ON THE EXISTENCE OF n BUT NOT n+1 EASY COMBINATORS

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Research Report No. 94–165 2
March, 1994

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NSF CCR 920 1893
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(I) m-Easy Combinators

Given two combinators M and N we define a graph G(M,N) as follows. The points of G(M,N) are the combinators modulo beta conversion, and we make P adjacent to Q if there exists an R such that P=RM and Q=RN,or, P=RN and Q=RM. Now the proof theoretic properties of the equation M=N are reflected by the properties of G(M,N). For example, M=N is inconsistent <=> G(M,N) is connected <=> K and K* lie in the same G(M,N) component. In particular if we wish to count steps in proofs it is convenient to count edges in G(M,N).

Recall that M is easy if it is consistent with every combinator. We say that M is m-easy if there is no proof with < m+1 steps that M is inconsistent with any combinator i.e. if for each N the diameter of G(M,N) is at least m. Obviously if M is easy then it is m-easy for each m. Here we shall show that for infinitely many m there are m but not m+1 easy terms.

Define terms E(n),F(n),G(n) as follows:

E(0) := \x. K
E(n+1) := \x. xIE(n)x
F(n) := \x. xxIE(n)(xx) \x. xxIE(n)(xx)
G(n) := \x. F(n)(xx) \x. F(n)(xx).

We shall show that G(n) is n-easy but not 2n+5 easy.

(II) Lower Bounds on the Failure of Church-Rosser

Let M be given. A term X is said to be an F-term of type n,M,k if it has the form

F(n)IE(k1)X1..........IE(kt)Xt

where $-1<t$, $k-1<ki$, and each Xi is an F-term of type n,M,k. A term Y is said to be a G-term of type n,M,k if it has the form

(a) Y1(...(Yt(\x. Yt+1(xx) \x. Yt+1(xx)))...)

where $-1<t$, and each Yi is an F-term of type n,M,k or

(b) Y1(...(YtN)...)

where $0<t$, each Yi is an F-term of type n,M,k, and

BT(M) [BT(N)

We define relations |->(k), ||->(k), and >->(k) as follows:

X |->(k) Y <=> X is a G-term of type n,M,k and Y := M.
X ||->(k) Y <=> X := X[Z1,...,Zr], each Zi is a G-term of type n,M,k, and Y := X[M,...,M].
X >->(k) Y <=> there exists a Z such that X ->> Z ||->(k) Y.

For $0<t$ we define the relation >->>(k.t) by

X >->>(k,t) Y <=> there exists Z1,...,Zt-1 X >->(k) Z1 >->(k).........>->(k)Zt-1 >->(k) Y.

We shall prove the following
PROPOSITION: The diagrams

Q <<- P ||->(k) R

and

Q (r)<-|| P ||->(k) R

can be completed to

Q ||->(k) T <<- R

resp.

Q ->> Q* ||->(k-1) T (r-1)<-|| R* <<- R.

Therefore the diagram

T (r)<-|| Q <<- P ||->(k)

can be completed to

R ->> U
|||
Q ||->(k) U ->> X ||->(r-1) Y (k-1)<<-|| Z <<- T.

when 0<k and 0<r.
From this follows the diamond property
COROLLARY: Q (r)<-< P >->(k) R
can be completed to

Q >->(k-1) T (r-1)<-< R

when 0<r and 0<k.

(II) If X is an F-term of type n,M,k and X := F(n)IE(k1)X1.........IE(kt)Xt then X has for its head the head positions of the Xi together with the subterm occurrence F(n)I above.

MATCHING LEMMA: Suppose that Y is a G-term of type n,M,k. If Y is of the form (a) then all its G-subterms of type n,M,r have the form (a) and are among the Yi(...(Yt(\x. Yt+1(xx) \x. Yt+1(xx)))...) except in case M=I when they can have the form (b),the shape F(n)I, and occur at the head positions of the Yi.

PROOF: First let X be an F-term of type n,M,k. We will show that X has no G-subterms of type n,M,r except when M=I and these are at the head positions of X. This is proved by induction and toward this end we let Y be a G-term of type n,M,r of the form (a) or (b) above. Then

(i) The last component of Y1 is either \x. xxIE(n)(xx) or has no normal form; therefore it =/= I or E(r) for any r.

(ii) If Y := Y1L then L has no narmal form so it =/= I or E(r) for any r.

(iii) If Y is of the form (a) then \x. Yt+j(xx) has no normal form so it =/= I or E(r) for any r when j=1,2

(iv) If Y is of the form (a) then \x. Yt+j has order 1 so it is not an F-term or a G-term of type n,M,s when j=1,2.

Now suppose that Y is a subterm of X. First suppsoe that Y has the form (a). If t=0 then by (iii) and (iv) Y is a subterm of Xi for some i. If 0<t then by (i) and (ii) Y is a subterm of some Xi. Next suppose that Y has the form (b). If t=1 then by (i) Y is a subterm of

Xi for some i unless Y := F(n)I and Y occupies the leftmost head position of X. But in this case we have N := I = M . If 1 < t then by (i) and (ii) Y is a subterm of some Xi. In conclusion our claim follows by induction.

To prove the lemma simply apply the above claim to the Yi after using (iv), and the unsolvability of G-terms of type n,M,s. This completes the proof of the lemma.

REPLACEMENT LEMMA: Let X be a G-term of type n,M,k. Then the replacement of any proper G-subterm of type n,M,r by M results in a G-term of type n,M,k except in case M=I when if 0<k it results in a term which ->> to G-term of type n,M,k-1.

PROOF: Suppose first that M=/=I. Let Y be a G-term of type n,M,k with a proper G-subterm Z of type n,M,r. By the Matching Lemma if Y has the form (a) then the replacement of Z by M is a G-term of type n,M,k and of the form (b). If Y is of the form (b) then the result of replacing Z by M remains of the form (b) since Z is unsolvable and its replacement in N yields a term whose Bohm tree still] the Bohm tree of M. Now if M = I then Z can occur at the head positions of the Yi if Z := F(n)I. These Yi are F-terms of type n,M,k and of the form

F(n)IE(k1)U1.........IE(kt)Ut

and the replacement of Z yields

IE(k1)U1.........IE(kt)Ut ->
E(k1)U1.........IE(kt)Ut ->
U1IE(k1-1)U1.........IE(kt)Ut

since 0< k and k < k1+1, which is an F-term of type n,M,k-1. This proves the lemma.

We can now proceed with the proof of the proposition. We remark here now that in case M=I ,k-1 and r-1 can be replaced in the corollary by k and r. In this case >->>(k) is Church-Rosser. However this already follows from the Replacemant Lemma by the theorem of Mitchke.

PROOF OF PROPOSITION: First suppose that X := X[Z1,...,Zr] where the Zi are G-term occurrence of type n,M,k which are pairwise disjoint. We can follow each Zi is a reduction X ->>Y. It can be copied,projected (deleted), and beta reduced internally. Thus we can write Y := Y[Z11,...,Z1s(1),...,Zr1,...Zrs(r)] where Zi ->>Zij for -1<j<s(i)+1 so that

X[x1,...,xr] ->> Y[x1,...x1,...,xr,...xr]

thus we have

Y[M,...,M,...,M,...,M] <-II Y <<- X II-> X[M,...,M] ->> Y[M,...,M,...,M,...,M]

Next suppose X'[U1,...,Up] := X := X"[V1,...,Vs] where the Ui are G-subterm occurrences of type n,M,k and pairwise disjoint, and the Vj are G-subterm occurrences of type n,M,r and also pairwise disjoint.. Each Ui can contain one or more Vj say Ui := Ui[Vi1,...,Vit(i)] and by the Replacement Lemma Ui[M,...,M] is a G-term of type n,M,k unless M=I in which case Ui[I,...,I] ->> to a G-term of type n,M,k-1. Similar remarks hold for the Vj. Let Z1,...,Zq be the maximal occurrences in the union of the two sets {U1,...,Up} and {V1,...,Vs}. Then we have X := X'"[Z1,...,Zq] and

X'"[M,...,M] (r-1)<-< X'[M,...,M] <-II X II-> X"[M,...,M] >->(k-1) X'"[M,...,M].

This completes the proof of the proposition.

COROLLARY (strip lemma): The diagram Z (k)<-< X >->>(k,t) Y can be completed to

Z >->>(k-1,t) U (k-t)<-< Y provided 0<t<k+1.

(III)

REDUCTION LEMMA: Suppose that P and Q are connected in G(G(n),I) by a path of length k<n+1 then there exists an R such that

P >->>(n-k,k+1) R (n-k,k+1)<<-< Q

where M := I.

PROOF: By induction on k. When k=0 the lemma follows from the Church-Rosser theorem. Suppose now that we have a path P := P(0), P(1),...,P(k) := Q where k < n+1. We have by our induction hypothesis that there exists an R such that P >->>(n-(k-1),k) R (n-(k-1),k)<<-< P(k-1). We distinguish two cases.

Case 1 ; P(k-1) = TI and Q = TG(n). We are assuming that k>0 so TG(n) >->(n-(k-1)) TI. By the Proposition there exists an R* such that P >->>(n-(k-1),k) R* (n-(k-1),k)<<-< TI. Again by the Proposition there exists an R** such that

P >->>(n-(k-1),k+1) R** (n-(k-1),k+1)<<-< Q.

This completes the proof for this case.

Case 2 ; P(k-1) = TG(n) and Q = TI. By the Proposition there exists an R* such that

P >->>(n-(k-1),k) R* (n-(k-1),k)<<-< TG(n) .

By the strip lemma corollary to the Proposition there exists an R** for the following diagram

TG(n) >->(n) TI >->>(n-k,k) R** (n-k)<-< R* (n-(k-1),k)<<-< P.

Finally by the Proposition there exists an R*** such that

Q >->>(n-k,k+1) R*** (n-k,k+1)<<-< P

and this completes the proof of the lemma.

COROLLARY: Suppose that k<n+1. Then there is no path in G(G(n),I) connecting the combinators K and K* of length <k+1.

PROOF : K and K* are >->(n-k) normal.

We can now prove the following

THEOREM : G(n) is n-easy but not 2n+5 easy.

PROOF : Suppose that K and K* are connected in G(G(n),M) by a path. If M=/=I then by the Replacement Lemma and the theorem of Mitchke >->>(n) is Church-Rosser ;so this is impossible. Thus M=I. But by the Corollary to the Reduction Lemma such a path must be longer than n. Thus G(n) is n-easy. Clearly there is such a path of length 2n+5 so G(n) is not 2n+5 easy.

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