LINEARLY INDUCED MAPPINGS BETWEEN CONES OF QUADRATIC FORMS

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CONTENTS

1. INTRODUCTION

Mappings between linear cones have been studied in [NS77a], [NS77b], [Sch78], [Sch79b], [Sch81], and [Sch79a]. The purpose of the present paper is twofold. First, we give proofs of two theorems that were announced without proof in [NS77b] (where they were designated as Theorems 1 and 3). These theorems deal with linear cones consisting of quadratic forms over finite-dimensional real-linear spaces. The first of these, called Theorem 1 here, states that there is an inclusion-reversing one-to-one correspondence beween the lattice of subspaces of a given space and the lattice of faces of the cone of quadratic forms on that space. The second, called Theorem 2 here, states that cone-isomorphisms between cones of quadratic forms are induced by linear isomorphisms between the underlying spaces (*i.e.*, the linear spaces which are the domains of the forms).

The second purpose of the paper is to show (Theorem 3 here) that a cone-linear mapping F from one cone of quadratic forms to another is induced by a linear mapping between the underlying spaces if and only if both F and its cone-transpose are face-preserving mappings.[1]

One of the reasons for studying the mappings described above is the possibility of generalizations to include the case when the cones of quadratic forms are replaced by

[1]The proof of Thm. 2 presented here was found by R.A; it is different and more direct that the one W.N. had in mind when [NS77a] was published. Thm. 3 was discovered recently by R.A.

Hermitian forms over complex-linear spaces, which arise in the study of quantum mechanics (see [Art78]). These genereralizations, as well generalizations which apply to forms over quaternionic-linear spaces, will be deferred to future papers.

The notation and terminology of [Nol87] is used in this paper. In particular, $\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{P}$ the set of all *positive* real numbers (both including zero). A superscript $^\times$ indicates the removal of zero; in particular $\mathbb{P}^\times$ denotes the set of all *strictly positive* real numbers. Given $n \in \mathbb{N}^\times$, we denote by $n^]$ the set consisting of the first n non-zero natural numbers. The collection of all subsets of a given set $\mathcal{S}$ is denoted by $\operatorname{Sub}\mathcal{S}$. Given a mapping ϕ and subsets $\mathcal{A}$ of its *domain* $\operatorname{Dom}\phi$ and $\mathcal{B}$ of its *codomain* $\operatorname{Cod}\phi$, we denote the *image* of $\mathcal{A}$ under ϕ by $\phi_>(\mathcal{A}) := \{\phi(x) \mid x \in \mathcal{A}\}$ and the *pre-image* of $\mathcal{B}$ under ϕ by $\phi^<(\mathcal{B}) := \{x \in \operatorname{Dom}\phi \mid \phi(x) \in \mathcal{B}\}$. If $\phi_>(\mathcal{A}) \subset B$, we define the *adjustment* $\phi|_{\mathcal{A}}^{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ of ϕ by

$$\phi|_{\mathcal{A}}^{\mathcal{B}}(x) := \phi(x) \qquad \text{for all } x \in \mathcal{A}.$$

If ϕ is bijective, we denote its *inverse* by $\phi^{\leftarrow} : \operatorname{Cod}\phi \to \operatorname{Dom}\phi$.

Let a set $\mathcal{A}$ and $\mathcal{S} \in \operatorname{Sub}\mathcal{A}$ be given. We denote by $\mathbf{1}_{\mathcal{S}\subset\mathcal{A}} \in \operatorname{Map}(\mathcal{S},\mathcal{A})$ the *inclusion mapping, i.e.*, the mapping which satisfies $\mathbf{1}_{\mathcal{S}\subset\mathcal{A}}(x) = x$ for all $x \in \mathcal{S}$. We abbreviate $\mathbf{1}_{\mathcal{A}} := \mathbf{1}_{\mathcal{A}\subset\mathcal{A}}$, so $\mathbf{1}_{\mathcal{A}}$ is the *identity* mapping on $\mathcal{A}$.

When we say "let a linear space be given" (or equivalent language), we mean "let a finite-dimensional real linear space be given". When dealing with linear spaces, we extensively use the notation, terminology, and results of Chapters 1 and 2 of [Nol87]. In particular, when a linear space $\mathcal{V}$ is given, we identify $\mathcal{V}^{**} \cong \mathcal{V}$.

Let a linear space $\mathcal{V}$ and a subset $\mathcal{S}$ of $\mathcal{V}$ be given. We note that the *closure*[2] of $\mathcal{S}$ remains unchanged if $\mathcal{V}$ is replaced by a subspace of $\mathcal{V}$ that includes $\mathcal{S}$. The *interior* of $\mathcal{S}$, however, depends not only on $\mathcal{S}$ but also on the imbedding space $\mathcal{V}$; we write $\operatorname{Int}_{\mathcal{V}}\mathcal{S}$ for the interior of $\mathcal{S}$ to make this dependence on $\mathcal{V}$ explicit (See [Nol87], Sect. 53).

The collection of all subspaces of $\mathcal{V}$ is denoted by $\operatorname{Subsp}\mathcal{V}$. For each $\mathcal{U} \in \operatorname{Subsp}\mathcal{V}$, we denote by $\Omega_{\mathcal{V}/\mathcal{U}} \in \operatorname{Lin}(\mathcal{V}, \mathcal{V}/\mathcal{U})$ the *quotient mapping*; , *i.e.*, the mapping which satisfies $\Omega_{\mathcal{V}/\mathcal{U}}\boldsymbol{v} := \boldsymbol{v} + \mathcal{U}$ for all $\boldsymbol{v} \in \mathcal{V}$.

Let a second linear space $\mathcal{W}$ and a (not necessarily linear) mapping $\phi : \mathcal{V} \to \mathcal{W}$ be given. We call $\operatorname{Null}\phi := \phi^<(\{\mathbf{0}\})$ the **nullset of** ϕ.[3] We record several elementary facts for later reference:

Proposition 1.1. *Let sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a surjective mapping $\alpha : \mathcal{C} \to \mathcal{A}$ be given. Then the mapping*

$$(\phi \mapsto \phi \circ \alpha) : \operatorname{Map}(\mathcal{A},\mathcal{B}) \to \operatorname{Map}(\mathcal{C},\mathcal{B}) \tag{1.1}$$

is injective.

Proposition 1.2. *Let $\mathcal{U} \in \operatorname{Subsp}\mathcal{V}$ be given. Then $\boldsymbol{\mu}\Omega_{\mathcal{V}/\mathcal{U}} \in \mathcal{U}^\perp$ for all $\boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*$ and*

$$(\boldsymbol{\mu} \mapsto \boldsymbol{\mu}\Omega_{\mathcal{V}/\mathcal{U}}) : (\mathcal{V}/\mathcal{U})^* \to \mathcal{U}^\perp \tag{1.2}$$

[2] All topological terms are to be understood in the context of the usual topologies for finite-dimensional spaces; see Chapter 5 of [Nol87].

[3] In [Nol87], this notation was used only for the case in which ϕ is a linear mapping; in that case $\operatorname{Null}\phi$ is a subspace and is called the null*space* of ϕ.

is a linear isomorphism.

Proposition 1.3. *Let L be a linear mapping. Then there is exactly one linear isomorphism* $\tilde{L}\colon \operatorname{Dom} L/\operatorname{Null} L \to \operatorname{Rng} L$ *such that*

$$L = \Omega_{\operatorname{Dom} L/\operatorname{Null} L}\,\tilde{L}\,\mathbf{1}_{\operatorname{Rng} L \subset \operatorname{Cod} L}\,. \tag{1.3}$$

Corollary 1. *Let L and L' be surjective linear mappings such that* $\operatorname{Dom} L = \operatorname{Dom} L'$ *and* $\operatorname{Null} L = \operatorname{Null} L'$. *Then there is exactly one linear isomorphism* $A\colon \operatorname{Cod} L \to \operatorname{Cod} L'$ *such that* $L' = AL$.

Corollary 2. *Let L and L' be injective linear mappings such that* $\operatorname{Rng} L = \operatorname{Rng} L'$. *Then there is exactly one linear isomorphism* $A\colon \operatorname{Dom} L \to \operatorname{Dom} L'$ *such that* $L = L'A$.

2. Linear cones

Throughout this section, we assume that a linear space $\mathcal{W}$ is given. A subset $\mathcal{P}$ of $\mathcal{W}$ is called a **linear cone** in $\mathcal{W}$ if it is stable under addition and under scalar multiplication by *strictly positive* real numbers, *i.e.*, if $\mathcal{P} + \mathcal{P} \subset \mathcal{P}$ and $\mathbb{P}^{\times}\mathcal{P} \subset \mathcal{P}$. Linear cones are convex sets. The interior and the closure of a given linear cone are again linear cones. The intersection of a collection of linear cones is again a linear cone. Hence, for every subset $\mathcal{S}$ of $\mathcal{W}$, there is exactly one smallest linear cone that includes $\mathcal{S}$; we call this linear cone the **cone-span** of $\mathcal{S}$ and denote it by $\operatorname{Csp}\mathcal{S}$ (see [Nol87], Sect. 03)

For the remainder of this section, we assume that a linear cone $\mathcal{P} \in \mathcal{W}$ is given.

If $\mathcal{P}$ is not empty, then its *linear span* is given by $\operatorname{Lsp}\mathcal{P} = \mathcal{P} - \mathcal{P}$. Thus $\mathcal{P}$ *spans* $\mathcal{W}$ if and only if $\mathcal{P} - \mathcal{P} = \mathcal{W}$. The **dimension** of a linear cone $\mathcal{P}$, denoted by $\dim\mathcal{P}$, is defined to be the dimension $\dim \operatorname{Lsp}\mathcal{P}$ of its linear span.

A linear cone that is included in $\mathcal{P}$ is called a **subcone** of $\mathcal{P}$. We say that a subcone $\mathcal{F}$ of $\mathcal{P}$ is a **face** of $\mathcal{P}$ if it includes $\mathcal{P} \cap \{\mathbf{0}\}$[4] and

$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{F} \implies \boldsymbol{u}, \boldsymbol{v} \in \mathcal{F} \qquad \text{for all } \boldsymbol{u},\, \boldsymbol{v} \in \mathcal{P}. \tag{2.1}$$

The intersection of a collection of faces of $\mathcal{P}$ is again a face of $\mathcal{P}$. Hence, for a given subset $\mathcal{S}$ of $\mathcal{P}$, there is exactly one smallest face of $\mathcal{P}$ that includes $\mathcal{S}$; we call this face the **facial span** of $\mathcal{S}$, and denote it by $\operatorname{Fcsp}\mathcal{S}$. Of course, $\mathcal{P}$ is a face of itself. A face of $\mathcal{P}$ which is a proper subset of $\mathcal{P}$ is called a **proper face** of $\mathcal{P}$. A one-dimensional face of $\mathcal{P}$ is called an **extreme ray** of $\mathcal{P}$. The concepts of face, facial span, and extreme ray do not depend on the linear space in which $\mathcal{P}$ is considered to be a linear cone.

Proposition 2.1. *Assume that $\mathcal{P}$ is not empty. Then $\mathcal{P}$ spans $\mathcal{W}$ if and only if* $\operatorname{Int}_{\mathcal{W}}\mathcal{P}$ *is non-empty.*

Proof. Since $\mathcal{P}$ is convex, this result is an immediate consequence of Prop. 6 of Sect. 54 of [Nol87]. □

Proposition 2.2. *For every* $\boldsymbol{u} \in \operatorname{Int}_{\mathcal{W}}\mathcal{P}$, *we have* $\operatorname{Fcsp}\{\boldsymbol{u}\} = \mathcal{P}$.

[4] This definition is revised relative to the one presented in [NS77b]; the first condition has been added to insure that a face of a cone with zero contains the zero and is thus non-empty. This revision is necessary for Thm. 1 of [NS77b] (as well as Thm. 1 of this paper) to be correct as stated.

Proof. Let $\boldsymbol{u} \in \operatorname{Int}_{\mathcal{W}} \mathcal{P}$ be given and put $\mathcal{F} := \operatorname{Fcsp}\{\boldsymbol{u}\}$. Let $\boldsymbol{v} \in \mathcal{P}$ be given. Since $\boldsymbol{u} \in \operatorname{Int}_{\mathcal{W}} \mathcal{P}$, we may and do choose $\epsilon \in \mathbb{P}^{\times}$ such that $\boldsymbol{w} := -\epsilon\boldsymbol{v} \in \mathcal{P}$ and hence $\boldsymbol{w} + \epsilon\boldsymbol{v} = \boldsymbol{u}$. Since $\boldsymbol{u} \in \mathcal{F}$ and since $\mathcal{F}$ is a face, it follows from (2.1) that $\epsilon\boldsymbol{v} \in \mathcal{F}$. Hence, since $\mathcal{F}$ is a linear cone, we have $\boldsymbol{v} = \frac{1}{\epsilon}(\epsilon\boldsymbol{v}) \in \mathcal{F}$. Since $\boldsymbol{v} \in \mathcal{P}$ was arbitrary, it follows that $\mathcal{F} = \mathcal{P}$. □

Proposition 2.3. *Every non-empty face of $\mathcal{P}$ is the facial span of a singleton.*

Proof. Let a non-empty face $\mathcal{F}$ of $\mathcal{P}$ be given and put $\mathcal{U} := \operatorname{Lsp}\mathcal{F}$. Then $\mathcal{F}$ is a linear cone that spans $\mathcal{U}$. Hence, by Prop. 2.1, the interior $\operatorname{Int}_{\mathcal{U}} \mathcal{F}$ of $\mathcal{F}$ relative to $\mathcal{U}$ is not empty. Choose $\boldsymbol{u} \in \operatorname{Int}_{\mathcal{U}} \mathcal{F}$. By Prop. 2.2 we have $\operatorname{Fcsp}\{\boldsymbol{u}\} = \mathcal{F}$. □

We say that a family $(\mathcal{F}_i \mid i \in I)$ of faces of $\mathcal{P}$ is **facially independent** if the facial span of its union properly includes the facial span of the union of each of its proper subfamilies, *i.e.*, if

$$(2.2)\qquad \operatorname{Fcsp}\left(\textstyle\bigcup_{j\in J}\mathcal{F}_j\right) = \operatorname{Fcsp}\left(\textstyle\bigcup_{i\in I}\mathcal{F}_i\right) \implies J = I \quad \text{for all } J \in \operatorname{Sub} I.$$

A facially-independent family of faces of $\mathcal{P}$ is called a **facial decomposition** of $\mathcal{P}$ if the facial span of its union is $\mathcal{P}$. We note that a family $(\mathcal{F}_i \mid i \in I)$ of faces of $\mathcal{P}$ is a facial decomposition of $\mathcal{P}$ if and only if

$$(2.3)\qquad \operatorname{Fcsp}\left(\textstyle\bigcup_{j\in J}\mathcal{F}_j\right) = \mathcal{P} \iff J = I \quad \text{for all } J \in \operatorname{Sub} I.$$

The **dual** of the linear cone $\mathcal{P}$ is defined by

$$(2.4)\qquad \mathcal{P}^{\star} := \{\boldsymbol{\lambda} \in \mathcal{W}^* \mid \boldsymbol{\lambda}_{>}(\mathcal{P}) \subset \mathbb{P}\}.$$

It is easily seen that $\mathcal{P}^{\star}$ is a closed linear cone in $\mathcal{W}^*$.

Now let, in addition to $\mathcal{W}$ and $\mathcal{P}$, a linear space $\mathcal{W}'$ and a linear cone $\mathcal{P}'$ be given. Also, let a linear mapping $Q\colon \mathcal{W} \to \mathcal{W}'$ be given. We say that Q is **cone-compatible** (relative to $\mathcal{P}$ and $\mathcal{P}'$) if $Q_{>}(\mathcal{P}) \subset \mathcal{P}'$ and **cone-preserving** if $Q_{>}(\mathcal{P}) = \mathcal{P}'$. Now let a mapping $P\colon \mathcal{P} \to \mathcal{P}'$ be given. We say that P is **cone-linear** if it preserves addition and scalar multiplication by strictly positive numbers. If P is also invertible, then its inverse is also cone-linear and P is called a **cone-isomorphism**. When dealing with cone-linear mappings, we adopt the same "multiplicative" notation used in [Nol87] for linear mappings. In particular, if P is cone linear, then, given $\boldsymbol{x} \in \mathcal{P}$, we abbreviate $P\boldsymbol{x} := P(\boldsymbol{x})$; if P is also invertible we denote its inverse by $P^{-1} := P^{\leftarrow}$. We denote by $\operatorname{Lin}(\mathcal{P}, \mathcal{P}')$ the set of all cone-linear mappings from $\mathcal{P}$ to $\mathcal{P}'$.[5] If $Q\colon \mathcal{W} \to \mathcal{W}'$ is a cone-compatible linear mapping, then $Q|_{\mathcal{P}}^{\mathcal{P}'}\colon \mathcal{P} \to \mathcal{P}'$ is cone-linear. Conversely, if $P\colon \mathcal{P} \to \mathcal{P}'$ is cone-linear, then there is a cone-compatible linear mapping $Q\colon \mathcal{W} \to \mathcal{W}'$ such that $P = Q|_{\mathcal{P}}^{\mathcal{P}'}$; moreover, if $\mathcal{P}$ spans $\mathcal{W}$ then Q is uniquely determined by P; also, if $\mathcal{P}$ and $\mathcal{P}'$ span $\mathcal{W}$ and $\mathcal{W}'$ respectively, and P is a cone-isomorphism, then Q is a cone-preserving linear isomorphism.

Suppose that $\mathcal{P}$ spans $\mathcal{W}$ and that P is cone-linear, and denote the (cone-compatible) linear extension of P by $Q\colon \mathcal{W} \to \mathcal{W}'$. It is easy to see that $Q^{\mathrm{T}}\colon \mathcal{W}'^* \to \mathcal{W}^*$ is cone-compatible relative to the dual cones $\mathcal{P}'^{\star}$ and $\mathcal{P}^{\star}$ (defined by (2.4)).

[5] We note that if $\mathcal{P}$ and $\mathcal{P}'$ equal their linear spans, then the set of cone-linear mappings from $\mathcal{P}$ to $\mathcal{P}'$ and the set of linear mappings from $\mathcal{P}$ to $\mathcal{P}'$ are one and the same. Thus $\operatorname{Lin}(\mathcal{P}, \mathcal{P}')$ is not ambiguous.

Thus we may and do define the **cone-transpose** of P by $P^{\top} := Q^{\top}|_{\mathcal{P}'^{\star}}^{\mathcal{P}^{\star}}$. Clearly, $P^{\top} : \mathcal{P}'^{\star} \to \mathcal{P}^{\star}$ is cone-linear.

Again, suppose that P is cone-linear. It is not hard to show that the pre-image under P of each face of $\mathcal{P}'$ is a face of $\mathcal{P}$. We say that P is **face-preserving** if the *image* under P of each face of $\mathcal{P}$ is a face of $\mathcal{P}'$. Of course, every cone-isomorphism is face-preserving and the range of every face-preserving cone-linear mapping is a face. An injective cone-linear mapping is face-preserving if and only if its range is a face. We note that the transpose of a face-preserving cone-linear mapping need not be face-preserving. (See Sect. 6.)

3. Spaces and cones of quadratic forms

Throughout this section, we assume that a linear space $\mathcal{V}$ is given.

Our treatment of quadratic forms is based on Sect. 27 of [Nol87]. In particular, we define the space $\mathrm{Qu}\,\mathcal{V}$ of **quadratic forms** on $\mathcal{V}$ as the range of the injective linear mapping

$$S \mapsto S \circ (\mathbf{1}_{\mathcal{V}}, \mathbf{1}_{\mathcal{V}}) \colon \mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R}) \to \mathrm{Map}(\mathcal{V}, R), \tag{3.1}$$

and we shall make use of the natural isomorphism[6]

$$\phi \mapsto \phi^{\sqcup} \colon \mathrm{Qu}\,\mathcal{V} \to \mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R}) \cong \mathrm{Sym}(\mathcal{V}, \mathcal{V}^*) \tag{3.2}$$

characterized by

$$\phi^{\sqcup}(\boldsymbol{v}, \boldsymbol{v}) = \phi(\boldsymbol{v}) \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}. \tag{3.3}$$

Let $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in \mathcal{V}^*$ be given. The *tensor product* $\boldsymbol{\lambda} \otimes \boldsymbol{\mu} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}^*)$ is defined in Sect. 25 of [Nol87] by $(\boldsymbol{\lambda} \otimes \boldsymbol{\mu})\boldsymbol{v} := (\boldsymbol{\lambda}\boldsymbol{v})\boldsymbol{\mu}$ for all $\boldsymbol{v} \in \mathcal{V}$; it follows from Prop. 1 of the same section that the *symmetric tensor product* $\frac{1}{2}(\boldsymbol{\lambda} \otimes \boldsymbol{\mu} + \boldsymbol{\mu} \otimes \boldsymbol{\lambda})$ is a member of $\mathrm{Sym}(\mathcal{V}, \mathcal{V}^*)$. In this paper, we find it convenient to use the *value-wise product* $\boldsymbol{\lambda}\boldsymbol{\mu} \colon \mathcal{V} \to \mathbb{R}$ defined by

$$(\boldsymbol{\lambda}\boldsymbol{\mu})(\boldsymbol{v}) := (\boldsymbol{\lambda}\boldsymbol{v})(\boldsymbol{\mu}\boldsymbol{v}) \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}.$$

This function is the quadratic form on $\mathcal{V}$ which corresponds under the isomorphism (3.2) to the symmetric tensor product of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, *i.e.,*

$$\boldsymbol{\lambda}\boldsymbol{\mu} \in \mathrm{Qu}\,\mathcal{V} \qquad \text{and} \qquad (\boldsymbol{\lambda}\boldsymbol{\mu})^{\sqcup} = \tfrac{1}{2}(\boldsymbol{\lambda} \otimes \boldsymbol{\mu} + \boldsymbol{\mu} \otimes \boldsymbol{\lambda}). \tag{3.4}$$

In particular, we have $(\boldsymbol{\lambda}^2)^{\sqcup} = \boldsymbol{\lambda} \otimes \boldsymbol{\lambda}$ when $\boldsymbol{\lambda}^2$ denotes the value-wise square of $\boldsymbol{\lambda}$. It is not hard to show that

$$\mathrm{Qu}\,\mathcal{V} = \mathrm{Lsp}\{\boldsymbol{\lambda}\boldsymbol{\mu} \mid \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{V}^*\} = \mathrm{Lsp}\{\boldsymbol{\lambda}^2 \mid \boldsymbol{\lambda} \in \mathcal{V}^*\}, \tag{3.5}$$

where the symbol Lsp denotes linear span in the space $\mathrm{Map}(\mathcal{V}, \mathbb{R})$ (see Problem 8, Chapter 2 of [Nol87]).

We denote by[7]

$$\mathrm{Pqu}\,\mathcal{V} := \{\phi \in \mathrm{Qu}\,\mathcal{V} \mid \mathrm{Rng}\,\phi \subset \mathbb{P}\} \tag{3.6}$$

[6] Clearly, (3.2) is the inverse of the linear isomorphism obtained by adjusting the codomain of (3.1).

[7] In [NS77b], the symbol $\overline{\mathrm{Qu}^{+}}(\mathcal{V})$ was used for our $\mathrm{Pqu}\,\mathcal{V}$.

the set of all **positive** quadratic forms on $\mathcal{V}$. It is clear that $\operatorname{Pqu}\mathcal{V}$ is a closed linear cone in $\operatorname{Qu}\mathcal{V}$. Since $\boldsymbol{\lambda}^2 \in \operatorname{Pqu}\mathcal{V}$ for every $\boldsymbol{\lambda} \in \mathcal{V}^*$, it is clear from (3.5) that $\operatorname{Pqu}\mathcal{V}$ spans $\operatorname{Qu}\mathcal{V}$. Also, we have

$$\operatorname{Pqu}\mathcal{V} = \operatorname{Csp}\{\boldsymbol{\lambda}^2 \,|\, \boldsymbol{\lambda} \in \mathcal{V}^*\}. \tag{3.7}$$

(This fact can easily be inferred from [Nol87], Prop. 2 of Sect. 85 by introducing a genuine inner product in $\mathcal{V}$, using the resulting natural isomorphism from $\operatorname{Pqu}\mathcal{V}$ to $\operatorname{Pos}\mathcal{V}$, and then using the Spectral Theorem.) The interior of $\operatorname{Pqu}\mathcal{V}$ is the (open) linear cone[8]

$$\begin{aligned}\operatorname{Pqu}^+\mathcal{V} :&= \{\boldsymbol{\phi} \in \operatorname{Qu}\mathcal{V} \,|\, \boldsymbol{\phi}_>(\mathcal{V}^\times) \subset \mathbb{P}^\times\} \\ &= \{\boldsymbol{\phi} \in \operatorname{Pqu}\mathcal{V} \,|\, \operatorname{Null}\boldsymbol{\phi} = \{\mathbf{0}\}\}\end{aligned}$$

of all **strictly positive** quadratic forms on $\mathcal{V}$. (This fact can easily be inferred from the first statement in the Theorem on the Smoothness of the Strict Lineonic Square Root in Sect. 85 of [Nol87].) In view of Prop. 2.3, we have

$$\operatorname{Fcsp}\{\boldsymbol{\phi}\} = \operatorname{Pqu}\mathcal{V} \qquad \text{for all } \boldsymbol{\phi} \in \operatorname{Pqu}^+\mathcal{V}.$$

In view of [Nol87], Prop. 1 of Sect. 27, we have

$$\dim \operatorname{Qu}\mathcal{V} = \dim \operatorname{Pqu}\mathcal{V} = \dim \operatorname{Pqu}^+\mathcal{V} = \tfrac{\dim\mathcal{V}(1+\dim\mathcal{V})}{2}. \tag{3.8}$$

Proposition 3.1. *For all $\boldsymbol{\phi} \in \operatorname{Pqu}\mathcal{V}$, we have*[9]

$$\left|\boldsymbol{\phi}^{\sqcup}(\boldsymbol{u},\boldsymbol{v})\right|^2 \le \boldsymbol{\phi}(\boldsymbol{u})\boldsymbol{\phi}(\boldsymbol{v}) \qquad \textit{for all } \boldsymbol{u},\boldsymbol{v} \in \mathcal{V}. \tag{3.9}$$

Proof. Let $\boldsymbol{\phi} \in \operatorname{Qu}\mathcal{V}$ and $\boldsymbol{u},\ \boldsymbol{v} \in \mathcal{V}$ be given. Since $\operatorname{Rng}\boldsymbol{\phi} \subset \mathbb{P}$, we have

$$0 \le \boldsymbol{\phi}(\alpha\boldsymbol{u} - \beta\boldsymbol{v}) = \alpha^2\boldsymbol{\phi}(\boldsymbol{u}) + \beta^2\boldsymbol{\phi}(\boldsymbol{v}) - 2\alpha\beta\,\boldsymbol{\phi}^{\sqcup}(\boldsymbol{u},\boldsymbol{v}) \tag{3.10}$$

for all $\alpha,\ \beta \in \mathbb{R}$.

Suppose that $\boldsymbol{\phi}(\boldsymbol{u}) = \boldsymbol{\phi}(\boldsymbol{v}) = 0$. Then using (3.10) with $\alpha := \frac{1}{2}$, $\beta := -1$ yields $0 \le \boldsymbol{\phi}^{\sqcup}(\boldsymbol{u},\boldsymbol{v})$. Using (3.10) with $\alpha := \frac{1}{2}$, $\beta := 1$ yields $0 \le -\boldsymbol{\phi}^{\sqcup}(\boldsymbol{u},\boldsymbol{v})$, and we conclude that $\boldsymbol{\phi}^{\sqcup}(\boldsymbol{u},\boldsymbol{v}) = 0$, so that (3.9) holds because both sides are zero.

Suppose that one of $\boldsymbol{\phi}(\boldsymbol{u})$ and $\boldsymbol{\phi}(\boldsymbol{v})$, say $\boldsymbol{\phi}(\boldsymbol{u})$, is not zero. Using (3.10) with $\alpha := \boldsymbol{\phi}(\boldsymbol{v})^2$ and $\beta := \boldsymbol{\phi}^{\sqcup}(\boldsymbol{u},\boldsymbol{v})$ yields

$$0 \le \boldsymbol{\phi}(\boldsymbol{v})\left(\boldsymbol{\phi}(\boldsymbol{u})\boldsymbol{\phi}(\boldsymbol{v}) - \boldsymbol{\phi}^{\sqcup}(\boldsymbol{u},\boldsymbol{v})^2\right).$$

Since $\boldsymbol{\phi}(\boldsymbol{v}) > 0$, it follows that (3.9) holds. □

For each $\boldsymbol{u} \in \mathcal{V}$, we define $\boldsymbol{u}+ : \mathcal{V} \to \mathcal{V}$ by

$$(\boldsymbol{u}+)(\boldsymbol{v}) := \boldsymbol{u} + \boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}.$$

[8] In [NS77b], the symbol $\operatorname{Qu}^+(\mathcal{V})$ was used for our $\operatorname{Pqu}^+\mathcal{V}$.

[9] In the special case when $\boldsymbol{\phi}$ is strictly positive, it can be used to make $\mathcal{V}$ an inner-product space and (3.9) reduces to the Inner-Product Inequality (see Sect. 42 of [Nol87]).

Proposition 3.2. *For all* $\phi \in \operatorname{Pqu} \mathcal{V}$, *we have*[10]

$$\text{(3.11)} \qquad \operatorname{Null} \phi = \operatorname{Null} \phi^{\sqcup} = \{ u \in \mathcal{V} \mid \phi \circ (u+) = \phi \}$$

when $\phi^{\sqcup}$ *is regarded as an element of* $\operatorname{Sym}(\mathcal{V}, \mathcal{V}^*)$.

Proof. Let $\phi \in \operatorname{Pqu} \mathcal{V}$ and $u \in \mathcal{V}$ be given.

On the one hand, suppose that $u \in \operatorname{Null} \phi$, so that that $\phi(u) = 0$. By Prop. 3.1 we then have $(\phi^{\sqcup} u)v = \phi^{\sqcup}(u, v) = 0$ for all $v \in \mathcal{V}$ and hence $\phi^{\sqcup} = \mathbf{0}$, showing that $u \in \operatorname{Null} \phi^{\sqcup}$. Since $u \in \operatorname{Null} \phi$ was arbitrary, it follows that $\operatorname{Null} \phi \subset \operatorname{Null} \phi^{\sqcup}$. On the other hand, suppose that $u \in \operatorname{Null} \phi^{\sqcup}$; then $0 = (\phi^{\sqcup} u)u = \phi(u)$ and hence $u \in \operatorname{Null} \phi$. It follows that $\operatorname{Null} \phi^{\sqcup} \subset \operatorname{Null} \phi$. It also follows that

$$\phi(w + u) \;=\; \phi(w) + \phi(u) + 2\phi^{\sqcup}(u)w \;=\; \phi(w) \qquad \text{for all } w \in \mathcal{V},$$

so that

$$\operatorname{Null} \phi \subset \{ u \in \mathcal{V} \mid \phi \circ (u+) = \phi \}.$$

Finally, suppose, instead, that $\phi \circ (u+) = \phi$. Then $\phi(u) = \phi(u + u) = 4\phi(u)$ and hence $\phi(u) = 0$, so $u \in \operatorname{Null} \phi$. It follows that

$$\operatorname{Null} \phi \supset \{ u \in \mathcal{V} \mid \phi \circ (u+) = \phi \}. \quad \square$$

As an immediate consequence of Prop. 3.2, the nullset $\operatorname{Null} \phi$ is a sub*space* for all $\phi \in \operatorname{Pqu} \mathcal{V}$.[11]

For every $\mathcal{U} \in \operatorname{Subsp} \mathcal{V}$, the set

$$\text{(3.12)} \qquad \Phi_{\mathcal{V}}(\mathcal{U}) := \{ \phi \in \operatorname{Pqu} \mathcal{V} \mid \phi|_{\mathcal{U}} = 0 \} = \{ \phi \in \operatorname{Pqu} \mathcal{V} \mid \mathcal{U} \subset \operatorname{Null} \phi \}$$

is clearly a subcone of $\operatorname{Pqu} \mathcal{V}$.

Proposition 3.3. *For every* $\mathcal{U} \in \operatorname{Subsp} \mathcal{V}$, *the subcone* $\Phi_{\mathcal{V}}(\mathcal{U})$ *of* $\operatorname{Pqu} \mathcal{V}$ *defined by* (3.12) *is actually a face of* $\operatorname{Pqu} \mathcal{V}$.

Proof. Let $\mathcal{U} \in \operatorname{Subsp} \mathcal{V}$ and $\phi_1, \phi_2 \in \operatorname{Pqu} \mathcal{V}$ be given such that

$$\phi := \phi_1 + \phi_2 \in \Phi_{\mathcal{V}}(\mathcal{U}).$$

By (3.12) this means that $\phi_1(u) + \phi_2(u) = 0$ for all $u \in \mathcal{U}$. Since $\operatorname{Rng} \phi_1 \subset \mathbb{P}$ and $\operatorname{Rng} \phi_2 \subset \mathbb{P}$ by (3.6), we conclude that $\phi_1(u) = 0 = \phi_2(u)$ for all u in $\mathcal{U}$, *i.e.*, that $\phi_1, \phi_2 \in \Phi_{\mathcal{V}}(\mathcal{U})$. $\square$

In view of Prop. 3.3, we may consider (3.12) to be the definition of a mapping

$$\text{(3.13)} \qquad \Phi_{\mathcal{V}} : \operatorname{Subsp} \mathcal{V} \to \operatorname{Face}(\operatorname{Pqu} \mathcal{V})$$

from $\operatorname{Subsp} \mathcal{V}$ to the lattice $\operatorname{Face}(\operatorname{Pqu} \mathcal{V})$ of all faces of $\operatorname{Pqu}(\mathcal{V})$.

Proposition 3.4. *Let* $\mathcal{U} \in \operatorname{Subsp} \mathcal{V}$ *be given. Then* $\rho \circ \Omega_{\mathcal{V}/\mathcal{U}} \in \Phi_{\mathcal{V}}(\mathcal{U})$ *for all* ρ *in* $\operatorname{Pqu}(\mathcal{V}/\mathcal{U})$ *and the mapping*

$$\text{(3.14)} \qquad \rho \mapsto \rho \circ \Omega_{\mathcal{V}/\mathcal{U}} : \operatorname{Pqu}(\mathcal{V}/\mathcal{U}) \to \Phi_{\mathcal{V}}(\mathcal{U})$$

is a cone-isomorphism.

---

[10] The second equality in (3.11) holds for all ϕ in $\operatorname{Qu} \mathcal{V}$; the first fails when $\phi \in \operatorname{Qu} \mathcal{V}$ is *double-signed*, *i.e.*, when neither ϕ nor $-\phi$ is positive.

[11] If $\phi \in \operatorname{Qu} \mathcal{V}$ is double-signed, then $\operatorname{Null} \phi$ is *not* a subspace.

Proof. Let $\rho \in \operatorname{Pqu}(\mathcal{V}/\mathcal{U})$ be given. Since the composite of a linear mapping with a positive quadratic form is again a positive quadratic form, it is clear that $\rho \circ \Omega_{\mathcal{V}/\mathcal{U}} \in \operatorname{Pqu}\mathcal{V}$. Also, since $\mathcal{U}$ is the zero-element of $\mathcal{V}/\mathcal{U}$ and since $\Omega_{\mathcal{V}/\mathcal{U}}\boldsymbol{u} = \mathcal{U}$ for all $\boldsymbol{u}$ in $\mathcal{U}$, it is clear that $(\rho \circ \Omega_{\mathcal{V}/\mathcal{U}})|_{\mathcal{U}} = \mathbf{0}$, so that $\rho \circ \Omega_{\mathcal{V}/\mathcal{U}} \in \Phi_{\mathcal{V}}(\mathcal{U})$ in view of (3.12).

It is evident that the mapping (3.14) is cone-linear. Since $\Omega_{\mathcal{V}/\mathcal{U}}$ is surjective, the mapping

$$\phi \mapsto \phi \circ \Omega_{\mathcal{V}/\mathcal{U}}\colon \operatorname{Map}(\mathcal{V}/\mathcal{U}, \mathbb{R}) \to \operatorname{Map}(\mathcal{V}, \mathbb{R})$$

is injective by Prop. 1.1; it follows that (3.14) is injective.

Now let ϕ in $\Phi_{\mathcal{V}}(\mathcal{U})$ be given, so that $\mathcal{U} \subset \operatorname{Null}\phi$ by (3.12). It follows by Prop. 3.2 that

$$\phi_{>}(\boldsymbol{v} + \mathcal{U}) = \phi_{>}(\{\boldsymbol{v}\}) = \{\phi(\boldsymbol{v})\} \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}.$$

Thus we can determine $\rho\colon \mathcal{V}/\mathcal{U} \to \mathbb{R}$ by the condition $\rho(\mathcal{G}) \in \phi_{>}(\mathcal{G})$ for all $\mathcal{G}$ in $\mathcal{V}/\mathcal{U}$; it is clear that $\phi = \rho \circ \Omega_{\mathcal{V}/\mathcal{U}}$. Choosing a right inverse $L \in \operatorname{Lin}(\mathcal{V}/\mathcal{U}, \mathcal{V})$ for $\Omega_{\mathcal{V}/\mathcal{U}}$, we note that $\phi \circ L = \rho \circ \Omega_{\mathcal{V}/\mathcal{U}} \circ L = \rho$. It follows that $\rho \in \operatorname{Pqu}(\mathcal{V}/\mathcal{U})$. Since $\phi \in \Phi_{\mathcal{V}}(\mathcal{U})$ was arbitrary, if follows that (3.14) is surjective. □

Proposition 3.5. *For all $\phi \in \operatorname{Pqu}\mathcal{V}$, the facial span of the singleton $\{\phi\}$ is given by*

$$\operatorname{Fcsp}\{\phi\} = \Phi_{\mathcal{V}}(\operatorname{Null}\phi)\,. \tag{3.15}$$

Proof. We noted after proving Prop. 3.2 that $\operatorname{Null}\phi \in \operatorname{Subsp}\mathcal{V}$ for all $\phi \in \operatorname{Pqu}\mathcal{V}$. Let $\phi \in \operatorname{Pqu}\mathcal{V}$ be given and put $\mathcal{U} := \operatorname{Null}\phi$. It is clear from (3.12) and Prop. 3.3 that $\Phi_{\mathcal{V}}(\mathcal{U})$ is a face of $\operatorname{Pqu}\mathcal{V}$ that contains ϕ and hence that $\operatorname{Fcsp}\{\phi\} \subset \Phi_{\mathcal{V}}(\mathcal{U})$. In view of Prop. 3.4, we may determine ρ in $\operatorname{Pqu}(\mathcal{V}/\mathcal{U})$ such that $\rho \circ \Omega_{\mathcal{V}/\mathcal{U}} = \phi$, so $\rho(\boldsymbol{v} + \mathcal{U}) = \phi(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathcal{V}$. Since $\mathcal{U} = \operatorname{Null}\phi$, it follows that

$$\operatorname{Null}\rho = \{\boldsymbol{v} + \mathcal{U} \mid \boldsymbol{v} \in \operatorname{Null}\phi\} = \{\mathcal{U}\}.$$

Thus, since $\mathcal{U}$ is the zero-element of $\mathcal{V}/\mathcal{U}$, and since clearly $\rho \in \operatorname{Pqu}\mathcal{V}$, we have $\rho \in \operatorname{Pqu}^{+}(\mathcal{V}/\mathcal{U})$. Since $\operatorname{Pqu}^{+}(\mathcal{V}/\mathcal{U})$ is the interior of $\operatorname{Pqu}(\mathcal{V}/\mathcal{U})$, it follows by Prop. 2.2 that $\operatorname{Pqu}(\mathcal{V}/\mathcal{U}) = \operatorname{Fcsp}\{\rho\}$. Hence, by Prop. 3.4, we have $\Phi_{\mathcal{V}}(\mathcal{U}) = \operatorname{Fcsp}\{\phi\}$. □

It follows from Proposition 3.4 and (3.8) that

$$\dim\Phi_{\mathcal{V}}(\mathcal{U}) = \dim\operatorname{Pqu}(\mathcal{V}/\mathcal{U}) = \tfrac{(\dim\mathcal{V} - \dim\mathcal{U})(1 + \dim\mathcal{V} - \dim\mathcal{U})}{2} \quad \text{for all } \mathcal{U} \in \operatorname{Subsp}\mathcal{V}. \tag{3.16}$$

Theorem 1. *The mapping $\Phi_{\mathcal{V}}$ of* (3.13), *defined by* (3.12), *is an inclusion-reversing bijection.*

Proof. It is easily seen from (3.12) and (3.8) that $\Phi_{\mathcal{V}}$ is strictly inclusion-reversing in the sense that

$$\mathcal{U} \subsetneqq \mathcal{U}' \quad \Longrightarrow \quad \Phi_{\mathcal{V}}(\mathcal{U}) \supsetneqq \Phi_{\mathcal{V}}(\mathcal{U}') \tag{3.17}$$

for all $\mathcal{U}, \mathcal{U}' \in \operatorname{Subsp}\mathcal{V}$. Hence $\Phi_{\mathcal{V}}$ is injective. To show that $\Phi_{\mathcal{V}}$ is surjective, let a face $\mathcal{F}$ of the cone $\operatorname{Pqu}\mathcal{V}$ be given. By Prop. 2.3, we may choose $\phi \in \operatorname{Pqu}\mathcal{V}$ such that $\mathcal{F} = \operatorname{Fcsp}\{\phi\}$. By Prop. 3.5 above, we have $\mathcal{F} = \Phi_{\mathcal{V}}(\operatorname{Null}\phi)$ and hence $\mathcal{F} \in \operatorname{Rng}\Phi_{\mathcal{V}}$. Since the face $\mathcal{F}$ was arbitrary, it follows that $\Phi_{\mathcal{V}}$ is surjective. □

It follows from Theorem 1 that for every family $(\mathcal{U}_i \mid i \in I)$ of subspaces of $\mathcal{V}$, we have

$$\Phi_{\mathcal{V}}\left(\bigcap_{i\in I}\mathcal{U}_i\right) = \mathrm{Fcsp}\left(\bigcup_{i\in I}\Phi_{\mathcal{V}}(\mathcal{U}_i)\right), \tag{3.18}$$

$$\Phi_{\mathcal{V}}\left(\mathrm{Lsp}\left(\bigcup_{i\in I}\mathcal{U}_i\right)\right) = \bigcap_{i\in I}\Phi_{\mathcal{V}}(\mathcal{U}_i); \tag{3.19}$$

these formulas relate greatest lower bounds in the lattice Subsp $\mathcal{V}$ to least upper bounds in the lattice Face(Pqu $\mathcal{V}$) and *vice versa.*

4. Duality

Throughout this section, we assume that linear spaces $\mathcal{V}$ and $\mathcal{V}'$ are given. We note the identification $\mathcal{V}^{**} \cong \mathcal{V}$ which states that every vector $\boldsymbol{v} \in \mathcal{V}$ may be regarded as linear form on $\mathcal{V}^*$. Hence we consider the value-wise square $\boldsymbol{v}^2$ of a given $\boldsymbol{v} \in \mathcal{V}$ as a quadratic form on $\mathcal{V}^*$, so that $\boldsymbol{v}^2 \in \mathrm{Qu}\,\mathcal{V}^*$.

Proposition 4.1. *There is exactly one bilinear mapping*

$$\Gamma\colon \mathrm{Qu}\,\mathcal{V} \times \mathrm{Qu}\,\mathcal{V}^* \to \mathbb{R} \tag{4.1}$$

such that

$$\Gamma(\boldsymbol{\lambda}^2, \boldsymbol{v}^2) = (\boldsymbol{\lambda}\boldsymbol{v})^2 \qquad \text{for all } \boldsymbol{\lambda} \text{ in } \mathcal{V}^*,\ \boldsymbol{v} \in \mathcal{V}. \tag{4.2}$$

This mapping satisfies

$$\Gamma(\boldsymbol{\phi}, \boldsymbol{v}^2) = \boldsymbol{\phi}(\boldsymbol{v}) \qquad \text{for all } \boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V},\ \boldsymbol{v} \text{ in } \mathcal{V}, \tag{4.3}$$

$$\Gamma(\boldsymbol{\lambda}^2, \boldsymbol{f}) = \boldsymbol{f}(\boldsymbol{\lambda}) \qquad \text{for all } \boldsymbol{\lambda} \in \mathcal{V}^*,\ \boldsymbol{f} \text{ in } \mathrm{Qu}\,\mathcal{V}^*, \tag{4.4}$$

and the mappings

$$\boldsymbol{\phi} \mapsto \Gamma(\boldsymbol{\phi}, \cdot)\colon \mathrm{Qu}\,\mathcal{V} \to (\mathrm{Qu}\,\mathcal{V}^*)^*, \tag{4.5}$$

$$\boldsymbol{f} \mapsto \Gamma(\cdot, \boldsymbol{f})\colon \mathrm{Qu}\,\mathcal{V}^* \to (\mathrm{Qu}\,\mathcal{V})^* \tag{4.6}$$

are cone-preserving linear isomorphisms relative to the cones Pqu $\mathcal{V}$, $(\mathrm{Pqu}\,\mathcal{V}^*)^\star$ *and* Pqu $\mathcal{V}^*$, $(\mathrm{Pqu}\,\mathcal{V})^\star$, *respectively.*

Proof. Using [Nol87], Prop. 6 of Chap. 2, with the choices $\mathcal{V}_1 := \mathcal{V}_2 := \mathcal{V}^*$, $\mathcal{W} := \mathbb{R}$, and using the identifications $\mathcal{V}^{**} \cong \mathcal{V}$ and $\mathrm{Lin}(\mathcal{V}, \mathcal{V}^*) \cong \mathrm{Lin}_2(\mathcal{V}^2, \mathbb{R})$, we see that there is exactly one linear isomorphism

$$\Lambda\colon \mathrm{Lin}_2({\mathcal{V}^*}^2, \mathbb{R}) \to \left(\mathrm{Lin}_2(\mathcal{V}^2, \mathbb{R})\right)^*$$

such that $B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \Lambda(B)(\boldsymbol{\lambda}_1 \otimes \boldsymbol{\lambda}_2)$ for all $B \in \mathrm{Lin}_2({\mathcal{V}^*}^2, \mathbb{R})$,and all $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2 \in \mathcal{V}$.

Define

$$\overline{\Gamma}\colon \mathrm{Sym}_2({\mathcal{V}^*}^2, \mathbb{R}) \to \left(\mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R})\right)^*$$

by

$$\overline{\Gamma}(S) := \Lambda(S)|_{\mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R})} \qquad \text{for all } S \in \mathrm{Sym}_2({\mathcal{V}^*}^2, \mathbb{R}).$$

Then $\overline{\Gamma}$ is easily seen to be the only linear isomorphism from $\mathrm{Sym}_2({\mathcal{V}^*}^2, \mathbb{R})$ to $\left(\mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R})\right)^*$ which satisfies $S(\boldsymbol{\lambda}, \boldsymbol{\lambda}) = \overline{\Gamma}(S)(\boldsymbol{\lambda} \otimes \boldsymbol{\lambda})$ for all $S \in \mathrm{Sym}_2({\mathcal{V}^*}^2, \mathbb{R})$ and all $\boldsymbol{\lambda} \in \mathcal{V}^*$. Replacing the spaces $\mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R})$ and $\mathrm{Sym}_2({\mathcal{V}^*}^2, \mathbb{R})$ of bilinear mappings by the corresponding spaces $\operatorname{Qu}\mathcal{V}$ and $\operatorname{Qu}\mathcal{V}^*$ of quadratic forms yields the first statement of the proposition.

In view of (3.5) and the linearity of Γ in its first argument, (4.3) follows directly from (4.2); statement (4.4) is established similarly.

It follows from (4.3) that

$$\phi = \mathbf{0} \iff \Gamma(\phi, \cdot) = \mathbf{0} \qquad \text{for all } \phi \in \operatorname{Qu}\mathcal{V},$$

so that (4.5) is injective and hence a linear isomorphism. In view of (3.7) and the linearity of Γ in its first argument, it also follows from (4.3) that

$$\phi \in \operatorname{Pqu}\mathcal{V} \iff \Gamma(\phi, \cdot) \in (\operatorname{Pqu}\mathcal{V}^*)^{\bigstar} \qquad \text{for all } \phi \in \operatorname{Qu}\mathcal{V};$$

thus (4.5) is cone-preserving. The assertions that (4.6) is a linear isomorphism and cone-preserving are established similarly. □

The mappings (4.5) and (4.6) establish natural linear isormorphims between $\operatorname{Qu}\mathcal{V}$ and $(\operatorname{Qu}\mathcal{V}^*)^*$ and between $\operatorname{Qu}\mathcal{V}^*$ and $(\operatorname{Qu}\mathcal{V})^*$ which are compatible with the identification $(\operatorname{Qu}\mathcal{V})^{**} \cong \operatorname{Qu}\mathcal{V}$. They also establish cone-isomorphisms between $\operatorname{Pqu}\mathcal{V}$ and $(\operatorname{Pqu}\mathcal{V}^*)^{\bigstar}$ and between $\operatorname{Pqu}\mathcal{V}^*$ and $(\operatorname{Pqu}\mathcal{V})^{\bigstar}$. We do not treat these natural isomorphisms as identifications *per se*[12], but we re-interpret some notation and terminology (insofar as it applies to cones and spaces of quadratic forms) in order to let the four spaces and cones on the left-hand sides of these isomorphisms conveniently "stand in" for those on the right. In particular,

(1) we consider *annihilators* of subsets of $\operatorname{Qu}\mathcal{V}$ and $\operatorname{Qu}\mathcal{V}^*$ to be subsets of $\operatorname{Qu}\mathcal{V}^*$ and $\operatorname{Qu}\mathcal{V}$, respectively; thus

(4.7) $\mathcal{A}^{\perp} = \{\boldsymbol{f} \in \operatorname{Qu}\mathcal{V}^* \mid \Gamma(\phi, \boldsymbol{f}) = 0 \text{ for all } \phi \in \mathcal{A}\}$ for all $\mathcal{A} \in \operatorname{Sub}(\operatorname{Qu}\mathcal{V})$,

(4.8) $\mathcal{B}^{\perp} = \{\phi \in \operatorname{Qu}\mathcal{V} \mid \Gamma(\phi, \boldsymbol{f}) = 0 \text{ for all } \boldsymbol{f} \text{ in } \mathcal{B}\}$ for all $\mathcal{B}$ in $\operatorname{Sub}(\operatorname{Qu}\mathcal{V}^*)$;

(2) we consider the *transposes* of linear mappings $\operatorname{Qu}\mathcal{V} \to \operatorname{Qu}\mathcal{V}'$ to be linear mappings $\operatorname{Qu}\mathcal{V}'^* \to \operatorname{Qu}\mathcal{V}^*$, and transposes of cone-linear mappings $\operatorname{Pqu}\mathcal{V} \to \operatorname{Pqu}\mathcal{V}'$ to be cone-linear mappings $\operatorname{Pqu}\mathcal{V}'^* \to \operatorname{Pqu}\mathcal{V}^*$; thus, for all $Q \in \operatorname{Lin}(\operatorname{Qu}\mathcal{V}, \operatorname{Qu}\mathcal{V}')$ and all $P \in \operatorname{Lin}(\operatorname{Pqu}\mathcal{V}, \operatorname{Pqu}\mathcal{V}')$, the transpose Q^{T} and the cone-transpose $P^{\top}$ are determined by the properties[13]

(4.9) $\Gamma(Q\phi, \boldsymbol{f}) = \Gamma(\phi, Q^{\mathrm{T}}\boldsymbol{f})$ for all $\phi \in \operatorname{Qu}\mathcal{V}$, $\boldsymbol{f}$ in $\operatorname{Qu}\mathcal{V}'^*$,

(4.10) $\Gamma(P\phi, \boldsymbol{f}) = \Gamma(\phi, P^{\top}\boldsymbol{f})$ for all $\phi \in \operatorname{Pqu}\mathcal{V}$, $\boldsymbol{f}$ in $\operatorname{Pqu}\mathcal{V}'^*$;

(3) given a linear space $\mathcal{W}$, $\boldsymbol{w} \in \mathcal{W}$, $\phi \in \operatorname{Qu}\mathcal{V}$ and $\boldsymbol{f} \in \operatorname{Qu}\mathcal{V}^*$, we interpret the *tensor products* $\boldsymbol{w} \otimes \phi$ and $\boldsymbol{w} \otimes \boldsymbol{f}$ as members of $\operatorname{Lin}(\operatorname{Qu}\mathcal{V}^*, \mathcal{W})$ and $\operatorname{Lin}(\operatorname{Qu}\mathcal{V}, \mathcal{W})$, respectively.

[12] To do so would lead to awkward ambiguities, *e.g.*, identifying $\phi \in \operatorname{Qu}\mathcal{V}$ with $\Gamma(\phi, \cdot) \in (\operatorname{Qu}\mathcal{V}^*)^*$ would make "$\operatorname{Rng}\phi$" ambiguous.

[13] We note that the symbol Γ on the right side of the equalities denotes the bilinear mapping (4.1) with domain $\operatorname{Qu}\mathcal{V} \times \operatorname{Qu}\mathcal{V}^*$, while the same symbol Γ on the left side denotes the corresponding bilinear mapping with domain $\operatorname{Qu}\mathcal{V}' \times \operatorname{Qu}\mathcal{V}'^*$.

In addition, we feel free to speak of the spaces $\mathrm{Qu}\,\mathcal{V}$ and $\mathrm{Qu}\,\mathcal{V}^*$ (and of the cones $\mathrm{Pqu}\,\mathcal{V}$ and $\mathrm{Pqu}\,\mathcal{V}^*$) as being *dual* to each other.

Proposition 4.2. *Let a subspace $\mathcal{U}$ of $\mathcal{V}$ be given. Then we have*

$$\Phi_{\mathcal{V}}(\mathcal{U}) = \mathrm{Csp}\{\boldsymbol{\lambda}^2 \mid \boldsymbol{\lambda} \in \mathcal{U}^{\perp}\} \tag{4.11}$$

and

$$(\Phi_{\mathcal{V}}(\mathcal{U}))^{\perp} = \{\boldsymbol{f} \in \mathrm{Qu}\,\mathcal{V}^* \mid \boldsymbol{f}|_{\mathcal{U}^{\perp}} = 0\}, \tag{4.12}$$

where $\Phi_{\mathcal{V}}(\mathcal{U})$ is defined by (3.12).

Proof. Since

$$\boldsymbol{\mu}^2 \circ \Omega_{\mathcal{V}/\mathcal{U}} = (\boldsymbol{\mu}\Omega_{\mathcal{V}/\mathcal{U}})^2 \qquad \text{for all } \boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*, \tag{4.13}$$

it follows from Prop. 1.2 that

$$\mathrm{Csp}\{\boldsymbol{\mu}^2 \circ \Omega_{\mathcal{V}/\mathcal{U}} \mid \boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*\} = \mathrm{Csp}\{(\boldsymbol{\mu}\Omega_{\mathcal{V}/\mathcal{U}})^2 \mid \boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*\} = \mathrm{Csp}\{\boldsymbol{\lambda}^2 \mid \boldsymbol{\lambda} \in \mathcal{U}^{\perp}\}.$$

Since

$$\mathrm{Csp}\{\boldsymbol{\mu}^2 \mid \boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*\} = \mathrm{Pqu}\,(\mathcal{V}/\mathcal{U})$$

by (3.7), the assertion (4.11) follows by Prop. 3.4.

It follows from (4.4) that $\boldsymbol{f}|_{\mathcal{U}^{\perp}} = 0$ if and only if $\Gamma(\boldsymbol{f}, \boldsymbol{\lambda}^2) = 0$ for all $\boldsymbol{\lambda} \in \mathcal{U}^{\perp}$. By (4.11), this is the case if and only if $\Gamma(\boldsymbol{f})\boldsymbol{\phi} = 0$ for all $\boldsymbol{\phi} \in \Phi_{\mathcal{V}}(\mathcal{U})$, *i.e.*, in view of (4.7), if and only if $\boldsymbol{f} \in (\Phi_{\mathcal{V}}(\mathcal{U}))^{\perp}$. Since $\boldsymbol{f}$ in $\mathrm{Qu}\,\mathcal{V}^*$ was arbitrary, (4.12) follows. □

Proposition 4.3. *For each $\boldsymbol{\phi} \in (\mathrm{Pqu}\,\mathcal{V})^{\times}$, $\boldsymbol{\phi}$ belongs to an extreme ray of $\mathrm{Qu}\,\mathcal{V}$ if and only if $\boldsymbol{\phi} = \boldsymbol{\lambda}^2$ for some $\boldsymbol{\lambda}$ in $(\mathcal{V}^*)^{\times}$.*

Proof. Saying that $\boldsymbol{\phi}$ belongs to an extreme ray means that $\mathrm{Fcsp}\,\{\boldsymbol{\phi}\} = \mathbb{P}\,\boldsymbol{\phi}$. By Prop. 3.9, this is the case if and only if $\Phi_{\mathcal{V}}\,(\mathrm{Null}\,\boldsymbol{\phi}) = \mathbb{P}\,\boldsymbol{\phi}$. By Prop. 4.2, (4.11), this is the case if and only if $\mathbb{R}\,\boldsymbol{\phi} = \mathbb{R}\,\boldsymbol{\mu}^2$ for some $\boldsymbol{\mu} \in \mathcal{V}^*$ and hence $\boldsymbol{\phi} = \boldsymbol{\lambda}^2$ for some $\boldsymbol{\lambda} \in \mathcal{V}^*$. □

Let $n \in \mathbb{N}$ be given. We denote the set of all isotone pairs in $(n^{]})^2$ by $\mathrm{Ip}(n)$, *i.e.*,

$$\mathrm{Ip}(n) := \{(i,k) \in (n^{]})^2 \mid i \leq k\}. \tag{4.14}$$

Given any list $\boldsymbol{\beta} := (\boldsymbol{\beta}_i \mid i \in n^{]})$ in $(\mathcal{V}^*)^n$, we define the family $\boldsymbol{\beta}\Box\boldsymbol{\beta} \in (\mathrm{Qu}\,\mathcal{V})^{\mathrm{Ip}(n)}$ by

$$(\boldsymbol{\beta}\Box\boldsymbol{\beta})_{(i,k)} := \boldsymbol{\beta}_i\boldsymbol{\beta}_k \qquad \text{for all } (i,k) \in \mathrm{Ip}(n). \tag{4.15}$$

Proposition 4.4. *Let $n \in \mathbb{N}$, a linear space $\mathcal{W}$, and $\boldsymbol{\gamma} \in (\mathcal{W}^*)^n$ be given. Then the following are equivalent:*

(i) *$\boldsymbol{\gamma}$ is a basis of $\mathcal{W}^*$;*
(ii) *$\boldsymbol{\gamma}\Box\boldsymbol{\gamma}$ is a basis of $\mathrm{Qu}\,\mathcal{W}$;*
(iii) *$(\mathbb{P}\,{\boldsymbol{\gamma}_i}^2 \mid i \in n^{]})$ is a facial decomposition of $\mathrm{Pqu}\,\mathcal{W}$.*

Proof. We first prove the equivalence of (i) and (ii). On the one hand assume (i). Then

$$\left(\tfrac{1}{2}(\gamma_i \otimes \gamma_j + \gamma_j \otimes \gamma_i) \,|\, (i,j) \in \mathrm{Ip}(n)\right)$$

is a basis for $\mathrm{Sym}_2(\mathcal{W}^2, \mathbb{R})$ by an argument very similar to the one used to prove Prop. 1 of Sect. 27 of [Nol87]. In view of (3.4) and the fact that (3.2) is a linear isomorphism, we have (ii).

On the other hand, assume (ii). Let a list $t := (t_i \,|\, i \in n^]})$ be given such that

$$\sum_{i \in n^]} t_i \gamma_i = \mathbf{0}. \tag{4.16}$$

Then we have

$$\mathbf{0} = \gamma_1 \sum_{i \in n^]} t_i \gamma_i = \sum_{i \in n^]} t_i (\gamma_1 \gamma_i).$$

Since $(\gamma_1 \gamma_i \,|\, i \in n^])$ is a subfamily of the basis $\gamma \Box \gamma$ of $\mathrm{Qu}\,\mathcal{W}$, it is linearly independent. It follows that $t = 0$.

Since t was arbitrary, subject to (4.16), γ is linearly independent, and hence a basis.

We next prove the equivalence of (i) and (iii). In view of (2.3), (3.18), and the fact that $\mathrm{Pqu}\,\mathcal{W} = \Phi_{\mathcal{W}}(\{0\})$, (iii) holds if and only if

$$\Phi_{\mathcal{W}}\left(\textstyle\bigcap_{i \in J} \Phi_{\mathcal{W}}^{\leftarrow}(\mathbb{P}\,{\gamma_i}^2)\right) = \Phi_{\mathcal{W}}(\{0\}) \iff J = n^] \qquad \text{for all } J \in \mathrm{Sub}\, n^].$$

Since, in view (3.15), we have

$$\mathbb{P}\,{\gamma_i}^2 = \Phi_{\mathcal{W}}(\mathrm{Null}\,{\gamma_i}^2) = \Phi_{\mathcal{W}}(\mathrm{Null}\,\gamma_i) \qquad \text{for all } i \in n^],$$

it follows that (iii) holds if and only if

$$\textstyle\bigcap_{i \in J} \mathrm{Null}\,\gamma_i = \{0\} \iff J = n^] \quad \text{for all } J \in \mathrm{Sub}\, n^].$$

Since

$$\bigcap_{i \in J} \mathrm{Null}\,\gamma_i = \Big(\sum_{i \in J} \mathbb{R}\gamma_i\Big)^{\perp} = (\mathrm{Lsp}\{\gamma_i \,|\, i \in J\})^{\perp} \quad \text{for all } J \in \mathrm{Sub}\, n^]$$

by [Nol87], Prop. 5 of Sect. 21, it follows that (iii) holds if and only if

$$\mathrm{Lsp}\{\gamma_i \,|\, i \in J\} = \mathcal{V} \iff J = n^] \qquad \text{for all } J \in \mathrm{Sub}\, n^],$$

which is equivalent to (i) because bases can be characterised as minimal spanning families. □

Corollary 1. *Let $n \in \mathbb{N}$, $\mathcal{U} \in \mathrm{Subsp}\,\mathcal{V}$, and $\beta \in (\mathcal{V}^*)^n$ be given. Then the following are equivalent:*

(i) *β is a basis of $\mathcal{U}^{\perp}$;*
(ii) *$\beta \Box \beta$ is a basis of $\mathrm{Lsp}\,\Phi_{\mathcal{V}}(\mathcal{U})$;*
(iii) *$(\mathbb{P}\,{\beta_i}^2 \,|\, i \in n^])$ is a facial decomposition of $\Phi_{\mathcal{V}}(\mathcal{U})$.*

Proof. If $\operatorname{Rng}\beta \not\subset \mathcal{U}^{\perp}$, then none of (i), (ii), and (iii) are valid. Assume $\operatorname{Rng}\beta \subset \mathcal{U}^{\perp}$. In view of Prop. 1.2 we may determine $\gamma \in ((\mathcal{V}/\mathcal{U})^{*})^{n}$ such that $\beta_i = \gamma_i \Omega_{\mathcal{V}/\mathcal{U}}$ for all $i \in n^{]}$. We note that the conclusion of Prop. 4.4 holds for $\mathcal{W} := \mathcal{V}/\mathcal{U}$ and γ as just chosen. The equivalence of (i), (ii), and (iii) (of the present proposition) follows by appeal to the isomorphisms (1.2) and (3.14). □

5. Cone-isomorphisms

Throughout this section, we assume that non-zero linear spaces $\mathcal{V}$ and $\mathcal{V}'$ are given. It is clear that if $L\colon \mathcal{V}' \to \mathcal{V}$ is a linear isomorphism, then the mapping $P\colon \operatorname{Pqu}\mathcal{V} \to \operatorname{Pqu}\mathcal{V}'$ defined by

$$P(\phi) := \phi \circ L \qquad \text{for all } \phi \text{ in } \operatorname{Pqu}\mathcal{V} \tag{5.1}$$

is a cone-isomorphism. (Indeed, its inverse is given by $P^{-1}(\psi) = \psi \circ L^{-1}$ for all ψ in $\operatorname{Pqu}\mathcal{V}'$.)

Theorem 2. *For every cone-isomorphism* $P\colon \operatorname{Pqu}\mathcal{V} \to \operatorname{Pqu}\mathcal{V}'$ *there are exactly two linear isomorphisms* $L\colon \mathcal{V}' \to \mathcal{V}$ *such that* (5.1) *holds. If* L *is one of them, then* $-L$ *is the other.*

Proof. Let a cone-isomorphism $P\colon \operatorname{Pqu}\mathcal{V} \to \operatorname{Pqu}\mathcal{V}'$ be given and let $\overline{P}\colon \operatorname{Qu}\mathcal{V} \to \operatorname{Qu}\mathcal{V}'$ be the linear isomorphism that extends P.

Lemma 5.1. *Suppose that* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{V}^{*}$ *and* $\boldsymbol{\lambda}', \boldsymbol{\mu}' \in \mathcal{V}'^{*}$ *satisfy*

$$P(\boldsymbol{\lambda}^2) = \boldsymbol{\lambda}'^2 \quad \textit{and} \quad P(\boldsymbol{\mu}^2) = \boldsymbol{\mu}'^2. \tag{5.2}$$

Then

$$\overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) = \boldsymbol{\lambda}'\boldsymbol{\mu}' \quad \textit{or} \quad \overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) = -\boldsymbol{\lambda}'\boldsymbol{\mu}' \tag{5.3}$$

Proof (Lemma 5.1*).* The assertion is easily seen to be valid if one of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ is a scalar multiple of the other. We may assume, therefore, that $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is linearly independent. Put $\mathcal{U} := \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}^{\perp}$, $\mathcal{U}' := \{\boldsymbol{\lambda}', \boldsymbol{\mu}'\}^{\perp}$. It is clear that $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $(\boldsymbol{\mu}', \boldsymbol{\lambda}')$ are bases of $\mathcal{U}$ and $(\mathcal{U}')^{\perp}$, respectively. It follows by Cor. 1 of Prop. 4.4 that $\operatorname{Fcsp}\{\boldsymbol{\lambda}^2, \boldsymbol{\mu}^2\} = \Phi_{\mathcal{V}}(\mathcal{U})$ and $\operatorname{Fcsp}\{\boldsymbol{\gamma}'^2, \boldsymbol{\mu}'^2\} = \Phi_{\mathcal{V}}(\mathcal{U}')$. Since P is a cone-isomorphism, the images under P of facial spans in $\operatorname{Pqu}\mathcal{V}$ are corresponding facial spans in $\operatorname{Pqu}\mathcal{V}'$. In particular, we have $P_{>}\left(\operatorname{Fcsp}\{\boldsymbol{\lambda}^2, \boldsymbol{\mu}^2\}\right) = \operatorname{Fcsp}\{\boldsymbol{\lambda}'^2, \boldsymbol{\mu}'^2\}$, *i.e.*,

$$P_{>}\left(\Phi_{\mathcal{V}}(\mathcal{U})\right) = \Phi_{\mathcal{V}'}(\mathcal{U}'). \tag{5.4}$$

Hence, since $\overline{P}$ is linear

$$\overline{P}_{>}\left(\operatorname{Lsp}\Phi_{\mathcal{V}}(\mathcal{U})\right) = \operatorname{Lsp} P_{>}\left(\Phi_{\mathcal{V}}(\mathcal{U})\right) = \operatorname{Lsp}\Phi_{\mathcal{V}'}(\mathcal{U}'). \tag{5.5}$$

It follows from item (ii) of Cor. 1 of Prop. 4.4 that $\overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) \in \operatorname{Lsp}\Phi_{\mathcal{V}'}(\mathcal{U}')$ and that $(\boldsymbol{\lambda}'^2, \boldsymbol{\lambda}'\boldsymbol{\mu}', \boldsymbol{\mu}'^2)$ is a basis of $\operatorname{Lsp}\Phi_{\mathcal{V}'}(\mathcal{U}')$. Hence we can determine a, b, $c \in \mathbb{R}$ such that

$$\overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) = a\boldsymbol{\lambda}'^2 + b\boldsymbol{\lambda}'\boldsymbol{\mu}' + c\boldsymbol{\mu}'^2 \tag{5.6}$$

Now let $t \in \mathbb{R}$ be given and put $\boldsymbol{\gamma} := \boldsymbol{\lambda} + t\boldsymbol{\mu}$. On the one hand, since P is linear, it follows from (5.2) and (5.6) that

$$
\begin{aligned}
P(\gamma^2) &= \lambda'^2 + 2t\overline{P}(\lambda\mu) + t^2\mu'^2 \\
&= (1+2ta)\lambda'^2 + 2tb\lambda'\mu' + (t^2+2tc)\mu'^2.
\end{aligned} \tag{5.7}
$$

On the other hand, since γ^2 belongs, by Prop. 4.3, to an extreme ray of $\operatorname{Pqu}\mathcal{V}$, its image $P(\gamma^2)$ must belong to an extreme ray of $\operatorname{Pqu}\mathcal{V}'$, and hence we may choose $\gamma' \in \mathcal{V}'^*$ such that

$$P(\gamma^2) = \gamma'^2. \tag{5.8}$$

Since $\gamma^2 \in \Phi_{\mathcal{V}}(\mathcal{U})$, it follows from (5.4) that $\gamma'^2 \in \Phi_{\mathcal{V}'}(\mathcal{U}')$ and hence, by (4.11), that $\gamma' \in \mathcal{U}'^{\perp} = \operatorname{Lsp}\{\lambda', \mu'\}$. Thus we may determine s and $r \in \mathbb{R}$ such that $\gamma' = s\lambda' + r\mu'$; it follows by by (5.8) that

$$P(\gamma^2) = s^2\lambda'^2 + 2sr\lambda'\mu' + r^2\mu'^2. \tag{5.9}$$

Since $(\lambda'^2, \mu'^2, \lambda'\mu')$ is linearly independent, we conclude from (5.7) and (5.9) that

$$s^2 = (1+2ta), \qquad sr = tb, \qquad r^2 = (t^2+2tc),$$

and hence that

$$t^2b^2 \;=\; s^2r^2 \;=\; (1+2ta)(t^2+2tc) \;=\; 2t^3a + (1+4ac)t^2 + 2tc.$$

Since this equality must be valid for all $t \in \mathbb{R}$, it follows that $a = c = 0$ and $b^2 = 1$. Hence we must have $b = 1$ or $b = -1$ and (5.6) reduces to (5.7). □

We now put $n := \dim\mathcal{V} = \dim\mathcal{V}^*$ and choose a list-basis $\boldsymbol{\beta}$ of $\mathcal{V}^*$. We note that the conclusions of Prop. 4.4 apply. Hence $(\mathbb{P}\beta_i{}^2 \mid i \in n^]\,)$ is a facial decomposition for $\operatorname{Pqu}\mathcal{V}$. Since P preserves extreme rays, it follows from Prop. 4.3 and Prop. 4.4 that we may choose a list-basis $\boldsymbol{\beta}'$ of $\mathcal{V}'^*$ such that

$$P(\beta_i{}^2) = \beta_i'^2 \qquad \text{for all } i \text{ in } n^]. \tag{5.10}$$

Let $i \in n^]$ be given. By Lemma 5.1 applied to $\lambda := \beta_1$ and $\mu := \beta_i$, we see that we may choose $s_i \in \{1, -1\}$ such that

$$\overline{P}(\beta_1\beta_i) = s_i\beta_1'\beta_i'. \tag{5.11}$$

Hence we obtain a list $(s_i \mid i \in n^]) \in \{1, -1\}^n$ with $s_1 = 1$.

Lemma 5.2. *We have*

$$\overline{P}(\beta_i\beta_k) = s_is_k\beta_i'\beta_k' \qquad \text{for all } i,\ k \in n^]. \tag{5.12}$$

Proof (Lemma 5.2*).* Since $s_1 = 1$, it is clear from (5.10) and (5.11) that (5.12) holds when $i = 1$ or $k = 1$ or $i = k$. Suppose, then, that i and $k \in n^]$ are given such that $i \neq 1$, $k \neq 1$, and $i \neq k$. Then the triple $(\beta_1, \beta_i, \beta_k)$ is linearly independent and we may apply Cor. 1 of Prop. 4.4 to it. Using an argument similar to the one used in the proof of Lemma 5.1, we determine a, b, and $c \in \mathbb{R}$ such that

$$P\left((\beta_1 + \beta_i + \beta_k)^2\right) \;=\; (a\beta_1' + b\beta_i' + c\beta_k')^2.$$

Using the linearity of $\overline{P}$ and (5.10) and (5.11), we find that

$$\beta_1'^2 + \beta_i'^2 + \beta_k'^2 + 2s_i\beta_1'\beta_i' + 2s_k\beta_1'\beta_k' + 2\overline{P}(\beta_i\beta_k) = a^2\beta_1'^2 + b^2\beta_i'^2 + c^2\beta_k'^2 + 2ab\beta_1'\beta_i' + 2ac\beta_1'\beta_k' + 2bc\beta_i'\beta_k'.$$

Since $\overline{P}(\beta_i\beta_k) \in \mathbb{R}\beta_i'\beta_k'$ by Lemma 5.1 and since the sextuple

$$(\beta_1'^2, \beta_i'^2, \beta_k'^2, \beta_1'\beta_i', \beta_1'\beta_k', \beta_i'\beta_k')$$

is linearly independent, we conclude that $a^2 = b^2 = c^2 = 1$, $ab = s_i$, $ac = s_k$, and $\overline{P}(\beta_i\beta_k) = bc\beta_i'\beta_k'$. Hence $s_i s_k = (ab)(ac) = a^2bc = bc$ and $\overline{P}(\beta_i\beta_k) = s_i s_k\beta_i'\beta_k'$. □

Since β is a basis of $\mathcal{V}^*$ and $(s_i\beta_i' \mid i \in n^]})$ is a basis of $\mathcal{V}'^*$, we can determine a linear transformation $L\colon \mathcal{V}' \to \mathcal{V}$ whose transpose $L^{\mathrm{T}}\colon \mathcal{V}^* \to \mathcal{V}'^*$ satisfies

$$L^{\mathrm{T}}\beta_i = s_i\beta_i' \qquad \text{for all } i \text{ in } n^{]}.$$

By Lemma 5.2 we have

$$\overline{P}(\beta_i\beta_k) = (L^{\mathrm{T}}\beta_i)(L^{\mathrm{T}}\beta_k) = (\beta_i\beta_k)\circ L \qquad \text{for all } i,k \in n^{]}.$$

Since $\beta\Box\beta := (\beta_i\beta_k \mid (i,k) \in \mathrm{Ip}(n))$ is a basis of $\mathrm{Qu}\,\mathcal{V}$, we conclude that (5.1) holds. It is easy to see that (5.1) also holds when L is replaced by $-L$.

Now let a linear transformation $M : \mathcal{V} \to \mathcal{V}'$ be given such that

$$\phi \circ L = P(\phi) = \phi \circ M \qquad \text{for all } \phi \in \mathrm{Pqu}\,\mathcal{V}. \tag{5.13}$$

Using (5.13) with $\phi := \beta_1{}^2$, we find that $(M^{\mathrm{T}}\beta_1)^2 = (L^{\mathrm{T}}\beta_1)^2$. Hence we may choose $\epsilon \in \{1,-1\}$ such that $M^{\mathrm{T}}\beta_1 = \epsilon L^{\mathrm{T}}\beta_1$. Now let $k \in n^{]}$ be given. Using (5.13) with $\phi := \beta_k\beta_1$ we find that

$$(L^{\mathrm{T}}\beta_k)(L^{\mathrm{T}}\beta_1) = (M^{\mathrm{T}}\beta_k)(M^{\mathrm{T}}\beta_1) = (\epsilon M^{\mathrm{T}}\beta_k)(L^{\mathrm{T}}\beta_1).$$

Since $L^{\mathrm{T}}\beta_1 \neq 0$ and since $k \in n^{]}$ was arbitrary, it follows that

$$L^{\mathrm{T}}\beta_k = \epsilon M^{\mathrm{T}}\beta_k \qquad \text{for all } k \in n^{]}.$$

Since β is a basis of $\mathcal{V}^*$, we conclude that $L^{\mathrm{T}} = \epsilon M^{\mathrm{T}}$ and hence $M = \epsilon L$. □

6. Linearly induced mappings

Throughout this section, $\mathcal{V}$, $\mathcal{V}'$, and $\mathcal{V}''$ are given linear spaces. For a given $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$, we define $\mathrm{qu}(L)$ in $\mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}\,\mathcal{V}')$ and $\mathrm{pqu}(L) \in \mathrm{Lin}(\mathrm{Pqu}\,\mathcal{V}, \mathrm{Pqu}\,\mathcal{V}')$ by

$$\mathrm{qu}(L)\phi := \phi \circ L \qquad \text{for all } \phi \in \mathrm{Qu}\,\mathcal{V}; \tag{6.1}$$

$$\mathrm{pqu}(L)\phi := \phi \circ L \qquad \text{for all } \phi \text{ in } \mathrm{Pqu}\,\mathcal{V}. \tag{6.2}$$

It is clear that

$$\mathrm{pqu}(L) = \mathrm{qu}(L)|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'} \qquad \text{for all } L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V}). \tag{6.3}$$

If a given $Q \in \mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}\,\mathcal{V}')$ equals $\mathrm{qu}(L)$ for some L in $\mathrm{Lin}(\mathcal{V}', \mathcal{V})$, we say that Q is **linearly induced (by L)**; similarly, if a given $P \in \mathrm{Lin}(\mathrm{Pqu}\,\mathcal{V}, \mathrm{Pqu}\,\mathcal{V}')$ equals $\mathrm{pqu}(L)$ for some $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$, we say that P is **linearly induced (by L)**.

The following two results are immediate.

Proposition 6.1. *Let* $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ *and* $Q \in \mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}\,\mathcal{V}')$ *be given. Then* $Q = \mathrm{qu}(L)$ *if and only if* Q *is cone-compatible and* $Q|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'} = \mathrm{pqu}(L)$.

Proposition 6.2. *A composite of linearly induced mappings is itself linearly induced. Indeed, let* $L_1 \in \mathrm{Lin}(\mathcal{V}'', \mathcal{V}')$ *and* L_2 *in* $\mathrm{Lin}(\mathcal{V}', \mathcal{V})$ *be given; then*

$$\mathrm{qu}(L_1)\,\mathrm{qu}(L_2) = \mathrm{qu}(L_2 L_1), \qquad \mathrm{pqu}(L_1)\,\mathrm{pqu}(L_2) = \mathrm{pqu}(L_2 L_1). \tag{6.4}$$

Proposition 6.3. *The transpose of a linearly induced linear mapping and the cone-transpose of a linearly induced cone-linear mapping are both linearly induced. Indeed, let* $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ *be given; then*

$$(\mathrm{qu}(L))^{\mathrm{T}} = \mathrm{qu}(L^{\mathrm{T}}), \qquad (\mathrm{pqu}(L))^{\mathrm{T}} = \mathrm{pqu}(L^{\mathrm{T}}) \tag{6.5}$$

Proof. It is clear that $\mathrm{qu}(L^{\mathrm{T}})\boldsymbol{w}^2 = (\boldsymbol{w}L^{\mathrm{T}})^2 = (L\boldsymbol{w})^2$ for all $\boldsymbol{w} \in \mathcal{V}'$; it follows by (4.2) that

$$\Gamma\left(\boldsymbol{\mu}^2, \mathrm{qu}(L^{\mathrm{T}})\boldsymbol{w}^2\right) = \Gamma\left(\boldsymbol{\mu}^2, (L\boldsymbol{w})^2\right) = (\boldsymbol{\mu}L\boldsymbol{w})^2 = \Gamma\left((\boldsymbol{\mu}L)^2, \boldsymbol{w}^2\right) = \Gamma\left(\mathrm{qu}(L)\boldsymbol{\mu}^2, \boldsymbol{w}^2\right) \text{ for all } \boldsymbol{\mu} \in \mathcal{V}^*,\ \boldsymbol{w} \in \mathcal{V}'.$$

In view of (3.5) and the bilinearity of Γ, it follows that

$$\Gamma\left(\mathrm{qu}(L)\boldsymbol{\phi}, \boldsymbol{g}\right) = \Gamma(\boldsymbol{\phi}, \mathrm{qu}(L^{\mathrm{T}})\boldsymbol{g}) \qquad \text{for all } \boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V},\ \boldsymbol{g} \text{ in } \mathrm{Qu}\,\mathcal{V}'^*.$$

In view of the determining conditions (4.9) and (4.10) for the transposes, we have the first equality in (6.5). In view of (6.3), the second follows by adjustment. □

Proposition 6.4. *Let* $L \in \mathit{Lin}(\mathcal{V}', \mathcal{V})$ *be given. If* L *is surjective, then* $\mathrm{qu}(L)$ *and* $\mathrm{pqu}(L)$ *are injective. If* L *is injective, then* $\mathrm{qu}(L)$ *and* $\mathrm{pqu}(L)$ *are surjective.*

Proof. First, suppose that L is surjective. Then $\mathrm{qu}(L)$ and $\mathrm{pqu}(L)$ are injective by Prop. 1.1.

Next, suppose that L is injective, so L^{T} is surjective. Then, we have

$$\mathrm{Rng}\,\mathrm{pqu}(L) \supset \mathrm{Csp}\{\mathrm{pqu}(L)(\boldsymbol{\lambda}^2) \mid \boldsymbol{\lambda} \in \mathcal{V}^*\} = \mathrm{Csp}\{(L^{\mathrm{T}}\boldsymbol{\lambda})^2 \mid \boldsymbol{\lambda} \in \mathcal{V}^*\} = \mathrm{Csp}\{\boldsymbol{\mu}^2 \mid \boldsymbol{\mu} \in \mathcal{V}'^*\}.$$

It follows by (3.7) that $\mathrm{Rng}\,\mathrm{pqu}(L) = \mathrm{Pqu}\,\mathcal{V}'$, so that $\mathrm{pqu}(L)$ is surjective. Of course $\mathrm{Rng}\,\mathrm{qu}(L) = \mathrm{Lsp}\,(\mathrm{Rng}\,\mathrm{pqu}(L))$, so $\mathrm{qu}(L)$ is also surjective. □

Proposition 6.5. *Let* L *in* $\mathrm{Lin}(\mathcal{V}', \mathcal{V})$ *and* $\mathcal{U} \in \mathrm{Subsp}\,\mathcal{V}$ *be given. Then*

$$\mathrm{pqu}(L)_{>}\left(\Phi_{\mathcal{V}}(\mathcal{U})\right) = \mathrm{qu}(L)_{>}\left(\Phi_{\mathcal{V}}(\mathcal{U})\right) = \Phi_{\mathcal{V}'}\left(L^{<}(\mathcal{U})\right). \tag{6.6}$$

Proof. Let $\boldsymbol{\phi} \in \Phi_{\mathcal{V}}(\mathcal{U})$ be given. By (3.12), $\boldsymbol{\phi}|_{\mathcal{U}} = 0$. It follows immediately that $(\boldsymbol{\phi} \circ L)|_{L^{<}(\mathcal{U})} = 0$. Of course $\boldsymbol{\phi} \circ L$ is positive, so it follows by (3.12) that $\mathrm{qu}(L)\boldsymbol{\phi} = \boldsymbol{\phi} \circ L \in \Phi_{\mathcal{V}'}(L^{<}(\mathcal{U}))$. Since $\boldsymbol{\phi}$ in $\Phi_{\mathcal{V}}(\mathcal{U})$ was arbitrary, it follows that

$$\mathrm{qu}(L)_{>}\left(\Phi_{\mathcal{V}}(\mathcal{U})\right) \subset \Phi_{\mathcal{V}'}\left(L^{<}(\mathcal{U})\right).$$

We note that, by the Theorem on Annihilators and Transposes ([Nol87], Sect. 21) and by (6.5) and (4.12), we have

$$(6.7)\quad (\mathrm{qu}(L)_{>}(\Phi_{\mathcal{V}}(\mathcal{U})))^{\perp} = \mathrm{qu}(L^{\mathrm{T}})^{<}\left((\Phi_{\mathcal{V}}(\mathcal{U}))^{\perp}\right) = \left(\mathrm{qu}\left(L^{\mathrm{T}}\right)\right)^{<}(\{\boldsymbol{g} \in \mathrm{Qu}\,\mathcal{V}^* \mid \boldsymbol{g}|_{\mathcal{U}^\perp} = \mathbf{0}\}) = \{\boldsymbol{f} \in \mathrm{Qu}\,\mathcal{V} \mid (\boldsymbol{f} \circ L^{\mathrm{T}})|_{\mathcal{U}^\perp} = \mathbf{0}\}.$$

Now let $\boldsymbol{f} \in (\mathrm{qu}(L)_{>}(\Phi_{\mathcal{V}}(\mathcal{U})))^{\perp}$ be given. It follows by (6.7) that $(\boldsymbol{f} \circ L^{\mathrm{T}})|_{\mathcal{U}^\perp} = \mathbf{0}$. Using the Theorem on Annihilators and Transposes again, we obtain

$$\{\mathbf{0}\} = \left(\boldsymbol{f} \circ L^{\mathrm{T}}\right)_{>}(\mathcal{U}^{\perp}) = \boldsymbol{f}_{>}(L^{\mathrm{T}}{}_{>}(\mathcal{U}^{\perp})) = \boldsymbol{f}_{>}\left((L^{<}(\mathcal{U}))^{\perp}\right),$$

so $\boldsymbol{f}|_{(L^{<}(\mathcal{U}))^{\perp}} = 0$. In view of (4.12) again, it follows that $\boldsymbol{f} \in (\Phi_{\mathcal{V}'}(L^{<}(\mathcal{U})))^{\perp}$. Since $\boldsymbol{f} \in (\mathrm{qu}(L)_{>}(\Phi_{\mathcal{V}}(\mathcal{U})))^{\perp}$ was arbitrary, it follows that

$$(\mathrm{qu}(L)_{>}(\Phi_{\mathcal{V}}(\mathcal{U})))^{\perp} \subset (\Phi_{\mathcal{V}'}(L^{<}(\mathcal{U})))^{\perp}$$

and hence

$$(\mathrm{qu}(L)_{>}(\Phi_{\mathcal{V}}(\mathcal{U}))) \supset (\Phi_{\mathcal{V}'}(L^{<}(\mathcal{U}))).$$

This establishes the second equality in (6.6); the first follows by Prop. 6.1. □

Theorem 3. *Let $P \in \mathrm{Lin}(\mathrm{Pqu}\,\mathcal{V}, \mathrm{Pqu}\,\mathcal{V}')$ be given. Then P is linearly induced if and only if both P and P^{T} are face-preserving.*

Before proving this theorem, we note that the cone-transpose of a face-preserving cone-linear mapping need not be face-preserving. Indeed, suppose that $\dim \mathcal{V} \geq 2$ and that $\dim \mathcal{V}' \geq 1$, and choose $\boldsymbol{f} \in \mathrm{Pqu}^{+}\mathcal{V}^*$ and $\boldsymbol{\mu} \in (\mathcal{V}'^*)^{\times}$. Then the mapping $(\boldsymbol{\mu}^2 \otimes \boldsymbol{f})|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'}$ is face-preserving because the image of every non-zero face of $\mathrm{Pqu}\,\mathcal{V}$ under this mapping is the extreme ray $\mathbb{P}\,\boldsymbol{\mu}^2$ of $\mathrm{Pqu}\,\mathcal{V}'$. However, the range of the cone-transpose $(\boldsymbol{f} \otimes \boldsymbol{\mu}^2)|_{\mathrm{Pqu}\,\mathcal{V}'^*}^{\mathrm{Pqu}\,\mathcal{V}^*}$ of the mapping above is $\mathbb{P}\,\boldsymbol{f}$, which is not a face.

Proof (Thm. 3). On the one hand, suppose that P is linearly induced and choose $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ such that $P = \mathrm{qu}(L)$. Let a $\mathcal{F} \in \mathrm{Face}(\mathrm{Pqu}\,\mathcal{V})$ be given. In view of Thm. 1, we may determine $\mathcal{U} \in \mathrm{Subsp}\,\mathcal{V}$ such that $\mathcal{F} = \Phi_{\mathcal{V}}(\mathcal{U})$. Then $P_{>}(\mathcal{F}) = \Phi_{\mathcal{V}'}(L^{<}(\mathcal{U}))$ by Prop. 6.5, so $P_{>}(\mathcal{F})$ was a face of $\mathrm{Qu}\,\mathcal{V}'$ by Prop. 3.3. Since $\mathcal{F} \in \mathrm{Face}(\mathrm{Pqu}\,\mathcal{V})$ was arbitrary, it follows that the P is face-preserving. Indeed, since P was an arbitrary linearly induced cone-linear mapping, every linearly induced cone-linear mapping is face-preserving. In particular, in view of Prop. 6.3, P^{T} is face-preserving.

On the other hand, suppose that P and P^{T} preserve faces. Then their ranges are faces of $\mathrm{Pqu}\,\mathcal{V}'$ and $\mathrm{Pqu}\,\mathcal{V}^*$, respectively. It follows by Thm. 1 that subspaces $\mathcal{U}$ and $\mathcal{U}'$ of $\mathcal{V}$ and $\mathcal{V}'$, respectively, can be determined such that

$$(6.8)\qquad \mathrm{Rng}\,P = \Phi_{\mathcal{V}'}(\mathcal{U}');$$

$$(6.9)\qquad \mathrm{Rng}\,P^{\mathsf{T}} = \Phi_{\mathcal{V}^*}(\mathcal{U}^{\perp}).$$

Denote by $Q \in \mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}\,\mathcal{V}')$ the linear mapping determined by P, so that $Q|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'} = P$. It follows from (6.8) that

$$(6.10)\qquad \mathrm{Rng}\,Q = \mathrm{Lsp}\,\mathrm{Rng}\,P = \mathrm{Lsp}\,\Phi_{\mathcal{V}'}(\mathcal{U}'),$$

and from the Theorem on Annihilators and Transposes ([Nol87], Sect. 21), (6.9), and (4.12) that

$$(6.11)\quad \operatorname{Null} Q = \left(\operatorname{Rng} Q^{\mathrm{T}}\right)^{\perp} = \left(\operatorname{Rng} P^{\mathbf{T}}\right)^{\perp} = \left(\Phi_{\mathcal{V}^*}\left(\mathcal{U}^{\perp}\right)\right)^{\perp} = \{\phi \in \operatorname{Qu}\mathcal{V} \mid \phi|_{\mathcal{U}} = 0\}.$$

We will now describe and establish the validity of the following commutative diagram:

$$\begin{array}{ccccccc}
\operatorname{Dom} Q & \xrightarrow{\Omega_{\operatorname{Dom} Q/\operatorname{Null} Q}} & \operatorname{Dom} Q/\operatorname{Null} Q & \xrightarrow{\tilde{Q}} & \operatorname{Rng} Q & \xrightarrow{\mathbf{1}_{\operatorname{Rng} Q \subset \operatorname{Dom} Q}} & \operatorname{Cod} Q \\
\| & & \updownarrow & & \updownarrow & & \| \\
\operatorname{Qu}\mathcal{V} & \xrightarrow{\operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})} & \operatorname{Qu}\mathcal{U} & \xrightarrow{\tilde{\tilde{Q}}} & \operatorname{Qu}(\mathcal{V}'/\mathcal{U}') & \xrightarrow{\operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'})} & \operatorname{Qu}\mathcal{V}'
\end{array}$$

The upper line in the diagram represents Q as the composite of three linear mappings as described in Prop. 1.3. Two linear isomorphism-pairs, represented in the diagram by two-headed vertical arrows, will be determined below. Then, in turn, a linear isomorphism $\tilde{\tilde{Q}} \in \operatorname{Lin}(\operatorname{Qu}\mathcal{V}, \operatorname{Qu}(\mathcal{V}'/\mathcal{U}')$ will be determined such that

$$(6.12)\qquad Q = \operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'})\, \tilde{\tilde{Q}}\, \operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}).$$

It will be shown that $\tilde{\tilde{Q}}$ is linearly induced, and hence that the lower line in the diagram represents Q as a composite of three linearly induced mappings.

The linear mappings $\Omega_{\operatorname{Dom} Q/\operatorname{Null} Q}$ and $\operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})$ on the left side of the diagram are both surjective: the first because it is a quotient mapping and the second, since $\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}$ is injective, by Prop. 6.4. Of course $\operatorname{Null}\Omega_{\operatorname{Dom} Q/\operatorname{Null} Q} = \operatorname{Null} Q$, and, in view of (6.11),

$$\operatorname{Null}\operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}) = \{\phi \in \operatorname{Qu}\mathcal{V} \mid \phi|_{\mathcal{U}} = 0\} = \operatorname{Null} Q.$$

Thus $\Omega_{\operatorname{Dom} Q/\operatorname{Null} Q}$ and $\operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})$ are linear surjections with common domain and nullspace. It follows by Cor. 1 of Prop. 1.3 that the leftmost square in the commutative diagram determines a linear isomorphism-pair $\operatorname{Dom} Q/\operatorname{Null} Q \leftrightarrow \operatorname{Qu}\mathcal{V}$ as indicated.

Similarly, the linear mappings $\mathbf{1}_{\operatorname{Rng} Q\subset\operatorname{Cod} Q}$ and $\operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'})$ on the right side of the diagram are both injective: the first because it is an inclusion mapping and the second, since $\Omega_{\mathcal{V}'/\mathcal{U}'}$ is surjective, by Prop. 6.4. Of course $\operatorname{Rng}\mathbf{1}_{\operatorname{Rng} Q\subset\operatorname{Cod} Q} = \operatorname{Rng} Q$ and, in view of Prop. 3.4 and (6.10),

$$\operatorname{Rng}\operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'}) = \operatorname{Lsp}\Phi_{\mathcal{V}'}(\mathcal{U}') = \operatorname{Rng} Q.$$

Thus $\mathbf{1}_{\operatorname{Rng} Q\subset\operatorname{Cod} Q}$ and $\operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'})$ are linear injections with common range. It follows by Cor. 2 of Prop. 1.3 that the rightmost square in the commutative diagram determines a linear isomorphism-pair $\operatorname{Rng} Q \leftrightarrow \operatorname{Qu}(\mathcal{V}'/\mathcal{U}')$ as indicated.

Since the double-headed vertical arrows in the diagram indeed represent linear isomorphism-pairs, and since $\tilde{Q}$ is a linear isomorphism by Prop. 1.3, the center square in the diagram determines a linear isomorphism $\tilde{\tilde{Q}}$ as indicated.

We shall next show that $\tilde{\tilde{Q}}$ is cone-preserving relative to the cones $\operatorname{Pqu}\mathcal{V}$ and $\operatorname{Pqu}\mathcal{V}'$.

In view of (6.12) and (6.8), we have

$$\left(\operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'})\,\tilde{\tilde{Q}}\,\operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})\right)_{>}(\operatorname{Pqu}\mathcal{V}) = Q_{>}(\operatorname{Pqu}\mathcal{V}) = \Phi_{\mathcal{V}'}(\mathcal{U}')\,. \tag{6.13}$$

Since $\operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'})$ is injective, it follows immediately that

$$\tilde{\tilde{Q}}_{>}(\operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}))_{>}(\operatorname{Pqu}\mathcal{V}) = \operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'})^{<}(\Phi_{\mathcal{V}'}(\mathcal{U}'))\,. \tag{6.14}$$

Since $\operatorname{pqu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})$ is linearly induced, it preserves faces (as shown in the first part of this proof). It follows that the image under $\operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})$ of the cone $\operatorname{Pqu}\mathcal{V}$, which is spanning in $\operatorname{Qu}\mathcal{V}$, is a face of $\operatorname{Pqu}\mathcal{U}$ which is spanning in $\operatorname{Rng}P = \operatorname{Qu}\mathcal{U}$; hence

$$\operatorname{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})_{>}(\operatorname{Pqu}\mathcal{V}) = \operatorname{Pqu}\mathcal{U}. \tag{6.15}$$

By Prop. 3.4 we have

$$\operatorname{qu}(\Omega_{\mathcal{V}'/\mathcal{U}'})^{<}(\Phi_{\mathcal{V}'}(\mathcal{U}')) = \operatorname{Pqu}(\mathcal{V}'/\mathcal{U}')\,. \tag{6.16}$$

Substitution of (6.15) and (6.16) into (6.14) yields

$$\tilde{\tilde{Q}}_{>}(\operatorname{Pqu}\mathcal{U}) = \operatorname{Pqu}(\mathcal{V}'/\mathcal{U}')\,;$$

so the linear isomorphism $\tilde{\tilde{Q}}$ is cone-preserving. It follows that $\tilde{\tilde{Q}}|_{\operatorname{Pqu}\mathcal{V}}^{\operatorname{Pqu}\mathcal{V}'}$ is a cone-isomorphism; hence,in view of Thm. 2, there is a linear isomorphism, say $\tilde{\tilde{L}}\colon \mathcal{V}'/\mathcal{U}' \to \mathcal{U}$, which induces $\tilde{\tilde{Q}}|_{\operatorname{Pqu}\mathcal{V}}^{\operatorname{Pqu}\mathcal{V}'}$ and hence $\tilde{\tilde{Q}}$ as well. Thus $\tilde{\tilde{Q}} = \operatorname{qu}(\tilde{\tilde{L}})$. Putting $L := \mathbf{1}_{U\subset\mathcal{V}}\,\tilde{\tilde{L}}\Omega_{\mathcal{V}'/\mathcal{U}'} \in \operatorname{Lin}(\mathcal{V}',\mathcal{V})$, it follows, in view of (6.12) and Prop. 6.2, that $Q = \operatorname{qu}(L)$ and hence, in view of Prop. 6.1, that $P = \operatorname{pqu}(L)$. □

Proposition 6.6. *Let* $L, M \in \operatorname{Lin}(\mathcal{V}',\mathcal{V})$ *be given. Then* $\operatorname{pqu}(L) = \operatorname{pqu}(M)$ *(equivalently,* $\operatorname{qu}(L) = \operatorname{qu}(M)$*) if and only if* $M = L$ *or* $M = -L$.

Proof. It is easy to see that $\operatorname{pqu}(L) = \operatorname{pqu}(-L)$. It is also clear that the result holds if $L = \mathbf{0}$. Assume that $L \neq \mathbf{0}$, put $n := \dim\mathcal{V}$, and choose a basis $\boldsymbol{\beta} = (\boldsymbol{\beta}_i \mid i \in n^{]})$ of $\mathcal{V}'$ such that $\boldsymbol{\beta}_1 L \neq \mathbf{0}$. By the same argument used in the last part of the proof of Thm. 2, we may determine $\epsilon \in \{1,-1\}$ such that $L^{\mathrm{T}}\boldsymbol{\beta}_k = M^{\mathrm{T}}\boldsymbol{\beta}_k$ for all $k \in n^{]}$. It follows that $L = \epsilon M$. □

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