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The Dynamics of Solid-Solid Phase Transitions
2. Incoherent Interfaces

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THE DYNAMICS OF SOLID-SOLID PHASE TRANSITIONS
2. INCOHERENT INTERFACES

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Abstract. Incoherent phase transitions are more difficult to treat than their coherent counterparts. The interface, which appears as a single surface in the deformed configuration, is represented in its undeformed state by a separate surface in each phase. This leads to a rich but detailed kinematics, one in which defects such as vacancies and dislocations are generated by the moving interface. In this paper we develop a complete theory of incoherent phase transitions in the presence of deformation and mass transport, with phase interface structured by energy and stress. The final results are a complete set of interface conditions for an evolving incoherent interface.

CONTENTS

FREQUENTLY USED SYMBOLS

A_i, C_i	generic subsurface of S_i	§3.4, §5.1c
B_i	undeformed phase-i region	§3.2
$\mathbf{C}$	configurational bulk stress, Eshelby tensor	§6, (7.10)
$\mathbf{F}$	deformation gradient	§3.2
$\mathbf{G}$	inverse deformation gradient	(3.1)
$\mathbf{H}$	relative deformation gradient	(3.11)
J	bulk jacobian of the deformation	(3.3)
$\bar{K}$, K_i	total (twice the mean) curvature of $\mathcal{S}$ and S_i	§3.2
$\mathrm{Lin}(U,V)$	linear transformations from U into V	
Lin^+	linear transformations of $\mathbb{R}^3$ with positive determinant	§2
Orth^+	rotations of $\mathbb{R}^3$	§2
Q^α	external bulk mass supply of species α	§6
$\bar{\mathbf{S}}$	bulk Cauchy stress tensor	§6
$\mathbf{S}$	bulk Piola-Kirchhoff stress tensor	§6
S_i	undeformed phase i interface	§3.2
U_i	relative velocity of S_i	(4.5)
Unim^+	linear transformations of $\mathbb{R}^3$ with unit determinant	§2
$\bar{V}$, V_i	normal velocity of $\mathcal{S}$ and S_i	§4.1
$V_{(\partial\mathcal{G})\tan}$, $V_{(\partial A_i)\tan}$	intrinsic edge velocity of $\partial\mathcal{G}\subset\mathcal{S}$ and $\partial A_i\subset S_i$	§4.2b
W_i	volume flow across the phase-i interface	(4.6)
$\mathbf{X}$	material point	
$\mathbf{b}$	external body force	§6
$\mathbf{e}$	internal bulk configurational force	§6
$\mathbf{f}_i$	external interfacial force (configurational)	§6
$\bar{\mathbf{g}}$	external interfacial force (deformational)	§6
grad, div	spatial gradient and divergence	§2
$\mathrm{grad}_{\mathcal{S}}$, $\mathrm{div}_{\mathcal{S}}$	gradient and divergence on $\mathcal{S}$	
$\mathbf{h}$	relative deformation	(3.10)
$\mathbf{h}^\alpha$, $\mathfrak{h}$	diffusive mass flux of species α and list of mass fluxes	§6, (9.1)
$\bar{\mathbf{m}}$	outward unit normal to a spatial control volume	§4.6
$\bar{\mathbf{n}}$, $\mathbf{n}_i$	unit normal to $\mathcal{S}$ and S_i	§3.2
$\mathbf{n}^\perp$	subspace of $\mathbb{R}^3$ orthogonal to $\mathbf{n}$	
$\bar{q}^\alpha$	external interfacial mass supply of species α	§6
$\mathbf{s}$		(13.8)
$\bar{\mathbf{v}}$, $\mathbf{v}_i$	compatible velocity fields of $\mathcal{S}$ and S_i	§4.2a
$\bar{\mathbf{w}}$, $\mathbf{w}_i$	compatible edge velocity fields for $\partial\mathcal{G}$ and ∂A_i	§4.2b
$\mathbf{x}$	spatial point	
$\mathbf{y}_i$	deformation or motion of phase i	§3.2, §4.1
$\mathbf{y}^\cdot$	material velocity	§4.1
$\mathcal{G}$, $\mathcal{C}$	generic subsurfaces of $\mathcal{S}$	§3.4, §5.1c
$\mathcal{B}$, $\mathcal{B}_i$	deformed body and deformed phase-i region	§3.2
$\mathcal{E}(\mathcal{R})$	energy supplied to $\mathcal{R}$ by mass transport	(8.1)
$\mathcal{G}$	symmetry group of the lattice	§5.1a
$\mathcal{J}_i$, $\mathcal{H}$	surface jacobians	(3.4), (3.5)
$\mathcal{L}$	lattice	§3.1, §5.1a

$\mathcal{P}(\mathcal{R})$	power expended on $\mathcal{R}$	(8.2), (13.1)
$\mathcal{R}$	spatial control volume	§4.6
$\mathcal{S}$	deformed phase interface	§3.2
ℓ	lattice point density	§5.2
$\bar{\mathfrak{p}}$, $\mathfrak{p}$	interfacial power density	(8.13), (13.10), (16.8)
$\bar{\mathbf{A}}$, $\mathbf{A}$	total surface stress	(13.3), (16.4)
$\mathbf{C}$	configurational surface stress for phase 1 (material)	§12
$\bar{\mathbf{C}}_i$	configurational surface stress (spatial)	§12
$\mathbf{F}_i$	tangential deformation gradient	(3.9)
$\mathbf{S}_i$	inverse tangential deformation gradient	(3.9)
$\mathbf{H}$	incoherency tensor	(3.12)
$\bar{\mathbf{1}}(\mathbf{x})$, $\mathbf{1}_i(\mathbf{X})$	inclusions of $\bar{\mathbf{n}}^{\perp}(\mathbf{x})$ and $\mathbf{n}_i^{\perp}(\mathbf{X})$ into $\mathbb{R}^3$	§3.2
$\mathbf{K}$	configurational surface stress for phase 2 (material)	§12
$\mathbf{L}$, $\mathbf{L}_i$	curvature tensor of $\mathcal{S}$ and S_i	§3.2
$\bar{\mathbf{P}}(\mathbf{x})$, $\mathbf{P}_i(\mathbf{X})$	projections of $\mathbb{R}^3$ onto $\bar{\mathbf{n}}^{\perp}(\mathbf{x})$ and $\mathbf{n}_i^{\perp}(\mathbf{X})$	§3.2
$\bar{\mathbf{S}}$, $\mathbf{S}$	deformational surface stress (spatial and material)	§12
$\bar{\mathfrak{a}}$, $\mathfrak{a}$	normal part of total surface stress	(13.11), §16.1
$\mathfrak{c}$	normal part of configurational surface stress for phase 1 (material)	(14.10)
$\mathfrak{e}_i$	internal interfacial configurational force	§6
$\bar{\boldsymbol{\nu}}$, $\boldsymbol{\nu}_i$	unit normal to $\partial\mathcal{G}$ and ∂A_i	§4.2b
$\bar{\Lambda}(\mathbf{x})$, $\Lambda_i(\mathbf{X})$	projections of $\mathbb{R}^3$ onto $\bar{\mathbf{n}}^{\perp}(\mathbf{x})$ and $\mathbf{n}_i^{\perp}(\mathbf{X})$	(3.6)
Π_i	normal internal force (material)	(8.11)
Ψ	bulk free energy	§6
γ	slip velocity	(4.26)
$\delta_i = (-1)^i$		
λ_i		(3.2)
μ^{α}, μ	chemical potential of species α and list of potentials	§6, (9.1)
ρ^{α}, ρ	bulk molar density of species α and list of molar densities	§6, (9.1)
π_i	normal internal force (spatial)	(11.3)
σ	surface tension,	(13.4)
τ, τ_i	effective shear	(8.6), (8.9)
$\bar{\varphi}$	referential-to-spatial transform of a field φ (not all quantities with overbars are transforms)	§3.4
ψ	interfacial energy	§13.3
ω	grand canonical potential	(7.9)
$\mathbf{1}$	unit tensor in $\mathbb{R}^3$	
$\times$, $\otimes$	vector and tensor product in $\mathbb{R}^3$	
$(\ldots)^{\cdot}$, $\partial_t(\ldots)$	material and spatial time derivative	§2
∇, Div	material gradient and divergence	§2
∇_{S_i}, Div_{S_i}	gradient and divergence on S_i	
$(\ldots)^{r}$, $(\ldots)^{\circ}$	normal time derivative following $\mathcal{S}$ and S_i	(4.19), (4.21)
$(\ldots)_i$	limit of a bulk field as $\mathbf{x} \to \mathcal{S}$, $\mathbf{x} \in \mathcal{B}_i$	§2
$[\ldots]$, $\langle\ldots\rangle$	jump and average of a bulk field across the interface	§2
$(\ldots)_{ext}$	extension of a surface tensor to $\mathbb{R}^3$	(12.4)
$(..)_{tan\mathcal{S}}$, $(..)_{tanS_i}$	tangential part of a vector (tensor) on $\mathcal{S}$ and S_i	(3.7)

1. INTRODUCTION.

This paper concludes a two-part series on phase transitions in deformable solids, with and without mass transport, with the interface between phases sharp and capable of supporting energy and stress. Part 1, which was restricted to coherent phase transitions, is here extended to phase transitions that are incoherent. As in Part 1, the discussion is limited to a theory that neglects the flow of heat, concentrating instead on deformation and mass transport. In addition, *inertia is neglected.*

In a coherent phase transition the body B occupies a fixed region of space in a uniform reference configuration, the individual phases, which we label $i=1,2$, occupy complementary subregions $B_i(t)$ of B, and motions are continuous across the undeformed phase interface $S(t) = B_1(t) \cap B_2(t)$. As is clear from the statical treatments of Cahn and Larché [1982], Larché and Cahn [1985], and Leo and Sekerka [1989], incoherent phase transitions are far more complicated. The interface, which appears as a single surface in the deformed body, is represented in its undeformed state by a *separate* surface $S_i(t)$ for each phase i, even though we choose uniform reference configurations for the two phases with corresponding reference lattices coincident. Such complications lead to a rich but detailed kinematics, one in which defects such as dislocations, vacancies, and interstitials may be generated by the moving interface.[1]

We begin with a discussion of the underlying kinematics and with a systematic treatment of two-phase motions $\mathbf{y}=(\mathbf{y}_1,\mathbf{y}_2)$: at each time t, $\mathbf{y}_i$ maps material points $\mathbf{X}$ in the undeformed region B_i for phase i into points $\mathbf{x} = \mathbf{y}_i(\mathbf{X},t)$ in the deformed body. We write $\mathbf{F}$ for the deformation gradient: $\mathbf{F}=\nabla\mathbf{y}_1$ in phase 1, $\mathbf{F}=\nabla\mathbf{y}_2$ in phase 2; in addition, we denote by $\mathbf{F}_i$ the limit of $\mathbf{F}$ as the interface is approached from phase i.

Associated with each two-phase motion are three basic kinematical quantities:

(1) The *incoherency tensor* $\mathbf{H}$, which measures the stretching and twisting of one phase relative to the other at the interface. $\mathbf{H}$ is the tangential part of the relative deformation gradient

[1] Dislocations are discussed by Brooks [1952], Nye [1953], Frank [1955], Bilby [1955], Bilby, Bullough, and De Grinberg [1964], Christian [1965,1985], Bollman [1967], Christian and Crocker [1980], Pond [1985,1989]. Christian and Crocker [1980], p. 181 and Larché and Cahn [1985], p. 1587 note the possibility of vacancies and dislocations.

$$\mathbf{H} = \mathbf{F}_2^{-1}\mathbf{F}_1. \tag{1.1}$$

For any point $\mathbf{x}$ of the deformed interface, $\mathbf{H}$ is a linear transformation $d\mathbf{X}_2 = \mathbf{H}d\mathbf{X}_1$ between infinitesimal line segments $d\mathbf{X}_i$ on $\mathcal{S}_i$ that coincide at $\mathbf{x}$ when deformed. If, for all such line segments, $d\mathbf{X}_2 = d\mathbf{X}_1$ (or $d\mathbf{X}_2 = \mathbf{Q}d\mathbf{X}_1$ with $\mathbf{Q}$ a symmetry rotation of the lattice), then the deformed lattices fit together and the interface is *infinitesimally coherent* at $\mathbf{x}$.

(2) The *production-rate of lattice points,* as measured by the jump $[W]$ in the interfacial volume flows $W_i = V_i/\jmath_i$, where, for each i, V_i is the normal velocity of $\mathcal{S}_i$, while $\jmath_i$ is the surface Jacobian for $\mathbf{y}_i$ considered as a deformation of $\mathcal{S}_i$.

(3) The *slip,* as measured by the difference $(\mathbf{y}_2)^\circ - (\mathbf{y}_1)^\circ$, where $(\mathbf{y}_i)^\circ$ is the time derivative of $\mathbf{y}_i$ following the normal trajectories of $\mathcal{S}_i(t)$.

The incoherency tensor, the lattice-point production, and the slip completely characterize incoherency: *an initially coherent motion is coherent for all time if and only if, at each time, the interface is infinitesimally coherent and the slip and lattice-point production vanish identically.*[2]

The basic physical principles upon which our theory is based are balance of forces, balance of mass, and a version of the second law of thermodynamics appropriate to a mechanical theory.[3] The standard forces associated with continua arise as a response to the motion of material points. The mechanical description of a phase transition requires additional forces[4] that act in response to microstructural changes at the phase interface. We refer to the former as **deformational forces**, to the latter as **configurational forces.**[5] What is most important is that, in addition to the usual force and moment balances for deformational forces, *we postulate an additional balance for configurational forces.*

We assume that there are $\mathfrak{A}$ species, $\alpha = 1,2,\ldots,\mathfrak{A}$, of mobile atoms

[2] Cermelli and Gurtin [1993].

[3] Cf. Gurtin [1991].

[4] Cf. the discussion given in the Introduction of [Gu]. Throughout we write [Gu] for the reference "Gurtin [1993]", which is Part 1 of this series, and [GS] for the reference "Gurtin and Struthers [1990]".

[5] Here we depart from terminology introduced in [GS] and [Gu], where the term *accretive forces* was used.

with *molar* densities ρ^α and corresponding diffusive mass fluxes $\mathbf{h}^\alpha$.[6] Bulk fields that strongly influence the motion of the interface are the *grand canonical potential* ω and the *Eshelby tensor* $\mathbf{C}$ defined by

$$\omega = \Psi - \sum_{\alpha=1}^{\mathfrak{A}} \rho^\alpha \mu^\alpha, \qquad \mathbf{C} = \omega \mathbf{1} - \mathbf{F}^T \mathbf{S}, \tag{1.2}$$

with Ψ the bulk energy, μ^α the chemical potential of species α, $\mathbf{S}$ the bulk stress, measured per unit undeformed area (Piola-Kirchhoff stress), and $\mathbf{1}$ the unit tensor.

The final bulk relations are the balance laws

$$\mathrm{Div}\,\mathbf{S} = \mathbf{0}, \qquad (\rho^\alpha)^\cdot = -\mathrm{Div}\,\mathbf{h}^\alpha, \tag{1.3}$$

supplemented by constitutive equations

$$\Psi = \hat{\Psi}_i(\mathbf{F},\rho), \qquad \mathbf{S} = \partial_{\mathbf{F}} \hat{\Psi}_i(\mathbf{F},\rho), \qquad \mu = \partial_\rho \hat{\Psi}_i(\mathbf{F},\rho),$$
$$\mathfrak{h} = -\mathbf{D}_i(\mathbf{F},\rho)\nabla\mu, \tag{1.4}$$

for each phase i, with

$$\rho = (\rho^1,\ldots,\rho^{\mathfrak{A}}), \qquad \mu = (\mu^1,\ldots,\mu^{\mathfrak{A}}), \qquad \mathfrak{h} = (\mathbf{h}^1,\ldots,\mathbf{h}^{\mathfrak{A}}),$$

and with *mobility* $\mathbf{D}_i(\mathbf{F},\rho)$ a linear transformation compatible with the inequality $\Sigma_\alpha \mathbf{h}^\alpha \cdot \nabla\mu^\alpha \le 0$. (For completeness we assume elastic behavior in bulk, but the interface conditions we derive are independent of the particular choice of bulk constitutive equations.)

We turn next to the development of appropriate interface conditions. To best illustrate the basic ideas, we begin with a theory that neglects interfacial energy and stress, but includes interface kinetics. The resulting interface conditions consist of an equation

$$[\mathbf{y}^\cdot]\cdot\bar{\mathbf{n}} = -[JW] \tag{1.5}$$

expressing kinematical compatibility at the interface, a jump condition

$$[\mathcal{J}^{-1}\mathbf{S}\mathbf{n}] = \mathbf{0} \tag{1.6}$$

[6] Cf. Gurtin and Voorhees [1993], [Gu].

balancing forces across the interface, equations

$$\delta_i \mathbf{n}_i \cdot \mathbf{C}_i \mathbf{n}_i = (\beta_{i1} W_1 + \beta_{i2} W_2) \tag{1.7}$$

(i=1,2) balancing normal configurational forces on each phase at the interface, equations

$$(\mathbf{F}_i^{\mathsf{T}} \mathbf{S}_i \mathbf{n}_i)_{\tan S_i} = \mathbf{0} \tag{1.8}$$

(i=1,2) characterizing the vanishing of the tangential traction in each phase at the interface, a relation

$$[\rho^\alpha W] = [\gamma^{-1} \mathbf{h}^\alpha \cdot \mathbf{n}] \tag{1.9}$$

expressing mass balance for each species α, and a condition of local equilibrium

$$[\mu^\alpha] = 0 \tag{1.10}$$

for each species α. Here $\bar{\mathbf{n}}$ is the unit normal to the deformed interface $\mathcal{S}$, $\mathbf{n}_i$ is the unit normal to the undeformed phase i interface S_i, $[f]$ denotes the jump in a bulk field f across the interface, f_i denotes the interfacial limit of f from phase i, and β_{ij} are kinetic coefficients.

In the derivation of these interface conditions the slip was not included among the independent constitutive variables,[7] a direct consequence of this assumption is (1.8). The local equilibrium condition (1.10) is an assumption made from the outset.[8]

The balances (1.6)-(1.8) can be expressed more succintly as a normal force balance

$$[\gamma^{-1} \mathbf{S} \mathbf{n}] \cdot \bar{\mathbf{n}} = \mathbf{0} \tag{1.11}$$

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[7] Although we do discuss the form the basic equations take when the slip is included as a constitutive variable.

[8] Gurtin and Voorhees [1994] develop a theory in which this assumption is dropped. Their theory neglects deformation.

and a partial balance

$$\delta_i C_i \mathbf{n}_i = (\beta_{i1} W_1 + \beta_{i2} W_2)\mathbf{n}_i \tag{1.12}$$

for each phase i.

We turn next to a theory that includes interfacial energy and stress, but neglects mass flow within the interface. Here it is convenient to choose phase 1 as a reference for the interface, to measure interfacial fields relative to S_1, and to use the abbreviations

$$S = S_1, \qquad \mathbf{n} = \mathbf{n}_1.$$

We consider a single interfacial energy ψ, but endow the interface with three stress fields:

- a deformational stress $\mathbf{S}$ that represents the (Piola-Kirchhoff) stress in the surface and acts in response to the stretching of the phase 1 interface;
- a configurational stress $\mathbf{C}$ that represents microstructural forces in the phase 1 interface;
- a configurational stress $\mathbf{K}$ that acts in response to the stretching and rotation of the phase 2 lattice relative to that of phase 1.

We show, as a consequence of thermodynamics, that the tangential part of the total surface stress

$$\mathbf{A} = \mathbf{C} + \mathbf{F}_1^{\mathsf{T}}\mathbf{S} + \mathbf{H}^{\mathsf{T}}\mathbf{K}, \tag{1.13}$$

which represents the net configurational contribution of the stresses to the rate of working, is a surface tension whose value is the interfacial energy ψ.

Among the constitutive equations considered for the interface are relations giving the interfacial energy ψ, the surface stresses $\mathbf{S}$ and $\mathbf{K}$, and the normal part $\mathbf{a}=\mathbf{A}^{\mathsf{T}}\mathbf{n}$ of the total surface stress as functions of the limiting value $\mathbf{F}=\mathbf{F}_1$ of the deformation gradient, the relative deformation gradient $\mathbf{H}$, the limiting values ρ_1 and ρ_2 of the list of densities, the normal $\mathbf{n}$ to $S=S_1$, and the volume flows W_1 and W_2. We show, as a consequence of the second law, that ψ, $\mathbf{S}$, $\mathbf{K}$, and $\mathbf{a}$ are independent of ρ_1, ρ_2, W_1, and W_2; and that the energy

$$\psi = \hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}) \tag{1.14}$$

generates the stresses through the relations[9]

$$\mathbf{S} = \partial_{\mathbf{F}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}), \qquad \mathbf{K} = \partial_{\mathbf{H}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}), \qquad \mathbf{e} = -\partial_{\mathbf{n}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}). \tag{1.15}$$

We show further that ψ, $\mathbf{S}$, and $\mathbf{K}$ depend on $\mathbf{F}$ and $\mathbf{H}$ through the tangential deformation gradient $\mathbf{F}$ and the incoherency tensor $\mathbf{H}$, that

$$\mathbf{S} = \partial_{\mathbf{F}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}), \qquad \mathbf{K} = \partial_{\mathbf{H}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}), \tag{1.16}$$

and that $\mathbf{c}=\mathbf{C}^{\mathrm{T}}\mathbf{n}$ is given by

$$\mathbf{c} = -D_{\mathbf{n}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}), \tag{1.17}$$

with $D_{\mathbf{n}}$ the derivative following $\mathbf{n}$.

The final results — which form a complete set of conditions for an incoherent interface — consist of the compatibility condition (1.5), the mass balance (1.9), the local equilibrium condition (1.10), an equation

$$\psi K - (\mathbf{F}^{\mathrm{T}}\mathbf{S} + \mathbf{H}^{\mathrm{T}}\mathbf{K})\cdot\mathbf{L} + \mathrm{Div}_S\mathbf{c} - \mathbf{n}\cdot\mathbf{C}_1\mathbf{n} = \beta_{11}W_1 + \beta_{12}W_2 \tag{1.18}$$

that represents a normal configurational balance for phase 1, an equation

$$\mathrm{Div}_S\mathbf{K} + \mathcal{H}\mathbf{C}_2\mathbf{n}_2 = \mathcal{H}(\beta_{21}W_1 + \beta_{22}W_2)\mathbf{n}_2 \tag{1.19}$$

that represents a configurational balance for phase 2, a deformational force balance

$$\mathrm{Div}_S\mathbf{S} + \mathcal{H}\mathbf{S}_2\mathbf{n}_2 - \mathbf{S}_1\mathbf{n}_1 = \mathbf{0}, \tag{1.20}$$

and the constitutive relations (1.16) and (1.17). Here $\mathcal{H} = \wp_1/\wp_2$, while

[9] The stresses $\mathbf{S}$ and $\mathbf{K}$ are surface tensors whose values at a point $\mathbf{X}\in S$ are linear transformations from the tangent space at $\mathbf{X}\in S$ into $\mathbb{R}^3$. In the formulas (1.15) these stresses should be interpreted as their trivial extension to linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$.

$\mathbf{L} = \mathbf{L}_1 = -\nabla_S \mathbf{n}$ and $K = K_1 = \mathrm{tr}\mathbf{L}$, respectively, are the curvature tensor and the total (twice the mean) curvature for S.

We also discuss the form the general theory takes when described spatially (in the deformed configuration), and we discuss the reduced form the interface conditions take when the interfacial energy $\bar{\psi}$, per unit deformed area, is noninteractive; that is, $\bar{\psi} = \bar{\psi}_1 + \bar{\psi}_2$ with the energy $\bar{\psi}_i$ for phase i dependent on kinematic quantities for that phase only.[10]

[10] Our discussion here is based on work of Leo and Sekerka [1989], who use the adjective "greased" to signify an interface whose energy is noninteractive; for statical situations our results are consistent with results derived variationally by Leo and Sekerka.

2. GENERAL NOTATION

We will frequently refer to the papers — and will generally follow the notation — of Gurtin and Struthers [1990] and Gurtin [1993]; for that reason we use the abbreviations:

[GS] = Gurtin and Struthers [1990],
[Gu] = Gurtin [1993].

In particular, our notation concerning tensors in $\mathbb{R}^3$ is given in §2 of [Gu], as are definitions of: deformation; evolving two-phase region; interface; control volume; jump $[\varphi]$, average $\langle\varphi\rangle$, and limit φ_i (from phase i) of a bulk field φ at the interface, a notation we will also use for arbitrary fields φ_1 and φ_2 on the interface, the subscripts here denoting the phase to which the field is associated (cf. the paragraph containing (4.1)). Our terminology involving surfaces and smoothly evolving surfaces can be found in Appendices A1 and A2 of [Gu]; specifically, defined there are the notions of superficial field, surface gradient, and surface divergence. As in [Gu], we write

$$\delta_i = (-1)^i, \tag{2.1}$$

and, for linear spaces U and V, we let

$\mathrm{Lin}(U,V)$ = space of linear transformations from U into V.

In addition, we use the specific abbreviations

$$\begin{aligned} \mathrm{Lin}^+ &= \{\mathbf{F}\in\mathrm{Lin}(\mathbb{R}^3,\mathbb{R}^3) : \det\mathbf{F} > 0\}, \\ \mathrm{Unim}^+ &= \{\mathbf{F}\in\mathrm{Lin}^+ : \det\mathbf{F} = 1\}, \\ \mathrm{Orth}^+ &= \{\mathbf{Q}\in\mathrm{Lin}^+ : \mathbf{Q}^T\mathbf{Q} = \mathbf{1}\}, \end{aligned}$$

We label material points by their positions $\mathbf{X}$ in a fixed homogeneous reference configuration. We label spatial points (points in the deformed body) by $\mathbf{x}$. We use an overbar to denote the referential-to-spatial transformation; i.e., e.g., $\mathbf{S}$ is the Piola-Kirchoff stress, $\mathbf{T} = \bar{\mathbf{S}}$ is the Cauchy stress; ρ is the mass density in the undeformed body, $\bar{\rho}$ is the density in the deformed body. Finally, we use the following notation for

derivatives:

$(\ldots)^{\cdot}$	material time-derivative (holding $\mathbf{X}$ fixed);
$\partial_t(\ldots)$	spatial time-derivative (holding $\mathbf{x}$ fixed);
∇, Div	material gradient and divergence (with respect to $\mathbf{X}$),
grad, div	spatial gradient and divergence (with respect to $\mathbf{x}$).

A. KINEMATICS

3. TWO-PHASE DEFORMATIONS

3.1. REFERENCE CONFIGURATION

The description of a solid undergoing a phase transformation is more complicated than the more classical descriptions encountered in continuum mechanics, one reason being the role played by the lattice structure of the material, especially at an incoherent phase interface. Within our theory the reference configuration for each phase should be envisioned as a copy of $\mathbb{R}^3$ together with a lattice $\mathcal{L}(\mathbf{X})$, which, for each $\mathbf{X}\in\mathbb{R}^3$, models the microstructure of the material at $\mathbf{X}$. Here we assume that each phase is referred to a *uniform reference configuration* in which the lattice does not vary from point to point, with the

> *reference configurations chosen so that*
> *the lattices of the two phases coincide.*

Thus, in effect, we refer the phases to a *single* uniform reference configuration with lattice $\mathcal{L}$.

3.2. BASIC DEFINITIONS

We label the phases 1 and 2, and reserve the letter i for the label when the particular phase is unimportant. By a **two-phase body** we mean a pair $B=(B_1,B_2)$ with B_i a closed region in $\mathbb{R}^3$. Given a two-phase body B, let $\mathbf{y}=(\mathbf{y}_1,\mathbf{y}_2)$ be a pair of mappings with $\mathbf{y}_i$ a mapping $\mathbf{x}=\mathbf{y}_i(\mathbf{X})$ of B_i into $\mathbb{R}^3$, and let

$$\mathcal{B}_i = \mathbf{y}_i(B_i), \qquad \mathcal{S} = \mathcal{B}_1\cap\mathcal{B}_2.$$

Then $\mathbf{y}$ is a **two-phase deformation** of B if:
(i) $\mathbf{y}_i$ is a deformation of B_i for each i;
(ii) $\mathcal{S}$ is a smooth surface in $\mathbb{R}^3$.
$\mathcal{B}_i$ represents the deformed phase i region;

$$\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2$$

represents the **deformed body**; $\mathcal{S}$ represents the **deformed interface**; the inverse image

$$S_i = y_i^{-1}(\mathcal{S})$$

represents the **undeformed phase-i interface**, it being tacit that the phases not separate at the interface. We emphasize that — to allow for incoherency — we do not require coincidence of the undeformed interfaces S_1 and S_2, nor do we require that B_1 and B_2 be disjoint.

Let $\mathbf{y}$ be a two-phase deformation. We write:

$$F_i(X) = \nabla y_i(X)$$

for the **deformation gradient** for phase i, and

$$G_i(\mathbf{x}) = F_i^{-1}(X), \qquad \mathbf{x} = y_i(X) \tag{3.1}$$

for the **inverse deformation gradient**.

We use the invertibility of the deformation to consider fields $\varphi_i(X)$ associated with B_i (i=1,2) as fields $\tilde{\varphi}_i(\mathbf{x})$ on the deformed regions $\mathcal{B}_i$; in fact, we write $\varphi_i(\mathbf{x})$ rather than $\tilde{\varphi}_i(\mathbf{x})$, and let $\varphi(\mathbf{x})$ denote the combined field given by $\varphi_1(\mathbf{x})$ for $\mathbf{x}$ in the interior of $\mathcal{B}_1$ and $\varphi_2(\mathbf{x})$ for $\mathbf{x}$ in the interior of $\mathcal{B}_2$. This convention allows us to consider the deformation gradient as a spatial field $\mathbf{F}(\mathbf{x})$, and to reserve the subscripted symbol F_i for the limit of $\mathbf{F}$ as $\mathcal{S}$ is approached from phase i.

More generally, $\Phi_i(\mathbf{z})$ denotes the limit of a field $\Phi(\mathbf{x})$ at $\mathbf{z}\in\mathcal{S}$ as $\mathbf{x}\to\mathbf{z}$ from $\mathcal{B}_i$, while $[\Phi]$ is the **jump in** Φ across the interface:

$$[\Phi] = \Phi_2 - \Phi_1.$$

The undeformed interfaces S_1 and S_2 are in one-to-one correspondence with $\mathcal{S}$ via the mappings y_1 and y_2; we say that $\mathbf{x}\in\mathcal{S}$, $X_1\in S_1$, and $X_2\in S_2$ are **compatible** if

$$\mathbf{x} = y_1(X_1) = y_2(X_2).$$

In relations involving fields defined on the deformed interface as well as fields defined on the undeformed interfaces, it will always be understood that the fields are to be evaluated at compatible points.

We orient the deformed and undeformed interfaces by the following choice of unit normal fields:

$\bar{\mathbf{n}}$	unit normal to $\mathcal{S}$ *outward* from $\mathcal{B}_1$,
$\mathbf{n}_1$	unit normal to S_1 *outward* from B_1,
$\mathbf{n}_2$	unit normal to S_2 *inward* from B_2;

these normals are related by

$$\bar{\mathbf{n}} = \lambda_i \mathbf{F}_i^{-T} \mathbf{n}_i, \qquad \lambda_i = |\mathbf{F}_i^{-T} \mathbf{n}_i|^{-1} = |\mathbf{F}_i^T \bar{\mathbf{n}}|. \tag{3.2}$$

We write

$$J = \det \mathbf{F} \tag{3.3}$$

for the bulk Jacobian; the superficial field

$$\jmath_i = J_i / \lambda_i \tag{3.4}$$

then represents the Jacobian of the mapping that carries the undeformed phase i interface into the deformed interface. We will also use the surface Jacobian $\mathcal{H}$ of the mapping that carries S_1 into S_2:

$$\mathcal{H} = \jmath_1 / \jmath_2. \tag{3.5}$$

We denote by $\mathbf{L}_i(\mathbf{X})$ and $K_i(\mathbf{X})$ the curvature tensor and total (twice the mean) curvature for S_i, and by $\bar{\mathbf{L}}(\mathbf{x})$ and $\bar{K}(\mathbf{x})$ the corresponding quantities for $\mathcal{S}$.

The inclusions — into $\mathbb{R}^3$ — of the tangent spaces at $\mathbf{X} \in S_i$ and $\mathbf{x} \in \mathcal{S}$ will be denoted by $\mathbb{1}_i(\mathbf{X})$ and $\bar{\mathbb{1}}(\mathbf{x})$, respectively, while the corresponding projections will be denoted by $\mathbb{P}_i = \mathbb{1}_i^T$ and $\bar{\mathbb{P}} = \bar{\mathbb{1}}^T$. We shall also make use of the projection operators

$$\Lambda_i = 1 - \mathbf{n}_i \otimes \mathbf{n}_i = \mathbb{1}_i \mathbb{P}_i, \qquad \overline{\Lambda} = 1 - \bar{\mathbf{n}} \otimes \bar{\mathbf{n}} = \overline{\mathbb{1}}\,\overline{\mathbb{P}}. \tag{3.6}$$

Then

$$\mathbb{1}_i(\mathbf{X}) \in \mathrm{Lin}(\mathbf{n}_i(\mathbf{X})^{\perp}, \mathbb{R}^3), \qquad \overline{\mathbb{1}}(\mathbf{x}) \in \mathrm{Lin}(\bar{\mathbf{n}}(\mathbf{x})^{\perp}, \mathbb{R}^3),$$
$$\mathbb{P}_i(\mathbf{X}) \in \mathrm{Lin}(\mathbb{R}^3, \mathbf{n}_i(\mathbf{X})^{\perp}), \qquad \overline{\mathbb{P}}(\mathbf{x}) \in \mathrm{Lin}(\mathbb{R}^3, \bar{\mathbf{n}}(\mathbf{x})^{\perp}),$$
$$\Lambda_i(\mathbf{X}) \in \mathrm{Lin}(\mathbb{R}^3, \mathbb{R}^3), \qquad \overline{\Lambda}(\mathbf{x}) \in \mathrm{Lin}(\mathbb{R}^3, \mathbb{R}^3).$$

We write $\mathbf{u}_{\mathrm{tan}S_i}$ and $\mathbf{u}_{\mathrm{tan}\mathcal{S}}$ for the tangential components with respect to S_i and $\mathcal{S}$, respectively, of an interfacial vector field $\mathbf{u}$:

$$\mathbf{u}_{\mathrm{tan}S_i} = \Lambda_i \mathbf{u}, \qquad \mathbf{u}_{\mathrm{tan}\mathcal{S}} = \overline{\Lambda}\mathbf{u}. \tag{3.7}$$

We then have the useful result:

$$(\mathbf{G}_i^{\mathsf{T}}\mathbf{u})_{\mathrm{tan}\mathcal{S}} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{u}_{\mathrm{tan}S_i} = \mathbf{0}. \tag{3.8}$$

To verify (3.8) we let $\mathbf{w} = \mathbf{u}_{\mathrm{tan}S_i}$, so that $\mathbf{u} = \mathbf{w} + \alpha \mathbf{n}_i$, and we conclude, with the aid of (3.2), that $\overline{\Lambda}\mathbf{G}_i^{\mathsf{T}}\mathbf{u} = \overline{\Lambda}\mathbf{G}_i^{\mathsf{T}}\mathbf{w}$. Assume that $\overline{\Lambda}\mathbf{G}_i^{\mathsf{T}}\mathbf{w} = \mathbf{0}$. Let $\mathbf{p} = \mathbf{F}_i\mathbf{w}$, so that $\mathbf{p}$ is tangential to $\mathcal{S}$. Then $\mathbf{0} = \mathbf{p} \cdot \overline{\Lambda}\mathbf{G}_i^{\mathsf{T}}\mathbf{w} = \mathbf{w} \cdot \mathbf{w}$; hence $\mathbf{u}_{\mathrm{tan}S_i} = \mathbf{0}$. The converse assertion is established similarly.

3.3. TANGENTIAL GRADIENTS. INCOHERENCY TENSOR

The **tangential** and **inverse-tangential deformation gradients** are defined by

$$\mathbb{F}_i = \nabla_{S_i} \mathbf{y}_i = \mathbf{F}_i \mathbb{1}_i, \qquad \mathbb{G}_i = \mathrm{grad}_{\mathcal{S}}(\mathbf{y}_i^{-1}) = \mathbf{G}_i \overline{\mathbb{1}}, \tag{3.9}$$

with ∇_{S_i} the material gradient (with respect to $\mathbf{X}$) on S_i and $\mathrm{grad}_{\mathcal{S}}$ the spatial gradient (with respect to $\mathbf{x}$) on $\mathcal{S}$. Then

$$\mathbb{F}_i(\mathbf{X}) \in \mathrm{Lin}(\mathbf{n}_i(\mathbf{X})^{\perp}, \mathbb{R}^3), \qquad \mathbb{G}_i(\mathbf{x}) \in \mathrm{Lin}(\bar{\mathbf{n}}(\mathbf{x})^{\perp}, \mathbb{R}^3),$$

but $\mathbb{F}_i(\mathbf{X})$ actually maps tangent vectors at $\mathbf{X} \in S_i$ to *tangent vectors* at $\mathbf{x} \in \mathcal{S}$, and vice versa for $\mathbb{G}_i(\mathbf{x})$. Further, $\mathbb{F}_i$ and $\mathbb{G}_i$ are generally not invertible, but this causes no problem as $\overline{\mathbb{P}}\,\mathbb{F}_i$ is invertible with inverse

$\mathbb{P}_i\mathbb{G}_i$.[11] Further, $\mathcal{J}_i = \det(\bar{\mathbb{P}}\mathbb{F}_i)$.

In discussing the interface it will be convenient to take phase 1 as reference and to consider the **relative deformation** $\mathbf{h}$ of phase 2 with respect to phase 1:

$$\mathbf{h}(\mathbf{X}) = (\mathbf{y}_2^{-1} \circ \mathbf{y}_1)(\mathbf{X}), \qquad \mathbf{X} \in S_1. \tag{3.10}$$

We will refer to

$$\mathbf{H} = \mathbf{G}_2\mathbf{F}_1 \tag{3.11}$$

as the **relative deformation gradient**; its tangential part is the surface gradient of the map $\mathbf{h}$:

$$\mathbb{H} = \nabla_{S_1}\mathbf{h} = \mathbf{H}\mathbb{1}_1. \tag{3.12}$$

We will refer to $\mathbb{H}$ as the **incoherency tensor**;[12] $\mathbb{H}(\mathbf{X}) \in \mathrm{Lin}\,(\mathbf{n}_1(\mathbf{X})^{\perp}, \mathbb{R}^3)$, although $\mathbb{H}(\mathbf{X})$ maps tangent vectors at $\mathbf{X} \in S_1$ to *tangent vectors* at $\mathbf{h}(\mathbf{X}) \in S_2$. The tensor $\mathbf{H}$ also relates the relative orientations of the undeformed interfaces:

$$\mathbf{n}_2 = \lambda_2^{-1}\lambda_1\mathbf{H}^{-\mathrm{T}}\mathbf{n}_1. \tag{3.13}$$

Given any $\mathbf{X} \in S_1$, the smoothness of $\mathbf{y}_1$ and $\mathbf{y}_2$ up to the interface and the restrictions $\det\mathbf{F}_1 > 0$, $\det\mathbf{F}_2 > 0$ allow us to extend $\mathbf{y}_1$ and $\mathbf{y}_2$ smoothly and invertibly to $\mathbb{R}^3$-neighborhoods of $\mathbf{X}$ and $\mathbf{h}(\mathbf{X})$; thus, given any point $\mathbf{X} \in S_1$,

$$\begin{aligned} &\mathbf{h} \text{ admits a smooth, invertible extension to} \\ &\text{an } \mathbb{R}^3\text{-neighborhood of } \mathbf{X}, \text{ and } \mathbf{H} = \nabla\mathbf{h}. \end{aligned} \tag{3.14}$$

The relative gradient $\mathbf{H}$, which measures both the relative strain and relative rotation between phases, is invariant under observer changes, which are here transformations carrying $(\mathbf{F}_1, \mathbf{F}_2)$ into $(\mathbf{Q}\mathbf{F}_1, \mathbf{Q}\mathbf{F}_2)$ with

[11] Cf. [GS], p. 106.

[12] Cf. Cermelli and Gurtin [1993].

$\mathbf{Q} \in \mathrm{Orth}^+$. Contrast this to classical continuum theories, where rotations are generally not invariant and consequently ruled out of constitutive relations.

3.4. REFERENTIAL-TO-SPATIAL TRANSFORMS

Given a bulk tensor field $\mathbf{T}$ and a bulk scalar field φ, we write

$$\bar{\mathbf{T}} = J^{-1}\mathbf{T}\mathbf{F}^{\mathsf{T}}, \qquad \bar{\varphi} = J^{-1}\varphi, \tag{3.15}$$

where J is the bulk Jacobian (3.3). If A is a surface oriented by the unit normal $\mathbf{m}$, if Q is a region, if both A and Q are contained in one of the undeformed phase regions, and if $\mathcal{A}$ — oriented by $\bar{\mathbf{m}}$ — and $\mathcal{Q}$ denote the images, at some fixed time, of A and Q under the motion, then

$$\int_{\mathcal{A}} \bar{\mathbf{T}}\bar{\mathbf{m}}\, da = \int_{A} \mathbf{T}\mathbf{m}\, da, \qquad \int_{\mathcal{Q}} \bar{\varphi}\, dv = \int_{Q} \varphi\, dv. \tag{3.16}$$

Given a superficial (scalar or vector) field φ_i for S_i, we define

$$\bar{\varphi}_i = \jmath_i^{-1}\varphi_i, \tag{3.17}$$

where $\jmath_i$ is the surface Jacobian (3.4). If A is a subsurface of the undeformed phase i interface, and $\mathcal{A}$ is the image of A under the motion, then

$$\int_{\mathcal{A}} \bar{\varphi}_i\, da = \int_{A} \varphi_i\, da. \tag{3.18}$$

By (3.2), (3.4), and (3.15), for $\mathbf{T}$ a bulk tensor field,

$$\bar{\mathbf{T}}_i\bar{\mathbf{n}} = \jmath_i^{-1}\mathbf{T}_i\mathbf{n}_i. \tag{3.19}$$

Also, applying $(3.16)_1$ to the boundary of a smooth region, we conclude, with the aid of the divergence theorem, that

$$\mathrm{div}\,\bar{\mathbf{T}} = J^{-1}\mathrm{Div}\,\mathbf{T}, \tag{3.20}$$

with div and Div, respectively, the spatial and referential divergences.

Similarly, given a superficial tensor field $\mathbf{T}$ on S_i, we write

$$\bar{\mathbf{T}} = \wr_i^{-1}\mathbf{T}\mathbf{F}_i^{\mathsf{T}}\bar{\mathbb{1}}; \tag{3.21}$$

then, if $\bar{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}_i$ are the unit normals to $\partial\mathcal{G}$ and ∂A_i respectively, we have

$$\int_{\partial\mathcal{G}} \bar{\mathbf{T}}\bar{\boldsymbol{\nu}}\,ds = \int_{\partial A_i} \mathbf{T}\boldsymbol{\nu}_i\,ds, \tag{3.22}$$

and the identity

$$\int_{\partial\mathcal{G}} \mathbf{q}\cdot\bar{\mathbf{T}}\bar{\boldsymbol{\nu}}\,ds = \int_{\partial A_i} \mathbf{q}\cdot\mathbf{T}\boldsymbol{\nu}_i\,ds \tag{3.23}$$

holds for any superficial vector field[13] $\mathbf{q}$ on $\mathcal{G}$. Applying the surface divergence theorem[14] to both sides of (3.22) yields

$$\mathrm{div}_{\wr}\,\bar{\mathbf{T}} = \wr_i^{-1}\mathrm{Div}_{S_i}\mathbf{T}\,. \tag{3.24}$$

If the unbarred fields in (3.15), (3.17), and (3.21) represent physical quantities measured in the *undeformed* body, then the barred quantities represent these same quantities measured in the *deformed* body. We will refer to the barred quantities as the **referential-to-spatial transforms** of the unbarred quantities. In using this notation it is important to note that $\bar{\mathbf{n}}$ is not the transform of $\mathbf{n}_i$.

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[13] On the left side of (3.23) $\mathbf{q}$ is considered as a function $\mathbf{x}\mapsto\mathbf{q}(\mathbf{x})$ on $\mathcal{G}$, on the right side as a function $\mathbf{X}\mapsto\mathbf{q}(\mathbf{y}_i(\mathbf{X}))$ on A_i.

[14] Cf. [Gu], eqt. (A10).

4. TWO-PHASE MOTIONS

4.1. BASIC DEFINITIONS

At each time t, let $\mathbf{y}(t)$ be a two-phase deformation of a two-phase body $(B_1(t),B_2(t))$, and write $\mathbf{y}=(\mathbf{y}_1,\mathbf{y}_2)$, with $\mathbf{y}_i$ the mapping $\mathbf{y}_i(\mathbf{X},t)=\mathbf{y}_i(t)(\mathbf{X})$. Then $\mathbf{y}$ is a **two-phase motion** if, for each phase i:

(i) $\{\mathcal{B}(t), \mathcal{B}_1(t), \mathcal{B}_2(t)\}$ is an evolving two-phase region[15];

(ii) away from $S_i(t)$, $\partial B_i(t)$ is independent of time;

(iii) $\mathbf{y}_i$ is a smooth mapping.

Let $\mathbf{y}$ be a two-phase motion. We use the following notation:

$$\mathbf{y}_i^{\cdot}(\mathbf{X},t) = (\partial/\partial t)\mathbf{y}_i(\mathbf{X},t)$$

is the **material velocity** of phase i; $V_i(\mathbf{X},t)$ and $\bar{V}(\mathbf{x},t)$, respectively, are the **normal velocities** of $S_i(t)$ in the direction $\mathbf{n}_i(\mathbf{X},t)$ and $\mathcal{S}(t)$ in the direction $\bar{\mathbf{n}}(\mathbf{x},t)$; $\mathbf{y}_i^{-1}(\mathbf{x},t)$ is the (fixed-time) inverse of $\mathbf{y}_i(\mathbf{X},t)$,

$$\mathbf{y}_i^{-1}(\mathbf{y}_i(\mathbf{X},t),t) = \mathbf{X}, \qquad \mathbf{X}\in B_i(t). \tag{4.1}$$

As in Section 3.2, we use the invertibility of the motion to consider fields $\varphi_i(\mathbf{X},t)$ associated with $B_i(t)$ (i=1,2) (and smooth up to the interface) as fields $\varphi_i(\mathbf{x},t)$ on the deformed regions $\mathcal{B}_i(t)$, and let $\varphi(\mathbf{x},t)$ denote the combined field given by $\varphi_1(\mathbf{x},t)$ for $\mathbf{x}$ in the interior of $\mathcal{B}_1(t)$ and $\varphi_2(\mathbf{x},t)$ for $\mathbf{x}$ in the interior of $\mathcal{B}_2(t)$. We will refer to such fields $\varphi(\mathbf{x},t)$ as **spatially described bulk fields**, a convention that allows us to consider the deformation gradient and material velocity as spatial fields $\mathbf{F}(\mathbf{x},t)$ and $\mathbf{y}^{\cdot}(\mathbf{x},t)$, and to use the notation defined in the paragraph of [Gu] containing (2.1) with $\{\mathcal{B}(t), \mathcal{B}_1(t), \mathcal{B}_2(t)\}$ as the evolving two-phase region.

A (local) **trajectory** for $\mathcal{S}$ is a vector function $\mathbf{z}$ of time (on some open time interval) with $\mathbf{z}(\tau)\in\mathcal{S}(\tau)$ at each τ; $\mathbf{z}$ passes through $\mathbf{x}$ at time t if $\mathbf{z}(t)=\mathbf{x}$; $\mathbf{z}$ is **normal** if $\mathbf{z}^{\cdot} = \bar{V}\bar{\mathbf{n}}$. (Since $\mathcal{S}$ is, by hypothesis, a smoothly evolving surface, given t and $\mathbf{x}\in\mathcal{S}(t)$, there is always a trajectory — even a normal trajectory — through $\mathbf{x}$ at time t.) Trajectories for S_i are defined analogously. We define **compatible trajectories for the interface** as trajectories $\mathbf{z}$, $\mathbf{Z}_1$, $\mathbf{Z}_2$ for $\mathcal{S}$, S_1, and S_2,

[15] Cf. Sect. 2 of [Gu].

respectively, such that

$$\mathbf{z}(t) = \mathbf{y}_1(Z_1(t),t) = \mathbf{y}_2(Z_2(t),t); \tag{4.2}$$

the velocities $\mathbf{z}^\cdot$, $Z_1^\cdot$, and $Z_2^\cdot$ of compatible trajectories satisfy

$$\begin{aligned} &Z_i^\cdot \cdot \mathbf{n}_i = V_i, \qquad \mathbf{z}^\cdot \cdot \bar{\mathbf{n}} = \bar{V}, \\ &\mathbf{z}^\cdot = \mathbf{y}_1^\cdot + F_1 Z_1^\cdot = \mathbf{y}_2^\cdot + F_2 Z_2^\cdot. \end{aligned} \tag{4.3}$$

The (fixed-time) invertibility of $\mathbf{y}_i$ allows us to construct compatible trajectories through compatible points; (3.2) and (4.3) therefore yield

$$\bar{V} = \lambda_i V_i + \mathbf{y}_i^\cdot \cdot \bar{\mathbf{n}}, \qquad i = 1,2, \tag{4.4}$$

a constraint that rules out the formation of voids at the interface.

We will also use, as an intrinsic measure of the motion of S_i, the **relative velocity**

$$U_i = \bar{V} - \mathbf{y}_i^\cdot \cdot \bar{\mathbf{n}} = \lambda_i V_i \tag{4.5}$$

and the **volume-flow**

$$W_i = V_i / \jmath_i = U_i / J_i \tag{4.6}$$

for phase i (cf. (3.4) and (4.5)). W_i represents the volume flow across the phase i interface in the direction $-\bar{\mathbf{n}}$, per unit deformed area (cf. Section 5.2). By (3.4), (4.4), (4.5), and (4.6), we have the **compatibility conditions**:

$$[\lambda V] = [JW] = [U] = -[\mathbf{y}^\cdot] \cdot \bar{\mathbf{n}}. \tag{4.7}$$

4.2. COMPATIBLE INTERFACE AND EDGE VELOCITIES

a. Interface velocities

(Local) **parametrizations** $\mathbf{x}=\mathbf{z}(p,t)$ for $\mathcal{S}(t)$ and $\mathbf{X}=Z_i(p,t)$ for $S_i(t)$, i=1,2, are *compatible* if

$$\mathbf{z}(p,t) = \mathbf{y}_1(Z_1(p,t),t) = \mathbf{y}_2(Z_2(p,t),t); \tag{4.8}$$

the trajectories defined by $\mathbf{z}(p,t)$ and $\mathbf{Z}_i(p,t)$ for p fixed are then compatible, and the corresponding superficial fields $\bar{\mathbf{v}}(\mathbf{x},t)=(\partial/\partial t)\mathbf{z}(p,t)$, $\mathbf{x}=\mathbf{z}(p,t)$, and $\mathbf{v}_i(\mathbf{X},t)=(\partial/\partial t)\mathbf{Z}_i(p,t)$, $\mathbf{X}=\mathbf{Z}_i(p,t)$, satisfy

$$\mathbf{v}_i\cdot\mathbf{n}_i = V_i, \qquad \bar{\mathbf{v}}\cdot\bar{\mathbf{n}} = \bar{V}, \tag{4.9}$$

as well as the *compatibility condition*

$$\bar{\mathbf{v}} = \mathbf{y}_1^{\bullet} + \mathbf{F}_1\mathbf{v}_1 = \mathbf{y}_2^{\bullet} + \mathbf{F}_2\mathbf{v}_2. \tag{4.10}$$

We will refer to superficial fields $\bar{\mathbf{v}}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ consistent with (4.9) and (4.10) as **compatible velocity fields** for the interface. (By (4.4), (4.9) are redundant, (4.10) and one of (4.9) impies the other.) Here it is important to note that $\mathbf{v}_1$ and $\mathbf{v}_2$ need not be normal to S_1 and S_2, respectively, but their tangential components are not independent, as $\mathbf{v}_1$ and $\mathbf{v}_2$ must induce the *same* velocity field $\bar{\mathbf{v}}$ for $\mathcal{S}$. Working with compatible velocity fields guarantees the equivalence of the description of the evolution process with respect to any of the interfaces $\mathcal{S}$, S_1, and S_2. To choose compatible velocity fields it sufficies to specify one of the fields $\bar{\mathbf{v}}$, $\mathbf{v}_1$, or $\mathbf{v}_2$, for then the other two may be computed using (4.10).

In what follows we will use two particular choices of compatible velocity fields $\bar{\mathbf{v}}$, $\mathbf{v}_1$, and $\mathbf{v}_2$: one choice is normalized spatially, while the other is normalized materially *using phase 1 as reference.*

(i) The **spatially normalized interface velocities** $\bar{\mathbf{v}}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ have $\bar{\mathbf{v}}$ normal,

$$\bar{\mathbf{v}} = \bar{V}\bar{\mathbf{n}}, \tag{4.11}$$

so that, by (4.10),

$$\mathbf{v}_i = \mathbf{G}_i(\bar{\mathbf{v}} - \mathbf{y}_i^{\bullet}), \qquad i = 1,2; \tag{4.12}$$

(ii) the **interface velocities** $\bar{\mathbf{v}}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ **materially normalized with respect to phase 1** have $\mathbf{v}_1$ normal,

$$\mathbf{v}_1 = V_1\mathbf{n}_1, \tag{4.13}$$

so that, by (4.10),

$$\begin{aligned} \bar{\mathbf{v}} &= \mathbf{y}_1^{\cdot} + \mathbf{F}_1\mathbf{v}_1, \\ \mathbf{v}_2 &= \mathbf{G}_2(-[\mathbf{y}^{\cdot}] + \mathbf{F}_1\mathbf{v}_1). \end{aligned} \tag{4.14}$$

It is important to note that $\bar{\mathbf{v}}$ and $\bar{V}$ are *not* the referential-to-spatial transforms of $\mathbf{v}_i$ and V_i.

b. Edge velocities

Let $A_i(t)=\mathbf{y}_i^{-1}(\mathcal{Q}(t),t)$, with $\mathcal{Q}(t)$ a smoothly evolving subsurface of $\mathcal{S}(t)$, let $\bar{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}_i$ denote the outward unit normals to the boundary curves $\partial\mathcal{Q}(t)$ and $\partial A_i(t)$, and let $V_{(\partial\mathcal{Q})\tan}$ and $V_{(\partial A_i)\tan}$ denote the **intrinsic tangential edge velocities** of the boundary curves $\partial\mathcal{Q}$ and ∂A_i (the normal velocities of ∂A_i and $\partial\mathcal{Q}$ in the tangent planes to S_i and $\mathcal{S}$).[16] We say that **parametrizations** $\mathbf{x}=\mathbf{z}(u,t)$ for $\partial\mathcal{Q}(t)$ and $\mathbf{X}=\mathbf{Z}_i(u,t)$ for $\partial A_i(t)$, $i=1,2$, are *compatible* if $\mathbf{z}(u,t)=\mathbf{y}_1(\mathbf{Z}_1(u,t),t)=\mathbf{y}_2(\mathbf{Z}_2(u,t),t)$. The corresponding superficial fields $\bar{\mathbf{w}}(\mathbf{x},t)=(\partial/\partial t)\mathbf{z}(u,t)$, $\mathbf{x}=\mathbf{z}(u,t)$, and $\mathbf{w}_i(\mathbf{X},t)=(\partial/\partial t)\mathbf{Z}_i(u,t)$, $\mathbf{X}=\mathbf{Z}_i(u,t)$, then satisfy

$$\begin{aligned} \mathbf{w}_i\cdot\mathbf{n}_i &= V_i, & \mathbf{w}_i\cdot\boldsymbol{\nu}_i &= V_{(\partial A_i)\tan}, \\ \bar{\mathbf{w}}\cdot\bar{\mathbf{n}} &= \bar{V}, & \bar{\mathbf{w}}\cdot\bar{\boldsymbol{\nu}} &= V_{(\partial\mathcal{Q})\tan}, \end{aligned} \tag{4.15}$$

in conjunction with the compatibility condition

$$\bar{\mathbf{w}} = \mathbf{y}_1^{\cdot} + \mathbf{F}_1\mathbf{w}_1 = \mathbf{y}_2^{\cdot} + \mathbf{F}_2\mathbf{w}_2 \tag{4.16}$$

(cf. (4.10)). More generally, let $\bar{\mathbf{w}}(\mathbf{x},t)$ and $\mathbf{w}_i(\mathbf{X},t)$, respectively, be defined for all $\mathbf{x}\in\partial\mathcal{Q}(t)$ and all $\mathbf{X}\in\partial A_i(t)$. Then $\bar{\mathbf{w}}$, $\mathbf{w}_1$, and $\mathbf{w}_2$ are **compatible edge velocities** if they are consistent with (4.15) and (4.16). As before, we shall use two specific **normalizations of the edge velocities**: the **spatial normalization**

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[16] Cf. Appendix A of [Gu].

$$\bar{w} = \bar{V}\bar{n} + V_{(\partial\mathcal{Q})\tan}\bar{\nu} = \bar{v} + V_{(\partial\mathcal{Q})\tan}\bar{\nu},$$
$$w_i = G_i(\bar{w} - y_i^{\cdot}) = v_i + G_i(V_{(\partial\mathcal{Q})\tan}\bar{\nu}) \tag{4.17}$$

(with $\bar{v}$ and v_i spatially normalized); and the **material normalization with respect to phase 1**

$$w_1 = V_1 n_1 + V_{(\partial A_1)\tan}\nu_1 = v_1 + V_{(\partial A_1)\tan}\nu_1,$$
$$\bar{w} = y_1^{\cdot} + F_1 w_1 = \bar{v} + F_1(V_{(\partial A_1)\tan}\nu_1), \tag{4.18}$$
$$w_2 = G_2(-[y^{\cdot}] + F_1 w_1) = v_2 + H(V_{(\partial A_1)\tan}\nu_1)$$

(with $\bar{v}$ and v_i materially normalized with respect to phase 1).

4.3. NORMAL TIME DERIVATIVES FOLLOWING THE INTERFACE

The **normal time derivative** $\varphi^{\square}(x,t)$ **following** $\mathcal{S}(t)$ of a superficial field $\varphi(x,t)$ on $\mathcal{S}(t)$ is the superficial field defined as follows: for any normal trajectory $z(\tau)$ through x at time t:

$$\varphi^{\square}(x,t) = (d/d\tau)\{\varphi(z(\tau),\tau)\}|_{\tau=t}. \tag{4.19}$$

When the superficial field is the limit $\Phi_i(x,t)$ from $\mathcal{B}_i(t)$ of a bulk scalar field $\Phi(x,t)$,

$$(\Phi_i)^{\square} = (\partial_t\Phi)_i + \bar{V}\bar{n}\cdot(\mathrm{grad}\,\Phi)_i. \tag{4.20}$$

Analogously, let $\varphi(X,t)$ be a superficial field on $S_i(t)$. Then its **normal time derivative** $\varphi^{\circ}(X,t)$ **following** $S_i(t)$ is the superficial field on $S_i(t)$ given by

$$\varphi^{\circ}(X,t) = (d/d\tau)\{\varphi(Z_i(\tau),\tau)\}|_{\tau=t}, \tag{4.21}$$

with $Z_i(\tau)$ the normal trajectory through X at time t. As before, when the superficial field is the limit $\Phi_i(X,t)$ from $B_i(t)$ of a bulk scalar field $\Phi(X,t)$,

$$(\Phi_i)^{\circ} = (\Phi^{\cdot})_i + V_i n_i\cdot(\nabla\Phi)_i. \tag{4.22}$$

The relations (4.20) and (4.22) have obvious analogs for vector and tensor fields. Note that the symbol $(\cdot)^\circ$ defines the normal time derivative on both $S_1(t)$ and $S_2(t)$; when we use this symbol the underlying surface should be clear from the context. (In most cases $(\cdot)^\circ$ will denote the normal time derivative with respect to S_1.)

The above definitions, the discussion in the paragraph containing (4.8) and (4.19), (4.21) yield the identities:

$$\begin{aligned} &(\mathbf{y}_i^{-1})^\square = \mathbf{v}_i, \\ &\mathbf{y}_i^\square = \bar{\mathbf{v}} = \bar{V}\bar{\mathbf{n}} \end{aligned} \tag{4.23}$$

when the interface velocities are spatially normalized, and

$$\begin{aligned} &\mathbf{y}_1^\circ = \bar{\mathbf{v}} = \mathbf{y}_1^{\cdot} + F_1 V_1 \mathbf{n}_1, \\ &(\mathbf{y}_2^{-1} \circ \mathbf{y}_1)^\circ = \mathbf{h}^\circ = \mathbf{v}_2 \end{aligned} \tag{4.24}$$

when the interface velocities are materially normalized with respect to phase 1 and $(\cdot)^\circ$ is the normal time derivative with respect to S_1.

Further, for φ a superficial scalar field

$$\varphi^\circ = \varphi^\square + \mathbf{y}_i^\circ \cdot \operatorname{grad}_{\mathcal{S}} \varphi, \tag{4.25}$$

with $(\cdot)^\circ$ the normal time derivative with respect to S_i; and analogous relations hold also for vector and tensor fields. We will verify (4.25) for i=1. Choose $\mathbf{x} \in \mathcal{S}$ and t arbitrarily. Let $\mathbf{z}(\tau)$ be a normal trajectory for $\mathcal{S}(\tau)$ through $\mathbf{x}$ at t, let $\mathbf{Z}(\tau)$ be a normal trajectory for $S_1(\tau)$ through $\mathbf{X} = \mathbf{y}_1^{-1}(\mathbf{x},t)$ at t, and, without loss in generality, let t=0. Then, for φ described spatially,

$$(\varphi^\square - \varphi^\circ)(\mathbf{x},0) = \Phi(0^+), \quad \Phi(\tau) = \tau^{-1}\{\varphi(\mathbf{z}(\tau),\tau) - \varphi(\mathbf{y}_1(\mathbf{Z}(\tau),\tau),\tau)\},$$

since $\mathbf{z}(0) = \mathbf{y}_1(\mathbf{Z}(0),0) = \mathbf{x}$. Writing $\mathbf{u}(\tau) = \operatorname{grad}_{\mathcal{S}} \varphi(\mathbf{z}(\tau),\tau)$, it follows that

$$\Phi(\tau) = \tau^{-1}\{\mathbf{z}(\tau) - \mathbf{z}(0) + \mathbf{y}_1(\mathbf{Z}(0),0) - \mathbf{y}_1(\mathbf{Z}(\tau),\tau)\} \cdot \mathbf{u}(\tau) + o(1),$$

$$\Phi(0^+) = \{\mathbf{z}^\cdot(0) - \mathbf{y}_1^\cdot(\mathbf{X},0) - \mathbf{F}_1(\mathbf{X},0)\mathbf{Z}^\cdot(0)\}\cdot\mathbf{u}(0).$$

On the other hand, $\mathbf{z}^\cdot = \bar{V}\bar{\mathbf{n}}$, $\mathbf{Z}^\cdot = V_1\mathbf{n}_1$, and (4.25) follows from $(4.24)_1$. This result also holds for vector and tensor fields.

4.4 SLIP

A basic feature of incoherent motions is that the phases are allowed to slip along the interface. We introduce, as an intrinsic measure of this phenomenon, the **slip velocity**

$$\gamma = (\mathbf{y}_2)^\circ - (\mathbf{y}_1)^\circ, \tag{4.26}$$

with $(\mathbf{y}_2)^\circ$ the normal time derivative following S_2 and $(\mathbf{y}_1)^\circ$ the normal time derivative following S_1. Then γ *is tangential to* $\mathcal{S}$ and admits the representation

$$\gamma = [\mathbf{y}^\cdot] + [V\mathbf{F}\mathbf{n}]. \tag{4.27}$$

This identity follows from property (4.22) of the normal time derivative; (4.4) then yields the tangency of γ to $\mathcal{S}$

In terms of the interface velocities materially normalized with respect to phase 1,

$$\gamma = -\mathbf{F}_2(\mathbf{v}_2)_{\tan S_2}. \tag{4.28}$$

Indeed, by (3.6), (4.13), and (4.14),

$$\begin{aligned}(\mathbf{v}_2)_{\tan S_2} &= \mathbf{v}_2 - (\mathbf{v}_2\cdot\mathbf{n}_2)\mathbf{n}_2 \\ &= \mathbf{G}_2(\mathbf{y}_1^\cdot - \mathbf{y}_2^\cdot + \mathbf{F}_1\mathbf{v}_1) - V_2\mathbf{n}_2 \\ &= \mathbf{G}_2(\mathbf{y}_1^\cdot - \mathbf{y}_2^\cdot + V_1\mathbf{F}_1\mathbf{n}_1 - V_2\mathbf{F}_2\mathbf{n}_2) \\ &= -\mathbf{G}_2([\mathbf{y}^\cdot] + [V\mathbf{F}\mathbf{n}]),\end{aligned}$$

and (4.28) follows from (4.27).

Our definition (4.26) of the slip velocity γ characterizes it as the

"deviation rate" of the images in the deformed configuration of trajectories for S_1 and S_2 that are both normal. The identity (4.28) yields another interpretation. Consider a normal S_1-trajectory. The image of this trajectory in the reference configuration for phase 2 is an S_2-trajectory, but in general it is not normal to S_2. It describes the motion on S_2 of a time-dependent compatible point that moves normally on S_1. The tangential part of the velocity field for this S_2-trajectory is a measure of the shear rate across the interface; the multiplication by $\mathbf{F}_2$ in (4.28) converts this shear rate to one measured in the deformed configuration.

An alternative representation of the slip is furnished by the identity

$$\gamma = [\mathbf{F}\{V\mathbf{n} - (\mathbf{y}^{-1})^{\circ}\}], \tag{4.29}$$

which is a consequence of (4.12), $(4.23)_1$, and (4.27).

4.5. SOME IDENTITIES

We begin with identities in which the interface velocities are spatially normalized. Here $\mathbf{u}$ is a bulk vector field; and, for $i=1,2$, $\mathbf{a}_i$ is a superficial vector field on S_i.

$$\mathrm{grad}_{\mathcal{S}}\bar{\mathbf{v}} = -(\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}^{\circ})\bar{\mathbb{1}} - \bar{V}\bar{\mathbb{L}}, \tag{4.30}$$

$$\mathrm{grad}_{\mathcal{S}}\mathbf{v}_i = \mathbf{G}_i^{\circ}\bar{\mathbb{1}} + \mathbf{G}_i\,\mathrm{grad}_{\mathcal{S}}\bar{\mathbf{v}}, \tag{4.31}$$

$$\mathrm{grad}_{\mathcal{S}}(\mathbf{u}_i^{\circ}) = (\mathrm{grad}\mathbf{u})_i^{\circ}\bar{\mathbb{1}} + (\mathrm{grad}\mathbf{u})_i(\mathrm{grad}_{\mathcal{S}}\bar{\mathbf{v}}), \tag{4.32}$$

$$\mathbf{a}_i\cdot\mathbf{v}_i = (\bar{\mathbf{n}}\cdot\mathbf{G}_i^{\mathsf{T}}\mathbf{a}_i)U_i - (\mathbf{G}_i^{\mathsf{T}}\mathbf{a}_i)_{\tan\mathcal{S}}\cdot(\mathbf{y}_i^{\cdot})_{\tan\mathcal{S}}, \tag{4.33}$$

In the next set of identities the interface velocities are materially normalized with respect to phase 1, the normal time derivative is with respect to S_1, and, for $i=1,2$, $\mathbf{a}_i$ is a superficial vector field on S_i.

$$\nabla_{S_1}\mathbf{v}_1 = -(\mathbf{n}_1\otimes\mathbf{n}_1^{\circ})\mathbb{1}_1 - V_1\mathbb{L}_1, \tag{4.34}$$

$$\nabla_{S_1}\mathbf{v}_2 = \mathbf{H}^{\circ}\mathbb{1}_1 + \mathbf{H}\nabla_{S_1}\mathbf{v}_1, \tag{4.35}$$

$$\nabla_{S_1}\bar{\mathbf{v}} = \mathbf{F}_1^{\circ}\mathbb{1}_1 + \mathbf{F}_1\nabla_{S_1}\mathbf{v}_1, \tag{4.36}$$

$$\nabla_{S_1}(\mathbf{u}_1^{\circ}) = (\nabla\mathbf{u})_1^{\circ}\mathbb{1}_1 + (\nabla\mathbf{u})_1(\nabla_{S_1}\mathbf{v}_1), \tag{4.37}$$

$$\sum_{k=1,2} \mathbf{a}_k \cdot \mathbf{v}_k = \sum_{k=1,2} (\mathbf{a}_k \cdot \mathbf{n}_k) V_k - (\mathbf{G}_2^{\mathsf{T}} \mathbf{a}_2)_{\tan\mathcal{S}} \cdot \boldsymbol{\gamma}. \tag{4.38}$$

The identities (4.30) and (4.34) can be found in [GS] (cf. the proof of (3.29) in [GS]).

To prove (4.32), let $\mathbf{A}=\operatorname{grad}\mathbf{u}$. Then, by (4.20),

$$\begin{aligned} \operatorname{grad}_{\mathcal{S}}(\mathbf{u}_i^{\circ}) &= \operatorname{grad}_{\mathcal{S}}\{(\partial_t \mathbf{u})_i + \mathbf{A}_i \bar{\mathbf{v}})\} \\ &= (\partial_t \mathbf{A})_i \bar{\mathbb{1}} + ((\operatorname{grad}\mathbf{A})_i \bar{\mathbf{v}}) \bar{\mathbb{1}} + \mathbf{A}_i (\operatorname{grad}_{\mathcal{S}} \bar{\mathbf{v}}); \end{aligned}$$

the first two terms on the right equal $\mathbf{A}_i^{\circ}$. The identity (4.31) follows from (3.1), $(4.23)_1$, and (4.32), while (4.37) is the material counterpart of (4.32).

Next, note that, by (4.5), (4.11), and (4.12),

$$\begin{aligned} \mathbf{a}_i \cdot \mathbf{v}_i &= \mathbf{G}_i^{\mathsf{T}} \mathbf{a}_i \cdot (\bar{V} \bar{\mathbf{n}} - \mathbf{y}_i^{\cdot}) \\ &= (\bar{\mathbf{n}} \cdot \mathbf{G}_i^{\mathsf{T}} \mathbf{a}_i)(\bar{V} - \mathbf{y}_i^{\cdot} \cdot \bar{\mathbf{n}}) - (\mathbf{G}_i^{\mathsf{T}} \mathbf{a}_i)_{\tan\mathcal{S}} \cdot (\mathbf{y}_i^{\cdot})_{\tan\mathcal{S}} \\ &= (\bar{\mathbf{n}} \cdot \mathbf{G}_i^{\mathsf{T}} \mathbf{a}_i) U_i - (\mathbf{G}_i^{\mathsf{T}} \mathbf{a}_i)_{\tan\mathcal{S}} \cdot (\mathbf{y}_i^{\cdot})_{\tan\mathcal{S}} \end{aligned}$$

and (4.33) follows.

Equation (4.35) follows from (4.37) with $\mathbf{u}=\mathbf{h}$, (3.14), and $(4.24)_2$, while (4.36) is a consequence of (4.37) with $\mathbf{u}=\mathbf{y}_1$ and $(4.24)_1$. Finally, to prove (4.38), we decompose $\mathbf{v}_2$ into tangential and normal parts with respect to S_2 and then appeal to (4.28).

4.6. SPATIAL CONTROL VOLUMES

We will formulate balance laws using spatial control volumes, which are fixed regions of space through which the deformed material flows. Precisely, a **spatial control volume** is a *fixed region* $\mathcal{R}$ with the property that $\mathcal{R}$ is a control volume with respect to $\{\mathcal{B}(t), \mathcal{B}_1(t), \mathcal{B}_2(t)\}$. The time-dependent set $\mathcal{B}_i(t) \cap \mathcal{R}$ then represents the *portion of* $\mathcal{R}$ *consisting of phase i material.*

Let $\bar{\Phi}$ be a spatial field and let $\mathcal{D}(t)$ be a (possibly) time-dependent region in the deformed body. Then we write

$$(d/dt)\{\int_{\mathcal{D}} \bar{\Phi}\, dv\}(t) = (d/dt)\{\int_{\mathcal{D}(t)} \bar{\Phi}(\mathbf{x},t)\, dv(\mathbf{x})\}.$$

Given a spatial control volume $\mathcal{R}$ (with $\bar{\mathbf{m}}$ the outward unit normal to $\partial\mathcal{R}$), we write

$$\{\int_{\mathcal{R}} \bar{\Phi}\, dv\}^{\cdot} = (d/dt)\{\int_{\mathcal{R}} \bar{\Phi}\, dv\} + \int_{\partial\mathcal{R}} \bar{\Phi}\mathbf{y}^{\cdot}\cdot\bar{\mathbf{m}}\, da, \tag{4.39}$$

for the *time derivative following the material currently in* $\mathcal{R}$. Then, for $\mathcal{R}_\varepsilon$ a family of spatial control volumes that shrinks to an arbitrary regular interfacial set $\mathcal{G}$, and for $\bar{\Phi}$ the referential-to-spatial transform of a bulk field Φ,

$$\{\int_{\mathcal{R}_\varepsilon} \bar{\Phi}\, dv\}^{\cdot} \to -\int_{\mathcal{G}} [\bar{\Phi}U]\, da = -\int_{\mathcal{G}} [\Phi W]\, da; \tag{4.40}$$

the first limit follows from (4.5) and Lemma B1 of [Gu], while the final relation is a consequence of (3.15) and (4.6).

Similarly, if $\bar{\mathbf{T}}$ is a spatial tensor field, then, by (3.19),

$$\int_{\partial\mathcal{R}_\varepsilon} \bar{\mathbf{T}}\bar{\mathbf{m}}\, da \to \int_{\mathcal{G}} [\jmath^{-1}\mathbf{T}\mathbf{n}]\, da. \tag{4.41}$$

Throughout our discussion of incoherent interfaces *we will state and localize balance laws using spatial control volumes in conjunction with the referential-to-spatial transforms defined in Section* 3.4.

5. COHERENCY. FORMATION OF DEFECTS[17]

5.1. COHERENCY

a. Symmetry group of the lattice

We introduce a **symmetry group** $\mathcal{G}$ for the reference lattice $\mathcal{L}$. Unless specified to the contrary, $\mathcal{G}$ is a subgroup of Unim^+, so that all deformations that leave $\mathcal{L}$ invariant may be represented by $\mathcal{G}$. However, our results concerning the relation between incoherency and coherency require that we exclude from our discussion arbitrarily large shears. In fact, this discussion will require that $\mathcal{G}$ be a *finite* subgroup of Orth^+.[18]

In any case, we will refer to the tensors $A \in \mathcal{G}$ as **symmetry transformations**, and to mappings $f: \mathbb{R}^3 \to \mathbb{R}^3$ of the form

$$f(X) = AX + q \qquad (A \in \mathcal{G},\ q \in \mathbb{R}^3) \tag{5.1}$$

as **material isomorphisms**. It is important to distinguish the translation q, which may be any vector, from the translations which — on a microscopic scale — leave the lattice invariant; on a macroscopic scale the set of microscopic translations is not distinguishable from $\mathbb{R}^3$.

b. Infinitesimal coherency

Consider a two-phase deformation. Choose compatible points $\mathbf{x} \in \mathcal{S}$, $X_1 \in S_1$, and $X_2 \in S_2$. Further, let dX_i be an "infinitesimal line segment" on S_i at X_i, and let $d\mathbf{x}_i = F_i(X_i)dX_i$. If $d\mathbf{x}_1 = d\mathbf{x}_2$, then dX_1 and dX_2 coincide when deformed. In this case $dX_2 = \mathbb{H}(X_1)dX_1$; thus $\mathbb{H}(X_1)$ relates line segments on the undeformed interfaces S_1 and S_2 that coincide when deformed. Since the reference lattices are coincident, if $dX_2 = dX_1$ for all

[17] This material is taken from Cermelli and Gurtin [1993].

[18] Reducing the "effective" symmetry group of the lattice to a subgroup of rotations requires restricting the deformation of each phase to a sufficiently small neighborhood of the identity, an idea due to Ericksen [1980,1984,1989] (see also Parry [1976,1982]). Ericksen conjectured and Pitteri [1984] proved that such a neighborhood — with suitable properties — can always be found. A clear, concise proof of the Ericksen-Pitteri Theorem is given by Ball and James [1992]. See also Cermelli and Gurtin [1993] for a treatment of Ericksen's ideas within the framework of this paper. A slight generalization would be to require that the deformation gradients ∇y_i be confined to suitable "Ericksen-Pitteri neighborhoods" $\mathcal{N}_i$, but to let the transition strain between phases be large. This can be handled within the present framework with only minor modifications.

such line segments at $\mathbf{X}_1$ and $\mathbf{X}_2$ that coincide when deformed, then the deformed lattices are also coincident at $\mathbf{x}$. In this case the interface is — in some sense — coherent at $\mathbf{x}$, and the same can be said if, for some symmetry transformation $\mathbf{A}$, $d\mathbf{X}_2 = \mathbf{A}\,d\mathbf{X}_1$ for all such line segments. In this case we refer to the interface as infinitesimally coherent; precisely, the interface is **infinitesimally coherent** at $\mathbf{x}\in\mathcal{S}$ if there is a symmetry transformation $\mathbf{A}$ such that

$$\mathbf{H}(\mathbf{X}_1) = \mathbf{A}\mathbb{1}_1(\mathbf{X}_1), \tag{5.2}$$

so that $\mathbf{H}(\mathbf{X}_1)$ and $\mathbf{A}$ coincide on vectors tangent to S_1 at $\mathbf{X}_1 = \mathbf{y}_1^{-1}(\mathbf{x})$. A condition equivalent to infinitesimal coherency at $\mathbf{x}$ is that, for some vector $\mathbf{b}$,

$$\mathbf{H}(\mathbf{X}_1) = \mathbf{A} + \mathbf{b}\otimes\mathbf{n}_1(\mathbf{X}_1). \tag{5.3}$$

If, in addition, $\mathbf{A}$ is a rotation, then

$$\boldsymbol{\jmath}_2(\mathbf{X}_2) = \boldsymbol{\jmath}_1(\mathbf{X}_1), \qquad \mathbf{n}_2(\mathbf{X}_2) = \mathbf{A}\mathbf{n}_1(\mathbf{X}_1). \tag{5.4}$$

c. Coherent subsurfaces

We continue to consider a two-phase deformation. Let $\mathcal{C}$ be a subsurface of $\mathcal{S}$, and write

$$C_i = \mathbf{y}_i^{-1}(\mathcal{C}) \tag{5.5}$$

for the subsurface of S_i that transforms to $\mathcal{C}$. Then $\mathcal{C}$ is *infinitesimally coherent* if the interface is infinitesimally coherent at each $\mathbf{x}\in\mathcal{C}$. A much stronger restriction is the content of the next definition. We say that $\mathcal{C}$ is **coherent** if there is a material isomorphism $\mathbf{f}$ such that

$$\mathbf{X}_2 = \mathbf{f}(\mathbf{X}_1) \text{ whenever } \mathbf{X}_1\in C_1 \text{ and } \mathbf{X}_2\in C_2 \text{ are compatible.} \tag{5.6}$$

Thus infinitesimal coherence at $\mathbf{x}$ is the requirement that infinitesimal segments of the lattices for the two phases fit together at $\mathbf{x}$, while coherency for $\mathcal{C}$ is the requirement that the lattices fit together over all of

$\mathcal{C}$. Note that (5.6) is equivalent to the assertion that the relative deformation $\mathbf{h}$, defined by (3.10), when restricted to C_1, is the restriction of a material isomorphism: for some $A \in \mathcal{G}$

$$\mathbf{h}(Z) - \mathbf{h}(X) = A[Z - X] \qquad \text{for all } X, Z \in C_1. \tag{5.7}$$

Theorem. *Let* $\mathcal{C}$ *be a subsurface of* $\mathcal{S}$. *Then*

$$\mathcal{C} \text{ coherent} \Rightarrow \mathcal{C} \text{ infinitesimally coherent.} \tag{5.8}$$

Conversely, if the symmetry group is a finite subgroup of Orth^+, *and if* $\mathcal{C}$ *is connected, then*

$$\mathcal{C} \text{ infinitesimally coherent} \Rightarrow \mathcal{C} \text{ coherent.} \tag{5.9}$$

Proof. The implication (5.8) is immediate. To prove the converse assertion assume that the symmetry group is a finite subgroup of Orth^+, and that $\mathcal{C}$ is connected and infinitesimally coherent. Then, for each $X \in C_1$,

$$\mathbf{H}(X) = \mathbf{Q}(X)\mathbb{1}_1(X), \tag{5.10}$$

for some $\mathbf{Q}(X) \in \mathcal{G}$. Choose arbitrary points $Z, \bar{Z} \in C_1$. Since C_1 is connected we can find a smooth curve W in C_1 from Z to $\bar{Z}$. Let $\mathcal{X}$ denote the set of all points $X \in W$ with $\mathbf{Q}(X) = \mathbf{Q}(Z)$.

Then $\mathcal{X}$ is closed. To verify this, choose a sequence X_n, with $X_n \in \mathcal{X}$. Since W is compact, this sequence converges to some $\bar{X} \in W$. By (5.10), $\mathbf{Q}(X)\mathbb{1}_1(X)$ is continuous on W; hence $\mathbf{Q}(X_n)\mathbb{1}_1(X_n) \to \mathbf{Q}(\bar{X})\mathbb{1}_1(\bar{X})$. But, since $\mathbb{1}_1(X)$ is continuous, $\mathbf{Q}(X_n)\mathbb{1}_1(X_n) = \mathbf{Q}(Z)\mathbb{1}_1(X_n) \to \mathbf{Q}(Z)\mathbb{1}_1(\bar{X})$; thus, as both $\mathbf{Q}(\bar{X})$ and $\mathbf{Q}(Z)$ are orthogonal, $\mathbf{Q}(\bar{X}) = \mathbf{Q}(Z)$ and $\bar{X} \in \mathcal{X}$.

Assume, for the purpose of contradiction, that $\mathcal{X} \neq W$. As $\mathcal{X}$ is closed, there is a point $\tilde{X} \in \partial\mathcal{X}$, $\tilde{X} \notin \partial W$, such that $\mathbf{Q}(\tilde{X}) = \mathbf{Q}(Z)$, and, since $\tilde{X} \notin \partial W$, there is a sequence $X_n \to \tilde{X}$, $X_n \in W$, such that, for each n, $\mathbf{Q}(X_n) \neq \mathbf{Q}(Z)$. Again by the continuity of $\mathbf{Q}(X)\mathbb{1}_1(X)$, $\mathbf{Q}(X_n)\mathbb{1}_1(X_n) \to \mathbf{Q}(\tilde{X})\mathbb{1}_1(\tilde{X})$. But since $\mathcal{G}$ is a *finite* group with *orthogonal* elements, and since $\mathbb{1}_1(X)$ is continuous, this can happen only if $\mathbf{Q}(X_n) = \mathbf{Q}(\tilde{X}) = \mathbf{Q}(Z)$ for all sufficiently large n, a contradiction. Therefore $\mathcal{X} = W$ and $\mathbf{Q}(Z) = \mathbf{Q}(\bar{Z})$; hence $\mathbf{Q}$ is constant

on C_1. Finally, choosing $\mathbf{X},\mathbf{Z}\in C_1$ and integrating $(d/d\sigma)h(\hat{\mathbf{Z}}(\sigma))$ along a smooth path $\hat{\mathbf{Z}}(\sigma)\in C_1$ with $\hat{\mathbf{Z}}(0)=\mathbf{Z}$ and $\hat{\mathbf{Z}}(1)=\mathbf{X}$ yields $h(\mathbf{X}) - h(\mathbf{Z}) = \mathbf{Q}[\mathbf{X}-\mathbf{Z}]$, so that, by (5.7), $\mathcal{C}$ is coherent.

5.2. PRODUCTION OF LATTICE POINTS BY THE MOVING INTERFACE

We let ℓ (=constant) denote the molar density of lattice points in the reference lattice per unit volume. The density $\bar{\ell}$ of lattice points per unit volume in the deformed body is then given by

$$J\bar{\ell} = \ell, \tag{5.11}$$

so that

$$L(\mathcal{R}) = \{\int_{\mathcal{R}} \bar{\ell}\, dv\}^{\cdot} \tag{5.12}$$

represents the rate at which lattice points are produced in a spatial control volume $\mathcal{R}$. $L(\mathcal{R})=0$ if $\mathcal{R}$ lies solely in one phase; thus lattice points are produced, at most, at the interface. If we apply (5.12) to a family $\mathcal{R}_\varepsilon$ of spatial control volumes that shrinks to an arbitrary regular interfacial set $\mathcal{G}$, we find that, by (4.40) and (5.11),

$$L(\mathcal{R}_\varepsilon) \to -\int_{\mathcal{G}} [\bar{\ell}U]da = -\ell\int_{\mathcal{G}} [W]da. \tag{5.13}$$

Thus

> the **production-rate of lattice points**, per unit deformed interfacial area, is $-\ell[W] = -[\bar{\ell}U]$. (5.14)

Coherent interfaces conserve lattice points, since $\vartheta_1=\vartheta_2$ and $V_1=V_2$. This feature of coherent interfaces is generally not shared by their incoherent counterparts.

The (positive or negative) production of lattice points by a moving interface induces defects: since atoms are conserved, a positive production should induce vacancies, while a negative production should induce interstitials.

5.3. WHEN ARE INTERFACES COHERENT?

We now give a characterization of incoherent behavior using three fundamental kinematical measures: the incoherency tensor, the slip velocity, and the production-rate of lattice points.

We will refer to the interface $\mathcal{S}$ as *coherent for all time* if $\mathcal{S}(t)$ is coherent at each t, and if the corresponding material isomorphism $\mathbf{f}$ for $\mathcal{S}(t)$ is independent of t. Granted this, we may change reference configuration for phase 1 so that the material isomorphism $\mathbf{f}$ is the identity. Therefore, without loss in generality, we may take $\mathbf{f}$ to be the identity in the definition above, and this we shall do. Also, for consistency, the assertion "$\mathcal{S}(0)$ is coherent" will have associated with it the requirement the material isomorphism corresponding to $\mathcal{S}(0)$ be the identity. Direct consequences of this definition, for an interface that is coherent for all time, are that

$$\begin{aligned} &S_1(t) = S_2(t) =: S(t), \\ &\mathbf{n}_1(\mathbf{X},t) = \mathbf{n}_2(\mathbf{X},t) =: \mathbf{n}(\mathbf{X},t), \\ &V_1(\mathbf{X},t) = V_2(\mathbf{X},t) =: V(\mathbf{X},t), \end{aligned} \tag{5.15}$$

and that the motion is continuous across the interface:

$$\mathbf{y}_1(\mathbf{X},t) = \mathbf{y}_2(\mathbf{X},t) \quad \text{for all } \mathbf{X}\in S(t). \tag{5.16}$$

Theorem. *If the interface $\mathcal{S}$ is coherent for all time, then, at each time:*

(a) *the interface is infinitesimally coherent;*

(b) *the interfacial volume-production rate vanishes identically;*

(c) *the interfacial slip velocity vanishes identically.*

Conversely, if the symmetry group is a finite subgroup of Orth^+, *if $\mathcal{S}(0)$ is coherent, and if* (a), (b), *and* (c) *are satisfied, then $\mathcal{S}$ is coherent for all time.*

Proof. Assume first that the interface is coherent: (5.8) then implies (a). Next, the standard compatibility conditions for a coherent interface, namely, $[\mathbf{F}] = [\mathbf{F}]\mathbf{n}\otimes\mathbf{n}$ and $[\mathbf{y}^\cdot] = -V[\mathbf{F}]\mathbf{n}$ (cf. (3.6) and (3.8) of [Gu]), yield,

by virtue of (4.27), conclusion (c). Finally, (4.6), $(5.4)_1$, (5.14), and (5.15) imply that $W_1=W_2$, which is (b).

Assume next that the hypotheses of the converse assertion are satisfied. By (a), (5.4) is satisfied. Thus (b), (4.6) and (5.14) imply that, for all compatible points $\mathbf{X}\in S_1$ and $\mathbf{X}_2\in S_2$,

$$V_2(\mathbf{X}_2,t) = V_1(\mathbf{X},t). \tag{5.17}$$

Assume first that $\mathcal{B}(t)$ is connected. By (a) and (5.9), $\mathcal{B}(t)$ is coherent at each t; thus the function $\mathbf{h}$ defined by (3.10) at each t is the restriction to $\mathcal{B}(t)$ of a material isomorphism

$$\mathbf{h}(\mathbf{X},t) = \mathbf{Q}\mathbf{X} + \mathbf{q}(t), \tag{5.18}$$

where $\mathbf{Q}$ is independent of t, since $\mathcal{G}$ is finite and $\mathbf{h}(\mathbf{X},t)$ continuous in t; in fact, the initial coherence of the interface and our agreement in the first paragraph of the section yields

$$\mathbf{Q} = \mathbf{1}, \quad \mathbf{q}(0) = \mathbf{0}, \tag{5.19}$$

so that, by $(5.4)_2$,

$$\mathbf{n}_2(\mathbf{X}_2,t) = \mathbf{n}_1(\mathbf{X},t) =: \mathbf{n}(\mathbf{X},t). \tag{5.20}$$

Next, let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be compatible trajectories; then, by definition, $\mathcal{Z}_1(t)$ and $\mathcal{Z}_2(t)$ coincide in the deformed configuration and

$$\mathcal{Z}_2(t) = \mathcal{Z}_1(t) + \mathbf{q}(t). \tag{5.21}$$

Assume further, that $\mathcal{Z}_1$ is normal, so that, by (4.28) with $\gamma=0$, $\mathcal{Z}_2$ is also normal. Thus $\mathbf{q}^\cdot$ is parallel to $\mathbf{n}$. On the other hand, (5.17), (5.20), and (5.21) yield $\mathbf{q}^\cdot\cdot\mathbf{n}=0$. Thus $\mathbf{q}^\cdot(t)=\mathbf{0}$ for all t. But the initial coherence of the interface yields $\mathbf{q}(0)=\mathbf{0}$; hence $\mathbf{q}(t)=\mathbf{0}$ for all t, and $\mathbf{h}(\mathbf{X},t)$ is the identity on $S_1(t)$ at each t. Thus $\mathcal{B}$ is coherent.

If $\mathcal{B}$ is not connected, then the foregoing argument applied to each connected component of $\mathcal{B}$ again renders $\mathbf{h}(\mathbf{X},t)$ the identity on $S_1(t)$.

This completes the proof.

The last theorem shows that an incoherent interface is associated with the *formation of defects.* It is generally believed that an absence of infinitesimal coherency leads to dislocations.[19] Here our theory is, in some sense, lacking, as it makes no provision for dislocations passed on the bulk material behind an advancing interface. Thus, at best, our results seem applicable to situations in which the interfacial production of dislocations has only minor effect on the bulk mechanics. Removing this restriction would be an interesting topic for future study.

[19] For discussions related to interfacial dislocations, in statics, cf. Brooks [1952], Nye [1953], Frank [1955], Bilby [1955], Bilby, Bullough, and De Grinberg [1964], Christian [1965,1985], Bollman [1967], Christian and Crocker [1980], and Pond [1985,1989].

B. SIMPLE THEORY WITHOUT INTERFACIAL STRUCTURE

We begin with a theory that neglects interfacial energy and stress, but includes internal configurational forces that act at the interface in response to the exchange of material between phases.

6. PRIMITIVE QUANTITIES

We consider $\mathfrak{A}$ species, $\alpha = 1,2,\ldots,\mathfrak{A}$, of mobile atoms, labelled so that — in the absence of defects — the atoms of species $\alpha \in \{1,2,\ldots,\mathfrak{b}\}$ *lie on lattice points,* while the atoms of species $\alpha \in \{\mathfrak{b}+1,\mathfrak{b}+2,\ldots,\mathfrak{A}\}$ are *interstitial.*

Let $\mathbf{y}$ be a two-phase motion. The following fields are the primitive quantities of the theory:

bulk fields

Ψ	**free energy**
ρ^α	**density**
$\mathbf{h}^\alpha$	**diffusive mass flux**
μ^α	**chemical potential**
$\mathbf{S}$	**deformational stress**
$\mathbf{C}$	**configurational stress**
$\mathbf{e}$	**internal configurational force**

interfacial field

$\mathbf{e}_i$	**internal configurational force** $(i=1,2)$

The field Ψ is the bulk free-energy per unit undeformed volume. The bulk densities ρ^α are atomic or *molar* densities, measured per unit undeformed volume. Since ℓ represents the number of lattice points, per unit undeformed volume,

$$d := \ell - (\rho^1 + \rho^2 + \ldots + \rho^{\mathfrak{b}}) \tag{6.1}$$

represents a density of *defects;* for $d > 0$, d is a density of vacancies, for $d < 0$, $|d|$ is the density of lattice-point atoms forced into interstitial positions. The bulk mass flux $\mathbf{h}^\alpha$ is measured in moles per unit undeformed area (and per unit time) and represents the transport of atoms of

species α *relative to the material* (lattice); the field

$$\mathbf{h}_{def} := -(\mathbf{h}^1 + \mathbf{h}^2 + \ldots + \mathbf{h}^b) \tag{6.2}$$

then represents a *flux of defects.*

A basic *hypothesis* of our theory concerns the chemical potentials and asserts that

$$\mu^\alpha \text{ is continuous across the interface.} \tag{6.3}$$

This, an assumption of **local-equilibrium**, allows us to consider the μ^α as chemical potentials for the interface as well as for the bulk material.

The deformational stress $\mathbf{S}$ is the standard Piola-Kirchhoff stress, measured per unit undeformed volume. The configurational fields $\mathbf{C}$, $\mathbf{e}$, and $\mathbf{e}_i$ (i=1,2), which are nonstandard, represent forces associated with the internal structure of the material; they act in response to the addition, removal, and rearrangement of atoms at points of the body. The stress $\mathbf{C}$, measured per unit undeformed area, represents configurational forces acting between neighboring parts of the body; $\mathbf{e}$, measured per unit undeformed volume, is an internal bulk force related to nonuniformities in the crystal lattice; $\mathbf{e}_i$, measured per unit undeformed area, acts at the interface in response to the exchange of material between phases.

We allow also for the following **external fields:**

bulk fields

Q^α	**mass supply**
$\mathbf{b}$	**deformational body force**

interfacial fields

$\bar{q}^\alpha$	**mass supply**
$\bar{\mathbf{g}}$	**deformational force**
$\mathbf{f}_i$	**configurational force**

Q^α, the external supply of species α to the bulk material, is measured in moles per unit undeformed volume. The external supply $\bar{q}^\alpha$ of species α to the interface represents a supply of material of *both phases,* and is

measured in moles per unit *deformed* area, as the deformed interface is where the two phases interact. Similarly, $\mathbf{b}$, measured per unit undeformed volume, and $\bar{\mathbf{g}}$, measured per unit deformed area, represent external forces applied to the bulk material and to the interface, respectively, while $\mathbf{f}_i$, measured per unit undeformed area, is an external configurational force on the phase i interface.

The external fields might be viewed as *virtual supplies* and *virtual forces;* they allow the primitive fields to be specified arbitrarily without violating the mass and force balances, a feature that facilitates our use of the "second law" in restricting constitutive equations.[20]

The following referential-to-spatial transformations will be used repeatedly:

$$\bar{\Psi} = J^{-1}\Psi, \quad \bar{\rho}^\alpha = J^{-1}\rho^\alpha, \quad \bar{e} = J^{-1}e, \quad \bar{b} = J^{-1}b, \quad \bar{Q}^\alpha = J^{-1}Q^\alpha,$$
$$\bar{\mathbf{S}} = J^{-1}\mathbf{S}\mathbf{F}^T, \quad \bar{\mathbf{C}} = J^{-1}\mathbf{C}\mathbf{F}^T, \quad \bar{\mathbf{h}}^\alpha = J^{-1}\mathbf{F}\mathbf{h}^\alpha, \tag{6.4}$$
$$\bar{\mathbf{e}}_i = \chi_i^{-1}\mathbf{e}_i, \quad \bar{f}_i = \chi_i^{-1}f_i.$$

[20] Cf. the paragraph labelled (2) in the Introduction of [Gu].

7. BALANCE LAWS FOR MASS AND FORCE

We formulate basic balance laws in the *deformed* body, where the two phases are comparable, but, as solids are best described materially, we express the resulting local relations in the *undeformed* body.

7.1. BALANCE OF MASS

Let $\mathcal{R}$ be an arbitrary spatial control volume. We write balance of mass for $\mathcal{R}$ following the material points currently in $\mathcal{R}$. Thus, since the mass flux $\mathbf{h}^\alpha$ is measured relative to the material, **balance of mass** is the requirement that

$$\{\int_{\mathcal{R}} \bar{\rho}^\alpha \, dv\}^\cdot = -\int_{\partial\mathcal{R}} \bar{\mathbf{h}}^\alpha \cdot \bar{\mathbf{m}} \, da + \int_{\mathcal{S}\cap\mathcal{R}} \bar{q}^\alpha \, da + \int_{\mathcal{R}} \bar{Q}^\alpha \, dv \tag{7.1}$$

for each species α, where we have used the referential-to-spatial transforms (6.4) to express the basic quantities in the spatial description. Here and in what follows, $\bar{\mathbf{m}}$ is the outward unit normal to $\partial\mathcal{R}$.

Using (4.40) and the analog of (4.41) for vector fields to localize (7.1), we arrive at the **interfacial mass balance**

$$[\rho^\alpha W] = [\jmath^{-1}\mathbf{h}^\alpha \cdot \mathbf{n}] - \bar{q}^\alpha. \tag{7.2}$$

We can rewrite the left side of this balance as $[\rho^\alpha]\langle W\rangle + [W]\langle\rho^\alpha\rangle$; the first term is analogous to the standard term $\jmath^{-1}[\rho^\alpha]V$ for a coherent interface, but the second term, which, by (5.14), represents a rate of mass transfer due to lattice-point production, is a direct consequence of incoherency.

The local areas of the undeformed interfaces are rendered compatible in this balance by the presence of the surface jacobians $\jmath_i$; in fact, by (4.6) we can rewrite (7.2) in the form

$$[\jmath^{-1}(\mathbf{h}^\alpha \cdot \mathbf{n} - \rho^\alpha V)] = \bar{q}^\alpha. \tag{7.3}$$

If we consider control volumes that exclude the interface, we are led to the standard relations

$$(\rho^\alpha)^\cdot = -\operatorname{Div}\mathbf{h}^\alpha + Q^\alpha \tag{7.4}$$

in bulk (in the interior of B_1 and B_2).

7.2. CONFIGURATIONAL AND DEFORMATIONAL FORCE BALANCES

We postulate, for each spatial control volume $\mathcal{R}$, the **force and moment balances**

$$\begin{aligned} &\int_{\partial\mathcal{R}} \bar{\mathbf{S}}\bar{\mathbf{m}}\,da + \int_{\mathcal{R}} \bar{\mathbf{b}}\,dv + \int_{\mathcal{S}\cap\mathcal{R}} \bar{\mathbf{g}}\,da = \mathbf{0}, \\ &\int_{\partial\mathcal{R}} \mathbf{r}\times\bar{\mathbf{S}}\bar{\mathbf{m}}\,da + \int_{\mathcal{R}} \mathbf{r}\times\bar{\mathbf{b}}\,dv + \int_{\mathcal{S}\cap\mathcal{R}} \mathbf{r}\times\bar{\mathbf{g}}\,da = \mathbf{0}, \end{aligned} \tag{7.5}$$

where $\mathbf{r}=\mathbf{x}-\mathbf{x}_0$ with $\mathbf{x}_0$ fixed, and where we have used the referential-to-spatial transforms (6.4) to express the stress and forces in the spatial description.

For an incoherent interface the lattices are independent; for that reason we do not postulate a configurational force balance for $\mathcal{R}$ as a whole, but instead characterize configurational forces by means of balances for the individual control volumes $\mathcal{B}_i(t)\cap\mathcal{R}$. Precisely, for each spatial control volume $\mathcal{R}$ and each phase i, we postulate a **configurational balance**

$$\int_{\mathcal{B}_i\cap\partial\mathcal{R}} \bar{\mathbf{C}}\bar{\mathbf{m}}\,da + \int_{\mathcal{S}\cap\mathcal{R}} (\bar{\mathbf{e}}_i + \bar{\mathbf{f}}_i)\,da + \int_{\mathcal{B}_i\cap\mathcal{R}} \bar{\mathbf{e}}\,dv = \mathbf{0}. \tag{7.6}$$

Shrinking $\mathcal{R}$ to the interface in $(7.5)_1$ and (7.6) using (4.41), we arrive at the **interfacial force balances**

$$[\jmath^{-1}\mathbf{S}\mathbf{n}] + \bar{\mathbf{g}} = \mathbf{0}, \qquad \delta_i\mathbf{C}_i\mathbf{n}_i + \mathbf{e}_i + \mathbf{f}_i = \mathbf{0}. \tag{7.7}$$

Similarly, using spatial control volumes $\mathcal{R}$ that do not intersect the interface yields the *bulk relations*

$$\operatorname{Div}\mathbf{S} + \mathbf{b} = \mathbf{0}, \qquad \mathbf{S}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{S}^{\mathsf{T}}, \qquad \operatorname{Div}\mathbf{C} + \mathbf{e} = \mathbf{0}. \tag{7.8}$$

7.3. IDENTIFICATION OF THE CONFIGURATIONAL STRESS WITH THE BULK ESHELBY TENSOR

It is convenient to introduce a **grand canonical potential**

$$\omega = \Psi - \Sigma_\alpha \rho^\alpha \mu^\alpha, \tag{7.9}$$

where here and in what follows Σ_α designates the sum over α from 1 to $\mathfrak{A}$.

A basic *hypothesis* of our theory is that *the configurational stress* $\mathbf{C}$ *be equal to the* **bulk Eshelby tensor:**

$$\mathbf{C} = \omega \mathbf{1} - \mathbf{F}^T \mathbf{S}. \tag{7.10}$$

This assumption is actually a *consequence* of a more general treatment[21] that allows for control volumes $\mathcal{R}(t)$ that evolve with time and thereby capture the mechanics and energetics associated with the addition and deletion of material points at control-volume boundaries. Note that by (3.15) the relation (7.10) has the alternative form

$$\mathbf{G}^T \bar{\mathbf{C}} = \bar{\omega} \mathbf{1} - \bar{\mathbf{S}}. \tag{7.11}$$

[21] Gurtin [1994]. A sketch is given here in Section 8.2.

8. ENERGETICS. DISSIPATION INEQUALITIES

8.1. STATEMENT OF THE SECOND LAW. GLOBAL DISSIPATION INEQUALITY

In the mechanical theory discussed here the *second law* is the requirement that the energy of material currently in a control volume $\mathcal{R}$ increase at a rate not greater than the expended power $\mathcal{P}(\mathcal{R})$ plus the energy $\mathcal{E}(\mathcal{R})$ supplied to $\mathcal{R}$ by mass transport.

As with balance of mass, we will write the second law for $\mathcal{R}$ following the material currently in $\mathcal{R}$. Thus, since the chemical potential μ^α represents the energy of a mole of species α,

$$\mathcal{E}(\mathcal{R}) = \Sigma_\alpha \{-\int_{\partial\mathcal{R}} \mu^\alpha \bar{h}^\alpha \cdot \bar{m}\, da + \int_{\mathcal{S}\cap\mathcal{R}} \mu^\alpha \bar{q}^\alpha\, da + \int_{\mathcal{R}} \mu^\alpha \bar{Q}^\alpha\, dv\}. \qquad (8.1)$$

Deformational forces act in the deformed body as a response to deformation. The stress $\mathbf{S}$ and the body force $\mathbf{b}$ act in the bulk material away from the interface, and we presume them *conjugate*[22] to the material velocity $\mathbf{y}^\cdot$. The external force $\bar{\mathbf{g}}$ acts at the interface, and we assume that $\bar{\mathbf{g}}$ is conjugate to the velocity $\bar{\mathbf{v}}$ of the deformed interface. Configurational forces are associated with the nondeformational kinetics of material points, and, for that reason, we assume that they expend power only at the interface, where the phase i regions undergo change. Thus bulk configurational forces do not expend power, but the external forces $\mathbf{f}_i$, which act at the interface, do. We assume that $\mathbf{f}_i$ is conjugate to the velocity $\mathbf{v}_i$ at which the phase i interface moves through the reference configuration. We therefore express the power expended on a spatial control volume $\mathcal{R}$ in the form

$$\mathcal{P}(\mathcal{R}) = \int_{\partial\mathcal{R}} \bar{\mathbf{S}}\bar{\mathbf{m}}\cdot\mathbf{y}^\cdot\, da + \int_{\mathcal{R}} \bar{\mathbf{b}}\cdot\mathbf{y}^\cdot\, dv + \int_{\mathcal{S}\cap\mathcal{R}} \{\bar{\mathbf{f}}_1\cdot\mathbf{v}_1 + \bar{\mathbf{f}}_2\cdot\mathbf{v}_2 + \bar{\mathbf{g}}\cdot\bar{\mathbf{v}}\}\, da, \qquad (8.2)$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\bar{\mathbf{v}}$ are compatible velocity fields for the interface. We do not allow $\mathbf{e}_i$ to contribute to $\mathcal{P}(\mathcal{R})$, since it acts internally to $\mathcal{R}$.

This discussion leads to a statement of the second law in the form of a

[22] A force f is "conjugate to" a velocity v if f "expends power over" v.

global dissipation inequality

$$\{\int_{\mathcal{R}} \bar{\Psi}\, dv\}^{\cdot} \le \mathcal{P}(\mathcal{R}) + \mathcal{E}(\mathcal{R}) \tag{8.3}$$

for all spatial control volumes $\mathcal{R}$.

8.2. ALTERNATIVE STATEMENT OF THE SECOND LAW[23]

For this section only, consider a spatial control volume $\mathcal{R}(t)$ that depends on t, let $R_i(t)$ denote the undeformed phase i region that deforms to $\mathcal{B}_i(t)\cap\mathcal{R}(t)$, let v_i and $\bar{v}$ denote the normal velocities of $\partial R_i(t)$ and $\partial\mathcal{R}(t)$, and let $\mathbf{q}_i$ and $\bar{\mathbf{q}}$ denote compatible velocity fields for $\partial R_i(t)$ and $\partial\mathcal{R}(t)$ (as in Section 4.2a).

The forms taken for balance of mass and the second law depend crucially on how we view the motion of $\partial R_i(t)$. A standard precept of continuum mechanics is that when writing basic laws for $R_i(t)$ the material external to $R_i(t)$ is irrelevant provided its action is suitably accounted for by the action of stresses on $\partial R_i(t)$. Thus the basic laws should be expressible in a form that does not distinguish between whether or not there is a different phase exterior to $\partial R_i(t)$. To accomplish this we might view the dependence of $R_i(t)$ on t as representing the addition of material to — or the deletion of material from — the boundary $\partial R_i(t)$. Based on this viewpoint, we write balance of mass and the dissipation inequality as

$$(d/dt)\{\int_{\mathcal{R}} \bar{\rho}^{a}\, da\} = M(\mathcal{R}) + \int_{\partial\mathcal{R}} \mathcal{h}^{a}\, da, \tag{7.1'}$$

$$(d/dt)\{\int_{\mathcal{R}} \bar{\Psi}\, dv\} \le \mathcal{P}^{*}(\mathcal{R}) + \mathcal{E}(\mathcal{R}) + \Sigma_a \int_{\partial\mathcal{R}} \mu^{a}\mathcal{h}^{a}\, da, \tag{8.3'}$$

where $\mathcal{h}^{a} = \bar{\rho}^{a}(\bar{v} - \mathbf{y}^{\cdot}\cdot\bar{\mathbf{m}})$, $M(\mathcal{R})$ is the right side of (7.1), and

$$\mathcal{P}^{*}(\mathcal{R}) = \int_{\partial\mathcal{R}} (\bar{\mathbf{S}}\bar{\mathbf{m}}\cdot\bar{\mathbf{q}} + \bar{\mathbf{C}}\bar{\mathbf{m}}\cdot\mathbf{q})\, da + \int_{\mathcal{R}} \bar{\mathbf{b}}\cdot\mathbf{y}^{\cdot}\, dv + \int_{\mathcal{S}\cap\mathcal{R}} \{\bar{\mathbf{f}}_1\cdot\mathbf{v}_1 + \bar{\mathbf{f}}_2\cdot\mathbf{v}_2 + \bar{\mathbf{g}}\cdot\bar{\mathbf{v}}\}\, da, \tag{8.4}$$

---

[23] Gurtin [1994].

where $\mathbf{q} = \mathbf{q}_i$ on $\mathcal{B}_i \cap \partial\mathcal{R}$, i=1,2.

In contrast to (7.1) and (8.3), (7.1') and (8.3') do not follow the material currently in $\mathcal{R}(t)$, but instead follow $\mathcal{R}(t)$ itself; thus the terms $(d/dt)\{\ldots\}$ rather than $\{\ldots\}^{\cdot}$ and the presence of the terms involving $\mathcal{A}^{\alpha}$. The form $\mathcal{P}^{*}(\mathcal{R})$ taken for the expended power is also based on following $\mathcal{R}(t)$, as the stresses are conjugate to velocities associated with $\partial\mathcal{R}(t)$ and $\partial R_i(t)$.

The two formulations of the basic laws are, in a sense, consistent: for stationary $\mathcal{R}$, (7.1') is equivalent to (7.1) and, granted the identification (7.10) of the configurational stress with the Eshelby tensor, (8.3') is equivalent to (8.3). In fact, (7.10) is a consequence of the following two assumptions: that (8.3') hold for all $\mathcal{R}(t)$; (ii) that (8.3') be independent of the choice of compatible velocity fields $\mathbf{q}_i$ and $\bar{\mathbf{q}}$ for $\partial R_i(t)$ and $\partial\mathcal{R}(t)$.

8.3. LOCAL FORM FOR THE EXPENDED POWER

a. Invariance under reparametrization. Effective shear

We require that $\mathcal{P}(\mathcal{R})$ be *independent* of the choice of (compatible) parametrizations for the deformed and undeformed interfaces, or equivalently, of the choice of compatible velocity fields. The Invariance Lemma of [GS] (Appendix C, p. 156) then yields the compatibility condition

$$(\bar{\mathbf{g}} + \mathbf{G}_1^{\mathsf{T}}\bar{\mathbf{f}}_1 + \mathbf{G}_2^{\mathsf{T}}\bar{\mathbf{f}}_2)_{\tan\mathcal{S}} = \mathbf{0}. \tag{8.5}$$

This relation is useful in developing an expression for the effective shear acting across the interface. Let

$$\tau_i = -(\mathbf{G}_i^{\mathsf{T}}\bar{\mathbf{e}}_i)_{\tan\mathcal{S}}. \tag{8.6}$$

Then (7.7) and (7.11) imply that

$$\tau_i = (\delta_i\mathbf{G}_i^{\mathsf{T}}\bar{\mathbf{C}}_i\bar{\mathbf{n}} + \mathbf{G}_i^{\mathsf{T}}\bar{\mathbf{f}}_i)_{\tan\mathcal{S}} = (-\delta_i\bar{\mathbf{S}}_i\bar{\mathbf{n}} + \mathbf{G}_i^{\mathsf{T}}\bar{\mathbf{f}}_i)_{\tan\mathcal{S}}, \tag{8.7}$$

and, by $(7.7)_1$ and (8.5),

$$\tau_2 = -\tau_1. \tag{8.8}$$

We will refer to

$$\tau = \tau_2 = -\tau_1 = -\delta_i(G_i^T \bar{\mathbf{e}}_i)_{\tan\mathcal{S}}. \tag{8.9}$$

as the **effective shear**. Note that, if $\mathbf{f}_i = \mathbf{0}$, then, by (8.7),

$$\tau = -(\bar{\mathbf{S}}_i \bar{\mathbf{n}})_{\tan\mathcal{S}}; \tag{8.10}$$

thus in the absence of external forces τ_i represents the shearing traction exerted across the interface on the bulk material of phase i.

We will refer to

$$\Pi_i = \mathbf{e}_i \cdot \mathbf{n}_i \tag{8.11}$$

as the **normal internal force**. Note that, by (3.8),

$$\tau = 0 \quad \Leftrightarrow \quad \mathbf{e}_i = \Pi_i \mathbf{n}_i. \tag{8.12}$$

More generally, τ, Π_1, and Π_2 uniquely determine $\mathbf{e}_1$ and $\mathbf{e}_2$.

b. Interfacial power density

If in (8.2) we let the control volume $\mathcal{R}$ shrink to a regular interfacial set $G \subset \mathcal{S}$, we find, using (4.41), that $\mathcal{P}(\mathcal{R})$ tends to

$$\begin{aligned} &\mathcal{P}_{loc}(G) = \int_G \bar{\mathfrak{p}}\, da, \\ &\bar{\mathfrak{p}} = [\bar{\mathbf{S}}\bar{\mathbf{n}} \cdot \mathbf{y}^{\cdot}] + \bar{\mathbf{f}}_1 \cdot \mathbf{v}_1 + \bar{\mathbf{f}}_2 \cdot \mathbf{v}_2 + \bar{\mathbf{g}} \cdot \bar{\mathbf{v}}. \end{aligned} \tag{8.13}$$

$\mathcal{P}_{loc}$ is the power directly relevant to the evolution of the interface; its integrand, $\bar{\mathfrak{p}}$, the **interfacial power density**, is the power expended on the interface, per unit deformed area. By (4.10), (7.7), and (7.11), and since $\mathbf{v}_i \cdot \mathbf{n}_i = V_i = \jmath_i W_i$, $\bar{\omega}_i F_i^T \bar{\mathbf{n}} = \jmath_i^{-1} \omega_i \mathbf{n}_i$, $\bar{\mathbf{e}}_i = \jmath_i^{-1} \mathbf{e}_i$, it follows that

$$\begin{aligned} \bar{\mathfrak{p}} &= -\sum_{i=1,2} (\delta_i (\mathbf{F}_i^T \bar{\mathbf{S}}_i + \bar{\mathbf{C}}_i)\bar{\mathbf{n}} + \bar{\mathbf{e}}_i) \cdot \mathbf{v}_i \\ &= -\sum_{i=1,2} (\delta_i \bar{\omega}_i F_i^T \bar{\mathbf{n}} + \bar{\mathbf{e}}_i) \cdot \mathbf{v}_i = -\sum_{i=1,2} (\delta_i \omega_i W_i + \bar{\mathbf{e}}_i \cdot \mathbf{v}_i). \end{aligned}$$

Thus $\mathbf{e}_i$ is conjugate to the velocity $\mathbf{v}_i$ of the phase i interface. In fact, if we choose the materially normalized velocities (4.13)-(4.14) for $\mathbf{v}_i$ and use (4.38), (8.9), and (8.11), we find that

$$\bar{\mathbf{e}}_1 \cdot \mathbf{v}_1 + \bar{\mathbf{e}}_2 \cdot \mathbf{v}_2 = \Pi_1 W_1 + \Pi_2 W_2 + \tau \cdot \gamma; \tag{8.14}$$

hence the power expended by the internal configurational forces is equal to the power expended by the normal internal forces Π_i over the volume flows W_i plus the power expended by the effective shear τ over the slip velocity γ. By (8.14),

$$\bar{\rho} = - \sum_{i=1,2} \{(\delta_i \omega_i + \Pi_i) W_i\} - \tau \cdot \gamma, \tag{8.15}$$

8.4. LOCAL DISSIPATION INEQUALITIES

The inequality (8.3), when applied to control volumes that exclude the interface, yields the *bulk dissipation inequality*

$$\Psi^{\cdot} \leq \mathbf{S} \cdot \mathbf{F}^{\cdot} + \Sigma_\alpha \mu^\alpha (\rho^\alpha)^{\cdot} - \Sigma_\alpha \mathbf{h}^\alpha \cdot \nabla \mu^\alpha. \tag{8.16}$$

To localize (8.3) to the interface we let the control volume $\mathcal{R}$ shrink to a regular interfacial set $\mathcal{G} \subset \mathcal{S}$. By (4.41) and (7.2), $\mathcal{E}(\mathcal{R})$ tends to

$$- \int_{\mathcal{G}} \Sigma_\alpha \mu^\alpha [\rho^\alpha W] \, da.$$

Thus, by (4.40), (7.9), (8.13), and (8.15), shrinking $\mathcal{R}$ to the interface in (8.3) yields the **interfacial dissipation inequality**

$$\Pi_1 W_1 + \Pi_2 W_2 + \tau \cdot \gamma \leq 0, \tag{8.17}$$

which is a central relation of our theory. Note that, by (8.14) and (8.17), *dissipation at the interface is due solely to the working of the internal configurational forces* $\mathbf{e}_i$.

9. CONSTITUTIVE EQUATIONS

9.1. BULK CONSTITUTIVE EQUATIONS

Let

$$\rho = (\rho^1,\dots,\rho^{\mathfrak{a}}), \qquad \mu = (\mu^1,\dots,\mu^{\mathfrak{a}}), \qquad \mathfrak{h} = (\mathbf{h}^1,\dots,\mathbf{h}^{\mathfrak{a}}), \tag{9.1}$$

(with $\mathfrak{h}$ and $\nabla\mu$ identified with vectors in $\mathbb{R}^{3\mathfrak{a}}$). We consider bulk constitutive equations

$$\begin{aligned} \mathbf{S} = \hat{\mathbf{S}}_i(\mathbf{F},\rho), \quad \mu = \hat{\mu}_i(\mathbf{F},\rho), \quad \Psi = \hat{\Psi}_i(\mathbf{F},\rho), \\ \mathfrak{h} = -\mathbf{D}_i(\mathbf{F},\rho)\nabla\mu, \end{aligned} \tag{9.2}$$

for each phase i, with

$$\hat{\mathbf{S}}_i(\mathbf{F},\rho) = \partial_{\mathbf{F}}\hat{\Psi}_i(\mathbf{F},\rho), \qquad \hat{\mu}_i(\mathbf{F},\rho) = \partial_{\rho}\hat{\Psi}_i(\mathbf{F},\rho), \tag{9.3}$$

and with *mobility* $\mathbf{D}_i(\mathbf{F},\rho)$ (a linear transformation of $\mathbb{R}^{3\mathfrak{a}}$ into itself) compatible with the inequality $\Sigma_\alpha \mathbf{h}^\alpha\cdot\nabla\mu^\alpha \le 0$, restrictions that ensure consistency with the bulk dissipation inequality. We assume, in addition, that the constitutive relation for the stress is consistent with $(7.8)_2$.

The Eshelby relation (7.10) yields an auxiliary constitutive relation giving $\mathbf{C}$ as a function of $(\mathbf{F},\rho)$. In conjunction with this, we consider $\mathbf{e}$ as *indeterminate;* in fact, as defined by the configurational balance $(7.8)_3$; (9.3) then yields $\mathbf{e} = -\Sigma_\alpha \rho^\alpha\cdot\nabla\mu^\alpha - \mathbf{F}^{\mathrm{T}}\mathbf{b}$, so that $\mathbf{e}$ here responds to spatial variations in the chemical potential.

9.2. CONSTITUTIVE EQUATIONS FOR THE INTERFACE.

We consider constitutive equations for the interface giving the normal internal forces Π_i and the effective shear τ as functions of the normal $\bar{\mathbf{n}}$, the volume flows W_1 and W_2, and the limiting values $\mathbf{F}_1$, $\mathbf{F}_2$, ρ_1, and ρ_2 of the deformation gradient and density (list):

$$\begin{aligned} &\Pi_i = \hat{\Pi}_i(\mathcal{Z}), \qquad \tau = \hat{\tau}(\mathcal{Z}), \\ &\mathcal{Z} = (\mathbf{F}_1,\mathbf{F}_2,\rho_1,\rho_2,\bar{\mathbf{n}},W_1,W_2), \end{aligned} \tag{9.4}$$

where $\mathcal{Z}$ is required to satisfy

$$\hat{\mu}_1(F_1,\rho_1) = \hat{\mu}_2(F_2,\rho_2) \tag{9.5}$$

to ensure consistency with (6.3). (We choose the normal $\bar{n}$, rather than the referential normals n_i, since the n_i are not independent.)

Suppose that we are given an arbitrary two-phase motion y and an arbitrary density field (list) ρ that is smooth away from, and consistent with (9.5) across, the interface. Then the constitutive equations may be used to compute a **constitutive process** consisting of Z as an *interfacial field,* the bulk fields S, Ψ, μ, and $\flat$, and the interfacial fields Π_i and τ. The balance laws for mass and force may then be used as follows to compute the mass supplies and external forces needed to support the process: the mass balances (7.2) and (7.4) are used to compute the mass supplies $\bar{q}^\alpha$ and Q^α for each α; the force balances $(7.7)_1$ and $(7.8)_1$ yield b and $\bar{g}$; Φ_i is determined from Π_i and τ using (8.9) and (8.11); and $(7.7)_2$ is used to compute f_i. The forces $\bar{g}$ and f_i computed in this manner satisfy the invariance requirement (8.5). Therefore

> *The mass supplies and external forces allow us to consider arbitrary constitutive processes with the assurance that the balance laws are satisfied.* (9.6)

This leaves only the second law — in the form of the global dissipation inequality (8.3) — to be satisfied in all constitutive processes. Since the bulk constitutive equations automatically satisfy the bulk dissipation inequality (8.16), we have only to satisfy the interfacial dissipation inequality (8.17). A basic hypothesis of our theory is that *all constitutive processes be compatible with* (8.17). The following results are consequences of this axiom:

(i) *The effective shear vanishes:*

$$\tau = 0. \tag{9.7}$$

(ii) *The following kinetic relations hold:*

$$\Pi_i = -\beta_{i1}W_1 - \beta_{i2}W_2, \tag{9.8}$$

where $\beta_{ij} = \hat{\beta}_{ij}(Z)$ *is consistent with*

$$\sum_{j,k=1}^{2} \hat{\beta}_{jk}(Z) W_j W_k \geq 0. \tag{9.9}$$

To establish (i) and (ii), note first that, since the dissipation inequality (8.17) is required to hold in all constitutive processes,

$$\hat{\Pi}_1(Z)W_1 + \hat{\Pi}_2(Z)W_2 + \hat{\tau}(Z)\cdot\gamma \leq 0 \tag{9.10}$$

in all two-phase motions.

To prove (9.7), it is sufficient to recognize that since, by hypothesis, the functions $\Phi_i(Z)$ and $\hat{\tau}(Z)$ in (9.10) do not depend on γ, if $\tau \neq 0$, an arbitrary choice of γ would lead to a term of arbitrary sign and size in the left side of (9.10).

Thus, letting $\xi = (W_1, W_2)$, $\varphi = -(\hat{\Pi}_1, \hat{\Pi}_2)$, and suppressing the remaining arguments in Z, we see that to establish (9.8) and (9.9) it suffices to show that $\varphi(\xi)\cdot\xi \geq 0$ for all ξ implies the existence, for each ξ, of a linear transformation $B(\xi)$ from $\mathbb{R}^2$ into $\mathbb{R}^2$ such that $\varphi(\xi) = B(\xi)\xi$ and $\xi\cdot B(\xi)\xi \geq 0$ for all ξ. Choose $\lambda > 0$. Then, since $\varphi(\lambda\xi)\cdot\lambda\xi \geq 0$, we have $\varphi(\lambda\xi)\cdot\xi \geq 0$, and letting $\lambda \to 0$, we see that $\varphi(0)\cdot\xi \geq 0$ for all ξ, so that $\varphi(0) = 0$. Thus

$$\varphi(\xi) = \left\{ \int_0^1 \nabla\varphi(s\xi)\, ds \right\} \xi, \tag{9.11}$$

which yields the desired conclusions with $B(\xi) = \{\ldots\}$.

The results (i) and (ii) are also sufficient that all constitutive relations be compatible with the reduced dissipation inequality (8.17).

By (8.12), the constitutive relations (9.7) and (9.8) may be written in the equivalent vectorial form

$$\mathbf{e}_i = -(\beta_{i1}W_1 + \beta_{i2}W_2)\mathbf{n}_i. \tag{9.12}$$

Thus the internal configurational force $\mathbf{e}_i$ is a drag force representing dissipation in the exchange of material between phases.

We may recast the constitutive relations using $\langle W \rangle$ and the lattice-point production-rate $[W]$ as independent variables:

$$2\langle \Pi \rangle = -\alpha_{11}\langle W \rangle - \alpha_{12}[W], \qquad (9.13)$$
$$[\Pi] = -\alpha_{21}\langle W \rangle - \alpha_{22}[W],$$

with α_{ij} suitable (new) kinetic coefficients.

9.3. OTHER CONSTITUTIVE THEORIES

a. Theory without lattice-point production

A theory in which the *production of lattice points is not allowed* may be obtained by assuming, from the outset, that

$$W_1 = W_2 =: W. \qquad (9.14)$$

Granted this constraint, $[\Pi]$ becomes *indeterminate* and (9.13) is replaced by

$$2\langle \Pi \rangle = -\alpha W, \qquad (9.15)$$

with $\alpha = \hat{\alpha}(Z) \geq 0$.

b. Theory without slip

Aternatively, a theory in which *slip is prohibited* may be based on the constraint

$$\gamma = 0. \qquad (9.16)$$

Granted this, τ becomes *indeterminate,* so that (9.7) is no longer valid, but (9.8) remains unchanged.

c. Theory with viscous friction between phases

The result $\tau = 0$ is a consequence of our failure to include γ in the constitutive equations. If we allow for a dependence on γ in the constitutive functions (9.4) we obtain additional terms in (9.8) proportional

to γ in conjunction with a nonvanishing constitutive relation, similar to the other relations, for τ. In this formulation the relation between τ and γ is smooth, so that the friction between phases is viscous, but a relaxation of the underlying smoothness assumptions would allow a nonsmooth dependence of τ on γ, for example of the type encountered in plasticity.[24]

[24] After completing this work, Gurtin attended a lecture of L. Truskinovsky discussing constitutive behavior of this type. See also the discussion of Larché and Cahn [1978], p. 1587.

10. EVOLUTION EQUATIONS IN THE ABSENCE OF EXTERNAL FORCES

The external forces $\mathbf{b}$, $\bar{\mathbf{g}}$, and $\mathbf{f}_i$ and mass supplies Q^α and $\bar{q}^\alpha$ allow for arbitrary processes without violation of the basic balance laws, but in actual applications these fields are generally unimportant. We now list the complete set of equations that hold when these external fields vanish. Here $\mathbf{C}$ is the Eshelby tensor:

$$\mathbf{C} = \omega\mathbf{1} - \mathbf{F}^T\mathbf{S}, \qquad \omega = \Psi - \Sigma_\alpha \rho^\alpha \mu^\alpha. \tag{10.1}$$

10.1. UNCONSTRAINED THEORY

a. Dynamics

The interface conditions consist of the *compatibility condition*

$$[\mathbf{y}^\cdot]\cdot\bar{\mathbf{n}} = -[JW], \tag{10.2}$$

the *normal force balance*

$$[\jmath^{-1}\mathbf{S}\mathbf{n}]\cdot\bar{\mathbf{n}} = 0, \tag{10.3}$$

the *partial balance*

$$\delta_i \mathbf{C}_i \mathbf{n}_i = (\beta_{i1}W_1 + \beta_{i2}W_2)\mathbf{n}_i \tag{10.4}$$

for each phase i, and the *mass balance*

$$[\rho^\alpha W] = [\jmath^{-1}\mathbf{h}^\alpha\cdot\mathbf{n}] \tag{10.5}$$

and *local equilibrium condition*

$$[\mu^\alpha] = 0 \tag{10.6}$$

for each species α (cf. (4.7), (6.3), (7.2), (7.7), and (9.12)).

The partial balances may be split into normal and tangential parts with respect to S_i, yielding, for each phase i, a *configurational balance* and a *tangential slip condition:*

$$\delta_i \mathbf{n}_i \cdot \mathbf{C}_i \mathbf{n}_i = (\beta_{i1} W_1 + \beta_{i2} W_2),$$
$$(\mathbf{F}_i^{\mathsf{T}} \mathbf{S}_i \mathbf{n}_i)_{\tan S_i} = \mathbf{0}. \qquad (10.7)$$

The relations $(10.7)_2$ together form the tangential part of (10.3).

These interface conditions with the *bulk equations*

$$\mathrm{Div}\,\mathbf{S} = \mathbf{0}, \qquad (\rho^\alpha)^\cdot = -\mathrm{Div}\,\mathbf{h}^\alpha, \qquad (10.8)$$

the bulk constitutive equations (9.2), and appropriate boundary and initial conditions form the basic free-boundary problem of the theory.

The grand canonical energy ω and hence the basic equations are invariant when the energy per unit mass is changed by an additive constant. Indeed, if $\rho = \Sigma_\alpha \rho^\alpha$, and if we replace the energy Ψ by $\Psi^* = \Psi + \kappa\rho$ with κ an arbitrary constant, then the chemical potential, given by $(9.3)_2$ with Ψ replaced by Ψ^*, is $(\mu^\alpha)^* = \mu^\alpha + \kappa$; this yields the desired conclusion: $\omega^* = \Psi^* - \Sigma_\alpha \rho^\alpha (\mu^\alpha)^* = \omega$.

A consequence of the mass balances are the following bulk and interfacial *balance laws for defects:*

$$d^\cdot = -\mathrm{Div}\,\mathbf{h}_{\mathrm{def}}, \qquad [dW] = [\vartheta^{-1}\mathbf{h}_{\mathrm{def}} \cdot \mathbf{n}] + \ell[W], \qquad (10.9)$$

where d, ℓ, and $\mathbf{h}_{\mathrm{def}}$ are as defined in the paragraph containing (6.1). The first of (10.9) follows directly from (6.1), (6.2), and (10.8); the second follows from (4.40) in conjunction with (6.1), (6.2), and (10.5). Comparing $(10.9)_2$ to (10.5), we see that the interface serves as a source for defects, $\ell[W]$ being the strength of the source.

b. Statics

When the interface is stationary and the deformation and densities independent of time, the interface equations have a simple form:

$$[\vartheta^{-1}\mathbf{S}\mathbf{n}] \cdot \bar{\mathbf{n}} = 0, \quad \mathbf{C}_i \mathbf{n}_i = 0, \quad [\vartheta^{-1}\mathbf{h}^\alpha \cdot \mathbf{n}] = 0, \quad [\mu^\alpha] = 0; \qquad (10.10)$$

as do the bulk equations:

$$\mathrm{Div}\,\mathbf{S} = \mathbf{0}, \qquad \mathrm{Div}\,\mathbf{h}^{\alpha} = 0. \tag{10.11}$$

10.2. CONSTRAINED THEORIES

For the *theory without lattice-point production* the basic equations consist of the constraint

$$W_1 = W_2 =: W, \tag{10.12}$$

the compatibility condition

$$[\mathbf{y}^{\cdot}]\cdot\bar{\mathbf{n}} = -[J]W, \tag{10.13}$$

the mass balance

$$[\rho^{\alpha}]W = [\mathcal{J}^{-1}\mathbf{h}^{\alpha}\cdot\mathbf{n}], \tag{10.14}$$

the local equilibrium condition (10.6), the normal force balance (10.3), the tangential slip condition $(10.7)_2$, and the *single* configurational balance

$$[\mathbf{n}\cdot\mathbf{C}\mathbf{n}] = \alpha W, \tag{10.15}$$

which should be compared to the analogous result in the coherent theory.[25]

For the *theory without slip* the tangential slip condition $(10.7)_2$ is dropped, and the basic equations consist of the constraint

$$[\mathbf{y}^{\cdot}] = -[\mathbf{V}\mathbf{F}\mathbf{n}] \tag{10.16}$$

(which implies the compatibility condition (10.2)), the force balance

$$[\mathcal{J}^{-1}\mathbf{S}\mathbf{n}] = \mathbf{0}, \tag{10.17}$$

the mass balance (10.5), the local equilibrium condition (10.6), and the configurational balance $(10.7)_1$.

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[25] [GS], eqt. (14.7); [Gu], eqt. (8.3).

11. RESULTS IN THE SPATIAL DESCRIPTION

The interface conditions take a particularly simple form in the spatial description; we now outline the form the theory would take were that description used from the outset.

The mass and force balances for the interface and bulk material take the form

$$\begin{aligned} &[\bar{\rho}^{\alpha} U] = [\bar{h}^{\alpha}]\cdot\bar{n} - \bar{q}^{\alpha}, \quad [\bar{S}]\bar{n} + \bar{g} = 0, \quad \delta_i \bar{C}_i \bar{n} + \bar{\mathbf{e}}_i + \bar{f}_i = 0, \\ &\partial_t \bar{\rho}^{\alpha} + \operatorname{div}(\bar{\rho}^{\alpha} \mathbf{y}^{\cdot}) = -\operatorname{div} \bar{h}^{\alpha} + \bar{Q}^{\alpha}, \qquad \operatorname{div} \bar{S} + \bar{b} = 0. \end{aligned} \tag{11.1}$$

The interfacial power density, whose derivation is analogous to that of (8.15) and uses (4.33) and the spatially normalized velocities (4.11)-(4.12), is

$$\bar{\mathfrak{p}} = -\sum_{i=1,2} \{ (\delta_i \bar{\omega}_i + \pi_i) U_i \} - \tau \cdot [\mathbf{y}^{\cdot}]_{\tan \mathcal{S}}, \tag{11.2}$$

with $\bar{\omega} = J^{-1}\omega$ and

$$\pi_i = (G_i^{\mathsf{T}} \bar{\mathbf{e}}_i) \cdot \bar{n} \tag{11.3}$$

the spatial counterpart of the normal internal force. This leads to the interfacial dissipation inequality

$$\pi_1 U_1 + \pi_2 U_2 + \tau \cdot [\mathbf{y}^{\cdot}]_{\tan \mathcal{S}} \leq 0. \tag{11.4}$$

The spatial constitutive theory is based on relations

$$\begin{aligned} &\pi_i = \tilde{\pi}_i(\mathbf{z}) \quad (i = 1,2), \qquad \tau = \tilde{\tau}(\mathbf{z}), \\ &\mathbf{z} = (F_1, F_2, \bar{\rho}_1, \bar{\rho}_2, \bar{n}, U_1, U_2), \end{aligned} \tag{11.5}$$

which, when restricted by the dissipation inequality (11.4), reduce to

$$\begin{aligned} &\tau = 0 \\ &\pi_i = -\eta_{i1} U_1 - \eta_{i2} U_2, \end{aligned} \tag{11.6}$$

$i = 1,2$, where $\eta_{ij} = \tilde{\eta}_{ij}(\mathbf{z})$ is consistent with

$$\sum_{j,k=1}^{2} \tilde{\eta}_{jk}(\mathbf{z}) U_j U_k \geq 0 \tag{11.7}$$

and is related to the β_{ij} of (9.8) through $\beta_{ij} = J_i^2 \eta_{ij}$. The equations (11.6) are equivalent to the vector equation

$$\mathbf{G}_i^T \bar{\mathbf{e}}_i = -(\eta_{i1} U_1 + \eta_{i2} U_2)\bar{\mathbf{n}}. \tag{11.8}$$

When the mass supplies and external forces vanish, the resulting interface conditions consist of the *kinematic compatibility condition*

$$[\mathbf{y}^\cdot] \cdot \bar{\mathbf{n}} = -[U], \tag{11.9}$$

the *normal force balance*

$$\bar{\mathbf{n}} \cdot [\bar{\mathbf{S}}] \bar{\mathbf{n}} = 0, \tag{11.10}$$

the *partial balance*

$$\delta_i(\bar{\omega}_i \bar{\mathbf{n}} - \bar{\mathbf{S}}_i \bar{\mathbf{n}}) = (\eta_{i1} U_1 + \eta_{i2} U_2)\bar{\mathbf{n}} \tag{11.11}$$

for each phase i, and the *mass balance*

$$[\bar{\rho}^\alpha U] = [\bar{\mathbf{h}}^\alpha] \cdot \bar{\mathbf{n}} \tag{11.12}$$

and local equilibrium condition (10.6) for each species α. The relations (11.11) may be split into normal and tangential parts (with respect to $\mathcal{S}$), yielding, for each phase i, a *normal configurational balance* and a *tangential slip condition:*

$$\begin{aligned} &\delta_i(\bar{\omega}_i - \bar{\mathbf{n}} \cdot \bar{\mathbf{S}}_i \bar{\mathbf{n}}) = \eta_{i1} U_1 + \eta_{i2} U_2, \\ &(\bar{\mathbf{S}}_i \bar{\mathbf{n}})_{\tan \mathcal{S}} = 0. \end{aligned} \tag{11.13}$$

When the interface is stationary and the deformation and densities independent of time, the interface conditions reduce to relations

$$\bar{\mathbf{n}}\cdot[\bar{\mathbf{S}}]\bar{\mathbf{n}} = 0, \qquad (\bar{\omega}_i\mathbf{1} - \bar{\mathbf{S}}_i)\bar{\mathbf{n}} = 0, \qquad [\bar{\mathbf{h}}^\alpha]\cdot\bar{\mathbf{n}} = 0, \qquad [\mu^\alpha] = 0 \tag{11.14}$$

that were derived variationally by Larché and Cahn[26] as necessary conditions for the equilibrium of incoherent precipitates. The corresponding bulk equations are

$$\operatorname{div}\bar{\mathbf{S}} = \mathbf{0}, \qquad \operatorname{div}\bar{\mathbf{h}}^\alpha = 0. \tag{11.15}$$

Note that (11.14) yields the additional relation $[\bar{\omega}] = 0$.

[26] [1978], eqts. (25)-(27); [1985], eqts. (319)-(321).

C. THEORY WITH INTERFACIAL STRUCTURE

We now extend the theory to include the energy and elasticity of the interface. We neglect mass transport within the interface, so that the mass balances are as derived in Section 7.1.

We will continue to formulate basic laws in the deformed body, but, as before, we will express local relations referentially; in fact, *we will express interface conditions in the reference configuration for phase* 1.

Throughout what follows $\mathcal{R}$ will be a spatial control volume, and we will consistently write

$$\mathcal{G}(t)=\mathcal{S}(t)\cap\mathcal{R}, \qquad A_i(t) = \mathbf{y}_i^{-1}(\mathcal{G}(t),t),$$
$$A(t) = A_1(t),$$
$$\bar{\mathbf{m}} = \text{outward unit normal to } \partial\mathcal{R},$$
$$\bar{\boldsymbol{\nu}} = \text{outward unit normal to } \partial\mathcal{G},$$
$$\boldsymbol{\nu} = \text{outward unit normal to } \partial A,$$

and $\bar{\mathbf{n}}$, $\mathbf{n}_i$ as in Section 3.2.

12. FORCE SYSTEMS. BALANCE LAWS

We add to the force systems introduced in Section 6 the following interfacial fields, measured per unit *deformed* area:

$\bar{\mathbf{S}}$	**deformational surface stress**
$\bar{\mathbf{C}}_1, \bar{\mathbf{C}}_2$	**configurational surface stresses**

The stress $\bar{\mathbf{S}}$ is associated with the response of the interface to deformation, while $\bar{\mathbf{C}}_i$ is a partial stress that accounts for the configurational response of the phase-i portion of the interface. Suppressing t, $\bar{\mathbf{S}}(\mathbf{x})$ and $\bar{\mathbf{C}}_i(\mathbf{x})$ are, at each $\mathbf{x}$, linear transformations from $\bar{\mathbf{n}}(\mathbf{x})^{\perp}$ into $\mathbb{R}^3$.

The **force and moment balances** take the form

$$\begin{aligned}
&\int_{\partial\mathcal{R}} \bar{\mathbf{S}}\bar{\mathbf{m}}\,da + \int_{\mathcal{R}} \bar{\mathbf{b}}\,dv + \int_{\mathcal{G}} \bar{\mathbf{g}}\,da + \int_{\partial\mathcal{G}} \bar{\mathbf{S}}\bar{\boldsymbol{\nu}}\,ds = \mathbf{0},\\
&\int_{\partial\mathcal{R}} \mathbf{r}\times\bar{\mathbf{S}}\bar{\mathbf{m}}\,da + \int_{\mathcal{R}} \mathbf{r}\times\bar{\mathbf{b}}\,dv + \int_{\mathcal{G}} \mathbf{r}\times\bar{\mathbf{g}}\,da + \int_{\partial\mathcal{G}} \mathbf{r}\times\bar{\mathbf{S}}\bar{\boldsymbol{\nu}}\,ds = \mathbf{0}
\end{aligned} \tag{12.1}$$

for each spatial control volume $\mathcal{R}$, where $\mathbf{r}=\mathbf{x}-\mathbf{x}_0$ with $\mathbf{x}_0$ fixed. In addition, for each $\mathcal{R}$ and each phase i, we postulate — in place of (7.6) — a **partial configurational balance**

$$\int_{\mathcal{B}_i\cap\partial\mathcal{R}} \bar{\mathbf{C}}\bar{\mathbf{m}}\,da + \int_{\mathcal{G}} (\bar{\mathbf{e}}_i + \bar{\mathbf{f}}_i)\,da + \int_{\mathcal{B}_i\cap\mathcal{R}} \bar{e}\,dv + \int_{\partial\mathcal{G}} \bar{\mathbf{C}}_i\bar{\boldsymbol{\nu}}\,ds = 0. \tag{12.2}$$

When $\mathcal{R}$ does not intersect $\mathcal{S}$ the balance laws yield the local bulk relations (7.8). On the other hand, for $\mathcal{G}\neq\varnothing$, shrinking $\mathcal{R}$ to the interface yields, by (4.41), the interfacial balances

$$\begin{aligned}
&\int_{\partial\mathcal{G}} \bar{\mathbf{S}}\bar{\boldsymbol{\nu}}\,ds + \int_{\mathcal{G}} [\bar{\mathbf{S}}\bar{\mathbf{n}}]\,da + \int_{\mathcal{G}} \bar{\mathbf{g}}\,da = 0,\\
&\int_{\partial\mathcal{G}} \mathbf{r}\times\bar{\mathbf{S}}\bar{\boldsymbol{\nu}}\,ds + \int_{\mathcal{G}} \mathbf{r}\times[\bar{\mathbf{S}}\bar{\mathbf{n}}]\,da + \int_{\mathcal{G}} \mathbf{r}\times\bar{\mathbf{g}}\,da = 0,\\
&\int_{\partial\mathcal{G}} \bar{\mathbf{C}}_i\bar{\boldsymbol{\nu}}\,ds + \int_{\mathcal{G}} (\delta_i\bar{\mathbf{C}}_i\bar{\mathbf{n}} + \bar{\mathbf{e}}_i)\,da + \int_{\mathcal{G}} \bar{\mathbf{f}}_i\,da = 0.
\end{aligned} \tag{12.3}$$

The balance laws (12.2) and (12.3) are spatial. Using (3.16), (3.18) and (3.22), we may rewrite these balances in the reference configuration for, say, phase 1. With this in mind, we introduce new notation for the stresses measured per unit area on the phase-1 interface:

$$\begin{aligned}
\mathbf{S} &= \jmath_1\bar{\mathbf{S}}\,\mathbf{G}_1^{\mathsf{T}}\mathbb{1}_1 && \text{deformational stress}\\
\mathbf{C} &= \jmath_1\bar{\mathbf{C}}_1\mathbf{G}_1^{\mathsf{T}}\mathbb{1}_1 && \text{configurational stress for phase 1}\\
\mathbf{K} &= \jmath_1\bar{\mathbf{C}}_2\mathbf{G}_1^{\mathsf{T}}\mathbb{1}_1 && \text{configurational stress for phase 2}
\end{aligned}$$

with $\mathbf{G}_1$ the inverse tangential deformation gradient $(3.9)_2$. We view $\mathbf{S}$ as acting in response to the deformation as measured from the reference configuration for phase 1; $\mathbf{C}$ as measuring the configurational stress for the phase-1 material in $\mathcal{S}$; $\mathbf{K}$ as representing the excess stress due to the presence in $\mathcal{S}$ of phase-2 material. Suppressing t, $\mathbf{S}(\mathbf{X})$, $\mathbf{C}(\mathbf{X})$, and $\mathbf{K}(\mathbf{X})$ are, at each $\mathbf{X}$, linear transformations from $\mathbf{n}_1(\mathbf{X})^\perp$ into $\mathbb{R}^3$. It is convenient also to use the (trivial) extensions of $\mathbf{S}$ and $\mathbf{K}$ to tensor fields with values in $\mathrm{Lin}(\mathbb{R}^3,\mathbb{R}^3)$:

$$\mathbb{S}_{ext} = \mathbb{S}\mathbb{P}_1, \qquad \mathbb{K}_{ext} = \mathbb{K}\mathbb{P}_1 \tag{12.4}$$

(cf. (A4) of [Gu]).

Using these stresses, $(12.3)_{1,3}$ may be written referentially with respect to phase 1:

$$\begin{aligned}
&\int_{\partial A}\mathbb{S}\boldsymbol{\nu}ds + \int_{A}(\mathcal{H}\mathbf{S}_2\mathbf{n}_2 - \mathbf{S}_1\mathbf{n}_1)da + \int_{A}\mathbf{g}\,da = \mathbf{0},\\
&\int_{\partial A}\mathbb{C}\boldsymbol{\nu}ds + \int_{A}(-\mathbf{C}_1\mathbf{n}_1 + \mathbf{e}_1 + \mathbf{f}_1)da = \mathbf{0},\\
&\int_{\partial A}\mathbb{K}\boldsymbol{\nu}ds + \int_{A}\mathcal{H}(\mathbf{C}_2\mathbf{n}_2 + \mathbf{e}_2 + \mathbf{f}_2)da = \mathbf{0},
\end{aligned} \tag{12.5}$$

with $\mathbf{S}=J\bar{\mathbf{S}}\mathbf{G}^{\mathsf{T}}$ the bulk Piola-Kirchhoff stress, $\mathbf{g}=\vartheta_1\bar{\mathbf{g}}$, $\mathbf{e}_i=\vartheta_i\bar{\mathbf{e}}_i$, $\mathbf{f}_i=\vartheta_i\bar{\mathbf{f}}_i$, and $\mathcal{H}=\vartheta_1/\vartheta_2$.

The balances (12.3) may be localized to yield[27]

$$\begin{aligned}
&\mathrm{div}_{\mathcal{S}}\bar{\mathbb{S}} + [\bar{\mathbf{S}}]\bar{\mathbf{n}} + \bar{\mathbf{g}} = \mathbf{0},\\
&\mathrm{div}_{\mathcal{S}}\bar{\mathbb{C}}_i + \delta_i\bar{\mathbf{C}}_i\bar{\mathbf{n}} + \bar{\mathbf{e}}_i + \bar{\mathbf{f}}_i = \mathbf{0},
\end{aligned} \tag{12.6}$$

and

$$\bar{\mathbb{S}}^{\mathsf{T}}\bar{\mathbf{n}} = \mathbf{0}, \qquad \bar{\mathbb{P}}\bar{\mathbb{S}} = \bar{\mathbb{S}}^{\mathsf{T}}\bar{\mathbb{I}}, \tag{12.7}$$

so that $\bar{\mathbb{S}}$ is tangential and symmetric. These balances are spatial; writing

$$S(t) = S_1(t) \tag{12.8}$$

and starting with (12.5) yields, instead, the interfacial force balances expressed referentially with respect to phase 1:

$$\begin{aligned}
&\mathrm{Div}_S\mathbb{S} + \mathcal{H}\mathbf{S}_2\mathbf{n}_2 - \mathbf{S}_1\mathbf{n}_1 + \mathbf{g} = \mathbf{0},\\
&\mathrm{Div}_S\mathbb{C} - \mathbf{C}_1\mathbf{n}_1 + \mathbf{e}_1 + \mathbf{f}_1 = \mathbf{0},\\
&\mathrm{Div}_S\mathbb{K} + \mathcal{H}(\mathbf{C}_2\mathbf{n}_2 + \mathbf{e}_2 + \mathbf{f}_2) = \mathbf{0}.
\end{aligned} \tag{12.9}$$

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[27] Using (A10) of [Gu] and eqt. (7.16) of [GS]. Cf. Gurtin and Murdoch [1975], p. 307.

Further, the moment balances (12.7) can be written as a single assertion:

$$\mathbf{S}_{ext}\, \mathbf{F}^{T} \quad \text{is symmetric.} \tag{12.10}$$

In the absence of external forces, the local forms of the force balances become

$$\begin{aligned} &\mathrm{Div}_S \mathbf{S} + \mathcal{H} S_2 \mathbf{n}_2 - S_1 \mathbf{n}_1 = \mathbf{0}, \\ &\mathrm{Div}_S \mathbf{C} - C_1 \mathbf{n}_1 + \mathbf{e}_1 = \mathbf{0}, \\ &\mathrm{Div}_S \mathbf{K} + \mathcal{H}(C_2 \mathbf{n}_2 + \mathbf{e}_2) = \mathbf{0}. \end{aligned} \tag{12.11}$$

13. ENERGETICS. POWER IDENTITY. DISSIPATION INEQUALITY

13.1. POWER

We assume that the **total power expended** on a spatial control volume $\mathcal{R}$ is given by

$$\mathcal{P}(\mathcal{R}) = \int_{\partial\mathcal{R}} \bar{\mathbf{S}}\bar{\mathbf{m}}\cdot\mathbf{y}^{\cdot}\,da + \int_{\mathcal{R}} \bar{\mathbf{b}}\cdot\mathbf{y}^{\cdot}\,dv + \int_{\partial\mathcal{G}} \{\bar{\mathbf{S}}\bar{\boldsymbol{\nu}}\cdot\bar{\mathbf{w}} + \bar{\mathbf{C}}_1\bar{\boldsymbol{\nu}}\cdot\mathbf{w}_1 + \bar{\mathbf{C}}_2\bar{\boldsymbol{\nu}}\cdot\mathbf{w}_2\}\,ds + \int_{\mathcal{G}} \{\bar{\mathbf{g}}\cdot\bar{\mathbf{v}} + \bar{\mathbf{f}}_1\cdot\mathbf{v}_1 + \bar{\mathbf{f}}_2\cdot\mathbf{v}_2\}\,da, \tag{13.1}$$

where $\bar{\mathbf{w}}$ and $\mathbf{w}_i$ are compatible edge velocities for $\partial\mathcal{G}$ and ∂A_i, while $\bar{\mathbf{v}}$ and $\mathbf{v}_i$ are compatible velocity fields for $\mathcal{S}$ and S_i.

If we shrink $\mathcal{R}$ to the interface, we find that $\mathcal{P}(\mathcal{R})$ has the limit

$$\mathcal{P}_{\text{loc}}(A) = \int_A \{\mathcal{H}\mathbf{S}_2\mathbf{n}_2\cdot\mathbf{y}_2^{\cdot} - \mathbf{S}_1\mathbf{n}_1\cdot\mathbf{y}_1^{\cdot} + \mathbf{g}\cdot\bar{\mathbf{v}} + \mathbf{f}_1\cdot\mathbf{v}_1 + \mathcal{H}\mathbf{f}_2\cdot\mathbf{v}_2\}\,da + \int_{\partial A} \{\mathbf{S}\boldsymbol{\nu}\cdot\bar{\mathbf{w}} + \mathbf{C}\boldsymbol{\nu}\cdot\mathbf{w}_1 + \mathbf{K}\boldsymbol{\nu}\cdot\mathbf{w}_2\}\,ds, \tag{13.2}$$

when expressed referentially with respect to phase 1.

If we use (4.16) to eliminate $\mathbf{w}_2$ and $\bar{\mathbf{w}}$ from the integral over ∂A in (13.2), we find terms involving $\mathbf{y}_1^{\cdot}$ and $\mathbf{y}_2^{\cdot}$ and a term

$$\int_{\partial A} \mathbf{A}\boldsymbol{\nu}\cdot\mathbf{w}_1\,ds, \qquad \mathbf{A} = \mathbf{C} + \mathbf{F}_1^{\mathsf{T}}\mathbf{S} + \mathbf{H}^{\mathsf{T}}\mathbf{K} \tag{13.3}$$

involving only $\mathbf{w}_1$.

We assume that the expended power is invariant under reparametrizations of the interface.[28] More precisely, we assume that $\mathcal{P}$ is independent of: (i) the choice of compatible interface velocities $\bar{\mathbf{v}}$ and $\mathbf{v}_i$ for $\mathcal{S}$ and S_i; and (ii) the choice of compatible edge velocities $\bar{\mathbf{w}}$ and $\mathbf{w}_i$ for $\partial\mathcal{G}$ and ∂A_i.

Invariance under (ii) is equivalent to the invariance of (13.3) under (ii), and this reduces[29] the tangential part of $\mathbf{A}$ to a **surface tension**:

[28] See [GS], Sect. 7.1.

[29] Using an argument of [GS]. The result is essentially Theorem 7A of [GS].

$$\Lambda_1 \mathbb{A} = \sigma \mathbb{1}_1; \tag{13.4}$$

σ represents the surface tension expressed per unit area on $S=S_1$.

Invariance of (13.2) under changes in $\mathbf{v}_1$ yields a result,

$$(\mathbf{f}_1 + \mathbf{F}_1^T \mathbf{g} + \mathcal{H}\mathbf{H}^T \mathbf{f}_2)_{tanS} = \mathbf{0}, \tag{13.5}$$

which is equivalent to (8.5) and, by (3.2) and (7.10), granted the balances (12.9), also to the condition

$$(\mathrm{Div}_S \mathbb{C} + \mathbf{F}_1^T \mathrm{Div}_S \mathbb{S} + \mathbf{H}^T \mathrm{Div}_S \mathbb{K})_{tanS} = -(\mathbf{e}_1 + \mathcal{H}\mathbf{H}^T \mathbf{e}_2)_{tanS}. \tag{13.6}$$

13.2. THE POWER IDENTITY

We continue to define τ_i by (8.6),

$$\tau_i = -(\mathbf{G}_i^T \bar{\mathbf{e}}_i)_{tan\mathcal{S}}, \tag{13.7}$$

but at this point τ_1 is not necessarily equal to $-\tau_2$. To some extent, (13.7) is motivated by our next result, the power identity, which shows that the traction

$$\mathbf{s} = \mathcal{S}_1 \tau_2 \tag{13.8}$$

(which is τ_2 expressed per unit area on S_1) is conjugate to $\mathcal{V}$.

We now further localize the power; the result is the **power identity**:

$$\mathcal{P}_{loc}(A) = \int_A \mathfrak{p}\, da + \int_{\partial A} \sigma V_{(\partial A)tan}\, ds, \tag{13.9}$$

$$\mathfrak{p} = \mathbb{S}_{ext} \cdot \mathbf{F}_1^\circ + \mathbb{K}_{ext} \cdot \mathbf{H}^\circ - \mathbf{e} \cdot \mathbf{n}_1^\circ - \sigma K_1 V_1 - \mathbf{s} \cdot \mathcal{V} - \mathcal{S}_1 \sum_{i=1,2} \{\delta_i \omega_i + \Pi_i\} W_i, \tag{13.10}$$

where $(\cdot)^\circ$ is the normal time-derivative with respect to S, Π_i is given by (8.11), and

$$\mathbf{e} = \mathbf{A}^{\mathsf{T}}\mathbf{n}_1. \tag{13.11}$$

Here:

- $\mathbf{S}_{ext}\cdot\mathbf{F}_1^{\circ}$ represents power expended in stretching the phase 1 interface.
- $\mathbf{K}_{ext}\cdot\mathbf{H}^{\circ}$ represents power expended in stretching and rotating S_2 relative to S_1 and in some sense represents an expense of power due to incoherency.
- $\sigma K_1 V_1$ represents power expended in creating new surface for S_1.
- $\mathbf{e}\cdot\mathbf{n}_1^{\circ}$ represents power expended in changing the orientation of S_1.

The vector field $\mathbf{e}$, which is tangential on $S = S_1$, represents the normal part of the "total surface stress" $\mathbf{A}$; $\mathbf{e}$ represents shearing forces exerted *within* the interface (in contrast to the $\boldsymbol{\tau}_i$ which represent shearing forces exerted *on* the interface by the bulk material).

We now prove the power identity (13.9), (13.10). By (4.18) and $(13.3)_2$, the integrand — of the integral in (13.2) over ∂A — has the form

$$\begin{aligned}&\mathbf{S}\boldsymbol{\nu}\cdot\bar{\mathbf{w}} + \mathbf{C}\boldsymbol{\nu}\cdot\mathbf{w}_1 + \mathbf{K}\boldsymbol{\nu}\cdot\mathbf{w}_2 =\\ &\quad = \mathbf{S}\boldsymbol{\nu}\cdot\bar{\mathbf{v}} + \mathbf{C}\boldsymbol{\nu}\cdot\mathbf{v}_1 + \mathbf{K}\boldsymbol{\nu}\cdot\mathbf{v}_2 + \boldsymbol{\nu}\cdot\mathbf{A}\boldsymbol{\nu}V_{(\partial A)tan},\end{aligned}$$

and, by (13.4), the last term yields the integral in (13.9) over ∂A. Further,

$$\begin{aligned}&\int_{\partial A}(\mathbf{S}\boldsymbol{\nu}\cdot\bar{\mathbf{v}} + \mathbf{C}\boldsymbol{\nu}\cdot\mathbf{v}_1 + \mathbf{K}\boldsymbol{\nu}\cdot\mathbf{v}_2)\,ds =\\ &\qquad\int_A\{\mathrm{Div}_S\mathbf{S}\cdot\bar{\mathbf{v}} + \mathrm{Div}_S\mathbf{C}\cdot\mathbf{v}_1 + \mathrm{Div}_S\mathbf{K}\cdot\mathbf{v}_2 + \mathbf{S}\cdot\nabla_S\bar{\mathbf{v}} + \mathbf{C}\cdot\nabla_S\mathbf{v}_1 + \mathbf{K}\cdot\nabla_S\mathbf{v}_2\}\,da.\end{aligned} \tag{13.12}$$

On the other hand, (4.10) yields

$$\begin{aligned}&\mathcal{H}\mathbf{S}_2\mathbf{n}_2\cdot\mathbf{y}_2^{\cdot} - \mathbf{S}_1\mathbf{n}_1\cdot\mathbf{y}_1^{\cdot} =\\ &\qquad(\mathcal{H}\mathbf{S}_2\mathbf{n}_2 - \mathbf{S}_1\mathbf{n}_1)\cdot\bar{\mathbf{v}} - \mathcal{H}\mathbf{F}_2^{\mathsf{T}}\mathbf{S}_2\mathbf{n}_2\cdot\mathbf{v}_2 + \mathbf{F}_1^{\mathsf{T}}\mathbf{S}_1\mathbf{n}_1\cdot\mathbf{v}_1;\end{aligned}$$

thus, if we add the integrands of the integrals over A in (13.2) to those in (13.12), and use (7.10) and (12.9), we find

$$\mathbf{S}\cdot\nabla_S\bar{\mathbf{v}} + \mathbf{C}\cdot\nabla_S\mathbf{v}_1 + \mathbf{K}\cdot\nabla_S\mathbf{v}_2 - \mathcal{H}(\omega_2\mathbf{n}_2 + \mathbf{e}_2)\cdot\mathbf{v}_2 - (-\omega_1\mathbf{n}_1 + \mathbf{e}_1)\cdot\mathbf{v}_1. \tag{13.13}$$

To complete the proof we have only to show that (13.13) reduces to $\mathcal{P}$.

In view of (3.5), (4.6), and (4.38), the last two terms in (13.13) have the form

$$-\mathcal{J}_1 \sum_{i=1,2} \{\delta_i \omega_i + \Pi_i\} W_i - \mathbf{s}\cdot\boldsymbol{\gamma}. \tag{13.14}$$

By (4.35), (4.36), (12.4), and $(13.3)_2$,

$$\mathbf{S}\cdot\nabla_S \bar{\mathbf{v}} + \mathbf{K}\cdot\nabla_S \mathbf{v}_2 + \mathbf{C}\cdot\nabla_S \mathbf{v}_1 = \mathbf{S}_{ext}\cdot\mathbf{F}_1^{\circ} + \mathbf{K}_{ext}\cdot\mathbf{H}^{\circ} + \mathbf{A}\cdot\nabla_S \mathbf{v}_1,$$

and, by (4.34) and (13.4),

$$\mathbf{A}\cdot\nabla_S \mathbf{v}_1 = -\sigma K_1 V_1 - \mathbf{e}\cdot\mathbf{n}_1^{\circ}.$$

Thus (13.13) reduces to $\mathcal{P}$.

13.3. ENERGETICS. DISSIPATION INEQUALITY

Introducing the **interfacial energy** $\bar{\psi}$, per unit deformed area, as a superficial field on $\mathcal{S}$, we write the **global dissipation inequality** in the form

$$\{\int_{\mathcal{R}} \bar{\Psi}\, dv + \int_{\mathcal{G}} \bar{\psi} da\}^{\cdot} \leq \mathcal{P}(\mathcal{R}) + \mathcal{E}(\mathcal{R}) \tag{13.15}$$

for all spatial control volumes $\mathcal{R}$, with $\bar{\Psi}$ the bulk energy density per unit deformed volume, $\mathcal{P}(\mathcal{R})$ the total power, and $\mathcal{E}(\mathcal{R})$ the energy supplied to $\mathcal{R}$ by mass transport (cf. (8.1), (13.1)). Here the time derivative involving the bulk field $\bar{\Psi}$ is the time derivative (4.39) following the material currently in $\mathcal{R}$, but the derivative of the surface energy has a more standard meaning:

$$\{\int_{\mathcal{G}} \bar{\psi} da\}^{\cdot}(t) = (d/dt)\{\int_{\mathcal{G}(t)} \bar{\psi}(\mathbf{x},t)\, da(\mathbf{x})\}.$$

It is convenient to introduce the surface energy

$$\psi = \mathcal{J}_1 \bar{\psi} \tag{13.16}$$

expressed per unit area on $S = S_1$. Localizing (13.15) using the power identity, (A15) and Lemma B1 of [Gu], and (ii) of the Invariance Lemma of [GS], yields the *equivalence of surface tension and energy,*[30]

$$\psi = \sigma, \tag{13.17}$$

and the **reduced dissipation inequality**

$$\psi^\circ - \mathbf{S}_{ext}\cdot\mathbf{F}_1^\circ - \mathbf{K}_{ext}\cdot\mathbf{H}^\circ + \mathbf{a}\cdot\mathbf{n}_1^\circ + \gamma_1\{\Pi_1 W_1 + \Pi_2 W_2\} + s\cdot\theta \leq 0, \tag{13.18}$$

with $(\cdot)^\circ$ the normal time-derivative following S.

Finally, we note that, by (13.4), (13.11), and (13.17),

$$\mathbf{A} = \psi\mathbf{1}_1 + \mathbf{n}_1\otimes\mathbf{a}. \tag{13.19}$$

[30] Cf. [GS], eq. (9.19).

14. CONSTITUTIVE THEORY

We need only consider the interface; the constitutive theory for the bulk material is as described in Section 9.1, although we will assume that each of the functions $\mu = \hat{\mu}_i(\mathbf{F},\rho)$ may be inverted to give the density ρ as a function

$$\rho = \hat{\rho}_i(\mathbf{F},\mu) \tag{14.1}$$

of the deformation gradient and chemical potential.

14.1. CONSTITUTIVE EQUATIONS FOR THE INTERFACE

We now develop a constitutive theory for the interface appropriate to a description with phase 1 as reference. Here, for convenience, we use the abbreviations:

$$\mathbf{F} = \mathbf{F}_1, \qquad \mathbf{n} = \mathbf{n}_1.$$

In addition, for any unit vector $\mathbf{q}$, we write $\mathbb{1}(\mathbf{q})$ for the inclusion of $\mathbf{q}^\perp$ into $\mathbb{R}^3$ and $\mathbb{P}(\mathbf{q})=\mathbb{1}(\mathbf{q})^T$ for the projection of $\mathbb{R}^3$ into $\mathbf{q}^\perp$ (so that $\mathbb{1}(\mathbf{n})=\mathbb{1}_1$ and $\mathbb{P}(\mathbf{n})=\mathbb{P}_1$).

Using the reduced dissipation inequality (13.18) to suggest appropriate constitutive variables,[31] we consider constitutive equations giving the interfacial energy ψ, the normal internal forces Π_i, the effective shear τ_2 (actually $\mathbf{s}=\mathbb{1}_1\tau_2$), and the surface stresses $\mathbf{S}$, $\mathbf{K}$, and $\mathbf{B}$ as functions of the limiting value $\mathbf{F}=\mathbf{F}_1$ of the deformation gradient, the relative gradient $\mathbf{H}$, the limiting values ρ_1 and ρ_2 of the density, the normal $\mathbf{n}$ to S_1, and the volume flows W_1 and W_2. Further, the bulk relations (14.1) and the requirement that the chemical potential be continuous across the interface allow us to replace ρ_1 and ρ_2 by the common value μ of the (list of) chemical potentials at the interface. We therefore consider constitutive relations of the form

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[31] Cf. Footnote 22 of [Gu].

$$\psi = \hat{\psi}(Z), \qquad \Pi_i = \hat{\Pi}_i(Z), \qquad \mathbf{s} = \hat{\mathbf{s}}(Z),$$
$$\mathbf{S} = \hat{\mathbf{S}}(Z), \qquad \mathbf{K} = \hat{\mathbf{K}}(Z), \qquad \mathbf{e} = \hat{\mathbf{e}}(Z), \tag{14.2}$$
$$Z = (F, H, \mu, \mathbf{n}, W_1, W_2).$$

We assume that the constitutive relation for $\mathbf{S}$ is consistent with the moment balance (12.10), and, to simplify the discussion, we introduce the extended functions induced by (12.4):

$$\hat{\mathbf{S}}(Z)_{ext} = \hat{\mathbf{S}}(Z)\mathbb{P}(\mathbf{n}), \qquad \hat{\mathbf{K}}(Z)_{ext} = \hat{\mathbf{K}}(Z)\mathbb{P}(\mathbf{n}). \tag{14.3}$$

Note that we do not write a constitutive equation for τ_1, or equivalently, for the tangential component $\Lambda_1\mathbf{e}_1$ of $\mathbf{e}_1$, as it does not appear in the dissipation inequality (13.18). We consider $\Lambda_1\mathbf{e}_1$ as indeterminate; in fact, as a solution of (13.6).

As in Part B, *the external fields allow us to consider arbitrary constitutive processes with the assurance that the balance laws for mass and force are satisfied.* To verify this, assume we are given an arbitrary two-phase motion $\mathbf{y}$ and an arbitrary field μ that is smooth away from and continuous across the interface. Then the constitutive equations may be used to compute a **constitutive process** consisting of Z as an interfacial field, the bulk fields $\mathbf{S}$, Ψ, ρ, and $\mathfrak{h}$, and the interfacial fields Π_i, $\mathbf{s}$, ψ, $\mathbf{S}$, $\mathbf{K}$, and $\mathbf{C}$ (with $\mathbf{C}$ computed using $(13.3)_2$ and (13.19). The balance laws for mass and force may then be used to compute the mass supplies and external forces needed to support the process: the mass balance (7.2) and (7.4) are used to compute $\bar{q}^\alpha$ and Q^α; the force balances $(7.8)_1$ and $(12.9)_1$ yield $\mathbf{b}$ and $\mathbf{g}$; the normal component of $\mathbf{e}_1$ is Π_1 and the tangential component is computed using (13.6); the normal component of $\mathbf{e}_2$ is Π_2 and the tangential component (with respect to S_2) is computed using (13.7) and (13.8); $\mathbf{f}_i$ are determined using $(12.9)_{2,3}$. The forces $\mathbf{g}$ and $\mathbf{f}_i$ computed in this manner satisfy the invariance requirement (13.5). This leaves only the reduced dissipation inequality (13.18) to be satisfied.

14.2. THERMODYNAMIC RESTRICTIONS

a. General restrictions

We require that all constitutive processes be compatible with the reduced dissipation inequality (13.18). We now state three important consequences of this hypothesis; their verification will be given in Section 14.3.

(i) *The interface is free to slip:*

$$\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \mathbf{0}. \tag{14.4}$$

(ii) *The interfacial energy is independent of* μ, W_1, *and* W_2, *so that*

$$\psi = \hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}),$$

and $\hat{\psi}$ *generates the surface stresses through the relations:*

$$\begin{gathered}\hat{\mathbf{S}}(\mathbf{F},\mathbf{H},\mathbf{n})_{\text{ext}} = \partial_{\mathbf{F}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}), \qquad \hat{\mathbf{K}}(\mathbf{F},\mathbf{H},\mathbf{n})_{\text{ext}} = \partial_{\mathbf{H}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}), \\ \hat{\mathbf{e}}(\mathbf{F},\mathbf{H},\mathbf{n}) = -\partial_{\mathbf{n}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n});\end{gathered} \tag{14.5}$$

(iii) *The kinetic relations* (9.8) *hold for the normal internal forces* Π_i, *but with* $\mathbf{Z}$ *as in* (14.2).

The conditions (i)-(iii) are also sufficient for all constitutive processes to be compatible with the reduced dissipation inequality (13.18).

b. Consequences

By (14.3), $\mathbf{S}_{\text{ext}}\mathbf{n} = \mathbf{K}_{\text{ext}}\mathbf{n} = \mathbf{0}$; thus (12.10) and (14.5) imply that

$$(\partial_{\mathbf{F}}\hat{\psi})\mathbf{n} = \mathbf{0}, \quad (\partial_{\mathbf{H}}\hat{\psi})\mathbf{n} = \mathbf{0}, \quad (\partial_{\mathbf{F}}\hat{\psi})\mathbf{F}^{\mathsf{T}} \text{ is symmetric.} \tag{14.6}$$

These restrictions select — among all energies $\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n})$ — those compatible with thermodynamics and balance of moments.

The restrictions $(14.6)_{1,2}$ have an important consequence. In conjunction with Lemmas (2D)-(2F) of [GS], they render $\hat{\psi}$, $\hat{\mathbf{S}}$, and $\hat{\mathbf{K}}$ independent of the normal components $\mathbf{Fn}$ and $\mathbf{Hn}$, and hence dependent on $\mathbf{F}$ and $\mathbf{H}$ at most through a dependence on the tangential deformation gradient $\mathbf{F}$ and the incoherency tensor $\mathbf{H}$; in fact,

$$\hat{\mathbf{S}}(\mathbf{F},\mathbf{H},\mathbf{n}) = \partial_{\mathbf{F}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}), \qquad \hat{\mathbf{K}}(\mathbf{F},\mathbf{H},\mathbf{n}) = \partial_{\mathbf{H}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}). \tag{14.7}$$

The first of (14.7) gives the stress $\mathbf{S}$ as a response to the deformation of the phase 1 interface, while the second gives the stress $\mathbf{K}$ in the phase 2 interface as a response to the incoherency between phases.[32]

A reduction similar to (14.7) is generally not possible for $\hat{\mathbf{c}}(\mathbf{F},\mathbf{H},\mathbf{n})$; in fact, the partial derivative of $\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n})$ with respect to $\mathbf{n}$ is not so easily defined, as both $\mathbf{F}$ and $\mathbf{H}$, being linear transformations on $\mathbf{n}^{\perp}$, depend on $\mathbf{n}$. What is well defined is the derivative[33] $D_{\mathbf{n}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}) \in \mathbf{n}^{\perp}$ *following* $\mathbf{n}$: given any unit vector $\mathbf{n}$, for any unit vector $\mathbf{q}$, $\mathbf{q} \neq \mathbf{n}$, let $\mathbb{Q}(\mathbf{q}) = \mathbb{P}(\mathbf{n})\mathbf{Q}(\mathbf{q})\mathbb{1}(\mathbf{q})$ with $\mathbf{Q}(\mathbf{q})$ the rotation of $\mathbf{q}$ into $\mathbf{n}$ about the axis orthogonal to $\mathbf{q}$ and $\mathbf{n}$, and let $\mathbb{Q}(\mathbf{n})$ be the identity on $\mathbf{n}^{\perp}$; then

$$D_{\mathbf{n}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n})\cdot\mathbf{b} = (d/d\beta)\,\hat{\psi}(\mathbf{F}\mathbb{Q}(\mathbf{q}(\beta)),\mathbf{H}\mathbb{Q}(\mathbf{q}(\beta)),\mathbf{q}(\beta))\,|_{\beta=0} \tag{14.8}$$

for any $\mathbf{b} \in \mathbf{n}^{\perp}$, where $\mathbf{q}(\beta)$ is any smooth curve on the unit sphere satisfying $\mathbf{q}(0) = \mathbf{n}$, $\mathbf{q}'(0) = \mathbf{b}$. A trivial generalization of (2.49) of [GS] in conjunction with $(14.6)_{1,2}$ then yields

$$D_{\mathbf{n}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}) = \partial_{\mathbf{n}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}) + \partial_{\mathbf{F}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n})^{\mathrm{T}}\mathbf{F}\mathbf{n} + \partial_{\mathbf{H}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n})^{\mathrm{T}}\mathbf{H}\mathbf{n}. \tag{14.9}$$

A consequence of this result is that the normal part of $\mathbb{C}$, namely,

$$\mathbf{c} = \mathbb{C}^{\mathrm{T}}\mathbf{n}, \tag{14.10}$$

is given by a constitutive relation $\mathbf{c} = \hat{\mathbf{c}}(\mathbf{F},\mathbf{H},\mathbf{n})$ with

$$\hat{\mathbf{c}}(\mathbf{F},\mathbf{H},\mathbf{n}) = -D_{\mathbf{n}}\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}). \tag{14.11}$$

To verify (14.11) we simply note that, by $(13.3)_2$ and (13.11),

$$\mathbf{e} = \mathbf{c} + \mathbf{S}^{\mathrm{T}}\mathbf{F}\mathbf{n} + \mathbf{K}^{\mathrm{T}}\mathbf{H}\mathbf{n}, \tag{14.12}$$

which, with (14.7) and (14.9), implies (14.11).

[32] In this sense these stresses are similar to stresses introduced by Cahn and Larché [1982], eqt. (4), in their statical treatment of incoherency at small strains.

[33] [GS], p. 111.

Invariance under changes in observer yields the requirement

$$\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}) = \hat{\psi}(\mathbf{QF},\mathbf{H},\mathbf{n}) \tag{14.13}$$

for all rotations $\mathbf{Q}$.[34] The condition (14.13) yields the following further reduction for the reduced response function $\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n})$:

$$\hat{\psi}(\mathbf{F},\mathbf{H},\mathbf{n}) = \check{\psi}(\mathbf{F}^T\mathbf{F},\mathbf{H},\mathbf{n}). \tag{14.14}$$

c. The force balances revisited

A useful consequence of $(13.3)_2$, (13.19), and (14.5) is that, in any constitutive process,

$$(\mathrm{Div}_S\,\mathbf{C} + \mathbf{F}^T\mathrm{Div}_S\,\mathbf{S} + \mathbf{H}^T\mathrm{Div}_S\,\mathbf{K})_{\tan S} = \mathbf{0}, \tag{14.15}$$

a result that will be verified in Section 14.3. Further, granted (14.15), we may use (3.8), (13.6), and (13.7) to conclude that

$$\tau_1 = -\tau_2. \tag{14.16}$$

(Were the slip included in the constitutive equations for the interface, then the effective shears would not vanish. On the other hand, (14.15) and (hence) (14.16) would be satisfied and the effective shears would be balanced, rendering the discussion of Section 9.3c applicable here also.)

Consider the system consisting of the balance laws (12.11), the definition (13.7) of the effective shears, and the constitutive equations as restricted by (i)-(iii). By (i), (13.7) holds with $\tau_i = 0$. This has two consequences: firstly, by (8.11) and (iii), (9.12) holds for both $\mathcal{B}_1$ and $\mathcal{B}_2$; secondly, by (14.15), the invariance requirement (13.6) is satisfied. A consequence of the second of these is that, without loss in generality, the tangential part (with respect to S) of any one of the equations (12.11) may be dropped. We will, in fact, *omit the tangential part of* $(12.11)_2$.

By $(13.3)_2$ and (13.19) in conjunction with (A6) and (A9) of [Gu],

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[34] The condition (14.13) actually follows from the moment-balance relation $(14.6)_3$. This is in accord with a result of Noll [1955] for standard continua.

$$\mathbf{n}\cdot\mathrm{Div}_S\mathbf{C} = \psi K - (\mathbf{F}^T\mathbf{S} + \mathbf{H}^T\mathbf{K})\cdot\mathbf{L} + \mathrm{Div}_S\mathbf{c},$$

where $\mathbf{L} = \mathbf{L}_1 = -\nabla_S\mathbf{n}$ and $K = K_1 = \mathrm{tr}\mathbf{L}$, respectively, are the curvature tensor and total (twice the mean) curvature for S. Thus, by (9.12), we can write the normal part of $(12.11)_2$ in the form

$$\psi K - (\mathbf{F}^T\mathbf{S} + \mathbf{H}^T\mathbf{K})\cdot\mathbf{L} + \mathrm{Div}_S\mathbf{c} - \mathbf{n}\cdot\mathbf{C}_1\mathbf{n} = \beta_{11}W_1 + \beta_{12}W_2. \qquad (14.17)$$

Note that $\psi K - (\mathbf{F}^T\mathbf{S})\cdot\mathbf{L}$ can be written as the inner product of the "surface Eshelby tensor" $\psi\mathbf{1}_1 - \mathbf{F}^T\mathbf{S}$ with $\mathbf{L}$.

The balance (14.17) can also be written in the form

$$\psi K - \mathbf{S}_{ext}\cdot\nabla_n\mathbf{F} - \mathbf{K}_{ext}\cdot\nabla_n\mathbf{H} + \mathrm{Div}_S\mathbf{e} + (\det\mathbf{H})\omega_2 - \omega_1 = B_1W_1 + B_2W_2, \qquad (14.18)$$

where $B_i = \beta_{1i} + (\det\mathbf{H})\beta_{2i}$, while $(\nabla_n\mathbf{F}) = (\nabla\mathbf{F})\mathbf{n}$ is the directional derivative of $\mathbf{F}$ in the direction $\mathbf{n}$. The balance (14.18) follows from the equations

$$(\mathbf{F}^T\mathbf{S} + \mathbf{H}^T\mathbf{K})\cdot\mathbf{L} = -\mathrm{Div}_S(\mathbf{S}^T\mathbf{F}\mathbf{n} + \mathbf{K}^T\mathbf{H}\mathbf{n}) + (\mathbf{F}^T\mathrm{Div}_S\mathbf{S} + \mathbf{H}^T\mathrm{Div}_S\mathbf{K})\cdot\mathbf{n} + \mathbf{S}_{ext}\cdot\nabla_n\mathbf{F} + \mathbf{K}_{ext}\cdot\nabla_n\mathbf{H},$$

$$\mathcal{H}(\mathbf{H}^T(\mathbf{C}_2\mathbf{n}_2 + \mathbf{e}_2) + \mathbf{F}^T\mathbf{S}_2\mathbf{n}_2)\cdot\mathbf{n} = \mathcal{H}(\omega_2\mathbf{n}_2 + \mathbf{e}_2)\cdot\mathbf{H}\mathbf{n} = \mathcal{H}\{(\mathbf{e}_2)_{\tan S_2} + (\Pi_2 + \omega_2)\mathbf{n}_2\}\cdot\mathbf{H}\mathbf{n} = -\Lambda_2\mathbf{F}_2^T\mathbf{s}\cdot\mathbf{H}\mathbf{n} + (\det\mathbf{H})(\Pi_2 + \omega_2),$$

and (9.8), $(12.11)_{1,3}$, $(13.3)_2$, (13.7), (13.8), (13.11), and (14.4).

14.3. VERIFICATION OF THE RESTRICTIONS (i)-(iii) AND (14.15)

a. Verification of (ii) and (iii)

Throughout this section, the derivative $(\cdot)^\circ$ is following S_1.

Let us agree to use the term *motion-potential pair* for a pair $(\mathbf{y},\mu)$ with $\mathbf{y}$ a two-phase motion and $\mu = (\mu^1,\ldots,\mu^\alpha)$ a chemical-potential field that is smooth away from, and continuous across, the interface. We begin with a **lemma**:[35] there is a motion-potential pair $(\mathbf{y},\mu)$ such that: $0\in S_1(0)$, $0\in S_2(0)$, and such that the following fields have arbitrarily pre-

[35] Cf. the Variation Lemma of [GS]

assigned values at (0,0):

$$\mathbf{n},\ \mathbf{n}^\circ,\ \mathbf{F},\ \mathbf{F}^\circ,\ \mathbf{H},\ \mathbf{H}^\circ,\ \mathbf{W}_i,\ \mathbf{W}_i^\circ,\ \boldsymbol{\gamma},\ \boldsymbol{\mu},\ \boldsymbol{\mu}^\circ \qquad (i=1,2). \tag{14.19}$$

The proof will be given in Subsection c.

We now turn to the verification of (ii) and (iii). By hypothesis, the reduced dissipation inequality (13.18) is required to hold in all constitutive processes. Thus

$$\hat{\psi}(Z)^\circ - \hat{\mathbf{S}}(Z)_{ext}\cdot\mathbf{F}^\circ - \hat{\mathbf{K}}(Z)_{ext}\cdot\mathbf{H}^\circ + \hat{\mathbf{e}}(Z)\cdot\mathbf{n}^\circ + \mathfrak{z}_1\{\hat{\Pi}_1(Z)\mathbf{W}_1 + \hat{\Pi}_2(Z)\mathbf{W}_2\} + \hat{\mathbf{s}}(Z)\cdot\boldsymbol{\gamma} \le 0 \tag{14.20}$$

for all motion-potential pairs $(\mathbf{y},\mu)$, or equivalently,

$$(\partial_F\hat{\psi}(Z) - \hat{\mathbf{S}}(Z)_{ext})\cdot\mathbf{F}^\circ + (\partial_H\hat{\psi}(Z) - \hat{\mathbf{K}}(Z)_{ext})\cdot\mathbf{H}^\circ + \partial_\mu\hat{\psi}(Z)\cdot\mu^\circ + (\partial_n\hat{\psi}(Z) + \hat{\mathbf{e}}(Z))\cdot\mathbf{n}^\circ + \partial_{\mathbf{W}_1}\hat{\psi}(Z)\cdot\mathbf{W}_1^\circ + \partial_{\mathbf{W}_2}\hat{\psi}(Z)\cdot\mathbf{W}_2^\circ + \mathfrak{z}_1\{\hat{\Pi}_1(Z)\mathbf{W}_1 + \hat{\Pi}_2(Z)\mathbf{W}_2\} + \hat{\mathbf{s}}(Z)\cdot\boldsymbol{\gamma} \le 0 \tag{14.21}$$

for all $(\mathbf{y},\mu)$. Thus, appealing to the lemma, we see that the coefficients of $\mathbf{F}^\circ$, $\mathbf{H}^\circ$, $\mathbf{n}^\circ$, μ°, and $\mathbf{W}_i^\circ$ must vanish. Therefore (ii) must hold and

$$\hat{\mathbf{s}}(Z) = 0, \qquad \hat{\Pi}_1(Z)\mathbf{W}_1 + \hat{\Pi}_2(Z)\mathbf{W}_2 \le 0; \tag{14.22}$$

steps analogous to those following (9.10) then yield (iii).

b. Verification of (14.15) and (i)

Let

$$\mathbf{u} = \mathrm{Div}_S\,\mathbf{C} + \mathbf{F}^T\mathrm{Div}_S\,\mathbf{S} + \mathbf{H}^T\mathrm{Div}_S\,\mathbf{K}. \tag{14.23}$$

For convenience we suppress the argument t. Choose an arbitrary point $\mathbf{X}_0\in S$ and an arbitrary vector $\mathbf{q}$ *tangent to* S *at* $\mathbf{X}_0$. To verify (14.15), it suffices to show that $\mathbf{u}\cdot\mathbf{q}=0$ at $\mathbf{X}_0$. Clearly,

$$\mathbf{u}\cdot\mathbf{q} = \mathrm{Div}_S(\mathbf{C}^T\mathbf{q}) + (\mathrm{Div}_S\,\mathbf{S})\cdot(\mathbf{F}\mathbf{q}) + (\mathrm{Div}_S\,\mathbf{K})\cdot(\mathbf{H}\mathbf{q}). \tag{14.24}$$

Let $\mathbf{q}_{tan}=(1-\mathbf{n}\otimes\mathbf{n})\mathbf{q}$, so that

$$\text{at } \mathbf{X}_0\text{:}\quad \mathbf{q}_{tan}=\mathbf{q},\ \operatorname{Div}_S\mathbf{q}_{tan}=0. \tag{14.25}$$

By $(13.3)_2$ and (13.19),

$$\mathbb{C}^T\mathbf{q}=\psi\mathbf{q}_{tan}+(\mathbf{n}\cdot\mathbf{q})\mathbf{a}-\mathbb{S}^T\mathbf{F}\mathbf{q}-\mathbb{K}^T\mathbf{H}\mathbf{q}, \tag{14.26}$$

so that, using the fact that $\nabla_S\mathbf{n}$ is tangential and symmetric,

$$\operatorname{Div}_S(\mathbb{C}^T\mathbf{q})=(\nabla_S\psi)\cdot\mathbf{q}+\mathbf{a}\cdot(\nabla_S\mathbf{n})\mathbf{q}-\operatorname{Div}_S(\mathbb{S}^T\mathbf{F}\mathbf{q}+\mathbb{K}^T\mathbf{H}\mathbf{q}) \tag{14.27}$$

at $\mathbf{X}_0$. On the other hand, by (14.5), at $\mathbf{X}_0$,

$$(\nabla_S\psi)\cdot\mathbf{q}=(\nabla\psi)\cdot\mathbf{q}=-\mathbf{a}\cdot(\nabla_S\mathbf{n})\mathbf{q}+\mathbb{S}_{ext}\cdot(\nabla\mathbf{F})\mathbf{q}+\mathbb{K}_{ext}\cdot(\nabla\mathbf{H})\mathbf{q}, \tag{14.28}$$

where $\nabla\mathbf{F}$ is, at each point, a linear transformation from $\mathbb{R}^3$ into $\operatorname{Lin}(\mathbb{R}^3,\mathbb{R}^3)$, and similarly for $\nabla\mathbf{H}$ (Cf. (3.14)). Further,

$$\operatorname{Div}_S(\mathbb{S}^T\mathbf{F}\mathbf{q})=(\operatorname{Div}_S\mathbb{S})\cdot(\mathbf{F}\mathbf{q})+\mathbb{S}\cdot\nabla_S(\mathbf{F}\mathbf{q}),$$

$$\mathbb{S}\cdot\nabla_S(\mathbf{F}\mathbf{q})=\mathbb{S}\cdot(\nabla(\mathbf{F}\mathbf{q})\mathbb{1}_1)=(\mathbb{S}\mathbb{P}_1)\cdot\nabla(\mathbf{F}\mathbf{q})$$
$$=\mathbb{S}_{ext}\cdot\nabla(\mathbf{F}\mathbf{q})=\mathbb{S}_{ext}\cdot(\nabla\mathbf{F})\mathbf{q},$$

$$\mathbb{K}\cdot\nabla_S(\mathbf{H}\mathbf{q})=\mathbb{K}_{ext}\cdot(\nabla\mathbf{H})\mathbf{q}.$$

The last set of identities in conjunction with (14.23), (14.24), (14.27), and (14.28) yield the desired result: $\mathbf{u}\cdot\mathbf{q}=0$ at $\mathbf{X}_0$.

Finally, to establish (i), we note that (13.8) and the first of (14.22) yield $\tau_2=0$. Further, because (14.15) is valid, so also is (14.16); hence $\tau_1=0$.

c. Proof of the lemma

Throughout this proof, the derivative $(\cdot)^\circ$ is following S_1.

We begin by constructing a two-phase motion with the desired properties. Here it is convenient to work backward from the deformed

configuration. We will choose the deformed interface $\mathcal{S}(t)$ such that

$$\mathcal{S}(t) \text{ is an evolving } \textit{plane} \text{ with } 0 \in \mathcal{S}(t) \text{ for all } t; \tag{14.29}$$

thus, granted a choice of the normal $\bar{n}(t)$, and assuming that the deformed body occupies all of $\mathbb{R}^3$, the deformed phase regions $\mathcal{B}_i(t)$ are specified. In this case, we may conclude, with the aid of (4.25), that

$$\bar{n}^{\circ} = \bar{n}^{\circ} = \bar{n}^{\cdot}, \quad \operatorname{grad}_{\mathcal{S}} \bar{n} = 0, \quad \bar{V}(0,0) = \bar{V}^{\circ}(0,0) = 0. \tag{14.30}$$

Assertion 1. We can assign values at $(0,0)$ for

$$\bar{n}, \ \bar{n}^{\cdot}, \ F_i, \ F_i^{\cdot}, \ y_i^{\cdot}, \ y_i^{\cdot\cdot}, \tag{14.31}$$

$i=1,2$, such that any two-phase motion consistent with this assignment, with (14.29), and with

$$\nabla F_i(0,0) = 0, \tag{14.32}$$

$i=1,2$, is consistent with the preassigned values at $(0,0)$ of

$$n, \ n^{\circ}, \ F, \ F^{\circ}, \ H, \ H^{\circ}, \ W_i, \ W_i^{\circ}, \ \gamma, \tag{14.33}$$

$i=1,2$.

To verify Assertion 1 note first that, by (3.11), (4.22), and (14.32), we can choose F_i, $F_i^{\cdot}$ at $(0,0)$ so that F, F°, H, H° have the required values at $(0,0)$; then, using (3.2), we can choose $\bar{n}$ and $\bar{n}^{\cdot}$ so that n and n° are as required. Note that the discussion thus far assigns values to

$$n_i, \ n_i^{\circ}, \ \vartheta_i, \ \lambda_i, \ J_i, \ \vartheta_i^{\circ}, \ \lambda_i^{\circ}, \ J_i^{\circ}, \ \operatorname{grad}_{\mathcal{S}}(y_i^{\cdot}), \tag{14.34}$$

$i=1,2$, and implies that, at $(0,0)$,

$$\operatorname{grad}_{\mathcal{S}} J_i = \operatorname{grad}_{\mathcal{S}} \lambda_i = \operatorname{grad}_{\mathcal{S}} \vartheta_i = 0, \quad \operatorname{grad}_{\mathcal{S}} F_i = 0, \tag{14.35}$$

where we have used (3.2), (3.4), (14.30), and (14.32). Next, by (4.4) and (4.6), granted (14.30), we can choose $y_i^{\cdot} \cdot \bar{n}$ at $(0,0)$ to render $W_i(0,0)$ as

required. Then

$$V_i(0,0) \text{ and } \operatorname{grad}_{\mathcal{S}} V_i(0,0) \text{ are also prescribed,} \tag{14.36}$$

where the specification of V_i follows from (4.6), while that of $\operatorname{grad}_{\mathcal{S}} V_i$ follows upon operating on (4.4) with $\operatorname{grad}_{\mathcal{S}}$ and then appealing to (14.30), (14.34), (14.35), and the identity $\operatorname{grad}_{\mathcal{S}} \bar{V} = -\bar{n}^{\square}$.

By (4.27), $[\mathbf{y}^{\cdot}(0,0)]_{\tan \mathcal{S}}$ can be chosen to give the requisite value of γ, and we choose the values $\mathbf{y}_i^{\cdot}(0,0)_{\tan \mathcal{S}}$ consistent with the chosen value of $[\mathbf{y}^{\cdot}(0,0)]_{\tan \mathcal{S}}$. The $\mathbf{y}_i^{\cdot}(0,0)$ are now prescribed, and, by (4.22) and (14.36), so also is $\mathbf{y}_1{}^{\circ}(0,0)$. Finally, if we differentiate (4.4) following $\mathcal{S}(t)$, and then use (4.25), and (14.30), we see that (since $(\cdot)^{\circ}$ is with respect to S_1)

$$0 = (\lambda_i V_i)^{\circ} - \mathbf{y}_1{}^{\circ} \cdot \operatorname{grad}_{\mathcal{S}} (\lambda_i V_i) + (\mathbf{y}_i^{\cdot})^{\square} \cdot \bar{n} + \mathbf{y}_i^{\cdot} \cdot \bar{n}^{\cdot},$$

at $^{\cdot}(0,0)$, and, since $\lambda_i V_i = J_i W_i$, we may use (14.34)-(14.36) and the analog of (4.20) for $\mathbf{y}_i^{\cdot}$ to see that, at $(0,0)$,

$$J_i W_i{}^{\circ} = -(\mathbf{y}_i^{\cdot\cdot}) \cdot \bar{n} + \text{prescribed quantities.}$$

Thus we can assign $(\mathbf{y}_i^{\cdot\cdot}) \cdot \bar{n}$ a value at $(0,0)$ that gives $W_i{}^{\circ}$ its pre-assigned value. This establishes Assertion 1.

Assertion 2. There is a two phase motion consistent with (14.29) and (14.32) that has $0 \in S_1(0)$, $0 \in S_2(0)$, and is such that the fields (14.31) have arbitrarily preassigned values at $(0,0)$.

We begin the proof by choosing $\bar{n}(t)$ consistent with the prescription of $\bar{n}(0)$ and $\bar{n}^{\cdot}(0)$ (the choice for $t > 0$ is irrelevant). We will construct the motion by considering, at each t, mappings $\mathbf{x} \mapsto \mathbf{g}_i(\mathbf{x},t)$ from $\mathcal{B}_i(t)$ into regions $B_i(t)$: if each of the $\mathbf{g}_i$'s is, at each time, a bijection with strictly positive determinant, then their fixed-time inverses $\mathbf{y}_i$ define a two-phase motion. Repeated differentiations of $\mathbf{g}_i(\mathbf{y}_i(\mathbf{X},t),t) = \mathbf{X}$ confirm that there are vectors $\mathbf{a}_i$ and $\mathbf{e}_i$ and tensors $\mathbf{A}_i$ and $\mathbf{E}_i$ with $\det \mathbf{A}_i > 0$ such that, if the $\mathbf{g}_i$'s are as specified above, and if

$$\mathbf{g}_i(\mathbf{0},0) = \mathbf{0}, \quad \partial_t \mathbf{g}_i(\mathbf{0},0) = \mathbf{a}_i, \quad (\partial_t)^2 \mathbf{g}_i(\mathbf{0},0) = 2\mathbf{e}_i,$$
$$\operatorname{grad} \mathbf{g}_i(\mathbf{0},0) = \mathbf{A}_i, \qquad \operatorname{grad} \partial_t \mathbf{g}_i(\mathbf{0},0) = \mathbf{E}_i, \tag{14.37}$$
$$\operatorname{grad}^2 \mathbf{g}_i(\mathbf{0},0) = \mathbf{0},$$

$i=1,2$, then the corresponding $\mathbf{y}_i$'s are consistent with an arbitrary constraint of the type (14.31) and (14.32). Choose $f(t)$ with $f(0)=0$ and $(df/dt)(0)=1$, and with $f(t)$ sufficiently small that $\det(\mathbf{A}_i + f(t)\mathbf{E}_i) > 0$, $i=1,2$, for all t. We define $\mathbf{g}_i$ $(i=1,2)$ by the requirement that, at each t, $\mathbf{g}_i(\cdot,t)$ be the restriction to $\mathcal{B}_i(t)$ of the function

$$\mathbf{x} \mapsto t\mathbf{a}_i + t^2\mathbf{e}_i + (\mathbf{A}_i + f(t)\mathbf{E}_i)\mathbf{x}.$$

This yields (14.37) and the proof of the second assertion is complete.

Assertions 1 and 2 yield the existence of a two-phase motion with the desired properties as stated in the lemma.

To complete the proof of the lemma, we must construct a corresponding field $\mu = (\mu^1, \ldots, \mu^{\alpha})$ that is smooth away from, and continuous across, the interface, and has arbitrary preassigned of μ and μ° at $(\mathbf{0},0)$. Let $\mathbf{u}$ and $\mathbf{d}$ denote these assigned values; then the field

$$\mu(\mathbf{x},t) = \mathbf{u} + t\mathbf{d},$$

since it is independent of $\mathbf{x}$, has the desired properties.

15. EVOLUTION EQUATIONS IN THE ABSENCE OF EXTERNAL FORCES

We now list the complete set of equations that hold when the mass supplies and external forces vanish. We shall continue to use the abbreviations $S=S_1$, $\mathbf{L}=\mathbf{L}_1$, $K=K_1$, $\mathbf{n}=\mathbf{n}_1$, and $\mathbf{F}=\mathbf{F}_1$, and, as before, $\mathbf{C}$ is the Eshelby tensor (10.1).

For the general *dynamical* theory the interface conditions consist of the *compatibility condition* (10.2), the *mass balance* (10.5), the *local equilibrium condition* (10.6), the *normal partial balance*

$$\psi K - (\mathbf{F}^T\mathbf{S} + \mathbf{H}^T\mathbf{K})\cdot\mathbf{L} + \mathrm{Div}_S\mathbf{c} - \mathbf{n}\cdot\mathbf{C}_1\mathbf{n} = \beta_{11}W_1 + \beta_{12}W_2 \qquad (15.1)$$

for phase 1, the *partial balance*

$$\mathrm{Div}_S\mathbf{K} + \mathcal{H}C_2\mathbf{n}_2 = \mathcal{H}(\beta_{21}W_1 + \beta_{22}W_2)\mathbf{n}_2 \qquad (15.2)$$

for phase 2, the *force balance*

$$\mathrm{Div}_S\mathbf{S} + \mathcal{H}S_2\mathbf{n}_2 - S_1\mathbf{n}_1 = 0, \qquad (15.3)$$

and the constitutive relations (14.7) and (14.11). (The balances (15.1) and (15.3) are (14.17) and $(12.11)_1$, while (15.2) is $(12.11)_3$ with $\mathbf{c}_2$ given by (9.12).)

The relation (15.1) has an equivalent form

$$\psi K - \mathbf{S}_{ext}\cdot\nabla_n\mathbf{F} - \mathbf{K}_{ext}\cdot\nabla_n\mathbf{H} + \mathrm{Div}_S\mathbf{c} + (\det\mathbf{H})\omega_2 - \omega_1 = B_1W_1 + B_2W_2 \qquad (15.4)$$

(cf. (14.18)) with $\mathbf{c}$ given by $(14.5)_3$.

In *statical* situations the interface conditions reduce to $(10.10)_{3,4}$ and

$$\begin{aligned} &\psi K - (\mathbf{F}^T\mathbf{S} + \mathbf{H}^T\mathbf{K})\cdot\mathbf{L} + \mathrm{Div}_S\mathbf{c} - \mathbf{n}\cdot\mathbf{C}_1\mathbf{n} = 0, \\ &\mathrm{Div}_S\mathbf{K} + \mathcal{H}C_2\mathbf{n}_2 = \mathbf{0}, \\ &\mathrm{Div}_S\mathbf{S} + \mathcal{H}S_2\mathbf{n}_2 - S_1\mathbf{n}_1 = 0. \end{aligned} \qquad (15.5)$$

16. RESULTS IN THE SPATIAL DESCRIPTION

16.1. GENERAL THEORY

The interfacial force and moment balances are (12.6) and (12.7), and, in the absence of external forces, have the form

$$\begin{aligned} &\mathrm{div}_{\mathcal{S}}\,\bar{\mathbf{S}} + [\bar{\mathbf{S}}]\bar{\mathbf{n}} = \mathbf{0}, \\ &\mathrm{div}_{\mathcal{S}}\,\bar{\mathbf{C}}_i + \delta_i \bar{\mathbf{C}}_i \bar{\mathbf{n}} + \bar{\mathbf{e}}_i = \mathbf{0}. \end{aligned} \tag{16.1}$$

The limit $\mathcal{P}_{\mathrm{loc}}(A)$, which is now considered as a function $\mathcal{P}_{\mathrm{loc}}(\mathcal{G})$ of the deformed subsurface $\mathcal{G}\subset\mathcal{S}$, is given by

$$\mathcal{P}_{\mathrm{loc}}(\mathcal{G}) = \int_{\mathcal{G}} \{[\bar{\mathbf{S}}\bar{\mathbf{n}}\cdot\mathbf{y}^{\cdot}] + \bar{\mathbf{g}}\cdot\bar{\mathbf{v}} + \bar{\mathbf{f}}_1\cdot\mathbf{v}_1 + \bar{\mathbf{f}}_2\cdot\mathbf{v}_2\}\,da + \int_{\partial\mathcal{G}} \{\bar{\mathbf{S}}\bar{\boldsymbol{\nu}}\cdot\bar{\mathbf{w}} + \bar{\mathbf{C}}_1\bar{\boldsymbol{\nu}}\cdot\mathbf{w}_1 + \bar{\mathbf{C}}_2\bar{\boldsymbol{\nu}}\cdot\mathbf{w}_2\}\,ds. \tag{16.2}$$

Invariance of the power, as asserted in the paragraph following (13.3), yields (8.5), or equivalently

$$(\mathrm{div}_{\mathcal{S}}\,\bar{\mathbf{S}} + \mathbf{G}_1^{\mathsf{T}}\mathrm{div}_{\mathcal{S}}\,\bar{\mathbf{C}}_1 + \mathbf{G}_2^{\mathsf{T}}\mathrm{div}_{\mathcal{S}}\,\bar{\mathbf{C}}_2)_{\tan\mathcal{S}} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2, \tag{16.3}$$

with $\boldsymbol{\tau}_i$ given by (13.7), and the conclusion that the field

$$\bar{\mathbf{A}} = \bar{\mathbf{S}} + \mathbf{G}_1^{\mathsf{T}}\bar{\mathbf{C}}_1 + \mathbf{G}_2^{\mathsf{T}}\bar{\mathbf{C}}_2 \tag{16.4}$$

has the form

$$\bar{\mathbf{A}} = \bar{\sigma}\bar{\mathbf{1}} + \bar{\mathbf{n}}\otimes\bar{\mathbf{e}}, \tag{16.5}$$

with $\bar{\mathbf{e}}=\bar{\mathbf{A}}^{\mathsf{T}}\bar{\mathbf{n}}$ and $\bar{\sigma}=\sigma/\jmath_1$. ($\bar{\mathbf{A}}$ is generally not the referential-to-spatial transform of $\mathbf{A}$.)

Let

$$(\bar{\mathbf{C}}_i)_{\mathrm{ext}} = \bar{\mathbf{C}}_i\bar{\mathbf{P}}, \qquad (\bar{\mathbf{S}})_{\mathrm{ext}} = \bar{\mathbf{S}}\bar{\mathbf{P}}, \tag{16.6}$$

and let π_i and $\boldsymbol{\tau}_i$ given by (11.3) and (13.7). Then

$$\mathcal{P}_{loc}(\mathcal{G}) = \int_{\partial\mathcal{G}} \bar{\sigma} V_{(\partial\mathcal{G})tan} ds + \int_{\mathcal{G}} \bar{\rho} da, \tag{16.7}$$

$$\bar{\rho} = - \bar{\mathbf{e}} \cdot \bar{\mathbf{n}}^{\square} - \bar{\sigma} \bar{K} \bar{V} + \sum_{i=1,2} \{ (\bar{\mathbf{C}}_i)_{ext} \cdot (\mathbf{G}_i)^{\square} - \boldsymbol{\tau}_i \cdot (\mathbf{y}^{\bullet}_i)_{tan\mathcal{S}} - (\delta_i \bar{\omega}_i + \pi_i) U_i \} \tag{16.8}$$

($\bar{\rho}$ is generally not the transform of ρ). The verification of (16.7), (16.8) parallels that of (8.13)-(8.15) and (13.9)-(13.10), and uses the identities (4.31) and (4.33).

Regarding the energetics, we are led to the conclusion

$$\bar{\sigma} = \bar{\psi} \tag{16.9}$$

and to a reduced dissipation inequality in spatial form

$$\bar{\psi}^{\square} + \bar{\mathbf{e}} \cdot \bar{\mathbf{n}}^{\square} - \sum_{i=1,2} \{ (\bar{\mathbf{C}}_i)_{ext} \cdot \mathbf{G}_i^{\square} - \pi_i U_i - \boldsymbol{\tau}_i \cdot (\mathbf{y}^{\bullet}_i)_{tan\mathcal{S}} \} \leq 0. \tag{16.10}$$

The spatial theory is based on constitutive equations of the form

$$\begin{aligned} &\bar{\psi} = \tilde{\psi}(\mathbf{z}), \quad \boldsymbol{\tau}_2 = \tilde{\boldsymbol{\tau}}_2(\mathbf{z}), \quad \pi_i = \tilde{\pi}_i(\mathbf{z}), \\ &\bar{\mathbf{C}}_i = \tilde{\mathbf{C}}_i(\mathbf{z}), \qquad\qquad \bar{\mathbf{e}} = \tilde{\mathbf{e}}(\mathbf{z}), \\ &\mathbf{z} = (\mathbf{F}_1, \mathbf{F}_2, \bar{\rho}_1, \bar{\rho}_2, \bar{\mathbf{n}}, U_1, U_2), \end{aligned} \tag{16.11}$$

with (9.5) tacit. By (16.4)-(16.5), these yield a similar relation $\bar{\mathbf{S}} = \tilde{\mathbf{S}}(\mathbf{z})$ for the deformational stress, and we assume that

$$\tilde{\mathbf{S}}(\mathbf{z})_{ext} \quad \text{is symmetric,} \tag{16.12}$$

an assumption that ensures consistency with both requirements of the moment balance (12.7). (Note that we do not introduce a constitutive equation for $\boldsymbol{\tau}_1$, but instead consider $\boldsymbol{\tau}_1$ as defined through (16.3).)

Consequences of the dissipation inequality are that: (i) the interface is free to slip (in the sense of (14.4)); (ii) the interfacial energy reduces to a function

$$\bar{\psi} = \tilde{\psi}(G_1, G_2, \bar{n})$$

and generates $\bar{C}_i$ and $\bar{\theta}$ through

$$\tilde{C}_i(G_1, G_2, \bar{n})_{ext} = \partial_{G_i}\tilde{\psi}(G_1, G_2, \bar{n}), \qquad \tilde{\theta}(G_1, G_2, \bar{n}) = -\partial_{\bar{n}}\tilde{\psi}(G_1, G_2, \bar{n}), \quad (16.13)$$

(iii) the kinetic relations $(11.6)_2$ hold for the normal internal forces π_i.

Equations (16.5), (16.6), (16.12), and (16.13) imply that $\tilde{C}_i$ and $\tilde{\psi}$ are independent of the normal components $G_i\bar{n}$, and that

$$\tilde{C}_i(\mathbb{G}_1, \mathbb{G}_2, \bar{n}) = \partial_{\mathbb{G}_i}\tilde{\psi}(\mathbb{G}_1, \mathbb{G}_2, \bar{n}), \qquad D_{\bar{n}}\tilde{\psi}(\mathbb{G}_1, \mathbb{G}_2, \bar{n}) = 0, \quad (16.14)$$

where $D_{\bar{n}}\tilde{\psi}$ is the derivative of $\tilde{\psi}$ following $\bar{n}$, as defined in (14.8). The second of (16.14) implies that the interfacial energy is invariant under changes in observer:

$$\tilde{\psi}(G_1, G_2, \bar{n}) = \tilde{\psi}(G_1Q^T, G_2Q^T, Q\bar{n}) \quad (16.15)$$

for any rotation Q.

When the mass supplies and external forces vanish, the resulting interface conditions consist of the *kinematic compatibility condition* (11.9), the *mass balance* (11.12), the *local equilibrium condition* (10.6), the *partial balances*

$$G_i^T \mathrm{div}_{\mathcal{S}}\bar{C}_i + \delta_i(\bar{\omega}_i 1 - \bar{S}_i)\bar{n} = (\eta_{i1}U_1 + \eta_{i2}U_2)\bar{n} \quad (16.16)$$

for each phase i, and the *normal force balance*

$$\bar{\mathbb{E}} \cdot \bar{\mathbb{L}} = -\bar{n}\cdot[\bar{S}]\bar{n}, \quad (16.17)$$

or equivalently

$$(\bar{\sigma}\bar{1} - G_1^T\bar{C}_1 - G_2^T\bar{C}_2)\cdot\bar{\mathbb{L}} = -\bar{n}\cdot[\bar{S}]\bar{n} \quad (16.18)$$

with $\bar{\mathbb{L}} = -\nabla_{\mathcal{S}}\bar{n}$ the curvature tensor for $\mathcal{S}$.

Note that the normal parts of (16.16) and (16.17) may be combined to

give

$$\bar{\psi}\bar{K} - (\bar{\mathbf{C}}_1)_{ext}\cdot\mathrm{grad}_{\bar{n}}G_1 - (\bar{\mathbf{C}}_2)_{ext}\cdot\mathrm{grad}_{\bar{n}}G_2 + \mathrm{div}_{\mathcal{S}}\,\bar{\mathbf{e}} + [\bar{\omega}] = \bar{B}_1 W_1 + \bar{B}_2 W_2, \tag{16.19}$$

with $\mathrm{grad}_{\bar{n}}G = (\mathrm{grad}G)\bar{n}$ and $\bar{B}_i = \eta_{1i} + \eta_{2i}$.

When the interface is stationary and the deformation and densities independent of time, the interface conditions reduce to $(11.14)_{3,4}$ and

$$G_i^{\mathsf{T}}\mathrm{div}_{\mathcal{S}}\bar{\mathbf{C}}_i + \delta_i(\bar{\omega}_i\mathbf{1} - \bar{\mathbf{S}}_i)\bar{n} = \mathbf{0}, \qquad \bar{\mathbf{S}}\cdot\bar{\mathbf{L}} = -\bar{n}\cdot[\bar{\mathbf{S}}]\bar{n}. \tag{16.20}$$

Granted (16.4) and the constitutive relations (16.13), (16.20) are equivalent to interface conditions obtained variationally by Leo and Sekerka [1989].

16.2. NONINTERACTIVE INTERFACIAL ENERGY

The interaction between phases manifests itself in the coupling inherent in the dependence of the interfacial energy $\bar{\psi} = \tilde{\psi}(G_1, G_2, \bar{n})$ on G_1 and G_2. Here we will discuss energies, which we call **noninteractive**, for which this coupling is absent:[36]

$$\tilde{\psi}(G_1, G_2, \bar{n}) = \tilde{\psi}_1(G_1, \bar{n}) + \tilde{\psi}_2(G_2, \bar{n}), \tag{16.21}$$

Granted (16.21), we can write

$$\bar{\psi} = \bar{\psi}_1 + \bar{\psi}_2, \qquad \bar{\psi}_i = \tilde{\psi}_i(G_i, \bar{n}), \tag{16.22}$$

with $\bar{\psi}_i$ the interfacial energy of phase i, measured per unit deformed area, and (16.13) yields

$$\tilde{\mathbf{C}}_i(G_i, \bar{n})_{ext} = \partial_{G_i}\tilde{\psi}_i(G_i, \bar{n}), \tag{16.23}$$

so that $\bar{\mathbf{C}}_i$ is independent of G_j, $j \neq i$.

In view of (3.2), the energy $\psi_i = \wp_i\bar{\psi}_i$, measured per unit referential area, is given by a constitutive equation

$$\psi_i = \hat{\psi}_i(\mathbf{F}_i, \mathbf{n}_i), \tag{16.24}$$

[36] Leo and Sekerka [1989] use the term "greased interface" when the interfacial energy has the form (16.21).

and this allows us to define a deformational stress for phase i through

$$\mathbf{S}_i = \hat{\mathbf{S}}_i(\mathbf{F}_i,\mathbf{n}_i) = \partial_{\mathbf{F}_i}\hat{\psi}_i(\mathbf{F}_i,\mathbf{n}_i)\mathbb{1}_i. \tag{16.25}$$

If we differentiate the identity

$$\tilde{\psi}_i(\mathbf{G}_i,\bar{\mathbf{n}}) = \wp_i^{-1}\hat{\psi}_i(\mathbf{F}_i,\mathbf{n}_i) \tag{16.26}$$

with respect to $\mathbf{G}_i$, regarding $\mathbf{n}_i$ and $\wp_i^{-1}$ as functions of $(\mathbf{G}_i,\bar{\mathbf{n}})$ through the relations

$$\mathbf{n}_i = \lambda_i^{-1}\mathbf{G}_i^{-T}\bar{\mathbf{n}} \qquad \wp_i^{-1} = J_i^{-1}\lambda_i, \qquad \lambda_i = |\mathbf{G}_i^{-T}\bar{\mathbf{n}}|,$$

we obtain

$$\wp_i(\partial_{\mathbf{G}_i}\tilde{\psi}_i)\mathbf{G}_i^T = \hat{\psi}_i\Lambda_i - \mathbf{F}_i^T(\partial_{\mathbf{F}_i}\hat{\psi}_i) - \mathbf{n}_i\otimes\partial_{\mathbf{n}_i}\hat{\psi}_i, \tag{16.27}$$

where we have used the identities (with $\mathbf{B}$ an arbitrary tensor)

$$\mathbf{B}\cdot\partial_{\mathbf{G}_i}\mathbf{n}_i = -\Lambda_i\mathbf{F}_i^T\mathbf{B}^T\mathbf{n}_i, \qquad \mathbf{B}\cdot\partial_{\mathbf{G}_i}\wp_i^{-1} = \wp_i^{-1}(\Lambda_i\mathbf{F}_i^T)\cdot\mathbf{B},$$
$$\partial_{\mathbf{G}_i}\hat{\psi}_i = -\mathbf{F}_i^T(\partial_{\mathbf{F}_i}\hat{\psi}_i)\mathbf{F}_i^T,$$

Now, appealing to (3.6), (3.21), (16.23), and (16.25), (16.27) becomes

$$(\mathbb{C}_i)_{\tan S_i} = \psi_i\mathbb{1}_i - (\mathbf{F}_i^T\mathbf{S}_i)_{\tan S_i}, \tag{16.28}$$

rendering the tangential accretive stress in each phase an Eshelby tensor for the interface.

Similarly, differentiating (16.26) with respect to $\mathbf{F}_i$, considering $\bar{\mathbf{n}}$ and $\wp_i$ as functions of $(\mathbf{F}_i,\mathbf{n}_i)$, we find that

$$\wp_i^{-1}(\partial_{\mathbf{F}_i}\hat{\psi}_i)\mathbf{F}_i^T = \tilde{\psi}_i\bar{\Lambda} - \mathbf{G}_i^T(\partial_{\mathbf{G}_i}\tilde{\psi}_i) - \bar{\mathbf{n}}\otimes(\partial_{\bar{\mathbf{n}}}\tilde{\psi}_i). \tag{16.29}$$

Further, by (16.6), $(\partial_{\mathbf{G}_i}\tilde{\psi}_i)\bar{\mathbf{n}}=\mathbf{0}$, so that, by (16.27), $(\partial_{\mathbf{F}_i}\hat{\psi}_i)\mathbf{n}_i=0$. Therefore

$$(\bar{\mathbf{S}}_i)_{\tan\mathcal{S}} = \bar{\psi}_i \bar{\mathbb{1}} - (\mathbf{G}_i^T \bar{\mathbf{C}}_i)_{\tan\mathcal{S}}, \qquad \bar{\mathbf{S}}_i^T \bar{\mathbf{n}} = -D_{\bar{\mathbf{n}}} \tilde{\psi}_i. \tag{16.30}$$

The first of (16.30) is an interesting counterpart of (16.28). By $(16.14)_2$, (16.21), and $(16.30)_2$,

$$(\bar{\mathbf{S}}_1 + \bar{\mathbf{S}}_2)^T \bar{\mathbf{n}} = \mathbf{0},$$

and this, $(12.7)_1$, (16.4), (16.5), (16.22), and (16.30) yield the conclusion

$$\bar{\mathbf{S}} = \bar{\mathbf{S}}_1 + \bar{\mathbf{S}}_2. \tag{16.31}$$

Note that, by (16.30), the $\bar{\mathbf{S}}_i$'s generally will be neither tangential nor symmetric, although balance of moments (16.12) requires that their sum $\bar{\mathbf{S}}$ is.

The balance equations that arise when the mass supplies and external forces vanish are (16.16) and (16.17), but the lack of coupling between phases, as expressed in (16.23), renders *the partial balances* (16.16) *independent*. Further, granted (16.29), an argument similar to the proof of (14.15) yields the identity

$$(\mathrm{div}_{\mathcal{S}} \bar{\mathbf{S}}_i)_{\tan\mathcal{S}} = -(\mathbf{G}_i^T \mathrm{div}_{\mathcal{S}} \bar{\mathbf{C}}_i)_{\tan\mathcal{S}}, \tag{16.32}$$

and we can rewrite the tangential part of the partial balance (16.16) in the form

$$(\mathrm{div}_{\mathcal{S}} \bar{\mathbf{S}}_i + \delta_i \bar{\mathbf{S}}_i \bar{\mathbf{n}})_{\tan\mathcal{S}} = \mathbf{0}. \tag{16.33}$$

Thus the interface conditions for a noninteractive energy are,[37] with the exception of the normal deformational balance, *totally uncoupled*.

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[37] For statical situations the interface conditions reduce to those derived variationally by Leo and Sekerka [1989].

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