COTOPOLOGICAL COMPLETIONS AND HULLS OF CONCRETE CATEGORIES

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Introduction

Categories in this paper will always be concrete over a fixed base category **S**, with faithful and amnestic forgetful functors to **S**. A completion of a concrete category **A** is a full and dense concrete embedding $G : \mathbf{A} \to \mathbf{B}$, with **B** complete in an appropriate sense. Completions of **A** with specified properties usually are the objects of a (quasi-)category, with full concrete embeddings as morphisms. An initial object of such a category is called a hull of **A**.

Topological completions and hulls with various properties have been studied intensively; we refer to the papers in the Bibliography for important contributions to this theory, and for further references. As H. HERRLICH [19] has pointed out, "smaller is better" for completions: a small completion of a category **A** is likely to retain more of the desirable properties of **A** than a large one. This makes hulls particularly desirable, and it suggests that the word "dense" should be used in a suitably restricted sense. Beginning with P. ANTOINE [13], "dense" has often been used in the least restrictive meaning of finally dense. J. PENON in [25] introduced epidense extensions, and the author in [32] discussed colimit-dense extensions over a monosieve-complete quasitopos **S** as base category. Colimit-dense extensions seem too restrictive for a more general base category; we replace them by quotient-dense extensions [33] which have almost all basic properties established in the literature for other types of dense extensions.

The existence of at least three different meanings for "dense extension" calls for a general theory of dense extensions, completions and hulls. This paper develop such a theory for an arbitrary concrete category **A**, based on a factorization structure for sinks in the base category **S**. Sections 1 through 3 present the general theory, with factorization structures reviewed in Section 1, **E**-dense extensions and completions studied in Section 2, and hulls discussed in Section 3. Section 4 deals with cartesian closed completions and hulls, and Section 5 with completions and hulls with partial morphisms represented. Section 6 deals with quasitopos completions and hulls, and with initial lifts for finite sources. We show that most results obtained in the literature for dense completions of some kind remain valid in our theory, but some results in Sections 4–6 have to be restricted to quotient-dense completions.

For every reasonable meaning of the word "dense", there is a largest dense completion of a concrete category **A**, a terminal object in the category of dense extensions of **A**, and completions of **A** can only be as complete as this largest completion. Thus dense completions are in general not topological; we can only expect them to be **E**-cotopological for a collection **E** of sinks in **S**, *i.e.* to admit final lifts for **E**-dense sinks (defined in

2.1). This raises the problem of existence of initial lifts for sources in dense completions. We discuss this for finite sources in 6.6; very little is known about the general situation.

Unlike some authors, we do not require that concrete categories have small fibres over the base category. Small fibres can be important; thus we obtain in Section 3 a criterion for hulls to be small-fibred. This result includes and generalizes all criteria for fibre-smallness of hulls which the author has seen in the literature.

Our terminology follows mostly [7] and [32].

1. Factorizations of sinks

1.1. Sinks. We define a *sink for a functor* $G : \mathbf{X} \to \mathbf{Y}$, with *codomain* an object E of $\mathbf{Y}$, or simply a G*-sink at* E, as a class of pairs (X, u) with X an object of $\mathbf{X}$ and $u : GX \to E$ in $\mathbf{Y}$. A sink for a functor $\mathrm{Id}\,\mathbf{S}$ will be called a sink *in* $\mathbf{S}$. Except for an empty sink with no pairs (X, u), the pairs (X, u) in a sink determine its codomain.

Sinks thus defined are structured sinks as defined dually to [7] 17.1. For sinks in a category $\mathbf{S}$, classes of pairs (X, u) with $u : X \to E$ in $\mathbf{S}$ may be replaced by classes of morphisms of $\mathbf{S}$ with codomain E.

We denote by $f\Phi$ the composition of a sink Φ at an object E of $\mathbf{Y}$ and a morphism $f : E \to E'$ of $\mathbf{Y}$; this is the sink at E' consisting of all pairs (X, fu) with (X, u) in Φ.

1.2. Sink factorization structures. Throughout this paper, categories will be concrete categories $(\mathbf{A}, P)$ over a fixed base category $\mathbf{S}$, with a faithful and amnestic forgetful functor P. For a collection $\mathbf{E}$ of sinks in $\mathbf{S}$ and a class $\mathcal{M}$ of morphisms of $\mathbf{S}$, both closed under composition with isomorphisms in $\mathbf{S}$, we say that $\mathbf{S}$ *has* $(\mathbf{E}, \mathcal{M})$*-factorizations*, or that $\mathbf{S}$ is an $(\mathbf{E}, \mathcal{M})$*-category*, if the following conditions are satisfied.

(1) Every sink Φ in $\mathbf{S}$ factors $\Phi = m\Phi'$ with Φ' in $\mathbf{E}$ and $m \in \mathcal{M}$.

(2) If $g\Phi = m\Psi$ for sinks Φ and Ψ in $\mathbf{S}$ and morphisms m and g of $\mathbf{S}$, with Φ in $\mathbf{E}$ and $m \in \mathcal{M}$, then $g = mt$ and $\Psi = t\Phi$ for a unique morphism t of $\mathbf{S}$.

(3) $\mathcal{M}$ consists of monomorphisms of $\mathbf{S}$.

1.3. Discussion. With our definition of sinks as classes, and not as families indexed by classes as in [7], the proof of the dual of [7] 15.4 is not valid. We do not know whether $\mathcal{M}$ must consist of monomorphisms of $\mathbf{S}$ if $\mathbf{E}$ and $\mathcal{M}$ satisfy 1.2.(1) and 1.2.(2), but we note the following result.

Proposition. *If* $\mathbf{E}$ *and* $\mathcal{M}$ *satisfy* 1.2.(1) *and* 1.2.(2), *then the following are equivalent.*

(i) $\mathcal{M}$ *consists of monomorphisms of* $\mathbf{S}$.

(ii) *Every morphism* f *of* $\mathbf{S}$ *factors* $f = me$ *in* $\mathbf{S}$, *with* $\{e\}$ *in* $\mathbf{E}$ *and* $m \in \mathcal{M}$.

(iii) $\{\mathrm{id}_E\}$ *is in* $\mathbf{E}$ *for every object* E *of* $\mathbf{S}$.

PROOF. If $\{f\} = m\Phi$ with m monomorphic, then Φ is a singleton; thus (i)$\Longrightarrow$(ii). If we factor $\mathrm{id}_E = me$ by (ii), then it follows easily from 1.2.(2) that e and m are inverse isomorphisms; thus (ii)$\Longrightarrow$(iii). If $ma = mb$ in $\mathbf{S}$ with m in $\mathcal{M}$, then we can factor

$m\{a,b\} = ma\{\mathrm{id}_E\}$ for the common domain E of a and b. If $\{\mathrm{id}_E\}$ is in $\mathbf{E}$, then $\{a,b\} = \{t\}$ for a morphism t with $mt = ma$. Thus $a = b$, and (iii)$\Longrightarrow$(i).

1.4. Examples and remarks. Every category $\mathbf{S}$ is an $(\mathbf{E},\mathcal{M})$-category for $\mathbf{E}$ consisting of all sinks in $\mathbf{S}$ and $\mathcal{M}$ the class of all isomorphisms of $\mathbf{S}$.

Topological categories over sets are $(\mathbf{E},\mathcal{M})$-categories for $\mathbf{E}$ the collection of all final sinks and $\mathcal{M}$ the class of all bijective morphisms, for $\mathbf{E}$ the collection of all episinks and $\mathcal{M}$ the class of all embeddings, and also for $\mathbf{E}$ the collection of all quotient sinks and $\mathcal{M}$ the class of all monomorphisms.

For an $(\mathbf{E},\mathcal{M})$-category structure of $\mathbf{S}$, the collection $\mathbf{E}$ is determined by $\mathcal{M}$; it consists of all sinks Φ which satisfy 1.2.(2) for all factorizations $g\Phi = m\Psi$ with $m \in \mathcal{M}$. A composition $g\Phi$ factors $g\Phi = m\Psi$ with m in $\mathcal{M}$ iff Ψ consists of morphisms h_u of $\mathbf{B}$, one for every u in Φ, such that $gu = mh_u$ in $\mathbf{S}$. If $mt = g$ in this situation, then $h_u = tu$ for every u in Φ, and $\Psi = t\Phi$. It follows easily that for sinks Φ and Φ' at an object of $\mathbf{S}$, with Φ in $\mathbf{E}$ and $\Phi \subset \Phi'$, the sink Φ' always is in $\mathbf{E}$. We also note that $\mathbf{E}$ is the class of all sinks Φ $\mathbf{S}$ such that m is an isomorphism for every factorization $\Phi = m\Phi'$ with m in $\mathcal{M}$, and $\mathcal{M}$ consists of all monomorphisms m of $\mathbf{S}$ such that every commutative square $mf = ge$ with $\{e\}$ in $\mathbf{E}$ has a diagonal. All isomorphisms of $\mathbf{S}$ are in $\mathcal{M}$, and $\mathcal{M}$ is pullback-stable. If mm' is defined in $\mathbf{S}$ with $m \in \mathcal{M}$, then $mm' \in \mathcal{M}$ iff $m' \in \mathcal{M}$.

1.5. Before discussing standard choices for $\mathbf{E}$ and $\mathcal{M}$, we extend the dual of [7] 15.7.

Proposition. *If* $\mathbf{S}$ *is an* $(\mathbf{E},\mathcal{M})$*-category, then the following are logically equivalent.*
(i) $\mathbf{E}$ *consists of episinks.*
(ii) $\mathbf{S}$ *has equalizers, and all strong monomorphisms of* $\mathbf{S}$ *are in* $\mathcal{M}$.
(iii) $\mathbf{S}$ *has equalizers, and all equalizers in* $\mathbf{S}$ *are in* $\mathcal{M}$.

PROOF. For parallel morphisms a and b in $\mathbf{S}$, consider the sink Φ of all f in $\mathbf{S}$ with $af = bf$, and factor it $\Phi = m\Phi'$, with m in $\mathcal{M}$ and Φ' in $\mathbf{E}$. If Φ' is an episink, then $am = bm$, and m is an equalizer of a and b. Now factor a strong monomorphism m as $m = m'e$, with m' in $\mathcal{M}$ and $\{e\}$ in $\mathbf{E}$. If e is epimorphic, then e is isomorphic, and $m \in \mathcal{M}$ follows. Thus (i)$\Longrightarrow$(ii).

(ii)$\Longrightarrow$(iii) trivially, and (iii)$\Longrightarrow$(i) dually to the proof of [7] 15.7.

1.6. Quotient sinks. We recall from [32] 63.1 that a sink Φ in a category $\mathbf{S}$ is called a *quotient sink* if for a factorization $g\Phi = m\Psi$ with m a monomorphism and Ψ a sink, the morphism g always factors $g = mt$ for a (unique) morphism t of $\mathbf{S}$. It follows that also $\Psi = t\Phi$. The induced sink of a colimit cone in $\mathbf{S}$ is always a quotient sink, and it is easily seen that every quotient sink in $\mathbf{S}$ is an episink if $\mathbf{S}$ has equalizers. We also note the following result.

Proposition. *If* $\mathbf{S}$ *is an* $(\mathbf{E},\mathcal{M})$*-category, then every quotient sink in* $\mathbf{S}$ *is in* $\mathbf{E}$*, and* $\mathbf{E}$ *is the class of all quotient sinks in* $\mathbf{S}$ *if and only if* $\mathcal{M}$ *is the class of all monomorphisms of* $\mathbf{S}$.

PROOF. Quotient sinks have the factorization property 1.2.(2) for all monomorphisms. Thus they are in $\mathbf{E}$, and every sink in $\mathbf{E}$ is a quotient sink if $\mathcal{M}$ consists of all monomorphisms of $\mathbf{S}$. Conversely, factor a monomorphism m as $m = m'e$ with $\{e\}$ in $\mathbf{E}$ and m' in $\mathcal{M}$. If $\mathbf{E}$ is the class of all quotient sinks, then $m' = mt$ and $te = \mathrm{id}$ for a morphism t, with t monomorphic since m' is monomorphic. But then e is an isomorphism, and $m \in \mathcal{M}$.

1.7. Episinks. We note the following result.

Proposition. *If* $\mathbf{S}$ *is an* $(\mathbf{E},\mathcal{M})$*-category, then* $\mathbf{E}$ *is the collection of all episinks of* $\mathbf{S}$ *if and only if* $\mathbf{S}$ *has equalizers and* $\mathcal{M}$ *is the class of all strong monomorphisms of* $\mathbf{S}$.

If $\mathbf{S}$ is balanced, with every monomorphism strong, it follows that every episink in $\mathbf{S}$ is a quotient sink.

PROOF. If $\mathbf{E}$ consists of all episinks, then $\mathbf{S}$ has equalizers, and all strong monomorphisms of $\mathbf{S}$ are in $\mathcal{M}$, by 1.5. If $m \in \mathcal{M}$ and $ge = mh$ in $\mathbf{S}$ with e epimorphic, then $g = mt$ and $f = te$ for a morphism t; thus m is a strong monomorphism. Conversely, if $\mathcal{M}$ is the class of strong monomorphisms and $\mathbf{S}$ has equalizers, then $\mathbf{E}$ consists of episinks. If an episink Φ factors $\Phi = m\Phi'$, then m is an epimorphism. If m is also a strong monomorphism, then m is an isomorphism; thus every episink is in $\mathbf{E}$.

1.8. Properties of $(\mathbf{E},\mathcal{M})$-categories. The duals of all results of [7] Section 15, are valid for an $(\mathbf{E},\mathcal{M})$-category $\mathbf{S}$. In particular, $\mathbf{S}$ has all possible pullbacks of morphisms in $\mathcal{M}$ and intersections for all sinks of morphisms in $\mathcal{M}$, and these pullbacks and intersections are again in $\mathcal{M}$. Also, if D and D_1 are diagrams in $\mathbf{S}$ with limit cones $\lambda : L \to D$ and $\lambda_1 : L_1 \to D_1$, and if $\mu : D \to D_1$ is a morphism of diagrams with all components in $\mathcal{M}$, then the unique morphism m with $\lambda_1 m = \mu\lambda$ is also in $\mathcal{M}$. In particular, products $m_1 \times m_2$ of morphisms in $\mathcal{M}$ are always in $\mathcal{M}$.

If $\mathbf{S}$ is an $(\mathbf{E},\mathcal{M})$-category and A an object of $\mathbf{S}$, then we have a slice category $\mathbf{S}/A$ with a domain functor $D_A : \mathbf{S}/A \to \mathbf{S}$. If $\mathbf{E}/A$ consists of all sinks Φ in $\mathbf{S}/A$ with $D_A\Phi$ in $\mathbf{E}$, and $\mathcal{M}_A$ of all morphisms m of $\mathbf{S}/A$ with $D_A m$ in $\mathcal{M}$, then $\mathbf{S}/A$ clearly is an $(\mathbf{E}_A,\mathcal{M}_A)$-category. We note that a pullback functor $f^* : \mathbf{S}/B \to \mathbf{S}/A$, for $f : A \to B$ in $\mathbf{S}$, always maps $\mathcal{M}_B$ into $\mathcal{M}_A$.

1.9. The following result applies in particular to classes $\mathcal{M}$ and $\mathcal{M}_1$ consisting of all isomorphisms, or all strong monomorphisms, or all monomorphisms. These classes are preserved by every right adjoint functor.

Proposition. *Let* $\mathbf{S}$ *be an* $(\mathbf{E},\mathcal{M})$*-category and* $\mathbf{T}$ *an* $(\mathbf{E}_1,\mathcal{M}_1)$*-category. If* $F \dashv G : \mathbf{S} \to \mathbf{T}$*, then* G *maps* $\mathcal{M}$ *into* $\mathcal{M}_1$ *if and only if* F *maps* $\mathbf{E}_1$ *into* $\mathbf{E}$.

PROOF. For $m \in \mathcal{M}$ and Φ in $\mathbf{E}_1$, we have a commutative square $g \cdot F\Phi = m\Psi$ in $\mathbf{S}$ iff we have an adjoint commutative square $\hat{g}\Phi = Gm \cdot \widehat{\Psi}$ in $\mathbf{T}$. One of the squares has a diagonal iff the other square has one, and these diagonals are adjoint morphisms. The Proposition follows immediately.

2. Dense extensions and completions

2.1. Dense sinks and sieves. We consider from now on a fixed concrete category $(\mathbf{A}, P)$ over the base category $\mathbf{S}$, and concrete full embeddings $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ over $\mathbf{S}$. We assume that the base category $\mathbf{S}$ has $(\mathbf{E}, \mathcal{M})$-factorizations, for a collection $\mathbf{E}$ of sinks in $\mathbf{S}$ and a class $\mathcal{M}$ of morphisms of $\mathbf{S}$, satisfying the conditions of 1.2. Concrete categories over $\mathbf{S}$ will have faithful and amnestic forgetful functors, but we do not assume that fibres for forgetful functors are small.

For a concrete functor $H : (\mathbf{B}, Q) \to (\mathbf{C}, R)$, every H-sink Φ at an object C of $\mathbf{C}$ has an underlying sink in $\mathbf{S}$, at RC. We say that Φ is $\mathbf{E}$-*dense* if this underlying sink is in $\mathbf{E}$.

P-*sieves* ([31], [32] 59.2) are special P-sinks Φ, with (B', vf) always in Φ for a pair (B, v) in Φ and $f : B' \to B$ in $\mathbf{B}$. Pairs $X = (|X|, \Phi_X)$ with $|X|$ an object of $\mathbf{S}$ and Φ_X a P-sieve at $|X|$ are the objects of a category $\mathbf{A}^{cr}$ of P-sieves. Morphisms $f : (E, \Phi) \to (F, \Psi)$ in $\mathbf{A}^{cr}$ are morphisms $f : E \to F$ of $\mathbf{S}$ with $f\Phi \subset \Psi$. With these morphisms, $\mathbf{A}^{cr}$ is a topological category over $\mathbf{S}$, a largest finally dense full extension of $\mathbf{A}$. We denote by $\mathbf{A}^{cd}$ the full subcategory of $\mathbf{A}^{cr}$ with $\mathbf{E}$-dense P-sieves as its objects.

2.2. E-dense extensions and E-cotopological categories. If a concrete full embedding $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ is given, then we say that an object B of $\mathbf{B}$ is $\mathbf{E}$-*dense* over $\mathbf{A}$ if B has the final structure in $\mathbf{B}$ for an $\mathbf{E}$-dense G-sink at B. In particular, every object GA of $\mathbf{B}$ is $\mathbf{E}$-dense. We say that G, or by *abus de langage* $(\mathbf{B}, Q)$, is an $\mathbf{E}$-*dense extension* of $(\mathbf{A}, P)$ if every object of $\mathbf{B}$ is $\mathbf{E}$-dense over $\mathbf{A}$. We say that G or $(\mathbf{B}, Q)$ is an $\mathbf{E}$-*cotopological completion* of $(\mathbf{A}, P)$ if G is $\mathbf{E}$-dense and $(\mathbf{B}, Q)$ is $\mathbf{E}$-*cotopological*, *i.e.* if every $\mathbf{E}$-dense Q-sink has a final lift in $\mathbf{B}$.

$\mathbf{E}$-dense extensions $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ of $(\mathbf{A}, \mathbf{S})$ are the objects of a category, with full concrete embeddings I satisfying $IG = H$ as morphisms $I : G \to H$. We note that $\mathrm{Id}\,\mathbf{A}$ is an initial object of this category.

For $\mathbf{E}$ the collection of all sinks, we get finally dense extensions, and $\mathbf{E}$-cotopological categories are topological categories. For $\mathbf{E}$ the collection of all episinks or all quotient sinks, we get epidense and quotient-dense extensions, and $\mathbf{E}$-cotopological categories are dual to $\mathbf{M}$-topological categories for $\mathbf{M}$ the collections of all monosinks and all strong monosinks respectively. For $\mathbf{E}$ all quotient sinks and $\mathcal{M}$ all monomorphisms, $\mathbf{E}$-cotopological categories and completions will be called *quotient-topological.*

We note the following useful result without proof.

Proposition. *Every* $\mathbf{E}$-*cotopological category* $(\mathbf{B}, Q)$ *has* $(\mathbf{E_B}, \mathcal{M}_\mathbf{B})$-*factorizations, where* $\mathbf{E_B}$ *consists of all* $\mathbf{E}$-*dense final sinks in* $\mathbf{B}$, *and* $\mathcal{M}_\mathbf{B}$ *of all morphisms* $m : B \to B'$ *in* $\mathbf{B}$ *with* m *in* $\mathcal{M}$.

2.3. Proposition ([7] 10.71). *For* $\mathbf{E}$-*dense extensions* $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ *and* $H : (\mathbf{A}, P) \to (\mathbf{C}, R)$, *every morphism* $I : G \to H$ *preserves initial lifts of sources.*

PROOF. If objects B_i of $\mathbf{B}$ and morphisms $f_i : E \to QB_i$ define a source for Q with initial lift B in $\mathbf{B}$, then $g : C \to IB$ in $\mathbf{C}$, for $g : RC \to E$ in $\mathbf{S}$, iff $gu : HA \to IB$ for every $u : HA \to C$ in $\mathbf{C}$. This is the case iff $gu : GA \to B$ in $\mathbf{B}$ for every such u, hence iff $f_i gu : GA \to B_i$ in $\mathbf{B}$ for every u and every f_i, hence iff $f_i gu : HA \to IB_i$ in $\mathbf{C}$ for every u and every f_i, hence iff $f_i g : C \to IB_i$ in $\mathbf{C}$ for every f_i. Thus IB is an initial lift in $\mathbf{C}$ for the source of morphisms $f_i : E \to RIB_i$.

2.4. Theorem. *For a morphism $I : G \to H$ of $\mathbf{E}$-dense extensions of $(\mathbf{A}, P)$, with $H : (\mathbf{A}, P) \to (\mathbf{C}, R)$ an $\mathbf{E}$-cotopological completion and $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$, the following are equivalent.*

(i) *G is an $\mathbf{E}$-cotopological completion of $(\mathbf{A}, P)$.*

(ii) *The functor I has a left-inverse concrete left adjoint J.*

(iii) *The functor I creates initial lifts,* i.e. *a source of objects B_i of $\mathbf{B}$ and morphisms $f_i : E \to QB_i$ of $\mathbf{S}$ has an initial lift in $\mathbf{B}$, preserved by I, if the source of objects IB_i of $\mathbf{C}$ and morphisms $f_i : E \to RIB_i$ has an initial lift in $\mathbf{C}$.*

PROOF. For an object C of $\mathbf{C}$, the Q-sink Σ of pairs (GA, u) with $u : HA \to C$ in $\mathbf{C}$ is a final $\mathbf{E}$-dense sink. If (i) holds, let JC be the final lift of this sink in $\mathbf{B}$. Then $f : JC \to B$ in $\mathbf{B}$, for $f : RC \to QB$ in $\mathbf{S}$, iff $fu : GA \to B$ in $\mathbf{B}$ for every pair (A, u) in Σ, hence iff $fu : HA \to IB$ for every (A, u) in Σ, and thus iff $f : C \to IB$ in $\mathbf{C}$. Thus objects JC define a concrete left adjoint J of I, with J left inverse to I because I is a full embedding and Q amnestic.

If morphisms $f_i : E \to RIB_i$ have an initial lift C in $\mathbf{C}$, and if (ii) is valid, then $f_i : JC \to B_i$; we claim that JC is an initial lift for this source. If $g : QX \to E$ in $\mathbf{S}$ with $f_i g : X \to B_i$ in $\mathbf{B}$ for every f_i, then also $f_i g : IX \to IB_i$ in $\mathbf{C}$ for every f_i, and thus $g : IX \to C$ in $\mathbf{C}$. But then $g : X \to JC$ in $\mathbf{B}$ since $JIX = X$.

For an $\mathbf{E}$-dense sink Σ of objects B_i of $\mathbf{B}$ and morphisms $f_i : QB_i \to E$ in $\mathbf{S}$, consider the source Σ' of all pairs (u, B) with $u : E \to QB$ in $\mathbf{S}$ and $uf_i : B_i \to B$ in $\mathbf{B}$ for each (B_i, f_i) in Σ, and the sink Σ'' of pairs (B', v) with $v : QB' \to E$ and $uv : B' \to B$ for every pair (u, B) in Σ'. Then Σ'' is an $\mathbf{E}$-dense sink containing Σ; let C be its final lift in $\mathbf{C}$. If $g : RX \to E$ in $\mathbf{S}$ with $ug : X \to IB$ in $\mathbf{C}$ for every pair (u, B) in Σ', then (GA, gx) is in Σ'', and $gx : HA \to C$ in $\mathbf{C}$, for every $x : HA \to X$ in $\mathbf{C}$, with A an object of $\mathbf{A}$. But then $g : X \to C$ in $\mathbf{C}$; thus C is an initial lift for the source of morphism $u : E \to RIB$ with (u, B) in Σ'. An initial lift of Σ' in $\mathbf{B}$ is clearly a final lift for Σ; thus (iii)$\Longrightarrow$(i).

2.5. The Antoine functor. For an object A of $\mathbf{A}$, we denote by $\mathsf{Y}A = (PA, \Upsilon A)$ the *Antoine sieve* with $(X, u) \in \Upsilon A$ iff $u : X \to A$ in $\mathbf{A}$. The sieve ΥA is $\mathbf{E}$-dense since $(A, \mathrm{id}_{PA}) \in \Upsilon A$. Antoine sieves clearly define a concrete functor $\mathsf{Y} : \mathbf{A} \to \mathbf{A}^{\mathrm{cd}}$, an $\mathbf{E}$-dense embedding since $(A, u) \in \Phi$ for a P-sieve (E, Φ) iff $u : \mathsf{Y}A \to (E, \Phi)$ in $\mathbf{A}^{\mathrm{cr}}$, and (E, Φ) has the final structure in $\mathbf{A}^{\mathrm{cr}}$ for these morphisms.

Theorem. *The Antoine functor $\mathsf{Y} : \mathbf{A} \to \mathbf{A}^{\mathrm{cd}}$ is an $\mathbf{E}$-cotopological completion, and a terminal object in the category of $\mathbf{E}$-dense extensions of $(\mathbf{A}, P)$.*

PROOF. For an $\mathbf{E}$-dense sink of morphisms $f_i : E_i \to E$ of $\mathbf{S}$ and $\mathbf{E}$-dense P-sieves (E_i, Φ_i), it is easily seen that the final lift (E, Φ) in $\mathbf{A}^{\mathrm{cr}}$, consisting of all pairs of the form $(A, f_i u)$ for some f_i, with $(A,u) \in \Phi_i$, is $\mathbf{E}$-dense. Thus Y is an $\mathbf{E}$-cotopological completion. For an $\mathbf{E}$-dense extension $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ and a full concrete embedding $I : \mathbf{B} \to \mathbf{A}^{cd}$ with $IG = \mathrm{Y}$, and for $u : PA \to QB$ in $\mathbf{S}$, we have

$$(1) \qquad (A,u) \in IB \iff u : \mathrm{Y}A \to IB \iff u : GA \to B.$$

This determines I uniquely, with each IB $\mathbf{E}$-dense since G is $\mathbf{E}$-dense, and (1) clearly defines a morphism $I : G \to \mathrm{Y}$ for every $\mathbf{E}$-dense extension G of $(\mathbf{A}, P)$.

2.6. If $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ is an $\mathbf{E}$-dense extension, with $IG = \mathrm{Y}$ for a concrete full embedding $I : \mathbf{B} \to \mathbf{A}^{cd}$, then every source of objects B_i of $\mathbf{B}$ and morphisms $f_i : E \to QB_i$ of $\mathbf{S}$ induces a source of objects IB_i of $\mathbf{A}^{cd}$ and morphisms $f_i : E \to |IB_i|$ of $\mathbf{S}$. We note the following result.

Proposition. *If $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ is an $\mathbf{E}$-cotopological completion, then the following are equivalent for a source of objects B_i of $\mathbf{B}$ and morphisms $f_i : E \to QB_i$ in $\mathbf{S}$.*

(i) *The source has a lift in $\mathbf{B}$.*
(ii) *The source has an initial lift in $\mathbf{B}$.*
(iii) *The induced source of morphisms $f_i : E \to |IB_i|$ has a lift in $\mathbf{A}^{cd}$.*
(iv) *The induced source has an initial lift in $\mathbf{A}^{cd}$.*

PROOF. If morphisms $f_i : B \to B_i$ form a lift of the given source, then consider the P-sieve of all morphisms $u : PA \to E$ of $\mathbf{S}$ with $f_i u : GA \to B_i$ in $\mathbf{B}$ for every f_i. This sieve includes all morphisms $u : GA \to B$ of $\mathbf{B}$; thus it is $\mathbf{E}$-dense. If B_0 is a final lift in $\mathbf{B}$ for this sieve and $g : QB' \to E$ satisfies $f_i g : B' \to B_i$ for every f_i, then $f_i g u : GA \to B_i$ for every f_i and every $u : GA \to B'$. But then $gu : GA \to B_0$ for every $u : GA \to B'$, and $g : B' \to B_0$ follows. Thus B_0 is the desired initial lift, and (i)$\Longrightarrow$(ii).

(ii)$\Longrightarrow$(iv) because the functor I preserves initial lifts, and (iv)$\Longrightarrow$(iii) trivially.

If X is a lift of the induced source, with $f_i : X \to IB_i$ in $\mathbf{A}^{cd}$ for each f_i, then $f_i : JX \to B_i$ for each f_i, for the concrete left adjoint J of I; thus (iii)$\Longrightarrow$(i).

2.7. Concrete monomorphisms, epimorphisms and colimits. All limits in $\mathbf{A}^{cr}$ are concrete, *i.e.* preserved by the forgetful functor to $\mathbf{S}$. Limits in $\mathbf{A}^{cd}$ are coreflections of limits in $\mathbf{A}^{cr}$, hence not necessarily concrete. Concrete limits in an $\mathbf{E}$-cotopological completion $\mathbf{B}$ of $\mathbf{A}$ are initial lifts of sources, and thus preserved and created by the full concrete embedding $\mathbf{B} \to \mathbf{A}^{cd}$.

Monomorphisms in an $\mathbf{E}$-cotopological category $\mathbf{B}$ are reflected, but not necessarily preserved, by the faithful forgetful functor of $\mathbf{B}$. The dual of the category of Hausdorff spaces is an example for this; see 5.2. We say that a monomorphism preserved by the forgetful functor is *concrete*; see 3.6 and 6.6 for related discussion. If $\mathbf{B}$ is an $\mathbf{E}$-dense completion of $\mathbf{A}$, then the concrete full embedding $\mathbf{B} \to \mathbf{A}^{cd}$ preserves and reflects monomorphisms and concrete monomorphisms.

For epimorphisms and colimits, we have the following result.

Proposition. *If* $(\mathbf{B},Q)$ *is an* $\mathbf{E}$*-cotopological category, then the forgetful functor* Q *of* $\mathbf{B}$ *preserves and reflects epimorphisms, and* Q *preserves and lifts colimits.*

PROOF. The faithful functor Q reflects monomorphisms and epimorphisms. If $e : B \to C$ is an epimorphism of $\mathbf{B}$ and $ae = be$ in $\mathbf{S}$, then we can factor $a = m\alpha$, $b = m\beta$, with α and β forming a Q-sink. This sink has a final lift C_1 in $\mathbf{B}$, and $\alpha e = \beta e$ in $\mathbf{B}$ for the lifted morphisms $\alpha, \beta : C \to C_1$. But then $\alpha = \beta$, and e is epimorphic in $\mathbf{S}$.

If D is a diagram in $\mathbf{B}$ with a colimit cone $\tau : D \to B$, factor $\tau = m\sigma$ in $\mathbf{S}$, with m in $\mathcal{M}$ and σ a sink in $\mathbf{E}$. Then σ has a final lift X to a cone $\sigma : D \to X$ in $\mathbf{B}$, with $m : X \to B$ in $\mathbf{B}$, and with $\sigma = s\tau$ for a morphism $s : B \to X$. But then $ms = \mathrm{id}_B$ in $\mathbf{B}$, and m is an isomorphism of $\mathbf{S}$. Thus the underlying sink of τ is in $\mathbf{E}$. If $\lambda : QD \to E$ is a cone in $\mathbf{S}$, factor $\lambda = m\rho$ with m monomorphic and ρ a quotient cone. Then ρ can be lifted to $\mathbf{B}$, and thus $\rho = r\tau$ for a morphism r, with $\lambda = mr \cdot Q\tau$. If also $\lambda = g \cdot Q\tau$, then $g = ms$ for a unique morphism s of $\mathbf{S}$, with $s \cdot Q\tau = \rho$ since m is monomorphic. But then $s = r$, and $g = mr$. Thus $Q\tau$ is a colimit cone in $\mathbf{S}$.

Conversely, if the diagram QD in $\mathbf{S}$ has a colimit cone $\sigma : QD \to E$ in $\mathbf{S}$ and we factor the underlying sink of σ as $m\sigma'$ with σ' in $\mathbf{E}$, then σ' is a cone with domain QD. Thus $\sigma' = t\sigma$ for a morphism t of $\mathbf{S}$, with $mt = \mathrm{id}_E$. Now m is an isomorphism, and the underlying sink of σ is in $\mathbf{E}$. But then σ has a final lift to a cone $\sigma : D \to B$ in $\mathbf{B}$, and this final lift is clearly a colimit cone of D in $\mathbf{B}$.

2.8. E-dense full sieves. We denote by Ω_E the full P-sieve at an object E of $\mathbf{S}$, consisting of all pairs (A,u) with A an object of $\mathbf{A}$ and $u : PA \to E$ in $\mathbf{S}$. Full P-sieves need not be $\mathbf{E}$-dense; we note however that if there is an $\mathbf{E}$-dense P-sieve Φ at an object E of $\mathbf{E}$, then every P-sieve at E coarser than Φ, and in particular Ω_E, is $\mathbf{E}$-dense. This is the case for every object PA of $\mathbf{S}$.

The following result shows that there is no essential loss of generality if we assume that all P-sieves Ω_E are $\mathbf{E}$-dense.

Proposition. *Objects* E *of* $\mathbf{S}$ *with an* $\mathbf{E}$*-dense* P*-sieve at* E *define an* $\mathcal{M}$*-coreflective full subcategory* $\mathbf{S}^{\mathrm{d}}$ *of* $\mathbf{S}$*, with objects including all objects* PA *for objects* A *of* $\mathbf{A}$.

PROOF. If $m : (E', \Phi') \to (E, \Omega_E)$ is a coreflection for $\mathbf{A}^{\mathrm{cd}}$, then clearly $\Phi' = \Omega_{E'}$, and we claim that $m : E' \to E$ is a coreflection for $\mathbf{S}^d$. If $g : F \to E$ in $\mathbf{S}$, then we have a factorization $gv = mv'$ for every pair (A,v) in Ω_F. If Ω_F is $\mathbf{E}$-dense, it follows that $g = mg'$ for a (unique) morphism $g' : F \to E'$.

2.9. We consider an $\mathbf{E}$-cotopological extension $G : \mathbf{A} \to \mathbf{B}$ over $\mathbf{S}$, and we denote by $\mathbf{B}^{\mathrm{d}}$ the full subcategory of $\mathbf{B}$ with objects C which have the final structure in $\mathbf{B}$ for an $\mathbf{E}$-dense sink of morphisms $u : GA \to C$ in $\mathbf{B}$. Thus we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{A} & \xrightarrow{G_1} & \mathbf{B}^{\mathrm{d}} & \xrightarrow{I} & \mathbf{A}^{\mathrm{cd}} \\ & & \downarrow{\scriptstyle H} & & \downarrow \\ & & \mathbf{B} & \longrightarrow & \mathbf{A}^{\mathrm{cr}} \end{array} \tag{1}$$

of concrete categories and concrete full embeddings over $\mathbf{S}$ with $HG_1 = G$ and $IG_1 = \mathsf{Y}$, and with $IC = (|C|, \Gamma C)$ for an object C of $\mathbf{B}^{\mathrm{d}}$, for the P-sieve ΓC of all pairs (A, u) with A an object of $\mathbf{A}$ and $u : GA \to C$ in $\mathbf{B}$.

Theorem. *If $G : A \to \mathbf{B}$ is a $\mathbf{E}$-cotopological extension, then the category $\mathbf{B}^{\mathrm{d}}$ in* (1) *is an $\mathbf{E}$-cotopological completion of $\mathbf{A}$, and an $\mathcal{M}$-coreflective subcategory of $\mathbf{B}$.*

PROOF. The subcategory $\mathbf{B}^{\mathrm{d}}$ of $\mathbf{B}$ is clearly closed under final lifts of $\mathbf{E}$-dense sinks in $\mathbf{B}$. For an object B of $\mathbf{B}$, factor the sieve of all morphisms $u : GA \to B$ in $\mathbf{B}$ as $\mu_B \cdot \Phi_B$, with Φ_B an $\mathbf{E}$-dense sieve and μ_B in $\mathcal{M}$. Let KB be the final lift for Φ_B in $\mathbf{B}$, hence in $\mathbf{B}^{\mathrm{d}}$; then $\mu_B : KB \to B$ in $\mathbf{B}$. An object C of $\mathbf{B}^{\mathrm{d}}$ has the final structure for the $\mathbf{E}$-dense sieve ΓC of morphisms $v : GA \to C$ of $\mathbf{B}$. If $f : C \to B$ in $\mathbf{B}$, then we have a commutative square $f \cdot \Gamma C = \mu_B \cdot \Psi$ of sieves and morphisms, with Ψ contained in Φ_B. Now $\Psi = g \cdot \Gamma C$ and $f = \mu_B \cdot g$ in $\mathbf{S}$ for a unique morphism g. Since C has the final structure for ΓC, we have $g : C \to KB$ in $\mathbf{B}^{\mathrm{d}}$. Thus μ_B is the desired coreflection.

2.10. Special case: constructs with constant maps. Concrete categories over sets have also been called *constructs.* We say that a construct $(\mathbf{A}, P)$ *has constant maps* ([31]) if for objects A, B of $\mathbf{A}$, every mapping $f : PA \to PB$ with its range $f^{\to}(PA)$ empty or a singleton lifts to a map $f : A \to A'$ in $\mathbf{A}$. This means that $\mathbf{A}$ has exactly one empty object, and for every singleton S exactly one object A with $PA = S$, and that these objects are both discrete and coarse.

For sets with the (episink, injective mapping) factorization of sinks, and for a construct $\mathbf{A}$ with constant maps, $\mathbf{A}^{\mathrm{cd}}$ is topological over sets, with exactly one object over a singleton, but with two empty objects $(\emptyset, \Phi)$, one with an empty sieve Φ and one with a full sieve Φ. If we remove the object $(\emptyset, \Phi)$ with an empty sieve Φ from $\mathbf{A}^{\mathrm{cd}}$, we obtain a full bireflective topological subcategory of $\mathbf{A}^{\mathrm{cd}}$. This subcategory, the category $\mathbf{A}^{\mathrm{c}}$ of [31], has constant maps, and the reflector $\mathbf{A}^{\mathrm{cr}} \to \mathbf{A}^{\mathrm{c}}$ preserves finite limits.

3. E-cotopological hulls

3.1. Definition. By 2.5, every $\mathbf{E}$-dense extension of $(\mathbf{A}, P)$ over $\mathbf{S}$ is concretely isomorphic to a full subcategory $\mathbf{B}$ of $\mathbf{A}^{\mathrm{cd}}$, containing all objects $\mathsf{Y}A$ for objects A of $\mathbf{A}$. If we restrict ourselves to full subcategories of $\mathbf{A}^{\mathrm{cd}}$, then morphisms $I : G \to H$ of $\mathbf{E}$-dense extensions become full subcategory embeddings. If $\mathcal{C}$ is a class of $\mathbf{E}$-dense P-sieves, then a smallest $\mathbf{E}$-cotopological completion of $(\mathbf{A}, P)$ with $\mathcal{C}$ contained in its class of objects is called an $\mathbf{E}$-*cotopological hull* of $\mathcal{C}$ in $\mathbf{A}^{\mathrm{cd}}$.

We show easily that $\mathbf{E}$-cotopological hulls in this sense always exist, and we discuss fibre-smallness for these completions.

3.2. Theorem. *For a concrete category $\mathbf{A}$ over an $(\mathbf{E}, \mathcal{M})$-category $\mathbf{S}$, every collection $\mathcal{C}$ of $\mathbf{E}$-dense P-sieves has an $\mathbf{E}$-cotopological hull. If every object $\mathsf{Y}A$ of $\mathbf{A}^{\mathrm{cd}}$ is an initial* *lift of a source of morphisms* $f_i : \mathsf{Y}A \to X_i$ *in* $\mathbf{A}^{\mathrm{cd}}$ *with every* X_i *in* $\mathcal{C}$*, then the objects of this hull are all initial lifts of sources* $f_i : E \to |X_i|$ *in* $\mathbf{A}^{\mathrm{cd}}$*, with each* X_i *in* $\mathcal{C}$*.*

PROOF. By 2.4 and 2.6, $\mathbf{E}$-cotopological completions of $\mathbf{A}$ are $\mathbf{E}$-dense extensions which, regarded as full subcategories of $\mathbf{A}^{\mathrm{cd}}$, are closed under initial lifts of sources. Intersections preserve this property; thus every class $\mathcal{C}$ of $\mathbf{E}$-dense P-sieves has an $\mathbf{E}$-cotopological hull. The collection of all objects X of $\mathbf{A}^{\mathrm{cd}}$ with the initial structure for a source of morphisms $f_i : X \to X_i$ of $\mathbf{A}^{\mathrm{cd}}$, with each X_i in $\mathcal{C}$, consists of objects of the $\mathbf{E}$-cotopological hull and is closed under initial lifts of sources. Thus it is the class of all objects of the $\mathbf{E}$-cotopological hull of $\mathcal{C}$ if it contains all objects $\mathsf{Y}A$.

3.3. Corollary. *Every concrete category* $\mathbf{A}$ *over an* $(\mathbf{E},\mathcal{M})$*-category* $\mathbf{S}$ *has an* $\mathbf{E}$*-cotopological hull.*

PROOF. This follows immediately from 3.2, with $\mathcal{C}$ the collection of all objects $\mathsf{Y}A$ for objects A of $\mathbf{A}$.

3.4. Fibres. We recall that the *fibre* of an object E of $\mathbf{S}$, in a concrete category $\mathbf{B}$ over $\mathbf{S}$, is the subcategory of all objects X of $\mathbf{B}$ with underlying object $|X| = E$, with morphisms $\mathrm{id}_E : X \to X'$. It is sometimes considered desirable that concrete categories should have small fibres. We investigate this property for $\mathbf{E}$-cotopological hulls.

We assume that every object $\mathsf{Y}A$ of $\mathbf{A}^{\mathrm{cd}}$ admits an intial source of morphisms $f_i : \mathsf{Y}A \to X_i$, with each X_i in $\mathcal{C}$. This is obviously no essential loss of generality.

For pairs (A,u) and (B,v), with A and B objects of $\mathbf{A}$ and with $u : PA \to E$ and $v : PB \to E$ in $\mathbf{S}$ for an object E of $\mathbf{S}$, we put $(B,v) \prec_{\mathcal{C}} (A,u)$ if for every pair (f,X) with X in $\mathcal{C}$ and $f : E \to |X|$ in $\mathbf{S}$, and with $fu : \mathsf{Y}A \to X$ in $\mathbf{A}^{\mathrm{cd}}$, we also have $fv : \mathsf{Y}B \to X$ in $\mathbf{A}^{\mathrm{cd}}$. This clearly defines a preorder $\prec_{\mathcal{C}}$.

For a fixed pair (A,u) with $u : PA \to E$ in $\mathbf{S}$, the pairs (B,v) with $(B,v) \prec_{\mathcal{C}} (A,u)$ clearly form a P-sieve (E,Φ). This sieve has the initial structure in $\mathbf{A}^{\mathrm{cr}}$ for the morphisms $f : E \to |X|$ in $\mathbf{S}$ with X in $\mathcal{C}$ and $fu : \mathsf{Y}A \to X$ in $\mathbf{A}^{\mathrm{cr}}$, but it need not be $\mathbf{E}$-dense.

We put $(A,u) \sim_{E,\mathcal{C}} (A',u')$ if $(A,u) \prec_{\mathcal{C}} (A',u')$ and $(A',u') \prec_{\mathcal{C}} (A,u)$, with $u : PA \to E$ and $u' : PA' \to E$. This obviously defines an equivalence relation, with $(A,u) \sim_{E,\mathcal{C}} (A',u')$ iff the P-sieve (E,Φ) constructed in the preceding paragraph is the same for (A,u) and for (A',u').

3.5. Proposition. *If the collection of equivalence classes for the relation* $\sim_{E,\mathcal{C}}$ *is small for every object* E *of* $\mathbf{S}$*, then the* $\mathbf{E}$*-cotopological hull of* $\mathcal{C}$ *in* $\mathbf{A}^{\mathrm{cd}}$ *has small fibres. Conversely, if* $\mathbf{S}$ *is well-powered for monomorphisms in* $\mathcal{M}$ *and the* $\mathbf{E}$*-cotopological hull of* $\mathcal{C}$ *in* $\mathbf{A}^{\mathrm{cd}}$ *has small fibres, then the collection of equivalence classes for* $\sim_{E,\mathcal{C}}$ *is small for every object* E *of* $\mathbf{S}$*.*

PROOF. For every object (E,Φ) of the $\mathbf{E}$-cotopological hull, the P-sieve Φ is a union of equivalence classes for $\sim_{E,\mathcal{C}}$. If the collection of equivalence classes is small, it follows that the collection of their unions, and hence the fibre of E in the $\mathbf{E}$-cotopological hull, is small.

Conversely, the equivalence classes for $\sim_{E,\mathcal{C}}$ correspond bijectively to the P-sieves (E,Φ) constructed in 3.4. If $\mathbf{S}$ is well-powered for monomorphism in $\mathcal{M}$, then there is a set-indexed family of morphism $m_i : E_i \to E$ in $\mathcal{M}$ such that for every P-sieve (E,Φ) of this kind there is a coarse morphism $m_i : (E_i,\Phi_i) \to (E,\Phi)$, with (E_i,Φ_i) $\mathbf{E}$-dense and hence in the $\mathbf{E}$-cotopological hull of $\mathcal{C}$. If this $\mathbf{E}$-cotopological hull has small fibres, it follows that the collection of sieves (E,Φ) is small.

3.6. Remarks. The results of Sections 2 and 3 are valid in particular for finally dense extensions, with $\mathbf{E}$ the collection of all sinks in $\mathbf{S}$. For this case, Theorem 2.4 shows that $\mathbf{E}$-cotopological completions are topological categories, with initial lifts for all sources, and with forgetful functors preserving and lifting monomorphisms and strong monomorphisms, as well as all limits and colimits. Thus the forgetful functor of a completion $\mathbf{C}$ preserves all limits and colimits. If D is a diagram in $\mathbf{C}$ for which the underlying diagram has a limit in $\mathbf{S}$, then the initial lift of the source induced by the limit cone is a limit of D in $\mathbf{C}$, and colimits in $\mathbf{C}$ are obtained dually.

These properties remain valid for colimits in an $\mathbf{E}$-cotopological completion $\mathbf{B}$ of $\mathbf{A}$ if full P-sieves are $\mathbf{E}$-dense, but limits of a diagram D in $\mathbf{B}$ are preserved and lifted by the forgetful functor of $\mathbf{B}$ only if the source induced by a limit cone of D in $\mathbf{S}$ has a lift in $\mathbf{A}^{\mathrm{cd}}$. By 2.6, $\mathbf{E}$-cotopological completions admit initial lifts only for those sources which admit a lift in $\mathbf{A}^{\mathrm{cd}}$.

We observe that collections $\mathcal{C}$ of $\mathbf{E}$-dense P-sieves can be replaced by collections of P-sieves, since every coreflection for $\mathbf{A}^{\mathrm{cd}}$ in $\mathbf{A}^{\mathrm{cr}}$ is a coarse monomorphism in $\mathbf{A}^{\mathrm{cr}}$. Theorem 3.2 then characterizes the $\mathbf{E}$-cotopological hull of the $\mathbf{A}^{\mathrm{cd}}$-coreflections of the P-sieves in $\mathcal{C}$. If $\mathbf{E}$ is the collection of all sinks in $\mathbf{S}$, then the hull of $\mathcal{C}$ in $\mathbf{A}^{\mathrm{cr}}$, for a class $\mathcal{C}$ of P-sieves, has small fibres iff the collection of equivalence classes is small for every equivalence relation $\sim_{E,\mathcal{C}}$, by 3.5 with $\mathcal{M}$ the class of all isomorphisms of $\mathbf{S}$. Every known criterion for fibre smallness of topological or $\mathbf{E}$-cotopological hulls follows directly from 3.5 and these observations.

4. Cartesian closed completions and hulls

4.1. Definitions. We say that a concrete category $(\mathbf{B},Q)$ has *concrete finite products* if $\mathbf{B}$ has finite products, and the functor Q preserves them. A *coarse object* of $(\mathbf{B},Q)$ is an object C of $\mathbf{B}$ for which every morphism $f : QB \to QC$ of $\mathbf{S}$ lifts to $f : B \to C$ in $\mathbf{B}$. Coarse objects define couniversal morphisms for the forgetful functor Q; we say that $(\mathbf{B},Q)$ *has coarse objects* if Q has a right adjoint. The category $\mathbf{A}^{\mathrm{cr}}$ always has coarse objects (S,Ω_S) with Ω_S the full P-sieve of all pairs (A,u) with A an object of $\mathbf{A}$ and $u : PA \to S$ in $\mathbf{S}$, and $\mathbf{A}^{\mathrm{cr}}$ has concrete finite products if $\mathbf{S}$ has finite products. Concrete finite products, and more generally all concrete limits, are initial lifts of sources, and thus preserved by $\mathbf{E}$-dense extensions, by 2.3. We note the following result.

Proposition. *The base category* $\mathbf{S}$ *is cartesian closed if and only if there is a cartesian closed concrete category* $(\mathbf{B}, Q)$ *over* $\mathbf{S}$ *with coarse objects and concrete finite products.*

PROOF. If $\mathbf{S}$ is cartesian closed, then $\mathbf{A}^{\mathrm{cr}}$ is cartesian closed by [32], 60.2. The converse follows immediately from the main result of [15], since coarse objects provide a full embedding of $\mathbf{S}$ into $\mathbf{B}$, right adjoint to the product preserving forgetful functor Q.

4.2. Standard assumptions. In view of 4.1, we assume in this Section that $\mathbf{S}$ is cartesian closed. *We must also assume the following equivalent conditions.*

(i) $\mathcal{M}$ is preserved by all functors $[S, —]$.

(ii) All product functors $— \times S$ preserve $\mathbf{E}$.

(iii) All pullback functors p^* for projections of products $S \times S'$ in $\mathbf{S}$ preserve $\mathbf{E}$.

This is automatically satisfied by the three main examples for $\mathcal{M}$: isomorphisms, strong monomorphisms and all monomorphisms.

Theorem. *Let* $\mathbf{S}$ *satisfy the assumptions stated above. If* $\mathbf{B}$ *is an* $\mathbf{E}$*-cotopological cartesian closed full extension of* $\mathbf{A}$ *with concrete finite products, and if finite products of objects* GA *in* $\mathbf{B}$ *are* $\mathbf{E}$*-dense, then the category* $\mathbf{B}^{\mathrm{d}}$ *of 2.9 is closed under finite products in* $\mathbf{B}$*, and cartesian closed. In this situation, function space objects* $[Y, Z]$ *in* $\mathbf{B}^{\mathrm{d}}$*, for objects* Y *and* Z *of* $\mathbf{B}^{\mathrm{d}}$*, are given by coreflections* $\mu_{Y,Z} : [Y, Z] \to Z^Y$ *of function space objects* Z^Y *in* $\mathbf{B}$*, with* $\mathrm{ev}_{Y,Z}(\mu_{Y,Z} \times \mathrm{id}_Y)$ *an evaluation in* $\mathbf{B}^{\mathrm{d}}$ *for an evaluation* $\mathrm{ev}_{Y,Z}$ *in* $\mathbf{B}$ *and the coreflection* $\mu_{Y,Z}$.

PROOF. For objects X and Y of $\mathbf{B}^{\mathrm{d}}$, we have pullback squares

$$\begin{array}{ccccc}
GA \times GB & \xrightarrow{u \times \mathrm{id}_{GB}} & X \times GB & \xrightarrow{q} & GB \\
\Big\downarrow \mathrm{id}_{GA} \times v & & \Big\downarrow \mathrm{id}_X \times v & & \Big\downarrow v \\
GA \times Y & \xrightarrow{u \times \mathrm{id}_Y} & X \times Y & \xrightarrow{q} & Y \\
\Big\downarrow p & & \Big\downarrow p & & \\
GA & \xrightarrow{u} & X & &
\end{array} ,$$

with projections p and q of products and with morphisms u and v forming $\mathbf{E}$-dense sinks, with final lifts X and Y. If $\mathbf{B}$ is cartesian closed, then morphisms $u \times \mathrm{id}_{GB}$ form an $\mathbf{E}$-dense sink in $\mathbf{B}$ with final lift $X \times GB$, by 1.9 for $\mathbf{S} = \mathbf{T} = \mathbf{B}$. In the same way, the sink of morphisms $\mathrm{id}_X \times v$ is $\mathbf{E}$-dense, with final lift $X \times Y$. Since objects $GA \times GB$ are in $\mathbf{B}^{\mathrm{d}}$, it follows that $X \times Y$ is an object of $\mathbf{B}^{\mathrm{d}}$.

Now $\mathbf{B}^{\mathrm{d}}$ is a full coreflective subcategory of a cartesian closed category $\mathbf{B}$, closed under products $X \times Y$ in $\mathbf{B}$. It is well known that $\mathbf{B}^{\mathrm{d}}$ is cartesian closed in this situation, with coreflections of function space objects in $\mathbf{B}^{\mathrm{d}}$ as function space objects. These coreflections are concrete monomorphisms $\mu_{Y,Z}$ by 2.9, with $\mu_{Y,Z}\hat{\varphi}$ exponentially adjoint to $\varphi : X \times Y \to Z$ in $\mathbf{B}$ if $\hat{\varphi}$ is exponentially adjoint to φ in $\mathbf{B}^{\mathrm{d}}$. Using this for $\hat{\varphi} = \mathrm{id}_{[Y,Z]}$, we get the last part of the Theorem.

4.3. Proposition. *Under the standard assumptions of 4.2 for* $\mathbf{S}$, *the following statements are logically equivalent.*

(i) $\mathbf{A}^{\mathrm{cd}}$ *has concrete finite products of objects* $\mathrm{Y}A$.

(ii) $\mathbf{A}^{\mathrm{cd}}$ *has concrete finite products.*

(iii) *Every* $\mathbf{E}$-*cotopological completion of* $\mathbf{A}$ *has concrete finite products.*

(iv) *For some* $\mathbf{E}$-*dense extension* $G : \mathbf{A} \to \mathbf{B}$ *of* $\mathbf{A}$, *finite products of objects* GA *in* $\mathbf{B}$ *are concrete.*

If these properties are satisfied, then $\mathbf{A}^{\mathrm{cd}}$ *is cartesian closed.*

PROOF. If (i) holds, then $\mathbf{A}^{\mathrm{cd}}$ is cartesian closed, and (ii) is valid, by 4.2. Also (ii) $\Longrightarrow$ (iii) by 2.6, and (iii) $\Longrightarrow$ (iv) trivially. Finally, (iv) $\Longrightarrow$ (i) by 2.3.

4.4. Theorem. *If* $\mathbf{S}$ *satisfies the standard assumptions of* 4.2 *and finite products of* P-*sieves* $\mathrm{Y}A$ *in* $\mathbf{A}^{\mathrm{cr}}$ *are* $\mathbf{E}$-*dense, then the following conditions are equivalent for an* $\mathbf{E}$-*cotopological completion* $(\mathbf{B}, Q)$ *of* $(\mathbf{A}, P)$.

(i) $\mathbf{B}$ *is cartesian closed, with function space objects preserved up to isomorphism by the full concrete embedding* $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cd}}$.

(ii) *Every product functor* $- \times C$ *in* $\mathbf{B}$ *preserves final lifts of* $\mathbf{E}$-*dense sinks.*

(iii) *The concrete left adjoint* $J : \mathbf{A}^{\mathrm{cd}} \to \mathbf{B}$ *of the full concrete embedding* $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cd}}$ *preserves products* $X \times Y$.

PROOF. If B has the final structure for an $\mathbf{E}$-dense sink of morphisms $u_i : B_i \to B$ in $\mathbf{B}$ and (i) holds, then Q preserves products, and the morphisms $u_i \times \mathrm{id}_C$ form an $\mathbf{E}$-dense sink. Now consider $\varphi : QB \times QC \to QX$ with $\varphi \cdot (u_i \times \mathrm{id}_C) : B_i \times C \to X$ in $\mathbf{B}$ for every u_i. Let $\psi_i : B_i \to [C, X]$ be exponentially adjoint to $\varphi \cdot (u_i \times \mathrm{id}_C)$ in $\mathbf{B}$, and let $\varphi^{\#} : QB \to QX^{QC}$ be exponentially adjoint to φ in $\mathbf{S}$. We have commutative diagrams

$$\begin{array}{ccc} QB_i & \xrightarrow{u_i} & QB \\ \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \varphi^{\#}} \\ Q[C,X] & \xrightarrow{\mu_{C,X}} & QX^{QC} \end{array}$$

in $\mathbf{S}$, one for each u_i. It follows that $\varphi^{\#}$ factors $\mu_{C,X}\hat{\varphi}$, with $\hat{\varphi} u_i = \psi_i : B_i \to [C, X]$ in $\mathbf{B}$. But then $\hat{\varphi} : B \to [C, X]$ in $\mathbf{B}$, exponentially adjoint to $\varphi : B \times C \to X$, and (ii) holds.

Every object X of $\mathbf{A}^{\mathrm{cd}}$ has the final structure for an $\mathbf{E}$-dense sink of morphisms $u_i : IB_i \to X$, and then JX has the final structure for the morphisms $u_i : B_i \to JX$. By (ii) for $\mathbf{A}^{\mathrm{cd}}$, $X \times IC$ has the final structure for the morphisms $u_i \times \mathrm{id}_C$ which form an $\mathbf{E}$-dense sink, and so $J(X \times IC)$ has the final structure for the morphisms $u_i \times \mathrm{id}_C$ with domains $B_i \times C$. If (ii) is valid for $\mathbf{B}$, then $JX \times C$ has the final structure in $\mathbf{B}$ for this sink; thus $J(X \times IC) = JX \times C$. Now if an object Y of $\mathbf{A}^{\mathrm{cd}}$ has the final structure for an $\mathbf{E}$-dense sink of morphisms $v_j : IC_j \to Y$, then $J(X \times Y)$ has the final structure for morphisms $\mathrm{id}_X \times v_j : J(X \times IC_j) \to J(X \times Y)$, and $JX \times JY$ for morphisms $\mathrm{id}_X \times v_j : JX \times C_j \to JX \times JY$. But these morphisms have the same domains, hence the same final structure, and (iii) follows.

The step (iii)$\Longrightarrow$(i) follows immediately from the main result of [15].

For our next theorem, we need the following two lemmas.

4.5. Lemma. *If an* $\mathbf{E}$*-cotopological completion* $(\mathbf{B},Q)$ *of* $(\mathbf{A},P)$ *satisfies the conditions of Theorem 4.4, and an object* X *of* $\mathbf{A}^{\mathrm{cd}}$ *has the final structure for an* $\mathbf{E}$*-dense sink of morphisms* $u_i : IB_i \to X$*, for objects* B_i *of* $\mathbf{B}$*, then* $[X,IC]$ *has the initial structure for the morphisms* $[u_i, \mathrm{id}_C] : [X,IC] \to I[B_i,C]$ *of* $\mathbf{A}^{\mathrm{cd}}$*, for every object* C *of* $\mathbf{B}$.

PROOF. Consider $\hat{\varphi} : |Y| \to |[X,IC]|$ in $\mathbf{S}$, with $[u_i, \mathrm{id}_C]\hat{\varphi} : Y \to I[B_i,C]$ in $\mathbf{A}^{\mathrm{cd}}$ for each u_i. If $\mu_{X,IC}\hat{\varphi}$ is exponentially adjoint to $\varphi : |Y| \times |X| \to QC$ in $\mathbf{S}$, then $\mu_{IB_i,IC}[u_i, \mathrm{id}_C]\hat{\varphi} = (\mathrm{id}_{QC})^{u_i}\mu_{X,IC}\hat{\varphi}$ is exponentially adjoint to $\varphi(\mathrm{id}_Y \times u_i)$. Since $Y \times X$ has the final structure for the morphisms $\mathrm{id}_Y \times u_i$, it follows that $\varphi : Y \times X \to IC$. The exponential adjoint of this morphism in $\mathbf{A}^{\mathrm{cd}}$ is $\hat{\varphi} : Y \to [X,IC]$. Thus $[X,IC]$ has the claimed initial structure.

4.6. Lemma. *For every object* Y *of* $\mathbf{A}^{\mathrm{cd}}$*, the endofunctor* $[Y,—]$ *of* $\mathbf{A}^{\mathrm{cd}}$ *preserves initial structures for sources.*

PROOF. Assume that B has the initial structure for morphisms $u_i : B \to B_i$ of $\mathbf{A}^{\mathrm{cd}}$, and consider $\hat{\varphi} : |X| \to |[Y,B]|$ in $\mathbf{S}$ with $[\mathrm{id}_Y, u_i]\hat{\varphi} : X \to [Y,B_i]$ in $\mathbf{A}^{\mathrm{cd}}$ for every u_i. If $\mu_{Y,B}\hat{\varphi}$ is the exponential adjoint of $\varphi : |X| \times |Y| \to |B|$, then $\mu_{Y,B_i}[\mathrm{id}_Y, u_i]\hat{\varphi} = u_i{}^{|Y|}\mu_{Y,B}\hat{\varphi}$ is the exponential adjoint of $u_i\varphi$ in $\mathbf{S}$, and it follows that $[\mathrm{id}_Y, u_i]\hat{\varphi}$ is the exponential adjoint of $u_i\varphi : X \times Y \to B_i$ in $\mathbf{A}^{\mathrm{cd}}$. But then $\varphi : X \times Y \to B$ in $\mathbf{A}^{\mathrm{cd}}$, with exponential adjoint $\hat{\varphi} : X \to [Y,B]$.

4.7. Theorem. *If* $\mathbf{S}$ *satisfies the standard assumptions of 4.2, and finite products of* P*-sieves* $\mathsf{Y}A$ *in* $\mathbf{A}^{\mathrm{cr}}$ *are* $\mathbf{E}$*-dense, then every* P*-sieve* $\mathsf{Y}A$ *has the initial structure for morphisms* $f : \mathsf{Y}A \to [\mathsf{Y}B, \mathsf{Y}C]$ *in* $\mathbf{A}^{\mathrm{cd}}$*, and the* $\mathbf{E}$*-cotopological hull of the class of objects* $[\mathsf{Y}B, \mathsf{Y}C]$ *of* $\mathbf{A}^{\mathrm{cd}}$*, for objects* B *and* C *of* $\mathbf{A}$*, is cartesian closed.*

We note that this $\mathbf{E}$-cotopological hull is the smallest cartesian closed $\mathbf{E}$-cotopological completion $G : (\mathbf{A},P) \to (\mathbf{B},Q)$ of $\mathbf{A}$ for which the full embedding $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cd}}$ with $IG = \mathsf{Y}$ preserves function space objects $[X,Y]$.

PROOF. Let $\mathbf{C}$ be the $\mathbf{E}$-cotopological hull of the Theorem. By assumption, the terminal object $(1,\Omega_1)$ of $\mathbf{A}^{\mathrm{cr}}$ is $\mathbf{E}$-dense, and thus a terminal object of $\mathbf{A}^{\mathrm{cd}}$, with the final structure for an $\mathbf{E}$-dense sink of morphisms $u_i : \mathsf{Y}A_i \to (1,\Omega_1)$. By 4.5, every object $[(1,\Omega_1), \mathsf{Y}A]$ is in $\mathbf{C}$, with the initial structure for the morphisms $[u_i, \mathrm{id}_A]$. This proves the first part of the Theorem since $\mathsf{Y}A$ is isomorphic to $[(1,\Omega_1), \mathsf{Y}A]$.

Again by 4.5, every function space object $[X, \mathsf{Y}C]$ is in $\mathbf{C}$. Since a product $X \times Y$ in $\mathbf{A}^{\mathrm{cd}}$ has the initial structure for its projections, $\mathbf{C}$ is closed under finite products in $\mathbf{A}^{\mathrm{cd}}$. Now if Y has the initial structure for morphisms $u_i : Y \to [\mathsf{Y}B_i, \mathsf{Y}C_i]$, then $[X,Y]$ has the initial structure for the morphisms $[\mathrm{id}_X, u_i]$ with codomains $[X, [\mathsf{Y}B_i, \mathsf{Y}C_i]]$, by 4.6. These codomains are isomorphic to the objects $[X \times \mathsf{Y}B_i, \mathsf{Y}C_i]$ of $\mathbf{C}$, and thus objects of $\mathbf{C}$. But then $[X,Y]$ is an object of $\mathbf{C}$ whenever Y is an object of $\mathbf{C}$, and $\mathbf{C}$ is cartesian closed.

4.8. Quotient-dense completions and hulls. We note two special properties of quotient-dense cartesian closed completions and hulls.

Theorem. *For* **E** *the collection of quotient sinks, and if the forgetful functor of* $\mathbf{A}^{\mathrm{cd}}$ *preserves monomorphisms, then the three conditions of* 4.4 *are also equivalent to:*

(iv) **B** *is cartesian closed,*

and the hull described in 4.7 *is the cartesian closed quotient-topological hull of* **A**.

PROOF. 4.4.(i)$\Longrightarrow$(iv) trivially. For the converse, let $H(Y,Z)$ denote a function space object in **B**. Commutative diagrams

$$\begin{array}{ccccc}
\mathbf{B}(X\times Y,Z) & \xrightarrow{I} & \mathbf{A}^{\mathrm{cd}}(IX,I(Y\times Z) & \xrightarrow{\mathrm{id}} & \mathbf{A}^{\mathrm{cd}}(IX,IY\times IZ) \\
\downarrow\cong & & & & \downarrow\cong \\
\mathbf{B}(X,H(Y,Z)) & \xrightarrow{I} & \mathbf{A}^{\mathrm{cd}}(IX,IH(Y,Z)) & \xrightarrow{\rho_{Y,Z}} & \mathbf{A}^{\mathrm{cd}}(IX,[IY,IZ])
\end{array},$$

with exponential adjunctions as vertical arrows, define natural maps $\rho_{Y,Z}: IH(Y,Z) \to [IY,IZ]$ in $\mathbf{A}^{\mathrm{cd}}$, by the general construction of adjoint natural transformations, with $\rho_{Y,Z}\hat{\alpha}: IX \to [IY,IZ]$ exponentially adjoint in $\mathbf{A}^{\mathrm{cd}}$ to $\alpha: IX \to IY\times IZ$ if $\hat{\alpha}: X \to H(Y,Z)$ is exponentially adjoint to $\alpha: X\times Y \to Z$ in **B**. Now if $\rho_{Y,Z}\hat{\alpha} = \rho_{Y,Z}\hat{\beta}$ with $\hat{\alpha},\hat{\beta}: X \to IH(Y,Z)$, then the equation remains valid for the morphisms $\hat{\alpha}$ and $\hat{\beta}$ from $IJX \to IH(Y,Z)$. But then $\hat{\alpha}$ and $\hat{\beta}$ from JX to $H(Y,Z)$ are adjoints of the same morphism $JX\times Y \to Z$, and $\hat{\alpha} = \hat{\beta}$ follows. Thus $\rho_{Y,Z}$ is monomorphic in $\mathbf{A}^{\mathrm{cd}}$. The quotient-dense sieve of monomorphisms $\mathsf{Y}A \to [IY,IZ]$ clearly factors through $\rho_{Y,Z}$; thus $\rho_{Y,Z}$ is an isomorphism if $\rho_{Y,Z}$ is monomorphic in **S**.

The hull **C** of 4.7 clearly is the smallest cartesian closed quotient-topological completion **C** of **A** for which the full embedding $I: \mathbf{C} \to \mathbf{A}^{\mathrm{cd}}$ preserves function space objects. By (iv), this is also the cartesian closed quotient-topological hull of **A** if the forgetful functor of $\mathbf{A}^{\mathrm{cd}}$ preserves monomorphisms.

4.9. Remarks. It is easily seen that the objects of the quotient-topological hull of the class of function spaces $[\mathsf{Y}B,\mathsf{Y}C]$ are the quotient-dense P-sieves which are power-closed in the sense of [12]; thus 2.3 becomes Theorem 1.13 of [12] for this case.

If **S** is the category of sets and **A** has constant maps (see 2.10) then $\mathbf{A}^{\mathrm{cd}}$ for the (epsisink, injective mapping) factorization structure of sets is topological, with a single structure for every singleton, and the underlying set of a function space object $[X,Y]$ in $\mathbf{A}^{\mathrm{cd}}$ is the set $\mathbf{A}^{\mathrm{cd}}(X,Y)$. If we remove the initial object $(\emptyset,\Phi)$ with Φ empty from $\mathbf{A}^{\mathrm{cd}}$, we obtain a cartesian closed quotient-topological extension of **A** which is a topological category over sets with constant maps. Thus the quotient-topological cartesian closed hull of **A** is topological over sets with constant maps.

Similar remarks apply to 5.9 and to quotient-dense quasitopos hulls.

5. Representing partial morphisms in completions

5.1. Basic definitions and results. We recall that a (strong) *partial morphism* in a category $\mathbf{C}$ is a span $A \overset{m}{\longleftarrow} \cdot \overset{f}{\longrightarrow} B$ in $\mathbf{C}$ with m a strong monomorphism. We call partial morphisms (m,f) and (m',f') *equivalent* (symbol: $\simeq$) if $m' = mu$ and $f' = fu$ for an isomorphism u of $\mathbf{C}$. For an object B of $\mathbf{C}$, we say that a partial morphism $(t,u) : \tilde{B} \to B$ *represents* partial morphisms with codomain B if diagrams

$$\begin{array}{ccccc} \cdot & \xrightarrow{g'} & \cdot & \xrightarrow{u} & B \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle t} & & \\ A & \xrightarrow{g} & \tilde{B} & & \end{array} ,$$

with $f = ug'$ and a pullback square at left, define a bijection between morphisms $g : A \to \tilde{B}$ and equivalence classes of partial morphisms $(m,f) : A \to B$ of $\mathbf{C}$, for every object A of $\mathbf{C}$. It follows easily ([32] 16.3) that u is an isomorphism, and we say that the strong monomorphism $\vartheta_B = tu^{-1} : B \to \tilde{B}$ *represents* partial morphisms with codomain B. We say that $\mathbf{C}$ has *representable partial morphisms* if partial morphisms with codomain B are represented for every object B of $\mathbf{C}$.

Categories with representable partial morphisms have been called *hereditary* [19] or *extensionable* [29]; we note that "hereditary" often has another meaning. We also note that $\mathbf{C}$ has all possible pullbacks of strong monomorphisms if $\mathbf{C}$ has representable partial morphisms. In the context of the present paper, we have the following basic result.

Proposition (J. PENON). *Partial morphisms in* $\mathbf{A}^{\mathrm{cr}}$ *are represented if and only if partial morphisms in* $\mathbf{S}$ *are represented, and then the forgetful functor* $\mathbf{A}^{\mathrm{cr}} \to \mathbf{S}$ *preserves representiations of partial morphisms.*

PROOF. If $\tau_E : E \to E^{\#}$ represents partial morphisms in $\mathbf{S}$, then $\tau_E : (E,\Phi) \to (E^{\#},\Phi^{\#})$ represents partial morphisms in $\mathbf{A}^{\mathrm{cr}}$ if $\Phi^{\#}$ consists of all pairs $(A,\bar{u})$ with $u : m^*\mathrm{Y}A \to (E,\Phi)$ in $\mathbf{A}^{\mathrm{cr}}$, *i.e.* if $(X,uv) \in \Phi$ for every $v : PX \to F$ in $\mathbf{S}$ with $mv : X \to A$ in $\mathbf{A}$, for a pullback

$$\begin{array}{ccc} F & \xrightarrow{u} & E \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \tau_E} \\ PA & \xrightarrow{\bar{u}} & E^{\#} \end{array}$$

in $\mathbf{S}$; see [32] 60.3. Conversely, if $\tau_E : (E,\Omega_E) \to (E^{\#},\bar{\Phi})$ represents partial morphisms in $\mathbf{A}^{\mathrm{cr}}$, then it is easily seen that $\bar{\Phi} = \Omega_{E^{\#}}$, and that $\tau_E : E \to E^{\#}$ represents partial morphisms in $\mathbf{S}$.

5.2. Embeddings. We recall that an *embedding* in a concrete category $\mathbf{C}$ over $\mathbf{S}$ is an initial lift $m : X \to Y$ of a strong monomorphism $m : |X| \to |Y|$ of $\mathbf{S}$, and we say that

$\mathbf{C}$ *has embeddings* if every strong monomorphism $m : E \to |Y|$ of $\mathbf{S}$, with Y an object of $\mathbf{C}$, has an initial lift $m : X \to Y$ in $\mathbf{C}$.

If the forgetful functor $\mathbf{C} \to \mathbf{S}$ preserves epimorphisms, then embeddings in $\mathbf{C}$ are strong monomorphisms. Conversely, if $\mathbf{S}$ has (epi, strong mono) factorizations and $\mathbf{C}$ has embeddings, then all strong monomorphisms in $\mathbf{C}$ are embeddings, and pullbacks

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle m} \\ X' & \xrightarrow{f} & X \end{array}$$

in $\mathbf{C}$ with $m : Y \to X$ an embedding are concrete, *i.e.* lifted from pullbacks in $\mathbf{S}$, with $m' : Y' \to X'$ an embedding.

Hausdorff spaces are an example of a concrete category over sets with embeddings, but with epimorphisms not preserved by the forgetful functor. In this example, only closed embeddings are strong monomorphisms.

We note that $\mathbf{A}^{\mathrm{cr}}$ always has embeddings, and that $\mathbf{A}^{\mathrm{cd}}$ has embeddings iff X is $\mathbf{E}$-dense for every embedding $m : X \to Y$ in $\mathbf{A}^{\mathrm{cr}}$ with Y $\mathbf{E}$-dense. We say that $\mathbf{A}^{\mathrm{cd}}$ is *closed under embeddings* in $\mathbf{A}^{\mathrm{cr}}$ if this is the case.

5.3. Standard assumptions. We assume in this Section that $\mathbf{S}$ has representable partial morphisms, and that the domain X of every embedding $m : X \to \mathsf{Y}A$ in $\mathbf{A}^{\mathrm{cr}}$ is $\mathbf{E}$-dense. The second assumption is satisfied in particular if $\mathbf{A}$ has embeddings. We must also assume that pullback functors m^*, for strong monomorphisms m in $\mathbf{S}$, preserve $\mathbf{E}$, or equivalently that the right adjoints of these functors preserve $\mathcal{M}$. This last assumption is always satisfied for the three main examples: all sinks, episinks and quotient sinks.

Theorem. *If the assumptions stated above are satisfied, then partial morphisms in* $\mathbf{A}^{\mathrm{cd}}$ *are represented,* $\mathbf{A}^{\mathrm{cd}}$ *is closed under embeddings in* $\mathbf{A}^{\mathrm{cr}}$*, and if* $\tau_Y : Y \to Y^{\#}$ *and* $\vartheta_Y : Y \to \tilde{Y}$ *represent partial morphisms in* $\mathbf{A}^{\mathrm{cr}}$ *and in* $\mathbf{A}^{\mathrm{cd}}$*, for an object* Y *of* $\mathbf{A}^{\mathrm{cd}}$*, then* $\tau_Y = \nu_Y \vartheta_Y$ *with* ν_Y *a coreflection for* $Y^{\#}$ *in* $\mathbf{A}^{\mathrm{cd}}$.

Proof. If $m : (E', m^*\Phi) \to (E, \Phi)$ is an embedding, then consider pullbacks

$$\begin{array}{ccc} X & \xrightarrow{u'} & (E', m^*\Phi) \\ \downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle m} \\ \mathsf{Y}A & \xrightarrow{u} & (E, \Phi) \end{array}$$

in $\mathbf{A}^{\mathrm{cr}}$ with $(A, u) \in \Phi$. The morphisms u in these pullback squares form an $\mathbf{E}$-dense sink if (E, Φ) is $\mathbf{E}$-dense, and so do their pullbacks u' by the strong monomorphism m. The morphisms $m' : X \to \mathsf{Y}A$ in the pullback squares are embeddings, with each object X $\mathbf{E}$-dense by assumption. But then morphisms $u'v : \mathsf{Y}A' \to m^*B$ with $v : \mathsf{Y}A' \to X$ in $\mathbf{A}^{\mathrm{cd}}$ form an $\mathbf{E}$-dense sink, and $(E', m^*\Phi)$ is $\mathbf{E}$-dense.

Now let $\tau_Y : Y \to Y^{\#}$ represent partial morphisms in $\mathbf{A}^{\mathrm{cr}}$, with coreflection $\nu_Y : \tilde{Y} \to Y^{\#}$ for $\mathbf{A}^{\mathrm{cd}}$. Every morphism $X \to Y^{\#}$ with X an object of $\mathbf{A}^{\mathrm{cd}}$ factors uniquely through ν_Y. In particular, τ_Y factors $\tau_Y = \nu_Y \vartheta_Y$, with ϑ_Y an embedding since τ_Y is one. If $\mathbf{A}^{\mathrm{cd}}$ is closed under embeddings in $\mathbf{A}^{\mathrm{cr}}$, then partial morphisms $X \xleftarrow{m} \cdot \xrightarrow{f} Y$ in $\mathbf{A}^{\mathrm{cd}}$ are partial morphisms in $\mathbf{A}^{\mathrm{cr}}$ with domain and codomain in $\mathbf{A}^{\mathrm{cd}}$. Now in a diagram

$$\begin{array}{ccccc} \cdot & \xrightarrow{f} & Y & \xrightarrow{\mathrm{id}_Y} & Y \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \vartheta_Y} & & \downarrow{\scriptstyle \tau_Y} \\ X & \xrightarrow{\bar{f}} & \tilde{Y} & \xrightarrow{\nu_Y} & Y^{\#} \end{array} ,$$

the righthand square is a pullback; thus the lefthand square is a pullback iff the outer rectangle is one. The morphism $X \to Y^{\#}$ in the diagram determines $\bar{f}$ uniquely; it follows that ϑ_Y represents partial morphisms in $\mathbf{A}^{\mathrm{cd}}$ if τ_Y represents partial morphisms in $\mathbf{A}^{\mathrm{cr}}$.

5.4. Corollary. *If the standard assumptions of 5.3 are satisfied, then every* $\mathbf{E}$*-cotopological completion* $(\mathbf{B}, Q)$ *of* $(\mathbf{A}, P)$ *has embeddings.*

PROOF. Since $\mathbf{A}^{\mathrm{cd}}$ has embeddings under the assumptions of 5.3, this follows immediately from the fact, proved in 2.3 and 2.4, that the full concrete embedding $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cd}}$ preserves and creates initial lifts for sources.

5.5. Theorem. *If the standard assumptions of 5.3 are satisfied, then the following conditions are equivalent for an* $\mathbf{E}$*-cotopological completion* $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ *of* $(\mathbf{A}, P)$.

(i) *Partial morphisms in* $\mathbf{B}$ *are represented, with representations preserved up to isomorphism by the full concrete embedding* $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cd}}$ *with* $IG = \mathsf{Y}$.

(ii) *Every pullback functor* m^* *by an embedding* m *in* $\mathbf{B}$ *preserves final lifts of* $\mathbf{E}$*-dense sinks.*

(iii) *The concrete left adjoint* $J : \mathbf{A}^{\mathrm{cd}} \to \mathbf{B}$ *of the full concrete embedding functor* $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cd}}$ *preserves embeddings.*

(iv) *For the concrete left adjoint* $J : \mathbf{A}^{\mathrm{cd}} \to \mathbf{B}$ *of the full concrete embedding functor* $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cd}}$ *with* $IG = \mathsf{Y}$, *and for every object* B *of* $\mathbf{B}$, *the morphism* $\vartheta_{IB} : B \to J\widetilde{IB}$ *of* $\mathbf{B}$ *is an embedding.*

PROOF. For (i)$\Longrightarrow$(ii), consider pullback squares

$$\begin{array}{ccc} C_i & \xrightarrow{v_i} & C \\ \downarrow{\scriptstyle m_i} & & \downarrow{\scriptstyle m} \\ B_i & \xrightarrow{u_i} & B \end{array} ,$$

for an $\mathbf{E}$-dense sink of morphisms u_i with final lift B and an embedding $m : C \to B$ in $\mathbf{A}^{\mathrm{cd}}$. These pullbacks lift pullbacks in $\mathbf{S}$, and the morphisms v_i form an $\mathbf{E}$-dense sink by assumption. For $f : |C| \to |D|$ in $\mathbf{S}$ with $fv_i : C_i \to D$ in $\mathbf{B}$ for every v_i, pullback squares

$$\begin{array}{ccc} C_i & \xrightarrow{v_i f} & D \\ \downarrow{\scriptstyle m_i} & & \downarrow{\scriptstyle \vartheta_D} \\ B_i & \xrightarrow{h_i} & \tilde{D} \end{array} \quad \text{and} \quad \begin{array}{ccc} |C| & \xrightarrow{f} & |D| \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \tau_D} \\ |B| & \xrightarrow{g} & |D^{\#}| \end{array}$$

determine morphisms $h_i : B_i \to \tilde{D}$ in $\mathbf{B}$ and $g : |B| \to |D^{\#}|$ in $\mathbf{S}$ uniquely, with $\nu_D h_i = gu_i$ for each u_i. It follows that g factors $g = \nu_D t$ in $\mathbf{S}$, with $h_i = tu_i$ for each u_i. But then $t : B \to \tilde{D}$ in $\mathbf{A}^{\mathrm{cd}}$. As $tm = \vartheta_D f$ and ϑ_D is an embedding, we have $f : C \to D$ in $\mathbf{A}^{\mathrm{cd}}$. Thus C has the final structure for the $\mathbf{E}$-dense sink of morphisms v_i.

Now let $m : Y \to X$ be an embedding in $\mathbf{A}^{\mathrm{cd}}$, with X the final lift in $\mathbf{A}^{\mathrm{cd}}$ for an $\mathbf{E}$-dense sink of morphisms $u_i : IGA_i \to X$ in $\mathbf{A}^{\mathrm{cd}}$. Then JX is the final lift in $\mathbf{B}$ for the $\mathbf{E}$-dense sink of morphisms $u_i : GA_i \to JX$ in $\mathbf{B}$. Pulling back the u_i by m, we get pullback squares

$$\begin{array}{ccc} IB_i & \xrightarrow{v_i} & Y \\ \downarrow{\scriptstyle m_i} & & \downarrow{\scriptstyle m} \\ \mathsf{Y}A_i & \xrightarrow{u_i} & X \end{array}$$

in $\mathbf{A}^{\mathrm{cd}}$, with an $\mathbf{E}$-dense sink of morphisms $v_i : IB_i \to Y$ in $\mathbf{A}^{\mathrm{cd}}$, by (ii) for $\mathbf{A}^{\mathrm{cd}}$ and 5.4. Now J preserves final lifts of $\mathbf{E}$-dense sinks; thus JX and JY are the final lifts in $\mathbf{B}$ for $\mathbf{E}$-dense sinks $u_i : GA_i \to JX$ and $v_i : B_i \to JY$. If Z is the initial lift in $\mathbf{B}$ for m and JX, then Z is the final lift for the $\mathbf{E}$-dense sink of morphisms $v_i : B_i \to Z$ in $\mathbf{B}$ if (ii) is valid for $\mathbf{B}$. But then $Z = JY$, and $\mathbf{B}$ satisfies (iii).

(iii) $\Longrightarrow$ (iv) trivially, since $JIB = B$. If (iv) is valid, then we have pullback squares

$$\begin{array}{ccc} IB & \xrightarrow{\mathrm{id}_{IB}} & IB \\ \downarrow{\scriptstyle \vartheta_{IB}} & & \downarrow{\scriptstyle \vartheta_{IB}} \\ IJ\widetilde{IB} & \xrightarrow{h} & \widetilde{IB} \end{array} \quad \text{and} \quad \begin{array}{ccc} QB & \xrightarrow{\mathrm{id}_{QB}} & QB \\ \downarrow{\scriptstyle \vartheta_{IB}} & & \downarrow{\scriptstyle \tau_{QB}} \\ |\widetilde{IB}| & \xrightarrow{\nu_{IB} h} & (QB)^{\#} \end{array},$$

in $\mathbf{A}^{\mathrm{cd}}$ and in $\mathbf{S}$. But then $\nu_{IB} h = \nu_{IB}$, and $h = \mathrm{id}_{|X|}$ follows for $X = \widetilde{IB}$. Thus $\mathrm{id}_{|X|} : IJX \to X$ for this X. As $\mathrm{id}_{|X|} : X \to IJX$ in any case, we get $IJ\widetilde{IB} = \widetilde{IB}$. Since I preserves embeddings and pullbacks by embeddings, it follows that (i) is valid, with partial morphisms in $\mathbf{B}$ represented by $\vartheta_{IB} : B \to J\widetilde{IB}$.

5.6. For a source of morphisms $u_i : X \to X_i$ in $\mathbf{A}^{\mathrm{cd}}$, we have pullback squares

$$(1) \qquad \begin{array}{ccccc} X & \xrightarrow{\vartheta_X} & \tilde{X} & \xrightarrow{\nu_X} & X^{\#} \\ \downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle \tilde{u}_i} & & \downarrow{\scriptstyle u_i^{\#}} \\ X_i & \xrightarrow{\vartheta_{X_i}} & \widetilde{X_i} & \xrightarrow{\nu_{X_i}} & (X_i)^{\#} \end{array}$$

in $\mathbf{A}^{\mathrm{cr}}$ if partial morphisms in $\mathbf{A}^{\mathrm{cd}}$ are represented.

Lemma. *If X has the initial structure for the morphisms u_i in pullback squares* (1), *then $\tilde{X}$ has the initial structure for the morphisms $\bar{u}_i$.*

Proof. Consider $f : |Y| \to |\tilde{X}|$, with $\bar{u}_i f : Y \to \widetilde{X_i}$ for every u_i. We have pullback squares

$$\begin{array}{ccccc} \cdot & \xrightarrow{g} & |X| & \xrightarrow{\mathrm{id}_{|X|}} & |X| \\ \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle \vartheta_X} & & \big\downarrow{\scriptstyle \tau_X} \\ |Y| & \xrightarrow{f} & |\tilde{X}| & \xrightarrow{\nu_X} & |X|^{\#} \end{array}$$

in $\mathbf{S}$, and it follows with (1) that we have pullbacks

$$\begin{array}{ccc} Z & \xrightarrow{g u_i} & X_i \\ \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle \vartheta_{X_i}} \\ Y & \xrightarrow{f \bar{u}_i} & \widetilde{X_i} \end{array}$$

in $\mathbf{A}^{\mathrm{cd}}$ for every u_i, with $m : Z \to Y$ an embedding. But then $g : Z \to X$ in $\mathbf{A}^{\mathrm{cd}}$, and it follows that $f : Y \to \tilde{X}$ in $\mathbf{A}^{\mathrm{cd}}$, corresponding to the partial morphism $Y \xleftarrow{m} Z \xrightarrow{g} X$.

5.7. Lemma ([1]). *If the standard assumptions of* 5.3 *are satisfied, then for every object Y of $\mathbf{A}^{\mathrm{cd}}$, the object $\tilde{\tilde{Y}}$ of $\mathbf{A}^{\mathrm{cd}}$ has the initial structure for a morphism $z_Y : \tilde{\tilde{Y}} \to \tilde{Y}$ of $\mathbf{A}^{\mathrm{cd}}$, with $z_Y \vartheta_{\tilde{Y}} = \mathrm{id}_{\tilde{Y}}$.*

Proof. The morphism z_Y is constructed by the pullback square at left in (1).

$$(1) \qquad \begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ \big\downarrow{\scriptstyle \vartheta_{\tilde{Y}}\vartheta_Y} & & \big\downarrow{\scriptstyle \vartheta_Y} \\ \tilde{\tilde{Y}} & \xrightarrow{z_Y} & \tilde{Y} \end{array} \qquad \begin{array}{ccccc} Y & \xrightarrow{\mathrm{id}_Y} & Y & \xrightarrow{\mathrm{id}_Y} & Y \\ \big\downarrow{\scriptstyle \vartheta_Y} & & \big\downarrow{\scriptstyle \vartheta_{\tilde{Y}}\vartheta_Y} & & \big\downarrow{\scriptstyle \vartheta_Y} \\ \tilde{Y} & \xrightarrow{\vartheta_{\tilde{Y}}} & \tilde{\tilde{Y}} & \xrightarrow{z_Y} & \tilde{Y} \end{array}.$$

The pullbacks at right show that then $\vartheta_{\tilde{Y}} z_Y = \mathrm{id}_{\tilde{\tilde{Y}}}$.

Now let Z be the initial lift of $|\tilde{\tilde{Y}}|$ for z_Y with codomain $\tilde{Y}$. Then $\vartheta_{\tilde{Y}} : \tilde{Y} \to Z$ is a strong monomorphism in $\mathbf{A}^{\mathrm{cd}}$ since $z_Y \vartheta_{\tilde{Y}} = \mathrm{id}_{\tilde{Y}}$. The partial morphism $(\vartheta_{\tilde{Y}}, \mathrm{id}_{\tilde{Y}})$ is represented in $\mathbf{S}$ by $\nu_{\tilde{Y}} : |Z| \to |\tilde{Y}|^{\#}$, and thus $(\vartheta_{\tilde{Y}}, \mathrm{id}_{\tilde{Y}}) : Z \to \tilde{Y}$ is represented in $\mathbf{A}^{\mathrm{cd}}$ by $\mathrm{id} : Z \to \tilde{\tilde{Y}}$. It follows that $Z = \tilde{\tilde{Y}}$, as claimed.

5.8. Theorem. *Under the standard assumptions of* 5.3, *the $\mathbf{E}$-cotopological hull of all objects $\widetilde{\mathsf{Y}A}$ in $\mathbf{A}^{\mathrm{cd}}$ — which includes all objects $\mathsf{Y}A$ — has partial morphisms represented.*

We note that this $\mathbf{E}$-cotopological hull is the smallest $\mathbf{E}$-cotopological completion $G : (\mathbf{A}, P) \to (\mathbf{B}, Q)$ of $\mathbf{A}$ with representations of partial morphisms for which the full embedding $I : \mathbf{B} \to \mathbf{A}^{\mathrm{cd}}$ with $IG = \mathsf{Y}$ preserves these representations.

PROOF. Let $\mathbf{C}$ be this $\mathbf{E}$-cotopological hull. If X has the initial structure in $\mathbf{A}^{\mathrm{cd}}$ for morphisms $u_i : X \to \widetilde{YA_i}$, then by 5.6 and 5.7, the object $\tilde{X}$ has the initial structure for the morphisms $z_{Y_i}\bar{u}_i$, with $Y_i = \mathrm{Y}A_i$ and with $\bar{u}_i$ constructed in 5.6. Thus $\mathbf{C}$ has partial morphisms represented, with representations inherited from $\mathbf{A}^{\mathrm{cd}}$.

5.9. The quotient-topological case. We note two additional results for this case.

Theorem. *Let* $\mathbf{S}$ *be a (quotient sink, mono)-category with partial morphisms represented. If the forgetful functor from* $\mathbf{A}^{\mathrm{cd}}$ *to* $\mathbf{S}$ *preserves monomorphisms, then the four conditions of* 4.2 *are equivalent to:*

(v) *Partial morphisms in* $\mathbf{B}$ *are represented,*

and the hull defined in 5.8 *is the quotient-topological hull of* $\mathbf{A}$ *for completions with partial morphisms represented.*

PROOF. 4.4.(i)$\Longrightarrow$(v) trivially. Conversely, if partial morphisms in $\mathbf{B}$ and in $\mathbf{A}^{\mathrm{cd}}$ are represented by $\tau_Y : Y \to Y^*$ and by $\vartheta_Y : Y \to \tilde{Y}$, then there are commutative squares

$$\begin{array}{ccc} IY & \xrightarrow{\mathrm{id}_Y} & IY \\ \downarrow{\scriptstyle \tau_Y} & & \downarrow{\scriptstyle \vartheta_{IY}} \\ IY^* & \xrightarrow{\sigma_Y} & \widetilde{IY} \end{array}$$

in $\mathbf{A}^{\mathrm{cd}}$. These squares are pullback squares; thus if a partial morphism $(m, f) : X \to Y$ is represented by $\bar{f} : X \to Y^*$ in $\mathbf{B}$, then $(m, f) : IX \to IY$ is represented by $\sigma_Y \bar{f}$ in $\mathbf{A}^{\mathrm{cd}}$. Now if $\sigma_Y \alpha = \sigma_Y \beta$ in $\mathbf{A}^{\mathrm{cd}}$ for morphisms $X \to IY$, then this remains true for $\alpha, \beta : IJX \to IY$. It follows that α and β represent the same partial morphism $(m, f) : JX \to Y$; thus σ_Y is monomorphic in $\mathbf{A}^{\mathrm{cd}}$. If σ_Y is monomorphic in $\mathbf{S}$, then it follows as in the proof of 4.8 that σ_Y is an isomorphism.

The second part follows immediately from the first part and the definitions.

6. Completions and hulls over a quasitopos

6.1. Results for quasitopos completions and quasitopos hulls are obtained by combining results of Sections 4 and 5. If $\mathbf{S}$ is a quasitopos satisfying the assumptions of 4.2 and 5.3, then every pullback functor f^* in $\mathbf{S}$ preserves $\mathbf{E}$, and its right adjoint preserves $\mathcal{M}$, because f factors $f = pm$ with m a strong monomorphism and p a projection of a product.

We prove three results of this kind which also exhibit the "side effects" of combining the assumptions of 4.2 and 5.3, and we append a useful result which does not quite fit into the scheme of this paper.

6.2. Theorem. *If* $\mathbf{S}$ *is a quasitopos satisfying the standard assumptions of* 4.2 *and* 5.3, *then* $\mathbf{A}^{\mathrm{cd}}$ *is a quasitopos, and the embedding* $\mathbf{A}^{\mathrm{cd}} \to \mathbf{A}^{\mathrm{cr}}$ *is the inverse image part* *of a geometric morphism. In this situation, the forgetful functor* $P^{cd} : \mathbf{A}^{cd} \to \mathbf{S}$ *preserves monomorphisms, strong monomorphisms, and finite limits.*

PROOF. Under the assumptions of the Theorem, $\mathbf{A}^{cd}$ is cartesian closed, and closed under finite products in $\mathbf{A}^{cr}$, by 4.2, and $\mathbf{A}^{cd}$ has partial morphisms represented and is closed under embeddings in $\mathbf{A}^{cr}$ by 5.3. Since finite limits in $\mathbf{A}^{cr}$ are strong subobjects of finite products, it follows that $\mathbf{A}^{cd}$ is closed under finite limits in $\mathbf{A}^{cr}$, and the embedding $\mathbf{A}^{cd} \to \mathbf{A}^{cr}$ preserves finite limits. It follows that the forgetful functor $\mathbf{A}^{cd} \to \mathbf{S}$ also preserves finite limits, and hence monomorphisms. Since $\mathbf{A}^{cd}$ has embeddings, this forgetful functor also preserves strong monomorphisms.

6.3. Theorem. *Under the standard assumptions of* 4.2 *and* 5.3, *the following conditions are equivalent for an* $\mathbf{E}$*-cotopological completion* $G : (\mathbf{B}, Q) \to (\mathbf{A}, P)$.

(i) $\mathbf{B}$ *is a quasitopos, and the full embedding* $I : \mathbf{B} \to \mathbf{A}^{cd}$ *with* $IG = \mathrm{Y}$ *preserves function space objects and representations of partial morphisms.*

(ii) *Every pullback functor* f^* *in* $\mathbf{B}$ *preserves final lifts of sinks in* $\mathbf{E}$.

(iii) *The concrete left adjoint* $J : \mathbf{A}^{cd} \to \mathbf{B}$ *of the full embedding* $I : \mathbf{B} \to \mathbf{A}^{cd}$ *preserves embeddings and finite products.*

If these conditions are satisfied, then the adjunction $J \dashv I$ *is a geometric morphism, and the functor* I *preserves function space objects and representations of partial morphisms, up to isomorphisms.*

PROOF. With the observation that the forgetful functor of $\mathbf{A}^{cd}$ preserves monomorphisms, this follows immediately from 4.4 and 5.5, using for (ii) the fact that every morphism of $\mathbf{A}^{cd}$ is the composition of an embedding and a projection of a product.

6.4. Theorem. *Under the standard assumptions of* 4.2 *and* 5.3, *the* $\mathbf{E}$*-cotopological hull of the objects* $[\mathrm{Y}A, \widetilde{\mathrm{Y}B}]$ *in* $\mathbf{A}^{cd}$, *for objects* A *and* B *of* $\mathbf{A}$, *is the smallest* $\mathbf{E}$*-cotopological completion of* $\mathbf{A}$ *which satisfies the conditions of* 6.3.

PROOF. Let $\mathbf{C}$ be the $\mathbf{E}$-cotopological hull described. Then $\mathbf{C}$ can be fully embedded into every $\mathbf{E}$-cotopological completion $\mathbf{B}$ of $\mathbf{A}$ satisfying the conditions of 6.3.

Since $\vartheta_B : B \to \tilde{B}$ is always an embedding, $\mathbf{C}$ contains all function space objects $[\mathrm{Y}A, \mathrm{Y}B]$, and hence all objects $\mathrm{Y}A$ of $\mathbf{A}^{cd}$. As in the proof of 4.7, it follows from 4.5 and 4.6 that $\mathbf{C}$ contains all objects $[X, \mathrm{Y}B]$ and $[X, \widetilde{\mathrm{Y}B}]$, and is cartesian closed.

Now put $T = [X, \tilde{Y}]$ for objects X, Y of $\mathbf{A}^{cd}$. By [10], we have a commutative diagram

$$(1) \qquad \begin{array}{ccccc} T \times X & \xrightarrow{\vartheta_T \times \mathrm{id}_X} & \tilde{T} \times X & \xrightarrow{k \times \mathrm{id}_X} & T \times X \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{ev}} \\ \tilde{Y} & \xrightarrow{\vartheta_{\tilde{Y}}} & \tilde{\tilde{Y}} & \xrightarrow{z_Y} & \tilde{Y} \end{array},$$

with h representing the partial morphism $(\vartheta_T \times \mathrm{id}_Y, \mathrm{ev})$, with z_Y given by 5.7, and with k exponentially adjoint to $z_Y h$. It follows from this and $z_Y \vartheta_{\tilde{Y}} = \mathrm{id}_{\tilde{Y}}$ that $k \vartheta_T = \mathrm{id}_T$. Now

$\tilde{T}$ has the initial structure for k, as in the proof of 5.7, and it follows as in the proof of 5.8 that partial morphisms in $\mathbf{C}$ are represented. Thus $\mathbf{C}$ is a quasitopos, and we are done.

6.5. Remarks. It may be noted that diagram 6.4.(1) can be constructed for objects X and Y of an arbitrary quasitopos, with the additional property that if h factors $h = h'e$, with e monomorphic and epimorphic, then e is an isomorphism. This can be used for proving that $\tilde{T}$ has the initial structure in $\mathbf{A}^{\mathrm{cd}}$ for h, and similarly that $\tilde{\tilde{Y}}$ in 5.7 has the initial structure for z_Y.

We also note that in the quotient-topological case, condition 6.3.(i) can be replaced by:

(iv) $\mathbf{B}$ *is a quasitopos,*

and the completion of 6.4 becomes the quotient-topological quasitopos hull of $\mathbf{A}$.

6.6. Lifting finite sources. It is well known, and easily seen, that the forgetful functor $\mathbf{A}^{\mathrm{cr}} \to \mathbf{S}$ preserves and creates limits. Limits in $\mathbf{A}^{\mathrm{cd}}$ are coreflections of limits in $\mathbf{A}^{\mathrm{cr}}$; it follows that a limit in $\mathbf{A}^{\mathrm{cd}}$ is preserved by the forgetful functor to $\mathbf{S}$ iff it is preserved by the embedding $\mathbf{A}^{\mathrm{cd}} \to \mathbf{A}^{\mathrm{cr}}$, and hence iff it is concrete.

Theorem. *If* $\mathbf{S}$ *has finite limits, then the following statements are logically equivalent for the forgetful functor* $P^{\mathrm{cd}} : \mathbf{A}^{\mathrm{cd}} \to \mathbf{S}$.

(i) *Every finite* P^{cd}*-source admits a lift in* $\mathbf{A}^{\mathrm{cd}}$.

(ii) *Every finite* P^{cd}*-source admits an initial lift in* $\mathbf{A}^{\mathrm{cd}}$.

(iii) $\mathbf{A}^{\mathrm{cd}}$ *has concrete finite products and embeddings, and full* P*-sieves are* $\mathbf{E}$*-dense.*

(iv) $\mathbf{A}^{\mathrm{cd}}$ *has concrete finite limits, and full* P*-sieves are* $\mathbf{E}$*-dense.*

(v) $\mathbf{A}^{\mathrm{cd}}$ *is closed under finite limits in* $\mathbf{A}^{\mathrm{cr}}$, *and full* P*-sieves are* $\mathbf{E}$*-dense.*

If these conditions are satisfied, then every $\mathbf{E}$*-cotopological completion* $\mathbf{B}$ *of* $\mathbf{A}$ *has concrete finite limits, and all monomorphisms of* $\mathbf{B}$ *are concrete.*

PROOF. (ii)$\Longrightarrow$(i) trivially, and if (i) is valid, then for finite P^{cd}-sources, initial lifts in $\mathbf{A}^{\mathrm{cr}}$ are quotient dense, and (i) holds. (ii)$\Longrightarrow$(iii) since concrete products, embeddings and full sieves in $\mathbf{A}^{\mathrm{cd}}$ are initial lifts of finite sources. Clearly (iv) $\Longleftrightarrow$ (v), and (iii)$\Longrightarrow$(iv) and (v) since concrete finite limits in $\mathbf{A}^{\mathrm{cr}}$ are objects embedded into finite products.

For a P^{cd}-source of morphisms $f_i : S \to |X_i|$, an initial lift in $\mathbf{A}^{\mathrm{cr}}$ is a limit of a diagram of morphism $f_i : \Omega_S \to \Omega_{|X_i|}$ and of morphisms $\mathrm{id}_{|X_i|} : X_i \to \Omega_{|X_i|}$. With this observation, (ii) follows immediately from (iv) and (v).

With the results of Section 2, the last part of the Theorem follows easily.

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