

$$\mathcal{A}_{ij}^{r+1} = \frac{\det\left(\mathcal{A}_{\mu\nu}^{r}\right)_2}{\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r}} \tag{3.8}$$

where the numerator is the determinant of the 2×2 matrix indexed by $\mu = k_{r+1}, i$ and $\nu = \ell_{r+1,j}$ and is, in fact, just the Lemma 3.1 with $k = 2$ after a change of indices. More generally, we can use Lemma 3.1 to prove inductively that, for $1 \le p \le r+1$,

$$\mathcal{A}_{ij}^{r+1} = \frac{\det\left(\mathcal{A}_{\mu\nu}^{r+1-\rho}\right)_{\rho+1}}{\mathcal{A}_{k_{r+2-\rho},\ell_{r+2-\rho}}^{r+1-\rho} \cdots \mathcal{A}_{k_{r+1},\ell_{r+1}}^{r}} \tag{3.9}$$

where the numerator is the determinant of the $(\rho+1)\times(\rho+1)$matrix indexed by $\mu = k_{r+2-\rho}, \ldots, k_{r+1}, i$ and $\nu = \ell_{r+1-\rho}, \ldots, \ell_{r+1}, j$. In particular, when $\rho = r+1$, (3.9) reduces, in view of (1.5), to

$$\mathcal{A}_{ij}^{r+1} = \frac{f_{r+2}\left(k_1, \ldots, k_{r+1},\ i : \ell_1, \ldots, \ell_{r+1}, j\right)}{\mathcal{A}_{k_{r+1},\ell_{r+1}}^{r}\mathcal{A}_{k_r,\ell_r}^{r-1} \cdots \mathcal{A}_{k_1,\ell_1}^{0}} \tag{3.10}$$

for any $i \in \mathcal{K}_{r+1}$, $j \in \mathcal{L}'_{r+1}$. In particular,

$$\mathcal{A}^{r+1}_{k_{r+2},\ell_{r+2}} = \frac{D_{r+2}}{\mathcal{A}^{r}_{k_{r+1},\ell_{r+1}} \dots \mathcal{A}^{0}_{k_1,\ell_1}} \tag{3.11}$$

or

$$D_{r+2} = \mathcal{A}^{0}_{k_1,\ell_1} \mathcal{A}^{1}_{k_2,\ell_2} \dots \mathcal{A}^{r+1}_{k_{r+2},\ell_{r+2}}. \tag{3.12}$$

Since this is true for each r, we have proved (1.7) with $i\, \mathcal{K}'_{r+1}$, $j \in \mathcal{L}'_{r+1}$ as a consequence of (3.10) and (3.11) with r replaced by $r-1$.

By eliminating $\mathcal{A}^{r+1}_{ij}$ between (3.9), (3.10), setting $i = k_{r+2,j} = \ell_{r+2}$, and using 1.6 for $\mathcal{A}^{r+1-\rho}_{\mu\nu}$, we obtain the interesting identity

$$\begin{aligned} & f_{r+2}\left(k_1, \dots, k_{r+2} : \ell_1, \dots, \ell_{r+2}\right)\left(D_{r+1-\rho}\right)^{\rho} \\ = \quad & \det\left(f_{r+2-\rho}\left(k_1, \dots, k_{r+1-\rho}, \mu : \ell_1, \dots, \ell_{r+1-\rho}, \nu\right)\right)_{\rho+1} \end{aligned} \tag{3.13}$$

with μ and ν ranging over the indices $k_{r+2-\rho}, \dots, k_{r+2}$ and $\ell_{r+2-\rho}, \dots, \ell_{r+2}$. The main use we make of this identity is:

Theorem 3.2. *Consider the identity (3.13) with $\rho = 1$. If three of the four minors comprising the determinant on the right have sign opposite the fourth then*
$D_r \neq 0$ and the sign of f_{r+2} is determined by the identity.

We shall also need the following identity

$$\begin{aligned} & f_{r+1}\left(k_1, \dots, k_{r+1} : \ell_1, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_r \ \right) f_r\left(k_1, \dots, k_r : \ell_i, \dots \ell_r\right) \\ - \quad & f_{r+1}\left(k_1, \dots, k_{r+1} : \ell_1, \dots, \ell_r, j\right) f_r\left(k_1, \dots, k_r : \ell_1, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_{r+1}\right) \\ + \quad & f_{r+1}\left(k_1, \dots, k_{r+1} : \ell_1, \dots, \dot{\ell}_{r+1}\right) f_r\left(k_1, \dots, k_r : \ell_1, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_r, j\right) = 0. \end{aligned} \tag{3.14}$$

If we suppress the dependence on $k_1, \dots, k_{r-1}$ and $\ell_1, \dots, \ell_{r-1}$, the left side of (3.14) is the 3×3 determinant of the matrix with rows indexed by (k_1, k_1, k_2) and column indexed by ℓ_r, ℓ_{r+1}, j. Since the first two rows are equal the determinant is zero.

By eliminating $\mathcal{A}^{1+i}_{ij}$ between (3.9), (3.10), setting $i = k_{r+2}$, $j = \ell_{r+2}$, and using 1.6 for $\mathcal{A}^{r+1-\rho}_{\mu\nu}$, we obtain the interesting identity

$$\begin{aligned} & f_{r+2}\left(k_1, \dots, k_{r+2} : \ell_1, \dots, \ell_{r+2}\right)\left(D_{r+1-\rho}\right)^{\rho} \\ = \quad & \det\left(f_{r+2-\rho}\left(k_1, \dots, k_{r+1-\rho}, \mu : \ell_1, \dots, \ell_{r+1-\rho}, \mu\right)\right)_{\rho+1} \end{aligned} \tag{3.15}$$

with μ and ν ranging over the indices $k_{r+2-\rho}, \dots, k_{r+2}$ and $\ell_{r+2-\rho}, \dots, \ell_{r+2}$. The main use we make of this identity is:

Theorem 3.3. *Consider the identity (3.13) with $\rho = 1$. If three of the four minors comprising the determinant on the right hand sign opposite the fourth then $D_r \neq 0$ and the sign of f_{r+2} is determined by the identity.*

We shall also need the following identity

$$\begin{aligned} & f_{r+1}(k_1, ..., k_{r+1} : \ell_1, ..., \ell_{i-1}, \ell_{i+1}, ..., \ell_r, j) f_r\left(k_1, ..., \dot{k}_r : \ell_i, ..., \ell_r\right) \\ - & f_{r+1}(k_1, ..., k_{r+1} : \ell_1, ..., \ell_r, j) f_r(k_1, ...k_r : \ell_1, \ell_r) \\ + & f_{r+1}(k_1, ..., k_{r+1} : \ell_1, ..., \ell_{r+1}) f_r(k_1, ..., k_r : \ell_1, ..., \ell_{i-1}, \ell_{i+1}, ..., \ell_r, j) \quad = 0. \end{aligned}$$

If we suppress the dependence on $k_1, ..., k_{r-1}$ and $\ell_1, ..., \ell_{r-1}$, the left side of (3.14) is the 3×3 determinant of the matrix with rows indexed by (k_i, k_1, k_2) and columns indexed by ℓ_r, ℓ_{r+1}, j. Since the first two rows are equal the determinant is zero.

4. Completion of the Proofs of the Identities.

We now have the main tools sufficient for the proofs of (1.7),...,(1.10) by induction. Note that we have proved (1.7) for all r and $i \notin \mathcal{K}_r$, $j \notin \mathcal{L}_r$, the proofs of the cases (1.8), (1.9), (1.10). Hence, we may use $i = k_{r+1}$, $j = \ell_{r+1}$ in (1.7) to express (2.1) as

$$\mathcal{A}^{r+1}_{k_{r+1},\ell_{r+1}} = \frac{f_r(k_1, ..., k_r : \ell_1, ..., \ell_r)}{D_{r+1}}. \tag{4.1}$$

This is the promoted version of (1.10) with $i = r+1$, $j = r+1$. By putting $i = k_{r+1}$, $j \notin \mathcal{L}_{r+1}$ into (1.7) and substituting (4.1) into (1.10), we obtain

$$\mathcal{A}^{r+1}_{k_{r+1},j} = -\frac{f_{r+1}(k_1, ..., k_{r+1} : \ell_1, ..., \ell_r, j)}{D_r} \tag{4.2}$$

which is the formula (1.9) corresponding to the pair $k_{r+1,j}$ with $j \notin \mathcal{L}_r$. It follows from (2.4) that for $k_i \in \mathcal{K}_r$, $j \notin \mathcal{L}_{r+1}$.

$$\mathcal{A}^{r+1}_{k_i,j} = \mathcal{A}^r_{k_i,j} - \frac{\mathcal{A}^r_{k_i,\ell_{r+1}} \mathcal{A}^r_{k_{r+1},j}}{\mathcal{A}^r_{k_{r+1},\ell_{r+1}}}. \tag{4.3}$$

After substituting (1.7) and (1.9), and setting the result equal to (1.9) with r replaced by $r+1$, we obtain the identity (3.14). This completes the proof of the remaining cases in (1.9). The proof of the promoted version of (1.8) is isomorphic.

There remains the case indexed by $k_i \in \mathcal{K}_r$ and $\ell_j \in \mathcal{L}_r$. We obtain from (2.4), (1.8), (1.9), (1.10) and (3.11) with r replaced by $r-1$ into (4.4), we obtain

$$\begin{aligned} \mathcal{A}^{r+1}_{k_i,k_j} &= \tfrac{(-1)^{i+j}}{D_r} \{f_{r-1}(k_1,...,k_{j-1},k_{i+1},...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...,\ell_r) \\ &+ \frac{f_r(k_1,...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...,\ell_{r-1})\, f_r(k_1,k_{i-1},k_{i+1},...,k_{i-1} : \ell_1,...,\ell_{r+1})}{f_{r+1}(k_1,...,k_{r+1} : \ell_1,...,\ell_{r+1})}\}. \end{aligned} \tag{4.4}$$

We now apply (3.13) in the form

$$\begin{aligned} & f_{r+1}(k_1,...,k_{r+1} : \ell_1,...,\ell_{r+1})\, f_{r-1}(k_1,...,k_{j-1},k_{j+1},...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...\ell_r) \\ = \; & f_r(k_1,...,k_{j-1},k_{j+1},...,k_{r+1} : \ell_1,...,\ell_{i-1},\ell_{i+1},\ell_{r+1})\, f_r(k_1,...,k_r : \ell_1,...,\ell_r) \\ & - f_r(k_1,...,k_{j-1},k_{j+1},...,k_{r+1} : \ell_1,...,\ell_r)\, f_r(k_1,...,k_r : \ell_1,...,\ell_{i-1},\ell_{i+1},...,\ell_r). \end{aligned} \tag{4.5}$$

After substituting (4.5) into (4.4), we have the promoted version of (1.10) for $k_i \in \mathcal{K}_i$, $\ell_j \in \mathcal{L}_j$. Since the case of $k_{r+1} \subset \mathcal{K}_{r+1}, \ell_{r+1} \in \mathcal{L}_{r+1}$ has already been disposed of, the proof is complete.

5. Duplications of Constraints.

The formulas (1.7) - (1.10) are derived under the assumption that the sets $(k_1,...,k_r)$ and $(\ell_1,...,\ell_r)$ are distinct. In particular the ℓ's are a subset of $(1,...,n)$ so we must have $r \leq n$. On the other hand, it follows from the recursion formulas (2.1) - (2.4) that we may, at any time, start over with a new matrix and continue until there is a duplication in either the k's or the ℓ's. In this section we resolve the question of such a duplication in the second step.

The new matrix coefficients, after interchanging the i^{th} nonbasic constraint with the ℓ^{th} basic constraint and then returning to reduced echelon form by the use of elementary column operations, are

$$\mathcal{A}'_{i\ell} = 1/\mathcal{A}_{i\ell} \tag{5.1}$$

$$\mathcal{A}'_{ij} = -\mathcal{A}_{ij}/\mathcal{A}_{i\ell}, \;\; j \neq \ell \tag{5.2}$$

and for $k \neq i$,

$$\mathcal{A}'_{k\ell} = \mathcal{A}_{k\ell}/\mathcal{A}_{i\ell}, \tag{5.3}$$

$$\mathcal{A}'_{kj} = \mathcal{A}_{kj} - \mathcal{A}_{k\ell}\, \mathcal{A}_{ij}/\mathcal{A}_{i\ell}, \;\; j \neq \ell. \tag{5.4}$$

Now let us interchange the new k^{th} constraint with the ℓ^{th} basic constraint. By analogy with (5.1), (5.2) the coefficients for the new k^{th} constraint are

$$\mathcal{A}''_{k\ell} = 1/\mathcal{A}'_{k\ell} \tag{5.5}$$

and

$$\mathcal{A}''_{kj} = \mathcal{A}'_{kj}/\mathcal{A}'_{k\ell}, \;\; j \neq \ell. \tag{5.6}$$

After substituting from (5.1) - (5.4) there becomes

$$\mathcal{A}''_{k\ell} = \mathcal{A}_{i\ell}/\mathcal{A}_{k\ell} \tag{5.7}$$

and

$$\mathcal{A}''_{kj} = \mathcal{A}_{ij} - \mathcal{A}_{i\ell}\;\mathcal{A}_{kj}/\mathcal{A}_{k\ell}, \;\; j \neq \ell. \tag{5.8}$$

These are just the parameters obtained after interchanging the i^{th} constraint with the ℓ^{th} and returning to reduced echelon form. But they are in the position of the k^{th}. The new coefficients for the i^{th} constraint are

$$\mathcal{A}''_{i\ell} = \mathcal{A}'_{i\ell}/\mathcal{A}'_{k\ell}, \tag{5.9}$$

$$\mathcal{A}''_{ij} = \mathcal{A}'_{ij} - \mathcal{A}'_{i\ell}\,\mathcal{A}'_{kj}/\mathcal{A}'_{k\ell}, \;\; j \neq \ell. \tag{5.10}$$

Again, after substituting from (5.1) - (5.4) and taking into account cancellations, there become

$$\mathcal{A}''_{i\ell} = 1/\mathcal{A}_{k\ell}, \tag{5.11}$$

$$\mathcal{A}''_{ij} = -\mathcal{A}_{kj}\,/\,\mathcal{A}'_{k\ell}, \;\; j \neq \ell. \tag{5.12}$$

They are the coefficients for the k^{th} constraint after interchanging the k^{th} constraint with the ℓ^{th}, and they are in the position of the i^{th}.

For $r \neq k$ or i, $r > n$, the new coefficients for the r^{th} constraint are

$$\mathcal{A}''_{r\ell} = \mathcal{A}'_{k\ell}\,/\,\mathcal{A}'_{k\ell}, \tag{5.13}$$

$$\mathcal{A}''_{rj} = \mathcal{A}'_{rj} - \mathcal{A}'_{r\ell}\,\mathcal{A}'_{kj}\,/\,\mathcal{A}'_{k\ell}, \;\; j \neq \ell \tag{5.14}$$

After substituting from (5.1) - (5.4), these become

$$\mathcal{A}''_{r\ell} = \mathcal{A}_{r\ell}\,/\,\mathcal{A}_{k\ell}, \tag{5.15}$$

$$\mathcal{A}''_{rj} = \mathcal{A}_{rj} - \mathcal{A}_{r\ell}\,\mathcal{A}_{rj}\,/\,\mathcal{A}_{k\ell}, \;\; j \neq \ell \tag{5.16}$$

which are just the coefficient obtained after interchanging the r^{th} constraint with the ℓ^{th} in the original matrix. This together with the remarks following (5.8) and (5.12) yields a proof of the following theorem.

Theorem 5.1. *Interchanging the i^{th} non-basic constraint with the ℓ^{th}, updating and then interchanging the k^{th} and updating is equivalent to merely interchanging the k^{th} with the ℓ^{th} in the original matrix, updating and then interchanging the i^{th} and k^{th}.*

Now let us determine the effect of interchanging one non-basic constraint with two different basic constraints. If after obtaining the formulas (5.1) - (5.4), we interchange the i^{th} constraint with the q^{th} basic constraint, $q \neq i$, the new parameter for the i^{th} constraint are

$$\mathcal{A}''_{iq} = 1/\mathcal{A}'_{iq}, \tag{5.17}$$

$$\mathcal{A}''_{i\ell} = -\mathcal{A}'_{i\ell} \,/\, \mathcal{A}'_{iq}, \tag{5.18}$$

and

$$\mathcal{A}'_{qj} = -\mathcal{A}'_{ij} \,/\, \mathcal{A}'_{iq}, \; j \neq q, \ell. \tag{5.19}$$

The formulas (5.17) - (5.19), after substituting from (5.1) - (5.4) are just the formulas obtained after interchanging the i^{th} with the q^{th} in the original matrix. For $k \neq q$,

$$\mathcal{A}''_{kq} = \mathcal{A}'_{kq} \,/\, \mathcal{A}'_{iq} \tag{5.20}$$

$$\mathcal{A}'_{k\ell} = \mathcal{A}'_{k\ell} - \mathcal{A}'_{kq}\, \mathcal{A}'_{i\ell} \,/\, \mathcal{A}'_{iq} \tag{5.21}$$

and

$$\mathcal{A}'_{kj} = \mathcal{A}'_{kj} - \mathcal{A}'_{kq}\, \mathcal{A}'_{ij} \,/\, \mathcal{A}'_{iq}, \;\; j \neq q, \ell. \tag{5.22}$$

Again, after substituting from (5.1) - (5.4), these are just the formula for the k^{th} constraint after interchanging the k^{th} with the q^{th} in the original matrix.

Theorem 5.2. *If we interchange the i^{th} non-basic constraint with the ℓ^{th} basic constraint, update and then interchange the new i^{th} constraint with the q^{th}, $q \neq i$, and update, this is equivalent to merely interchanging the i^{th} with the q^{th} and updating.*

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