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Young measure solutions for a nonlinear parabolic equation of forward-backward type

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Abstract

The scope is to study the nonlinear parabolic problem of forward-backward type

$$u_t = \nabla \cdot q(\nabla u) \qquad \text{on } Q_\infty \equiv \Omega \times \mathbf{R}^+$$

with initial data u_0 given in $H_0^1(\Omega)$. Here $\Omega \subset \mathbf{R}^N$ is open, bounded with mildly smooth boundary and $q \in C(\mathbf{R}^N; \mathbf{R}^N)$, an analogue to heat flux, satisfies $q = \nabla\phi$ with $\phi \in C^1(\mathbf{R}^N)$ of suitable growth. When ϕ is not convex classical solutions do not exist in general; the problem admits *Young measure* solutions. By that is meant a function $u \in H^1_{loc}(Q_\infty) \cap L^\infty(\mathbf{R}^+; H_0^1(\Omega))$ and a parametrized family of probability measures $\nu = (\nu_{x,t})_{(x,t)\in Q_\infty}$ related to u by $\nabla u = \int_{\mathbf{R}^N} \lambda\, \nu(d\lambda)$ a.e. in Q_∞; via ν the nonlinearity $q(\nabla u)$ is replaced by the moment $< \nu, q > = \int_{\mathbf{R}^N} q(\lambda)\, \nu(d\lambda)$ a.e. in Q_∞ and the equation is then interpreted in H^{-1}. The family ν is generated by the gradients of a sequence in $H^1_{loc}(Q_\infty)$, is non-unique, but through its first moment some of the classical properties are preserved: uniqueness of the function u is true; stability is reflected in a maximum principle and a comparison result. The asymptotic analysis yields, as time tends to infinity, a unique limit z and an associated Young measure ν^∞ such that the pair (z, ν^∞) is a Young measure solution of the steady-state problem $\nabla \cdot q(\nabla z) = 0$. The relevant energy function is shown to be monotone decreasing and asymptotically tending to its minimum, globally and locally in space.

Contents

1 Introduction

We study the nonlinear evolution problem

$$(1)\qquad u_t = \nabla \cdot q(\nabla u) \quad \text{on } Q_\infty \equiv \Omega \times \mathbb{R}^+$$

$$(2)\qquad u(\cdot, 0) = u_0 \quad \text{on } \Omega$$

$$(3)\qquad u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+$$

which will be denoted by $\mathcal{P}$. Here Ω is an open bounded subset of $\mathbb{R}^N$ such that the cone or the segment property is satisfied on the boundary (as for example in the case of a Lipschitz boundary), and $q : \mathbb{R}^N \to \mathbb{R}^N$ a nonlinear continuous potential gradient function, an analogue to heat flux, satisfying $q = \nabla\phi$, where $\phi \in C^1(\mathbb{R}^N)$ (the space of continuously differentiable functions on $\mathbb{R}^N$) is of suitable growth. The initial data function u_0 is given in $H_0^1(\Omega)$ and the zero boundary data (3) can be taken to be general time-homogeneous function $g \in H^1(\Omega)$. (Here $H^1(\Omega)$ is the Sobolev space of functions on Ω which together with their first order weak derivatives are in $L^2(\Omega)$ and $H_0^1(\Omega)$ is its subset consisting of the functions with zero trace on the boundary of Ω).

When ϕ is not convex, in which case the monotonicity condition $(q(x) - q(y)) \cdot (x - y) \geq 0$ for $x, y \in \mathbb{R}^N$ is violated on subsets of $\mathbb{R}^N$, the differential operator $-\nabla \cdot q(\nabla \cdot)$ fails to be accretive (e.g. monotone); equation (1) then constitutes of a *forward-backward* parabolic equation generally admitting no classical strong or distributional solutions. The non-convexity of the potential is compatible with the usual requirement that $q(\lambda) \cdot \lambda \geq 0$ imposed on a theory of thermal conductors by the Clausius-Duhem inequality.

A notion of solution appropriate for the study of $\mathcal{P}$ is that of a *measure-valued* or *Young measure* solution. By that is meant a function u in a Sobolev space and a parametrized family of probability measures $\nu = (\nu_{x,t})_{(x,t)\in Q_\infty}$ which is generated by the gradients of a sequence in the same space and satisfy

$$\int_0^{+\infty}\int_\Omega (<\nu, q> \nabla\zeta + \frac{\partial u}{\partial t}\zeta)\, dx\, dt = 0$$

where

$$<\nu, q> = \int_{\mathbb{R}^N} q(\lambda)\, \nu(d\lambda) \qquad \text{a.e. in } Q_\infty$$

for all ζ in an appropriate subspace of $H^1(Q_\infty)$. In addition, u and ν satisfy

$$\nabla u = <\nu, id> = \int_{\mathbf{R}^N} \lambda\, \nu(d\lambda) \qquad \text{a.e. in } Q_\infty$$

where $id(\lambda) = \lambda$. So to each point (x,t) in the domain Q_∞ associated is a probability measure $\nu_{x,t}$ on $\mathbf{R}^N$; via this parametrized measure the nonlinearity of $q(\nabla u)$ is replaced by the expected value of q, while the first moment of the measure is the gradient of the solution. To date the term 'Young measure solution', although strictly derived from the fundamental theorem of Young measures described in section 2, admits slightly different definitions by different authors: the decision of what is a Young measure solution of a problem must necessarily accommodate the way the generating sequence is chosen. The nature of the Young measure solutions is made precise in section 2.

The approach which we shall assume in this paper for the study of (1) - (3) is the one employed by Kinderlehrer and Pedregal in [KP1] to establish existence in the case of zero boundary data (immediately valid in the case of general time-homogeneous boundary conditions). The method incorporates the explicit methods for solutions of evolution equations (cf. [BC], [HK]) with variational methods used to accommodate and describe the oscillatory behavior (cf. [Ba], [E], [KP1], etc). The combination of the two methods leads to the existence of Young measure solutions of evolution problems which may be of forward-backward nature.

The analytical context of our approach to obtain existence is to approximate the dynamics of equation (1) with a sequence of stationary problems, the solutions of which are in turn interpreted as minimizers of variational principles. More precisely, the time-discretized version of (1) is the Euler equation of a non-convex variational principle which at each time step (of size h) is minimized. The minimizer solves the stationary problem and approximates the solution of $\mathcal{P}$ within time h. By taking arbitrarily small time steps we pass from the stationary to the evolution problem. The method is well known in the study of semigroups. It has been implemented recently by Horihata and Kikuchi in [HK] to construct weak solutions to a quasilinear parabolic problem associated with a convex variational principle. Further, this method has also been employed by Bethuel, Coron, Ghidaglia and Soyeur in [BC] to establish existence of weak solutions for a nonlinear heat equation associated to weakly harmonic maps in a

Sobolev-type space of functions of the unit ball into the sphere in $\mathbf{R}^3$.

In applying this method to treat $\mathcal{P}$ the difficulty which arises is twofold: firstly, the non-convexity of the potential ϕ implicating the minimization of a non-convex variational principle; secondly, the unwieldiness of the nonlinear dependence of the heat flow q on the gradient of the solution. Both situations call for sensible generalizations, in the former case that of a 'minimizer' of a variational principle and in the latter that of a 'weak solution' of a differential equation. Respectively, the pertinent themes implemented in [KP1] to overcome these impediments are: firstly, the *relaxation* of a non-convex functional and secondly, replacing the nonlinearity of $q(\nabla u)$ with the expected value of q against a *Young measure*. The sense in which $\mathcal{P}$ has a solution is then that of a Young measure solution.

We review the variational method of Kinderlehrer and Pedregal in [KP1] to obtain existence in the case of zero boundary data, which is immediately valid in the case of general time-homogeneous boundary conditions. Following, we investigate deeper the properties of the Young measure solution establishing a *uniqueness* result. It should be remarked that as a rule, non-uniqueness results appear in the literature (for example in [BC] or [HN]) regarding nonlinear parabolic problems especially of forward-backward nature. In our case, the uniqueness of the Young measure solution, although not directly dependent on the particular construction scheme of the solution, is contingent upon an *independence* property, namely that the heat flux q and the solution u be independent with respect to the Young measure ν and furthermore, a condition regarding the support of the Young measure ν, namely $<\nu, q> = <\nu, p>$ where $p = \nabla\phi^{**}$ and ϕ^{**} is the convexification of the potential ϕ. The function u is unique but the Young measure is not.

The solution u is also *continuously dependent on the initial data* in the L^2 norm. In addition, $u(\cdot, t)$ satisfies continuity properties in the L^2 norm, both as $t \to 0^+$ (monotone decreasing) and as $t \to +\infty$. Stability of the solution is reflected in the fact that it satisfies a *maximum principle* and a *comparison lemma*. A consequence of the comparison lemma is that the solution is also a *local* solution in the space variable with respect to its own initial-boundary data.

In section 3 we investigate the asymptotics of $\mathcal{P}$. As time tends to infinity, the solution $u(\cdot,t)$ converges to z strongly in L^2 (monotonically decreasing) and weakly in H^1 and the measure ν has a (weak) asymptotic limit ν^∞ such that the pair (z,ν^∞) constitutes a Young measure solution to the steady-state version of $\mathcal{P}$, $\nabla\cdot q(\nabla z) = 0$. This is achieved by showing that the set of all weak limit points of $(u(\cdot,t))_{t\geq 0}$ in H^1 is *invariant* under the operator $\mathcal{P}$ and further, there exists exactly one such weak limit point z. The asymptotic Young measure ν^∞ has restricted support satisfying $supp\,\nu^\infty \subseteq \{q(\lambda)\cdot\lambda = 0\}\cap\{\phi^{**} = \phi\}$.

In section 4 we introduce the relevant *energy* function $E(t) = \int_0^\infty \phi^{**}(\nabla u)(x,t)\,dx$. As time tends to infinity, the energy converges to zero monotonically decreasing globally in space. We show that it also vanishes locally in space, that is, on any subdomain $\omega \subseteq \Omega$ (although not monotonically on ω).

The forward-backward heat equation has also been studied by Höllig [H], Höllig and Nohel [HN] and Slemrod [Sl]. The treatment in [H] and [HN] concerns the Neumann initial value problem in the case of one spacial dimension ($\Omega = [0,1]$). It establishes in the model case of a piecewise linear heat flux q, decreasing on an interval $[a,b]\subset\Omega$, that a continuum of solutions exist for finite time satisfying (1) weakly in the sense of L^2. Each such solution is obtained as the sum of an explicitly constructed oscillating function and a smooth function which solves (weakly in L^2) an inhomogeneous heat equation.

The treatment in [Sl] involves Young measures but the spirit is different to that assumed here. $\mathcal{P}$ with Dirichlet or Neumann boundary conditions is approximated by a sequence of regular, singularly perturbed problems whose solutions are used to extract the Young measure solution. The differences between the Young measure solutions obtained in [Sl] and [KP1] are subtle. In [Sl] the heat flux q and the initial data u_0 are required to be sufficiently smooth, q must have strictly subquadratic growth and equation (1) is satisfied in the sense of distributions. In [KP1] q is continuous of linear growth, $u_0 \in H_0^1$ and equation (1) is satisfied in H^{-1}.

A one dimensional convex analogue to $\mathcal{P}$ associated with a potential of linear growth has been studied by Zhou [Z]. The approach in [Z] differs from a Young measure viewpoint but here also a variational technique is developed to solve the stationary and evolution problems.

2 Uniqueness, stability and properties of the Young measure solution

2.1 Background

We start with the notion of a $W^{1,p}$-gradient Young measure introduced in [KP2] and then describe some properties. Most statements appear in vectorial formulation although it is their scalar version which we shall make use of in this study.

Definition. A family of probability measures $\nu = (\nu_x)_{x\in\Omega}$ on $\mathbf{M}$, where Ω is an open set in $\mathbb{R}^N$, is a **$W^{1,p}$-gradient Young measure** for some $p \in [1,\infty]$ if
(i) $x \in \Omega \mapsto \int_{\mathbf{M}} f(A)\,\nu_x(dA) \in \mathbb{R}$ is a Lebesgue measurable function for all f bounded continuous on $\mathbf{M}$, the vector space $\mathbb{R}^{M\times N}$ of $M \times N$ matrices over the reals.
(ii) There is a sequence of functions $(u^k)_{k>0} \subset W^{1,p}(\Omega;\mathbb{R}^M)$ for which the representation formula

$$\lim_{k\to+\infty} \int_E \phi(\nabla u^k)(x)\,dx = \int_E\int_{\mathbf{M}} \phi(A)\,\nu_x(dA)\,dx \tag{4}$$

holds for all measurable $E \subseteq \Omega$ and all ϕ in the space

$$\mathcal{E}_0^p(\mathbf{M}) := \left\{ \phi \in C(\mathbf{M}) : \lim_{|A|\to+\infty} \frac{\phi(A)}{1+|A|^p} \text{ exists} \right\}$$

for $p < +\infty$, and for all functions ϕ continuous on $\mathbf{M}$ when $p = +\infty$. In the above, $C(\mathbf{M})$ denotes the space of continuous real valued functions on $\mathbf{M}$. We shall use the notation

$$\overline{\phi} = <\nu,\phi>:= \int_{\mathbf{M}} \phi(A)\,\nu(dA).$$

Property (i) above is equivalent to weak* measurability of $x \in \Omega \mapsto \nu_x \in Prob(\mathbb{R}^N)$ (the set of probability measures on $\mathbf{M}$), that is, measurability with respect to the weak* topology on $Prob(\mathbb{R}^N)$. Strong measurability usually will not be true. Property (ii) implies that there exists a sequence of functions $(u^k)_{k>0} \subset W^{1,p}(\Omega;\mathbb{R}^M)$ such that

$$\phi(\nabla u^k) \rightharpoonup <\nu,\phi> \qquad \text{in } L^1(\Omega;\mathbb{R}^M) \text{ as } k \to +\infty$$

for all $\phi \in \mathcal{E}_0^p$, (where the notation $\rightharpoonup$ is used to denote weak convergence in the space indicated). In particular,

$$|\nabla u^k|^p \rightharpoonup <\nu,|A|^p> \qquad \text{in } L^1(\Omega) \text{ as } k \to +\infty$$

(a condition not guaranteed for any subsequence by the uniform boundedness of the $(\|u^k\|_{W^{1,p}})_{k>0}$ alone).

As noted in [KP2] the space $\mathcal{E}_0^p(\mathbf{R}^N)$ is a separable Banach space in the norm

$$\|\phi\|_{\mathcal{E}^p} \;=\; \sup_{A\in \mathrm{M}} \frac{\phi(A)}{1+|A|^p} \;=\; \left\| \frac{\phi}{1+|\cdot|^p} \right\|_{L^\infty(\Omega)} ;$$

Separability is desirable when the duals of the spaces such as $L^1(\Omega, \mathcal{E}_0^p(\mathrm{M}))$ are considered and the representation formula (4) remains valid if $\mathcal{E}_0^p(\mathrm{M})$ is replaced by the inseparable space

$$\mathcal{E}^p(\mathrm{M}) \;=\; \left\{ \phi \in C(\mathrm{M}) : \sup_{A\in\mathrm{M}} \frac{\phi(A)}{1+|A|^p} < +\infty \right\}.$$

The guarantee for the existence of $W^{1,p}$-gradient Young measures draws upon the fundamental theorem of Young measures originally proved by Tartar [T] and built on ideas developed by Young[Y]; a version appears in Ball [Ba] and an extension was proved by Schonbek [Sc].

By the theorem, a sequence $(z^k)_{k\geq 1}$ of measurable, mildly bounded functions defined on a $\mathcal{L}^N$-measurable subset $S \subseteq \mathbf{R}^N$ into $\mathbf{R}^M$ has a subsequence $(z^j)_{j\geq 1}$ which generates a *Young measure*, that is, a parametrized family of probability measures $\boldsymbol{\nu} = (\nu_x)_{x\in S}$ on $\mathbf{R}^M$ via which certain weak limits can be characterized ($\mathcal{L}^N$ standing for the Lebesgue measure in $\mathbf{R}^N$). Namely, the weak limit of the $(f(z^j))_{j\geq 1}$ in $L^1(E)$ exists and is $< \nu_x, f > = \int_{\mathbf{R}^M} f(\lambda)\, \nu_x(d\lambda)$ for any $\mathcal{L}^N$-measurable $E \subseteq S$ and any continuous function f such that $(f(z^j))_{j\geq 1}$ are weakly sequentially precompact in $L^1(E)$.

Without any boundedness conditions on the $(z^j)_{j\geq 1}$, a subsequential convergence of the $(f(z^j))_{j\geq 1}$ is guaranteed only for $f \in C_0(\mathrm{M})$ (the continuous functions on M which vanish at infinity) weakly* in $L^\infty(S)$ and the $(\nu_x)_{x\in\Omega}$ are subprobability measures. With improved boundedness conditions on the $(z^j)_{j\geq 1}$ the convergence of the $(f(z^j))_{j\geq 1}$ is obtained for a larger class of functions f: for example, boundedness of $(z^j)_{j\geq 1}$ in L^∞ implies $f(z^j) \rightharpoonup < \boldsymbol{\nu}, f >$ in weakly* in L^∞ for any continuous f. Often however, the situation is that the generating sequence $(z^j)_{j\geq 1}$ is not bounded in L^∞ and the weak limit of the composition with a continuous, unbounded but of controlled growth function f is to be identified. In such a case it suffices to establish that the $(f(z^j))_{j\geq 1}$ are weakly sequentially precompact in L^1. When the domain is bounded, a general criterion is provided by de la Vallée Poussin: the $(f(z^j))_{j\geq 1}$ are weakly sequentially precompact in $L^1(E)$ for $E \subseteq \mathbb{R}^N$ bounded if and only if there exists $\psi : [0, +\infty) \mapsto \mathbb{R}$ with superlinear growth at infinity and such that

$$\sup_j \int_E \psi(f(z^j))\, dx < +\infty.$$

The following theorem of Ascerbi and Fusco in [AF] and Kinderlehrer and Pedregal in [KP1] also serves to characterize weak sequential precompactness in L^1 in a variational setting. It has an important application to minimization problems in variational calculus. The existence of (local) minimizers of a functional of the form

$$I(u) := \int_\Omega f(x, u, \nabla u)\, dx$$

over $W^{1,p}(\Omega; \mathbb{R}^M)$ is very closely related to the lower semicontinuity properties of I, which in turn are reflected in the quasiconvexity properties of f in the last argument. We recall the notion of quasiconvexity introduced by Morrey [Mo]: a Borel measurable function $f : \mathbf{M} \longrightarrow \mathbb{R}$ is *quasiconvex* if for all $A \in \mathbf{M}$,

$$f(A) \leq \frac{1}{\mathcal{L}^N(D)} \int_D f(A + \nabla\zeta)\, dx$$

for all $\zeta \in W^{1,p}(D; \mathbb{R}^M)$ and for all D open bounded sets in $\mathbb{R}^N$ with $\mathcal{L}^N(\partial D) = 0$. In general, convexity is a stronger condition than quasiconvexity but in the scalar case, that is when either $M = 1$ or $N = 1$, the two conditions are equivalent (cf. [D]).

Theorem 2.1 *Suppose $f \in \mathcal{E}^p(\mathbf{M})$, for some $1 \leq p \leq +\infty$, is quasiconvex and bounded below and let $u^k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^M)$. Then*

(i) For all measurable $E \subseteq \Omega$,

$$\int_E f(\nabla u)\, dx \leq \liminf_{k\to+\infty} \int_E f(\nabla u^k).$$

(ii) If in addition,

$$\int_\Omega f(\nabla u^k)\, dx \overset{k\to+\infty}{\longrightarrow} \int_\Omega f(\nabla u)\, dx$$

then the $(f(\nabla u^k))_{j>0}$ are weakly sequentially precompact in $L^1(\Omega)$ and a subsequence converges (weakly) to $f(\nabla u)$.

The proof can be found in [KP1]. Part *(ii)* is a consequence of *(i)* and it implies that if

$$f(\nabla u^{k_j}) \overset{j\to+\infty}{\rightharpoonup} f(\nabla u) \quad \text{in } L^1(\Omega)$$

then the $W^{1,p}$–gradient Young measure $\nu = (\nu_x)_{x\in\Omega}$ generated by $(\nabla u^{k_j})_{j>0}$ satisfies

$$< \nu, f > = f(\nabla u) \quad x \text{ a.e. in } \Omega.$$

The consequence of theorem 2.1 which we will have occasion to use directly in this paper occurs when a p-growth condition of the function f from below allows one to obtain information on the L^p norm of the gradients. This is described in the next result and provides a sufficient (but not necessary) condition for a sequence of functions in $W^{1,p}$ to generate a $W^{1,p}$-gradient Young measure.

Theorem 2.2 *Let f and $(u^k)_{k\geq 1}$ be as in theorem 2.1 and assume in addition that*

$$(c|A|^p - 1)^+ \leq \phi(A) \leq C|A|^p + 1$$

for $0 < c \leq C$. Let $\nu = (\nu_x)_{x\in\Omega}$ be generated by the gradients $(\nabla u^k)_{k\geq 1}$. Then ν is a $W^{1,p}$-gradient Young measure.

The proof can be found in [KP1].

2.2 The variational treatment and the existence of a Young measure solution

Assumptions. We define the two separable Banach spaces

$$\mathcal{E}_0 \equiv \mathcal{E}_0^2(\mathbf{R}^N) := \left\{ \psi \in C(\mathbf{R}^N) : \lim_{|A|\to+\infty} \frac{|\psi(A)|}{1+|A|^2} \text{ exists} \right\}$$

and

$$\mathcal{F}_0 \equiv \mathcal{E}_0^1(\mathbf{R}^N; \mathbf{R}^N) := \left\{ \psi \in C(\mathbf{R}^N; \mathbf{R}^N) : \lim_{|A|\to+\infty} \frac{|\psi(A)|}{1+|A|} \text{ exists} \right\}.$$

We assume the heat flux satisfies $q = \nabla\phi$ on $\mathbf{R}^N$ with $\phi \in C^1(\mathbf{R}^N)$. We impose the growth conditions $\phi \in \mathcal{E}_0$, $q \in \mathcal{F}_0$ and furthermore,

$$(c|a|^2 - 1)^+ \leq \phi(a) \leq C|a|^2 + 1 \quad \forall a \in \mathbf{R}^N \tag{5}$$

$$|q(a)| \leq C|a| \quad \forall a \in \mathbf{R}^N. \tag{6}$$

We let ϕ^{**} denote the *convexification* of ϕ, that is,

$$\phi^{**} = \sup\{f(x) : f \le \phi, \ f \text{ convex}\}.$$

Since ϕ is in $C^1(\mathbb{R}^N)$ so is ϕ^{**} and we set

$$p := \nabla \phi^{**}.$$

We note that $q = p$ on the set $\{\phi = \phi^{**}\}$ and that ϕ^{**} and p satisfy the same growth conditions as ϕ and q respectively. We assume

$$\phi^{**}(0) = 0 \quad \text{and} \quad p(0) = 0$$

which by the convexity of ϕ^{**} implies

$$p(\lambda) \cdot \lambda \ge 0 \quad \forall \lambda \in \mathbb{R}^N.$$

Under these hypotheses we fix ideas by agreeing on the following:

Definition. A measure solution to $\mathcal{P}$,

$$\begin{aligned} u_t &= \nabla \cdot q(\nabla u) \quad \text{in } Q_\infty := \Omega \times \mathbb{R}^+ \\ u(x,0) &= u_0(x) \quad \text{for } x \text{ in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+ \end{aligned}$$

with $u_0 \in H_0^1(\Omega)$ a given function, is a pair (u, ν) where $u \in H^1_{loc}(Q_\infty) \cap L^\infty(\mathbb{R}^+; H_0^1(\Omega))$ and $\nu = (\nu_{x,t})_{(x,t) \in \dot{Q}_\infty}$ a parametrized family of probability measures on $\mathbb{R}^N$ such that the equilibrium equation

$$(7) \qquad \int_0^{+\infty} \int_\Omega (< \nu, q > \cdot \nabla \zeta + \frac{\partial u}{\partial t} \zeta) \, dx \, dt = 0 \quad \forall \zeta \in H_0^1(Q_\infty)$$

(where $< \nu_{x,t}, q > = \int_{\mathbb{R}^N} q(\lambda) \nu_{x,t}(d\lambda)$) and

$$(8) \qquad < \nu_{x,t}, id > = \nabla u(x,t) \quad (x,t) \text{ a.e. in } Q_\infty$$

hold. Equivalently stated, equation (7) is

$$(9) \qquad u_t = \nabla \cdot < \nu, q > \quad \text{in } H^{-1}(Q_\infty).$$

If in addition ν is an $H^1_{loc}(Q_\infty)$-gradient Young measure then the pair (u,ν) is called a **Young measure** solution to $\mathcal{P}$. In the above, $\Omega \subset \mathbf{R}^N$ is an open, bounded set, $\partial\Omega \times \mathbf{R}^+$ is the lateral boundary of the parabolic cylinder $\Omega \times \mathbf{R}^+$ and id stands for the identity function. The notation H^1 stands for the Sobolev space $W^{1,2}$ and H^{-1} for its dual $W^{-1,2}$. We will say that (u,ν) is a *solution* or *solves* to indicate that the pair is a Young measure solution of $\mathcal{P}$. The function u is also called a solution and the use of the term is clear from the context.

The zero boundary data can be replaced by $g \in H^1(\Omega)$ (and the solution u is then sought in $g + H^1_0(\Omega)$) without any changes in the results that follow. For convenience in the sequel we suppose $g = 0$. A generalization to time-dependent data is not immediate.

Remarks. 1. It results from the existence proof below that a Young measure solution to $\mathcal{P}$ exists such that $u_t \in L^2(Q_\infty)$ and that (7) is satisfied also locally in time, that is, for test functions $\zeta \in H^1_{loc}(Q_\infty)$ with t-slices $\zeta(\cdot,t) \in H^1_0(\Omega)$ for $t > 0$ a.e. and in particular for $\zeta \in H^1_0(Q_T)$ $\forall T > 0$. This means that in addition to (9) it is true that

$$u_t = \nabla\cdot < \nu, q > \qquad \text{in } H^{-1}_{loc}(Q_\infty).$$

In particular, the solution u is an admissible test function.

2. It is a consequence of the above definition that an equilibrium equation is also satisfied pointwise in time in $H^{-1}(\Omega)$: indeed, for t a.e. in $[0,T]$ and for all $\zeta \in H^1_0(\Omega)$ we have

$$\int_0^T\int_\Omega < \nu_{x,t}, q > \cdot\nabla\zeta(x)\,dx\,dt = -\int_0^T\int_\Omega \frac{d}{dt}u(x,t)\zeta(x)\,dx\,dt;$$

differentiating in time we obtain

$$\int_\Omega < \nu_{x,t}, q > \cdot\nabla\zeta(x)\,dx = -\int_\Omega u_t(x,t)\zeta(x)\,dx \qquad t \text{ a.e. in } \mathbf{R}^+,\ \forall\zeta \in H^1_0(\Omega).$$

3. A classical solution to $\mathcal{P}$ which is bounded in time is a measure solution with $\nu = \delta_{\nabla u}$.

We establish existence:

Theorem 2.3 (Existence) *Under the assumptions stated above there exists a Young measure solution (u,ν) to $\mathcal{P}$. In addition, $u_t \in L^2(Q_\infty)$,*

$$supp\ \nu_{x,t} \subset \{a \in \mathbf{R}^N : \phi(a) = \phi^{**}(a)\} \qquad (x,t)\ a.e.\ in\ Q_\infty$$

and (u, ν) is also a Young measure solution of the relaxed problem

$$u_t = \nabla \cdot p(\nabla u) \qquad \textit{in } H^{-1}(Q_\infty)$$

with the same initial-boundary data.

Proof: The following existence proof is due to Kinderlehrer and Pedregal [KP1].

Step 1. Let $h > 0$ be fixed and for each $j \geq 0$ consider the functionals

$$\Phi_h(v; u^{h,j-1}) := \int_\Omega \phi(\nabla v) + \frac{1}{2h}(v - u^{h,j-1})^2 \, dx \qquad \text{for } v \in H_0^1(\Omega)$$

and

$$\Phi_h^{**}(v; u^{h,j-1}) := \int_\Omega \phi^{**}(\nabla v) + \frac{1}{2h}(v - u^{h,j-1})^2 \, dx \qquad \text{for } v \in H_0^1(\Omega).$$

We drop the explicit dependence on h. By relaxation,

$$I := \inf\left\{\Phi(v; u^{h,j-1}) : v \in H_0^1(\Omega)\right\} = \inf\left\{\Phi^{**}(v; u^{h,j-1}) : v \in H_0^1(\Omega)\right\}.$$

Let $(u^{h,j,k})_{k\geq 1} \subset H_0^1(\Omega)$ be a minimizing sequence for Φ (and Φ^{**}). By the growth condition (5) and the Rellich theorem, together with the H^1-weak sequential lower semicontinuity of Φ^{**}, there exist $u^{h,j} \in H_0^1(\Omega)$ and a subsequence,[1] not relabeled, such that

$$u^{h,j,k} \overset{k\to\infty}{\rightharpoonup} u^{h,j} \qquad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega),$$

$$I = \Phi^{**}(u^{h,j}; u^{h,j-1})$$

and therefore,

$$\text{(10)} \qquad \int_\Omega \phi^{**}(\nabla u^{h,j}) \, dx = \lim_{k\to+\infty} \int_\Omega \phi^{**}(\nabla u^{h,j,k}) \, dx = \lim_{k\to+\infty} \int_\Omega \phi(\nabla u^{h,j,}) \, dx.$$

Then by theorem 2.1

$$\text{(11)} \qquad \phi^{**}(\nabla u^{h,j,k}) \overset{k\to\infty}{\rightharpoonup} \phi^{**}(\nabla u^{h,j}) \qquad \text{in } L^1(\Omega).$$

Let $\nu^{h,j} = (\nu_x^{h,j})_{x\in\Omega}$ be the Young measure generated by the $(\nabla u^{h,j,k})_{k\geq 1}$. By theorem 2.2 $\nu^{h,j}$ is an H^1-gradient Young measure and by the representation formula (4) for L^1-weak limits and (10) and (11) we obtain

$$\int_\Omega \phi^{**}(\nabla u^{h,j}) \, dx = \int_\Omega < \nu, \phi^{**} > dx = \int_\Omega < \nu, \phi > dx$$

[1] In fact, the whole sequence converges (weakly) as the minimizer $u^{h,j}$ is unique (see remark 3 following theorem 2.8)

which together with $\phi^{**} \leq \phi$ implies

(12) $$supp\, \nu^{h,j} \subseteq \{\phi = \phi^{**}\}$$

and therefore,

(13) $$< \nu^{h,j}, \phi > = < \nu^{h,j}, \phi^{**} > = \phi^{**}(\nabla u^{h,j}) \quad x \text{ a.e. in } \Omega,$$

(14) $$\nabla u^{h,j} = < \nu^{h,j}, id > \quad x \text{ a.e. in } \Omega.$$

In addition,

(15) $$< \nu^{h,j}, q > = < \nu^{h,j}, p > \quad x \text{ a.e. in } \Omega,$$

(16) $$\nabla \cdot < \nu^{h,j}, q > = \nabla \cdot < \nu^{h,j}, p > = \nabla \cdot p(\nabla u^{h,j}) \quad \text{in } H^{-1}(\Omega)$$

hold.

Setting the Gâteaux derivative of Φ^{**} to zero at the minimizer $u^{h,j}$ we obtain the equilibrium equation

(17) $$\int_\Omega p(\nabla u^{h,j}) \cdot \nabla \zeta + \frac{1}{h}(u^{h,j} - u^{h,j-1})\zeta \, dx = 0 \quad \forall \zeta \in H_0^1(\Omega).$$

Let $I^{h,j} = [hj, h(j+1))$ and $\chi^{h,j}$ be the indicator function of $I^{h,j}$ and for $t > 0$ set

$$\lambda^{h,j}(t) = \begin{cases} \frac{t}{h} - j & \text{if } hj \leq t < h(j+1) \\ 0 & \text{otherwise.} \end{cases}$$

Define for x a.e. in Ω each $t \in \mathbb{R}^+$,

(18) $$u^h(x,t) := \sum_j \chi^{h,j}(t) \left\{ u^{h,j}(x) + \lambda^{h,j}(t)(u^{h,j+1}(x) - u^{h,j}(x)) \right\}$$

so that $u^h \in L^\infty(\mathbb{R}^+; H_0^1(\Omega))$ and also

$$u^h = 0 \text{ on } \partial\Omega \times \mathbb{R}^+.$$

We let

(19) $$w^h := \sum_j \chi^{h,j} u^{h,j} \quad \in L^\infty(\mathbb{R}^+; H_0^1(\Omega))$$

(20) $$\nu^h = (\nu^h_{x,t})_{(x,t) \in Q_\infty} := \sum_j \chi^{h,j} \nu^{h,j} \quad \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{E}_0')$$

probability measures on $\mathbb{R}^N$, (where $\mathcal{E}_0'$ is the dual space of $\mathcal{E}_0$). By (14) we know

$$\nabla w^h = < \nu^h, id > \qquad (x,t) \text{ a.e. in } Q_\infty. \tag{21}$$

We also let

$$\bar{q}^h := < \nu^h, q > = \sum_j \chi^{h,j} < \nu^{h,j}, q > \quad \in L^2(Q_\infty).$$

Differentiating (18) almost everywhere in time we have

$$\frac{\partial u^h}{\partial t} = \sum_j \chi^{h,j} \frac{1}{h}(u^{h,j+1} - u^{h,j}) \quad \in L^2(Q_\infty).$$

Then for each $t > 0$,

$$\frac{\partial u^h}{\partial t} = \nabla \cdot < \nu^h, q > \qquad \text{in } H^{-1}(\Omega)$$

or, equivalently, the equilibrium equation

$$\int_\Omega (\bar{q}^h \cdot \nabla \zeta + \frac{\partial u^h}{\partial t}\zeta)\, dx = 0 \quad \forall \zeta \in H_0^1(\Omega) \tag{22}$$

holds. From (22) it is easy to deduce that

$$\int_0^\tau \int_\Omega < \nu^h, q > \cdot \nabla \zeta + \frac{\partial u^h}{\partial t}\zeta \, dx\, dt = 0 \quad \forall \zeta \in H^1(Q_\tau)\, \forall \tau \in [0, +\infty] \tag{23}$$

(with $\zeta(\cdot, t) \in H_0^1(\Omega)$), that is

$$\frac{\partial u^h}{\partial t} = \nabla \cdot < \nu^h, q > \qquad \text{in } H^{-1}(Q_\tau)\, \forall \tau \in [0, +\infty].$$

By (15), (16) and (19),

$$< \nu^h, q > = < \nu^h, p > \qquad \text{for each } t > 0 \text{ and } x \text{ a.e. in } \Omega, \tag{24}$$

and

$$\nabla \cdot < \nu^h, q > = \nabla \cdot < \nu^h, p > = \nabla \cdot p(\nabla w^h) \tag{25}$$

both in $H^{-1}(\Omega)$ for each $t \geq 0$ and in $H^{-1}(Q_\tau)\, \forall \tau \in [0, +\infty]$.

Step 2. Using the growth conditions (5) and (6) on ϕ and q we obtain uniform estimates in h for the $(u^h)_{h>0}$ and $(w^h)_{h>0}$ in L^∞ $(\mathbb{R}^+;\, H_0^1(\Omega))$ and $(\bar{q}^h)_{h>0}$ in $L^2(Q_\infty)$. Further we obtain $(\frac{\partial u^h}{\partial t})_{h>0} \in L^2(Q_\infty)$ is bounded in h and the $H^1_{loc}(Q_\infty)$-gradient Young measures (ν^h) are bounded in $\mathrm{L}^\infty(\mathbb{R}^+; \mathcal{E}_0')$. Using weak compactness we may therefore extract weakly convergent subsequences indexed by $h' \to 0$ and a pair (u, ν) satisfying (7) and (8). Indeed, the two sequences $(u^{h'})_{h'>0}$ and $(w^{h'})_{h'>0}$ have the same $L^\infty(\mathbb{R}^+; H_0^1(\Omega))$ weak∗ limit[2] u satisfying $\frac{\partial u}{\partial t} \in L^2(Q_\infty)$; along h' (23) yields the equilibrium equation

$$(26) \qquad \int_0^T \int_\Omega (< \nu, q > \cdot \nabla \zeta + \frac{\partial u}{\partial t} \zeta)\, dx\, dt = 0 \quad \forall \zeta \in H_0^1(Q_T)\ \forall T \in [0, +\infty]$$

(and as in (23) all $\zeta \in H^1_{loc}(Q_\infty)$ with $\zeta(\cdot, t) \in H_0^1(\Omega)$ for almost all $t > 0$ are admissible). Further, by (24)

$$(27) \qquad < \nu, q > = < \nu, p > \qquad (x,t) \text{ a.e. in } Q_T\ \forall T \in [0, +\infty]$$

In addition, by (12) we have

$$(28) \qquad supp\, \nu \subseteq \{\phi = \phi^{**}\}.$$

The measure ν is extracted in [KP1] as the weak limit of the sequence of Young measures $(\nu^h)_{h>0}$ in $L^\infty(\mathbb{R}^+; \mathcal{E}_0')$. To complete the existence proof we must establish that ν is in fact an $H^1_{loc}(Q_\infty)$-gradient Young measure related to u via

$$< \nu_{x,t}, id > = \nabla u(x,t) \quad (x,t) \text{ a.e. in } Q_\infty.$$

This question falls in the general setting of the results in the following paragraph and is directly addressed in corollary 2.5. □

2.3 Some properties of sequences of gradient Young measures and completion of the existence proof

Given a sequence of Young measures we may often want to extract a (weakly) convergent subsequence using duality and at the same time ensuring that the limiting measure is itself a Young measure. The following lemma describes a situation in which this is true.

Lemma 2.4 *Suppose that $(\nu^\alpha)_{\alpha \geq 0}$ with $\nu^\alpha = (\nu^\alpha_{x,t})_{(x,t) \in Q_\infty}$ is a sequence of $H^1_{loc}(Q_\infty)$-gradient Young measures and for each α is generated by $(\nabla v^{\alpha,m})_{m \geq 0}$ where $(v^{\alpha,m})_{m \geq 0}$ is a sequence in $H^1_{loc}(Q_\infty)$ uniformly bounded in α and m. Then a subsequence (not*

[2] Weak∗ convergence in $L^\infty(\mathbb{R}^+; H_0^1(\Omega))$ is to imply that the sequence and the sequences of the N spacial partial derivatives converge weakly in $L^2(\mathbb{R}^+; L^2(\Omega))$.

relabeled) of the $(\nu^\alpha)_{\alpha>0}$ *and an* $H^1_{loc}(Q_\infty)$*-gradient Young measure* $\nu = (\nu_{x,t})_{(x,t)\in Q_\infty}$ *exist such that*

$$\nu^\alpha \overset{\alpha\to 0}{\Longrightarrow} \nu$$

weakly in $L^1(Q_T;\, \mathcal{E}_0')$*, weakly in* $L^2(Q_T;\, \mathcal{F}_0')$ *and weakly∗ in* $L^\infty\left(Q_T;\, M(\mathbb{R}^N)\right)$ *for each* $T \geq 0$.

That is,

$$< \nu^\alpha, \psi > \overset{\alpha\to 0}{\Longrightarrow} < \nu, \psi >$$

weakly in $L^1(Q_T)$ *for* $\psi \in \mathcal{E}_0$*, weakly in* $L^2(Q_T)$ *for* $\psi \in \mathcal{F}_0$ *and weakly∗ in* $L^\infty(Q_T)$ *for* $\psi \in C_0(\mathbb{R}^N)$.

Remarks. **1.** Recall that the assumption on the $(\nu^\alpha)_{\alpha\geq 0}$ implies that the representation formula

$$\int_0^T \int_\Omega \psi(\nabla v^{\alpha,m})(x,t)\theta(x,t)\, dx\, dt \overset{m\to\infty}{\Longrightarrow} \int_0^T \int_\Omega \int_{\mathbf{M}} \psi(A)\, d\nu^\alpha_{x,t}(A)\, \theta(x,t)\, dx\, dt$$

holds for all $\psi \in \mathcal{E}_0$, $\theta \in \mathrm{L}^1(Q_T)$, and for each $\alpha \geq 0$ (not necessarily uniformly in α). This in turn implies that the representation formula also holds for $\psi \in \mathcal{F}$ or $C_0(\mathbb{R}^N)$ weakly in L^2 or weakly∗ in L^∞ respectively. A converse of this statement is given by corollary 2.6.

2. Assume that a sequence of Young measures is bounded in $L^1_{loc}(Q_\infty;\, \mathcal{E}_0')$. Duality cannot be used here to ensure a limit point. However, we are able to reduce to the case of lemma 2.4 as follows:

Suppose $(\nu^\alpha)_{\alpha>0}$*, with* $\nu^\alpha = (\nu^\alpha_{x,t})_{(x,t)\in Q_\infty}$*, is a sequence of* $H^1_{loc}(Q_\infty)$*-gradient Young measures bounded in* $L^1_{loc}(Q_\infty; \mathcal{E}_0')$ *and for each* α *let* $(\nabla v^{\alpha,k})_{k>0}$ *be the generating gradients. Then* $(\nu^\alpha)_{\alpha>0}$ *is bounded in* $L^2_{loc}(Q_\infty; \mathcal{F}_0')$ *and* $L^\infty_{loc}(Q_\infty; M(\mathbb{R}^N))$ *and the sequence* $(v^{\alpha,k})_{\alpha,k}$ *is bounded in* $H^1_{loc}(Q_\infty)$ *uniformly in* α *and* k*. Hence lemma 2.4 applies.*

The proof of this remark is straightforward and is omitted.

Proof of lemma 2.4: *Step 1.* (Extract the subsequence of the measures) Fix $T > 0$. It is straightforward to see that $(\nu^\alpha)_{\alpha\geq 0}$ is bounded in the spaces $L^2(Q_T; \mathcal{F}_0')$ and $L^\infty(Q_T; M(\mathbb{R}^N))$ which are isomorphic to the dual spaces of $L^2(Q_T;\, \mathcal{F}_0)$ and $L^\infty(Q_T;\, C_0(\mathbb{R}^N))$

respectively. Using this we can extract a subsequence (not relabeled) $(\nu^\alpha)_{\alpha\geq 0}$ and a parametrized probability measure $\nu = (\nu_{x,t})_{(x,t)\in Q_\infty}$ such that

$$\nu^\alpha \overset{\alpha\to 0}{\rightharpoonup} \nu \qquad \text{weakly in } L^2(Q_T;\, \mathcal{F}_0') \text{ and weakly}* \text{ in } L^\infty(Q_T;\, M(\mathbb{R}^N)).$$

We show now that the convergence remains valid if we allow ψ to have higher growth provided we compensate by having uniformly bounded test functions.

Claim: $< \nu^\alpha, \psi > \overset{\alpha\to 0}{\rightharpoonup} < \nu, \psi >$ weakly in $L^1(Q_T)$ for all $\psi \in \mathcal{E}_0$.

Proof of Claim: We use the same cut-off functions used in Ball [Ba] and Slemrod [Sl]. Set

$$\eta^k(\lambda) := \begin{cases} 1 & \text{if } |\lambda| \leq k-1 \\ k - |\lambda| & \text{if } k-1 \leq |\lambda| \leq k \\ 0 & \text{if } |\lambda| \geq k. \end{cases}$$

Fix $T > 0$, $\psi \in \mathcal{E}_0$ and let $\theta \in L^\infty(Q_T)$. Define

$$\psi^k(\lambda) := \psi(\lambda)\eta^k(\lambda), \qquad \psi^k \in C_0(\mathbb{R}^N).$$

$$\begin{aligned}
&\left| \int_0^T \int_\Omega < \nu^\alpha_{x,t}, \psi > \theta(x,t)\, dx\, dt \; - \; \int_0^T \int_\Omega < \nu_{x,t}, \psi > \theta(x,t)\, dx\, dt \right| \\
&\leq \quad \|\theta\|_{L^\infty(Q_T)} \int_0^T \int_\Omega \left| < \nu^\alpha,\, \psi - \psi^k > \right| dx\, dt \\
&+ \quad \left| \int_0^T \int_\Omega \left(< \nu^\alpha,\, \psi^\alpha > - < \nu,\, \psi^k > \right) dx\, dt \right| \\
&+ \quad \|\theta\|_{L^\infty(Q_T)} \int_0^T \int_\Omega \left| < \nu,\, \psi - \psi^k > \right| dx \\
&= \quad I + II + III
\end{aligned}$$

Fix $\epsilon > 0$. It is a consequence of the Dunford-Pettis theorem that

$$\begin{aligned}
I \quad &= \quad \lim_{m\to+\infty} \|\theta\|_{L^\infty(Q_T)} \int_0^T \int_\Omega |\psi - \psi^k|(\nabla v^{\alpha,m})\, dx\, dt \\
&\leq \quad c \lim_{m\to+\infty} \int_{\{(x,t):|\nabla v^{\alpha,m}|\geq k\}} |\psi|(\nabla v^{\alpha,m})\, dx\, dt \\
&\leq \quad c \sup_{\alpha,m} \int_{\{(x,t):|\nabla v^{\alpha,m}|\geq k\}} \left(1 + c'|\nabla v^{\alpha,m}|^2\right) dx\, dt
\end{aligned}$$

$$\longrightarrow 0$$

as $k \to +\infty$ uniformly in α, because $\|\nabla v^{\alpha,m}\|_{L^2(Q_T)}$ is bounded independently of α, m and hence

$$meas\{(x,t) : \|\nabla v^{\alpha,m}\| \geq k\} \longrightarrow 0$$

as $k \to +\infty$, uniformly in α, m.

For each k, $\psi^k \in C_0(\mathbb{R}^N)$ and so $\exists \delta(k,\epsilon)$ such that $II < \epsilon \ \ \forall |\alpha| < \delta(k,\epsilon)$ (but not necessarily uniformly in k).

For III, assume $\psi \geq 0$; Then $0 \leq \psi^k \nearrow \psi$ pointwise and so $< \nu_{x,t}, \psi - \psi^k > \to 0$ as $k \to +\infty$. (For general ψ, write $\psi = \psi^+ - \psi^-$ and $(\psi^+)^k = \psi^+ \eta^k$, $(\psi^-)^k = \psi^- \eta^k$ and use the monotone convergence of each term). So $\exists K(\epsilon)$ such that $III < \epsilon \ \ \forall k \geq K(\epsilon)$.

We choose k for I and III which is independent of α; using this k we then find $\delta(\epsilon, k)$ for II. This shows that the sequence of ν^α converges in $L^\infty(Q_T; M(\mathbb{R}^N))$ to ν and proves the claim.

Step 2. We now show that the limit point ν is an $H^1_{loc}(Q_\infty)$-gradient Young measure. The idea is to find a sequence of gradients for which the representation formula holds for all functions in a dense set of $\mathcal{E}_0$ and show that the same sequence works for all ψ in $\mathcal{E}_0$. (It is obvious that for this argument one must work with the separable space $\mathcal{E}_0$ rather than the inseparable space $\mathcal{E}$).

Fix $T > 0$. Let $(\phi_n)_{n \geq 1}$ be dense in $\mathcal{E}_0$. For each $n \geq 1$ we have by Step 1,

$$\phi_n(\nabla v^{\alpha,m}) \quad \stackrel{m \to +\infty}{\rightharpoonup} \quad < \nu^\alpha, \phi_n > \qquad \text{in } L^1(Q_T)$$

and also,

$$< \nu^\alpha, \phi_n > \quad \stackrel{\alpha \to 0}{\rightharpoonup} \quad < \nu, \phi_n > \qquad \text{in } L^1(Q_T).$$

Therefore a diagonal subsequence indexed by $\mu(n)$ exists such that

$$\phi_n(\nabla v^{\mu(n)}) \quad \stackrel{\mu(n) \to +\infty}{\rightharpoonup} \quad < \nu, \phi^n > \qquad \text{in } L^1(Q_T).$$

This way we obtain the sequences $(\nabla v^{\mu(n)})_{\mu(n) \geq 1}$ for each n which we Cantor-diagonalize to obtain a single sequence $(\nabla v^\mu)_{\mu \geq 1}$ such that the representation formula holds for each ϕ_n, i.e.

$$\phi_n(\nabla v^\mu) \quad \stackrel{\mu \to +\infty}{\rightharpoonup} \quad < \nu, \phi^n > \qquad \text{in } L^1(Q_T) \text{ for all } n.$$

Now using density we show that the sequence of gradients just obtained is a generating sequence for the parametrized measure ν obtained in Step 1. For this, let $\phi \in \mathcal{E}_0$ and $\epsilon > 0$ be given. Find $N(\epsilon)$ such that $\|\phi - \phi_n\|_{\mathcal{E}_0} < \epsilon \; \forall n \geq N(\epsilon)$. Let $\theta \in L^\infty(Q_T)$.

$$\left| \int_0^T \int_\Omega \theta(x,t)\phi(\nabla v^\mu)(x,t)\, dx\, dt \; - \; \int_0^T \int_\Omega \theta(x,t) < \nu_{x,t}, \phi > \, dx\, dt \right|$$

$$\begin{aligned}
&\leq \; \| \theta \|_{L^\infty(\Omega)} \int_0^T \int_\Omega |\phi(\nabla v^\mu) - \phi_n(\nabla v^\mu)|\, dx\, dt \\
&+ \; \| \theta \|_{L^\infty(\Omega)} \int_0^T \int_\Omega |\phi(\nabla v^\mu) - < \nu, \phi_n > |\, dx\, dt \\
&+ \; \| \theta \|_{L^\infty(\Omega)} \int_0^T \int_\Omega | < \nu, \phi_n - \phi > |\, dx\, dt \\
&= \; I + II + III.
\end{aligned}$$

For each term we have,

$$I \;\leq\; c\, \|\phi - \phi_n\|_{\mathcal{E}_0} \int_0^T \int_\Omega (1 + \|\nabla v^\mu\|^2)_{L^2(\Omega)}\, dx\, dt \;\leq\; c\, \epsilon \, \forall n \geq N(\epsilon), \text{ uniformly in } \mu$$

$$II \;\leq\; c\, \epsilon \qquad \forall \mu \geq M(\epsilon, n)$$

$$III \;\leq\; c\, \|\phi - \phi_n\|_{\mathcal{E}_0} \int_0^T \int_\Omega < \nu, 1 + c'|id|^2 > \, dx\, dt \;\leq\; c\, \epsilon.$$

Thus we may choose n for I and III which is independent of μ and for this n we find M for II.

We conclude that

$$\phi(\nabla v^\mu) \quad \xrightarrow{\mu \to +\infty} \quad < \nu_{x,t}, \phi > \qquad \text{in } L^1(Q_T) \quad \forall \phi \in \mathcal{E}_0$$

and by remark 1 to the lemma 2.4 this finishes the proof. □

Next we apply lemma 2.4 to conclude the existence proof:

Corollary 2.5 *The measure ν obtained in the proof of the existence theorem 2.3 is an $H^1_{loc}(Q_\infty)$-gradient Young measure (and so the pair (u, ν) is indeed a Young measure solution of $\mathcal{P}$).*

Proof: In the notation of theorem 2.3 recall that

$$\nu^h_{x,t} := \sum_{j\geq 0} \chi_{[hj,h(j+1))}(t)\nu^{h,j}_x \qquad \text{for } x \text{ a.e. in } \Omega \text{ and } \forall t \geq 0.$$

Then for each $h > 0$ the sequence $(\nabla u^{h,k})_{k\geq 0}$, where

$$u^{h,k} := \sum_{j\geq 0} \chi_{[hj,h(j+1))}(t)u^{h,k}(x) \qquad \in H^1_{loc}(Q_\infty),$$

generates ν^h. We apply lemma 2.4 on ν^h to extract a subsequence indexed by $h' \to 0$ along which the ν^h converge to an $H^1_{loc}(Q_\infty)$-gradient Young measure $\nu = (\nu_{x,t})_{(x,t)\in Q_\infty}$ in the sense of the conclusion of the lemma. In particular,

$$< \nu^{h'}, id > \rightharpoonup < \nu, id > \qquad \text{in } L^2(Q_T)$$

(because $id \in \mathcal{F}_0$), and (21) then gives

$$< \nu_{x,t}, id > = \nabla u(x,t) \qquad (x,t) \text{ a.e. in } Q_\infty$$

since $\nabla w^{h'} \rightharpoonup \nabla u$ in $L^2(\Omega)$. □

The technique used in the claim of Step 1 above can be modified to prove a slightly more general statement and a partial converse to remark 1 following lemma 2.4. This is summarized in the following corollary.

Corollary 2.6 *Suppose that* $(z^k)_{k>0}$ *a bounded sequence in* $L^2(Q_T)$ *such that* $\psi(z^k)$ *converges weakly in* $L^2(Q_T)$ *for all* $\psi \in \mathcal{F}_0$ or *weakly* in* $L^\infty(Q_T)$ *for all* $\psi \in C_0(\mathbb{R}^N)$*; then the sequence converges also weakly in* $L^1(Q_T)$ *for* $\psi \in \mathcal{E}_0$.

To prove this simply consider the sequence $(\psi(z^k))_{k>0}$ in place of $< \nu^\alpha, \psi)_{\alpha>0}$ and show it is Cauchy weakly in $L^1(Q_T)$ by using the truncation of Step 1 above to pass from linear to quadratic growth.

2.4 Uniqueness and properties of the Young measure solution

The following lemma describes a property of the solution upon which the uniqueness proof relies.

Lemma 2.7 (Independence) *For* (u, ν) *a solution of* (1) *and* (2) *the equality*

$$(29) \qquad < \nu_{x,t}, q \cdot id > = < \nu_{x,t}, q > \cdot < \nu_{x,t}, id > \qquad (x,t) \text{ a.e. in } Q_\infty$$

holds, i.e. q *and* ∇u *are independent with respect to the Young measure* $\boldsymbol{\nu}$.

Proof: *Step 1.* (The time-discretized case) Fix $h > 0$.

Claim: $< \nu_x^{h,j}, q \cdot id > = < \nu_x^{h,j}, q > \cdot < \nu_x^{h,j}, id >$ $\quad x$ a.e. in Ω for $j = 0, 1, \dots$.

Proof of Claim: Let $(u^{h,j,k})_{k=1}^\infty$ be the minimizing sequence to the variational principle $\Phi^{**}(v; u^{h,j-1})$ converging to $u^{h,j}$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Recall that $(\nabla u^{h,j,k})_{k=1}^\infty$ generates $\nu^{h,j}$. For all $\zeta \in H_0^1(\Omega)$ we have,

$$\begin{aligned}
\int_\Omega p(\nabla u^{h,j,k}) \cdot \nabla\zeta + \frac{u^{h,j,k} - u^{h,j-1}}{h}\zeta\, dx \quad &\overset{k\to\infty}{\longrightarrow} \quad \int_\Omega < \nu_x^{h,j}, p > \cdot \nabla\zeta + \frac{u^{h,j} - u^{h,j-1}}{h}\zeta\, dx \\
&= \int_\Omega p(\nabla u^{h,j}) \cdot \nabla\zeta + \frac{u^{h,j} - u^{h,j-1}}{h}\zeta\, dx \\
&= 0
\end{aligned}$$

because $p(\nabla u^{h,j,k}) \overset{k\to\infty}{\rightharpoonup} < \nu_x^{h,j}, p >$ in $L^2(\Omega)$ and $\nabla\cdot < \nu_x^{h,j}, p > = \nabla \cdot p(\nabla u^{h,j})$ in $H^{-1}(\Omega)$. It follows easily that

(*i*) $\quad \nabla \cdot p(\nabla u^{h,j,k}) \overset{k\to\infty}{\longrightarrow} \nabla \cdot p(\nabla u^{h,j}) \qquad$ in $H^{-1}(\Omega)$

by the estimate

$$\begin{aligned}
\| \nabla \cdot p(\nabla u^{h,j,k}) - \nabla \cdot p(\nabla u^{h,j}) \|_{H^{-1}(\Omega)} \quad &= \quad \sup_{\|\zeta\|_{H_0^1(\Omega)}=1} \left| \int_\Omega \left[p(\nabla u^{h,j,k}) - p(\nabla u^{h,j}) \right] \cdot \nabla\zeta\, dx \right| \\
\text{(for all sufficiently large } k) \quad &\le \quad \sup_{\|\zeta\|_{H_0^1(\Omega)}=1} \left| \int_\Omega \frac{u^{h,j,k} - u^{h,j}}{h}\zeta\, dx \right| + \epsilon \\
&\le \quad \frac{1}{h} \| u^{h,j,k} - u^{h,j} \|_{L^2(\Omega)} + \epsilon \\
&\overset{k\to\infty}{\longrightarrow} \quad \epsilon \qquad \text{for any } \epsilon > 0
\end{aligned}$$

since $u^{h,j,k} \overset{k\to\infty}{\longrightarrow} u^{h,j}$ in $L^2(\Omega)$ strongly. Recalling remark 1 to lemma 2.4 and noting that p, $id \in \mathcal{F}_0$ and $p \cdot id \in \mathcal{E}_0$, we have as $k \to \infty$,

(*ii*) $\quad p(\nabla u^{h,j,k}) \cdot \nabla u^{h,j,k} \rightharpoonup < \nu_x^{h,j}, p > \qquad$ in $L^1(\Omega)$

$(iii)\quad p(\nabla u^{h,j,k}) \rightharpoonup < \nu_x^{h,j}, p > \qquad \text{in } L^2(\Omega)$

$(iv)\quad \nabla u^{h,j,k} \rightharpoonup < \nu_x^{h,j}, id > \qquad \text{in } L^2(\Omega).$

Now by the div-curl lemma (see [E], [T] or [Mu]), or by direct computation and using the H^1-strong convergence in (i), we have from $(i), (iii), (iv)$

$$p(\nabla u^{h,j,k}) \cdot \nabla u^{h,j,k} \overset{k \to +\infty}{\rightharpoonup} < \nu_x^{h,j}, p > \cdot < \nu_x^{h,j}, id >$$

in the sense of distributions; by (ii) above and recalling (15) we have the claim.

Step 2. (Passing to the limit).
By (20) and step 1 we have

$$< \nu^h, q \cdot id > = < \nu^h, q > \cdot < \nu^h, id > \qquad x \text{ a.e. in } \Omega\ \forall t \geq 0$$

We may apply lemma 2.4 to $(\nu^h)_{h>0}$ to pass to a limit point as $h \to 0$. We obtain a subsequence, not relabeled, such that for each $T \geq 0$,

$(v)\quad < \nu^h, q \cdot id > \rightharpoonup < \nu, q \cdot id > \qquad \text{in } L^1(Q_T)$

$(vi)\quad < \nu^h, q > \rightharpoonup < \nu, q > \qquad \text{in } L^2(Q_T)$

$(vii)\quad < \nu^h, id > \rightharpoonup < \nu, id > \qquad \text{in } L^2(Q_T).$

Using the div-curl lemma as in step 1 we obtain (29). □

Theorem 2.8 (Uniqueness and Continuity with respect to initial data) *There is a unique function $u : Q_\infty \longrightarrow \mathbf{R}$ with $u \in H^1_{loc}(Q_\infty)$ with $u(\cdot, 0) = u_0$ for which there exists a parametrized probability measure $\nu = (\nu_{x,t})_{(x,t)\in Q_\infty}$ so that* (7), (8), (27) *and* (29) *are true. Under the same conditions, $u_0 \mapsto u(\cdot, t)$ is continuous from $L^2(\Omega)$ into $L^2(\Omega)$ for each $t \geq 0$ (and also into $L^2(Q_T)$ for each $T > 0$).*

Proof: Suppose (u, ν) and (w, μ) are two Young measure solutions to $\mathcal{P}$ with initial data u_0 and w_0 respectively. Apply the equilibrium equation (7) using $(u - w)\chi_{[0,T]}$ as the test function[3] in the previous section and against $\overline{q}^\nu$ and $\overline{q}^\mu$ and subtract to obtain (where the shorthand notation $\overline{f}^\nu$ is used for $< \nu, f >$ and similarly for $\overline{f}^\mu$)

$$\begin{aligned}
\int_0^T \int_\Omega (\overline{q}^\nu - \overline{q}^\mu) \cdot (\overline{id}^\nu - \overline{id}^\mu)\, dx\, dt &= -\int_0^T \int_\Omega \frac{\partial(u-w)}{\partial t}(u-w)\, dx\, dt \\
&= -\frac{1}{2}\left(\| (u(\cdot,T) - w(\cdot,T) \|^2_{L^2(\Omega)} - \|u_0 - w_0\|^2_{L^2(\Omega)} \right). \qquad (30)
\end{aligned}$$

By lemma 2.7 and (27),

$$\begin{aligned}
\text{l.h.s.}\,(30) &= \int_0^T \int_\Omega \left(\overline{p \cdot id}^\nu + \overline{p \cdot id}^\mu - \overline{p}^\nu \cdot \overline{id}^\mu - \overline{p}^\mu \cdot \overline{id}^\nu \right) dx\, dt \\
&\geq 0
\end{aligned}$$

because the integrand above is precisely the quantity

$$\int_{\mathbf{R}^N} \int_{\mathbf{R}^N} (\nabla\phi^{**}(\alpha) - \nabla\phi^{**}(\beta)) \cdot (\alpha - \beta)\ \nu_{x,t}(d\alpha)\, \mu_{x,t}(d\beta)$$

which is non-negative by the convexity of ϕ^{**}. This implies for (30)

$$\| (u(\cdot,T) - w(\cdot,T) \|^2_{L^2(\Omega)} \leq \|u_0 - w_0\|^2_{L^2(\Omega)} \quad \forall T > 0 \qquad (31)$$

which is the continuity with respect to initial data. When $u_0 = w_0$ is used in (31) we have

$$u(\cdot,T) = w(\cdot,T) \quad x \text{ a.e. in } \Omega,\ \forall T > 0$$

and this shows uniqueness. □

Remarks. 1. The statement of uniqueness does not depend on the method of extracting a Young measure solution for $\mathcal{P}$ and it does not require that ν be an H^1-gradient Young measure, only that $\overline{q} = \overline{p}$ and the independence property of lemma 2.7 hold. In particular, if $(u, \delta_{\nabla u})$ is a classical solution to $\mathcal{P}$ satisfying $q(\nabla u) = p(\nabla u)$, a weaker condition than (28), by uniqueness it coincides with the Young measure solution provided by theorem 2.3 and (28) follows (the independence property is automatically satisfied by classical solutions). We note that there is no claim that the parametrized measure ν is unique: this is false in general.

[3] This is allowed by remark 1 to the definition of Young measure solutions to $\mathcal{P}$ and (26).

2. (Corollary to the uniqueness of solution). *There is a unique limit point of the sequences of the existence theorem 2.3* $(u^h)_{h>0}$ *and* $(w^h)_{h>0}$ *in the weak sense in* $H^1_{loc}(Q_\infty)$ *and strong sense in* $L^2_{loc}(Q_\infty)$ *and of* $(\frac{\partial u^h}{\partial t})_{h>0}$ *weakly in* $L^2(Q_\infty)$ *and therefore these sequences converge.*

Of course, there is no similar assertion for the Young measure ν.

The following Lemma gives some properties of the solution (u,ν) which are consequences of the convexity of ϕ^{**} and the independence property. Most will be useful in establishing the uniqueness of the asymptotic limit (in section 3).

Lemma 2.9 (Further properties of the Young measure solution) *Let* (u,ν) *be the solution to* $\mathcal{P}$ *and* $(u^h)_{h>0}$ *as in the proof of the existence theorem* 2.3. *Then the following are true:*

1. $\nabla\cdot p(\nabla u) = \nabla\cdot\overline{p} = \nabla\cdot\overline{q}$ *in* $H^{-1}_{loc}(Q_\infty)$.
   (Recall that by (27) $\overline{p}=\overline{q}\ (x,t)$ *a.e. in* $Q_\tau\ \forall\tau\in\mathbf{R}^+$.[4]*)*

2. *(i) For each* $T\geq 0$, $u^h\in C([0,T];\ L^2(\Omega))$ *and* $(u^h)_{h>0}$ *is Cauchy in* $C([0,T];\ L^2(\Omega))$.
   (ii) $u\in C([0,T];\ L^2(\Omega))$, *that is,* $u(\cdot,t)\to u(\cdot,t_0)$ *in* $L^2(\Omega)$ *as* $t\to t_0$, *for each* $t_0\geq 0$. *In particular,* $u(\cdot,t)\to u_0$ *in* $L^2(\Omega)$ *as* $t\to 0$.

3. *(i)* $t\mapsto \|u(\cdot,t)\|_{L^2(\Omega)}$ *is decreasing (and therefore has a limit as* $t\nearrow+\infty$*).*
   (ii) $t\mapsto \|u(\cdot,\delta+t)-u(\cdot,t)\|_{L^2(\Omega)}$ *is decreasing for each* $\delta\geq 0$*;*
   (iii) The integral
   $$\int_0^{+\infty}\int_\Omega \overline{q}\cdot\nabla u\,dx\,dt$$
   exists.

Proof: *1.* Fix $T>0$, let w^h be given by (19) and let $\zeta\in H^1(Q_T)$ with $\zeta(\cdot,t)\in H^1_0(\Omega)$, for t a.e. in $[0,T]$. By remark 1 to the definition (see section 3.1) $\zeta-w^h$ is an admissible test function in the equilibrium equation (23). Using the convexity of ϕ^{**} and (23) we know that

$$\int_0^T\int_\Omega p(\nabla\zeta)\cdot\nabla(\zeta-w^h)\,dx\,dt \;\;\geq\;\; \int_0^T\int_\Omega p(\nabla w^h)\cdot\nabla(\zeta-w^h)\,dx\,dt$$

[4]In fact, D. Kinderlehrer and N. Walkington have shown that $p(\nabla u)=\overline{p}=\overline{q}\ (x,t)$ a.e. in $Q_\tau\forall\tau\in\mathbf{R}^+$ is true.

$$= -\int_0^T\int_\Omega \frac{\partial u^h}{\partial t}(\zeta - w^h)\,dx\,dt.$$

Letting $h \to 0$ we obtain

$$\int_0^T\int_\Omega p(\nabla\zeta)\cdot\nabla(\zeta - u) \ \geq\ -\int_0^T\int_\Omega \frac{\partial u}{\partial t}(\zeta - u) \ =\ \int_0^T\int_\Omega \overline{q}\cdot\nabla(\zeta - u).$$

Choosing $\zeta = u + \lambda(\theta - u)$ for $\theta \in H_0^1(Q_T)$ and letting $\lambda \to 0^+$ we obtain

$$\int_0^T\int_\Omega p(\nabla u)\cdot\nabla(\theta - u) \ \geq\ -\int_0^T\int_\Omega \overline{q}\cdot\nabla(\theta - u) \quad \forall\theta \in H_0^1(Q_T).$$

Replacing $\theta - u$ with its negative we obtain equality above and this proves *1.* *2.* Fix $T > 0$. Recall that

$$u^h(x,t) = u^{h,j}(x) + (\frac{t}{h} - j)(u^{h,j+1} - u^{h,j})(x)$$

for $hj \leq t < h(j+1)$. When $|t-s| \leq h$,

$$\|u^h(\cdot,t) - u^h(\cdot,s)\|_{L^2(\Omega)} \ =\ \frac{|t-s|}{h}\,\|u^{h,j+1} - u^{h,j}\|_{L^2(\Omega)}.$$

Also, by the uniform estimates in [KP1],

$$\sup_{t\geq 0}\|u^h(\cdot,t)\|_{L^2(\Omega)} \ =\ \sup_{j\geq 0}\|u^{h,j}\|_{L^2(\Omega)} \ \leq\ M$$

which shows that $t \mapsto u^h(\cdot,t)$ is (uniformly) continuous and bounded on $\mathbb{R}^+$ into $L^2(\Omega)$.

Set $U^{h,h'} := u^h - u^{h'} \ \in H^1(Q_T)$; we have,

$$\|U^{h,h'}\|_{L^2(\Omega)}(T) \ =\ \int_0^T\int_\Omega 2U^{h,h'}U_t^{h,h'}\,dx\,dt$$

$$\leq\ \|U^{h,h'}\|_{L^2(Q_T)}\|U_t^{h,h'}\|_{H^1(Q_T)}$$

and since $(u^h)_{h>0}$ converges in $L^2(Q_T)$ and is bounded in $H^1(Q_T)$ we see

$$\lim_{h,h'\to 0}\ \sup_{t\geq 0}\|u^h - u^{h'}\|_{L^2(\Omega)}(t) \ =\ 0.$$

Therefore, $(u^h)_{h>0}$ is Cauchy in $C\left(\mathbb{R}^+; L^2(\Omega)\right)$. This shows *2.*

3. For $0 \leq s \leq t$ apply the equilibrium equation (7) with $u\chi_{[s,t]}$ as the test function and have

$$\int_s^t\int_\Omega \overline{q}\cdot\nabla u \ =\ -\int_s^t\int_\Omega \frac{\partial u}{\partial t}u$$

(32) $$:= -\frac{1}{2}\left(\|u(\cdot,t)\|^2_{L^2(\Omega)} - \|u(\cdot,s)\|^2_{L^2(\Omega)}\right).$$

Using (27), the convexity of ϕ^{**} and the independence relation (29) as in the proof of theorem 2.8, together with the assumption $p(0) = 0$, we conclude that the l.h.s. of (32) is non-negative and *(i)* follows. Letting $s = 0$ and $t \to +\infty$ in the l.h.s. of (32) we then obtain *(iii)*.

Notice that by the uniqueness of solution the pair $(u^\delta, \nu^\delta) \equiv \left(u(\cdot, \delta + \cdot),\, \nu_{(\cdot,\delta+\cdot)}\right)$ solves $\mathcal{P}$ with initial data $u(\cdot, \delta)$. For fixed $0 \le s \le t$ we apply the equilibrium equation to each of the solution pairs (u^δ, ν^δ) and (u, ν) using $(u^\delta - u)\chi_{[s,t]}$ as a test function and subtract the two equations. Arguing as in *(i)* yields *(ii)*. □

2.5 Stability: maximum and comparison principles. Localization

We investigate the stability of the Young measure solution. We show that a maximum principle and a comparison result are satisfied. We conclude the section with a localization property of the solution (u, ν), a corollary of the comparison principle.

Theorem 2.10 (Maximum principle) *Let (u, ν) and (w, μ) solve with initial data u_0 and w_0 respectively. Then (x,t) a.e. in $\overline{Q_\infty}$,*

(33) $$\text{ess sup}_{x\in\overline{\Omega}}\,(u_0 - w_0)^- \le u(x,t) - w(x,t) \le \text{ess sup}_{x\in\overline{\Omega}}\,(u_0 - w_0)^+.$$

Proof: Set

$$K := \text{ess sup}_{x\in\overline{\Omega}}(u_0 - w_0)^+.$$

We introduce auxiliary functions as in the proof for a maximum principle for the solution of the heat equation with H^1 data (cf. [Br]). Fix $G \in C^1(\mathbf{R})$ such that $G = 0$ on $(-\infty, 0]$ and strictly increasing with $0 < G' \le M$ on $(0, +\infty)$. For $t \ge 0$ define the functions

$$H(t) := \int_0^t G(s)\, ds$$

and

$$\psi(t) := \int_\Omega H\left(u(x,t) - w(x,t) - K\right)\, dx.$$

Then $\psi \in C(\mathbf{R}^+) \cap C^1(\mathbf{R}^+)$, $\psi(0) = 0$ and $\psi \geq 0$ on $\mathbf{R}^+$. Note that $G(u-w-K) \in H_0^1(\Omega)$ so that it is an admissible test function for the equilibrium equation. For all $T \geq 0$,

$$\begin{aligned}
\int_0^T \psi'(t)\, dt &= \int_0^T \int_\Omega G\left(u(x,t) - w(x,t) - K\right) \frac{\partial(u-w)}{\partial t}\, dx\, dt \\
&= -\int_0^T \int_\Omega (\overline{q}^\nu - \overline{q}^\mu) \cdot \nabla(u-w)\, G'\left(u - w - K\right) dx\, dt \\
&\leq 0
\end{aligned}$$

because $(\overline{q}^\nu - \overline{q}^\mu) \cdot \nabla(u-w) \geq 0$ (as in the proof of the uniqueness theorem 2.8), and $G' > 0$ on $[0,T]$. Hence $\psi \equiv 0$ and $H(u-w-K) = 0$ (x,t) a.e. in $\overline{Q_\infty}$, or

$$\int_0^{u(x,t)-w(x,t)-K} G(s)\, ds = 0 \qquad (x,t) \text{ a.e. in } \overline{Q_\infty}$$

which by the choice of G implies

$$u(x,t) - w(x,t) - K \leq 0 \quad (x,t) \text{ a.e. in } \overline{Q_\infty}.$$

Reversing the roles of u and w we obtain the lower bound in (33). □

Lemma 2.11 (Comparison principle) *Assume (u,ν) and (v,μ) are the solutions to $\mathcal{P}$ with to initial data u_0 and v_0 respectively. Assume further that*

$$u_0 \geq v_0 \qquad a.e. \text{ in } \Omega.$$

Then

$$u \geq v \quad (x,t) \text{ a.e. in } Q_\infty.$$

Proof: Let $w = \max(u,v)$ in Q_∞. It suffices to show that

$$(v-u)^+ = 0 \quad (x,t) \text{ a.e. in } Q_\infty.$$

We apply the equilibrium equation for each solution noting that $w - u = (v-u)^+$ is admissible as a test function.

$$\int_0^T \int_\Omega \overline{q}^\nu \cdot \nabla(w-u) + u_t(w-u)\, dx\, dt = 0$$

$$\int_0^T \int_\Omega \overline{q}^\mu \cdot \nabla(w-u) + v_t(w-u)\, dx\, dt = 0.$$

By subtraction we obtain

$$\begin{aligned}
\int_0^T \int_\Omega (\overline{q}^\mu - \overline{q}^\nu) \cdot \nabla (v-u)^+ &= -\frac{1}{2}\int_0^T \int_\Omega (v_t - u_t)(v-u)^+ \, dx\, dt \\
&= \int_0^T \frac{d}{dt}\int_\Omega (v-u)^+ \, dx\, dt \\
&= -\frac{1}{2}\{\|(v-u)^+\|^2_{L^2(\Omega)}(T) - \|(v_0-u_0)^+\|^2_{L^2(\Omega)}\} \\
&= -\frac{1}{2}\|(v-u)^+\|^2_{L^2(\Omega)}(T).
\end{aligned}$$

Since $(\overline{q}^\mu - \overline{q}^\nu)\cdot\nabla(v-u)^+ \geq 0$, we conclude that

$$\|(v-u)^+\|^2_{L^2(\Omega)}(T) = 0 \quad \forall T \geq 0,$$

that is, $v \leq u$, (x,t) a.e. in Q_∞. □

An immediate but rather noteworthy corollary of the comparison lemma is that (u,ν) solves also locally on subsets of the domain Ω. More precisely, we have the following property:

Corollary 2.12 (Localization) *Assume $\omega \subsetneq \Omega$ is open with Lipschitz boundary and let (u,ν) be the solution to $\mathcal{P}$. Let v be the restriction of u to ω. Then v is a solution with respect to initial data v_0 the restriction of u_0 to ω.*

Proof: Suppose χ is the solution on ω (with v_0 initial data). Apply the comparison result to the differences $\chi - v$ and $v - \chi$. □

3 Asymptotic analysis and the equilibrium Young measure solution

We investigate the asymptotic behavior of the solution as $t \to +\infty$ and establish the following:

Theorem 3.1 *Let (u,ν) be the (unique) solution of $\mathcal{P}$; there exists a (unique) $z \in H^1_0(\Omega)$ and an $H^1_{loc}(Q_\infty)$-gradient Young measure $\nu^\infty = (\nu^\infty_{x,t})_{(x,t)\in Q_\infty}$ such that*

(34) $\quad u(\cdot,t) \longrightarrow z \quad$ *weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$ as $t \to +\infty$*

(these limits exist without restricting to a subsequence in time).

$$t \mapsto \|u(\cdot,t) - z\|_{L^2(\Omega)} \quad \textit{is decreasing} \tag{35}$$

$$\nabla\cdot < \nu^\infty, q > = 0 \quad \textit{in } H^{-1}(Q_\infty) \textit{ and } H^{-1}_{loc}(Q_\infty) \tag{36}$$

$$\nabla z = < \nu^\infty, id > \quad \textit{a.e. in } \Omega \textit{ (independent of time)} \tag{37}$$

(so (z, ν^∞) is a Young measure solution of the steady-state version of $\mathcal{P}$).

$$< \nu^\infty, q(\lambda)\cdot\lambda > = 0 \quad \textit{a.e. in } \Omega \tag{38}$$

$$\textit{supp}\, \nu^\infty \subseteq \{\lambda : q(\lambda)\cdot\lambda = 0\} \cap \{\phi^{**} = \phi = 0\} \tag{39}$$

Definition. With z, ν^∞ as in the theorem, we call z the *asymptotic limit* of u and ν^∞ the *asymptotic Young measure*; we call the pair (z, ν^∞) the *equilibrium (Young measure) solution* of $\mathcal{P}$.

For the proof of the theorem we first establish two properties of $W_\omega(u_0)$, the set of weak limit points of $(u(\cdot,t))_{t\geq 0}$ in $H^1_0(\Omega)$. This is the content of lemma 3.2 below. Following, supposing $z \in W_\omega(u_0)$, we solve $\mathcal{P}$ with initial data z and obtain the Young measure solution (w, ν^∞). We show that (w, ν^∞) solves the stationary problem associated to $\mathcal{P}$ and thus infer that $w = z$. Furthermore, we show via the equilibrium equation that $W_\omega(u_0) = \{z\}$ and that the pair (z, ν^∞) satisfies the conclusions of the theorem.

We define

$$W_\omega(u_0) := \{z \in g(x,0) + H^1_0(\Omega) \mid \exists (t_n)_{n\geq 1} \nearrow +\infty \text{ with } u(\cdot,t_n) \xrightarrow{w-s} z\}.$$

The notation

$$u(\cdot,t_n) \xrightarrow{w-s} z \qquad \text{in } H^1 - L^2$$

indicates that the sequence converges weakly in H^1 and strongly in L^2 which we may always achieve by reducing to a subsequence using the Rellich theorem. Note that $W_\omega(u_0)$ is non-empty since $u \in L^\infty(\mathbb{R}^+; H^1_0(\Omega))$. Theorem 3.1 establishes that $W_\omega(u_0)$ consists of exactly one function.

We begin by describing some properties of all functions in $W_\omega(u_0)$.

Lemma 3.2 *Let $z \in W_\omega(u_0)$ and $t_n \to +\infty$ along which $u(\cdot, t_n) \overset{w-s}{\longrightarrow} z$. Then $\forall t \geq 0$*

(40) $$u(\cdot, t_n + t) \overset{w-s}{\longrightarrow} z \quad \text{in } H^1(\Omega) - L^2(\Omega)$$

and

(41) $$u(\cdot, t_n + \cdot) \overset{w-s}{\longrightarrow} z \quad \text{in } H^1_{loc}(Q_\infty) - L^2_{loc}(Q_\infty)$$

as $n \to +\infty$ (without restricting to subsequences).

Proof: Fix $t \geq 0$. Since $u \in L^\infty(\mathbb{R}^+; H^1_0(\Omega))$, the sequence $(u(\cdot, t_n + t))_{n \geq 1}$ is bounded in $H^1(\Omega)$; hence for a subsequence $(n_j)_{j \geq 1}$ there exists $y(\cdot, t) \in H^1(\Omega)$ such that as $j \to +\infty$,

$$u(\cdot, t_{n_j} + t) \overset{w-s}{\longrightarrow} y(\cdot, t) \quad \text{in } H^1(\Omega) - L^2(\Omega)$$

and of course,

$$u(\cdot, t_{n_j}) \overset{w-s}{\longrightarrow} z \quad \text{in } H^1(\Omega) - L^2(\Omega).$$

Note that $y(\cdot, t) \in W_\omega(u_0)$; we show that $y(\cdot, t) = z$. For all $\zeta \in H^1(\Omega)$,

$$\begin{aligned}
\int_\Omega (y(x,t) - z(x))\zeta(x)\,dx &= \lim_{j \to +\infty} \int_\Omega (u(x, t_{n_j + t}) - u(x, t_{n_j}))\zeta(x)dx \\
&= \lim_{j \to +\infty} \int_\Omega \int_{t_{n_j}}^{t_{n_j}+t} u_t(x,s)\zeta(x)\,ds\,dx \\
&\leq \|\zeta\|_{L^2(\Omega)} \limsup_{j \to +\infty} \|u_t\|^2_{L^2([t_{n_j}, t_{n_j}+t] \times \Omega)} \sqrt{t} \\
&= 0
\end{aligned}$$

since $u_t \in L^2(Q_\infty)$. This shows $y(\cdot, t) = z$ for x a.e. in Ω; as a result the whole sequence converges to z and (40) holds true.

Fix $T > 0$. The sequence $(u(\cdot, t_n + \cdot))_{t \geq 0}$ is bounded in $H^1(Q_T)$ and thus we can find $\chi \in H^1(Q_T)$ and a subsequence $(n_j)_{j \geq 1}$ along which

$$u(\cdot, t_{n_j} + \cdot) \overset{w-s}{\longrightarrow} \chi \quad \text{in } H^1(Q_T) - L^2(Q_T).$$

Choose $\zeta(x)\eta(t) \in H^1(Q_T)$ with $\eta(t)$ defined for each t. By (40),

$$\int_\Omega u(x, t_{n_j} + t)\zeta(x)\,dx \overset{j \to +\infty}{\longrightarrow} \int_\Omega z(x)\zeta(x)\,dx$$

for each $t > 0$. Thus

$$\eta(t)\int_\Omega u(x,t_{n_j}+t)\zeta(x)\,dx \overset{j\to+\infty}{\longrightarrow} \eta(t)\int_\Omega z(x)\zeta(x)\,dx \tag{42}$$

pointwise in t and the Lebesque Dominated Convergence theorem applies to (42) to give

$$\int_0^T\int_\Omega \eta(t)\zeta(x)u(x,t_{n_j}+t)\,dx\,dt \overset{n\to+\infty}{\longrightarrow} \int_0^T\int_\Omega \eta(t)\zeta(x)z(x)\,dx\,dt$$

$$\text{(by assumption)} \quad = \quad \int_0^T\int_\Omega \eta(t)\zeta(x)\chi(x,t)\,dx\,dt.$$

By the density of separable functions in L^2 this implies

$$\chi(\cdot,t) = z(\cdot) \quad \forall t>0 \quad x \text{ a.e. in } \Omega.$$

We conclude that no reduction to a subsequence is necessary and (41) obtains. □

Proof of theorem 3.1: Fix $z \in W_w(u_0)$ and $t_n \to +\infty$ along which $u(\cdot,t_n) \overset{w-s}{\longrightarrow} z$. We define

$$u^n(\cdot,\cdot) \ := \ u(\cdot,t_n+\cdot)$$

$$\nu^n \ = \ (\nu^n_{x,t})_{(x,t)\in Q_\infty} \ := \ (\nu_{x,t_n+t})_{(x,t)\in Q_\infty}.$$

Then (u^n,ν^n) is the solution with respect to initial data $u(\cdot,t_n)$ for $n \geq 0$. By lemma 3.2 we know that as $n \to +\infty$,

$$u^n(\cdot,t) \overset{w-s}{\longrightarrow} z \quad \text{in } H^1(\Omega) \ - \ L^2(\Omega)\ \forall t \in \mathbb{R}^+$$

$$u^n \overset{w-s}{\longrightarrow} z \quad \text{in } H^1_{loc}(Q_\infty) \ - \ L^2(Q_\infty).$$

Since z is independent of t it follows that

$$\frac{\partial u^n}{\partial t} \overset{n\to+\infty}{\longrightarrow} 0 \quad \text{in } L^2(Q_\infty).$$

In addition, note that $(\nu^n)_{n\geq 0}$ is bounded in $L^1(Q_\infty,\mathcal{E}_0')$. By remark 2 to lemma 2.4 there exists an $H^1_{loc}(Q_\infty)$-gradient Young measure, $\nu^\infty = (\nu^\infty_{x,t})_{(x,t)\in Q_\infty}$, satisfying

$$\nu^\infty \ \in \ L^\infty(Q_\infty; M(\mathbb{R}^N)) \ \cap \ L^2_{loc}(Q_\infty;\mathcal{F}_0') \ \cap \ L^1_{loc}(Q_\infty;\mathcal{E}_0').$$

For each n we apply the equilibrium equation

$$\int_0^\tau\int_\Omega <\nu^n,q> \nabla\zeta + \frac{\partial u^n}{\partial t}\zeta\,dx\,dt \ = \ 0 \quad \forall\zeta \in H^1(Q_\tau)\,\forall\tau \in [0,+\infty],$$

and we may pass to the limit as $n \to \infty$. (Note that the non-local convergence in n is not guaranteed by lemma 2.4 but by the boundedness of $(\frac{\partial u^n}{\partial t})_{n\geq 0}$ in $L^2(Q_\infty)$); we obtain

$$\int_0^\tau \int_\Omega <\nu^\infty, q> \nabla\zeta \, dx\, dt \;=\; 0 \quad \forall\zeta \in H^1(Q_\tau)\, \forall\tau \in [0,+\infty]$$

or, equivalently,

$$\nabla\cdot <\nu^\infty, q> \;=\; 0 \qquad \text{in } H^{-1}(Q_\infty) \text{ and in } H^{-1}_{loc}(Q_\infty).$$

Recall that $<\nu^n, id> = \nabla u^n$ and converges to ∇z weakly in $L^2_{loc}(Q_\infty)$; in addition, by lemma 2.4,

$$<\nu^n, id> \rightharpoonup <\nu^\infty, id> \qquad \text{in } L^2_{loc}(Q_\infty).$$

Thus,

$$\nabla z \;=\; <\nu^\infty, id> \qquad x \text{ a.e. in } \Omega.$$

From (36) and (37) we infer that (z, ν^∞) solves the stationary problem associated to $\mathcal{P}$ and so the independence lemma 2.7 implies

$$<\nu^\infty, q\cdot id> \;=\; <\nu^\infty, q>\cdot <\nu^\infty, id> \qquad (x,t) \text{ a.e. in } Q_\infty$$

and, as before,

(43) $$<\nu^\infty, q\cdot id> \;\geq\; 0 \qquad \text{a.e. in } \Omega.$$

On the other hand, for each $T \geq 0$, by lemma 2.9*(3i)*,

$$\int_0^T \int_\Omega <\nu^n_{x,t}, q> \cdot\nabla u^n(x,t)\, dx\, dt \quad = \quad -\frac{1}{2}\left(\|u(\cdot, T+t_n)\|^2 - \|u(\cdot, t_n)\|^2\right)$$

$$\overset{n\to+\infty}{\longrightarrow} \; 0$$

and applying the independence lemma 2.7 on the l.h.s. we obtain

$$\int_0^T \int_\Omega <\nu^n, q\cdot id> \; dx\, dt \;\overset{n\to+\infty}{\longrightarrow}\; 0$$

or,

$$\int_0^T \int_\Omega <\nu^\infty, q\cdot id> \; dx\, dt \;=\; 0.$$

This proves (38). By (28) and (43) we conclude

$$supp\, \nu^\infty \;\subseteq\; \{\lambda : q(\lambda)\cdot\lambda = 0\} \cap \{\phi^{**} = \phi\}.$$

The proof of equation (39) will be complete by corollary 4.2 in the next section.

It remains to show (34) and (35). Let $0 \le s \le t$ and apply the equilibrium equation to the solutions of $\mathcal{P}$ (u,ν) and (z,ν^∞) (corresponding to initial data u_0 and z respectively); we have (using the notation $\bar{q}^\infty =< \nu^\infty, q >$),

$$\begin{aligned}\int_s^t\int_\Omega (\bar{q}-\bar{q}^\infty)\cdot\nabla(u-z)\,dx\,dt &= -\int_s^t\int_\Omega \frac{\partial(u-z)}{\partial t}(u-z)\,dx\,dt \\ &= -\frac{1}{2}\left(\|u(\cdot,t)-z\|^2_{L^2(\Omega)} - \|u(\cdot,s)-z\|^2_{L^2(\Omega)}\right) \\ &\ge 0\end{aligned}$$

so that

$$t \mapsto \|u(\cdot,t)-z\|_{L^2(\Omega)}$$

is decreasing and its limit as $t \to +\infty$ exists: it is zero because $z \in W_w(u_0)$ and by the Rellich theorem a subsequence exists along which $u(\cdot,t_j) \overset{j\to\infty}{\longrightarrow} z$ in $L^2(\Omega)$. This finishes the proof of the theorem. □

Conclusion. Let (u,ν) be the Young measure solution to $\mathcal{P}$ with initial data u_0. Then $W_w(u_0) = \{z\}$, i.e. the $L^2(\Omega)$ asymptotic limit of u is unique and the equilibrium solution (z,ν^∞) solves the steady-state problem

$$\nabla\cdot < \nu^\infty, q > = 0 \quad \text{in } H^{-1}(Q_\infty) \text{ and } H^{-1}_{loc}(Q_\infty).$$

4 Energy

Define the *energy function*

$$E(t) := \int_\Omega \phi^{**}(\nabla u)(x,t)\,dx \qquad \text{for } t \ge 0. \tag{44}$$

In this section we justify the term *energy* for the function in (44); the energy vanishes at infinity globally in space and also locally (recall from lemma 2.12 that the solution solves also locally).

Throughout this section (u,ν) are the solution of $\mathcal{P}$ and (z,ν^∞) the equilibrium solution.

Theorem 4.1 *Let E be given by (44). Then $E \in L^1(\mathbb{R}^+)$ and E is a decreasing function of t. Moreover,*

(45) $$\int_\Omega \phi^{**}(\nabla u)(x,t)\,dx \searrow 0 \quad \text{as } t \nearrow +\infty$$

Proof: For $0 \le s \le t$,

$$\begin{aligned}
\int_s^t E(t)\,dt &= \int_s^t \int_\Omega \phi^{**}(\nabla u)(x,t)\,dx\,dt \\
(\text{since } \phi^{**} \text{ is convex and } \phi^{**}(0)=0) \quad &\le \int_s^t \int_\Omega p(\nabla u)\cdot \nabla u\,dx\,dt \\
&= -\int_s^t \int_\Omega u_t u\,dx\,dt \\
&= -\frac{1}{2}\left(\|u\|^2_{L^2(\Omega)}(t) - \|u\|^2_{L^2(\Omega)}(s)\right) \\
&\longrightarrow 0^+ \quad \text{as } s,t \to +\infty
\end{aligned}$$

by lemma 2.9 *(3i)* and *(3iii)*. Therefore,

$$\int_0^{+\infty} \int_\Omega \phi^{**}(\nabla u)(x,t)\,dx\,dt < +\infty$$

and this shows that the energy is integrable.

Next we give the proof due to P. Pedregal that the energy is decreasing. For $T \ge 0$ fixed we have for all $t > 0$,

$$\begin{aligned}
\int_T^{T+t} (E(s) - E(T))\,ds &= \int_T^{T+t} \int_\Omega (\phi^{**}(\nabla u(x,s)) - \phi^{**}(\nabla u(x,T)))\,dx\,ds \\
&\le \int_T^{T+t} \int_\Omega p(\nabla u(x,s))\cdot(\nabla u(x,s) - \nabla u(x,T))\,dx\,ds \\
&= -\int_T^{T+t} \int_\Omega \frac{\partial(u(x,s)-u(x,T))}{\partial s}(u(x,s)-u(x,T))\,dx\,ds \\
&= -\frac{1}{2}\|u(\cdot,T+t) - u(\cdot,T)\|^2_{L^2(\Omega)} \\
&\le 0.
\end{aligned}$$

By the continuity of E this implies

$$E(T+t) \leq E(T)$$

for all $t > 0$ sufficiently small. Since T is arbitrary, this shows that E is in $L^1(\mathbb{R}^+)$ and decreasing so (45) follows. $\square$

Corollary 4.2 *The energy converges as $t \to +\infty$ and attains its minimum, i.e.,*

$$\lim_{t\to+\infty} \int_\Omega \phi^{**}(\nabla u)(x,t)\,dx = \int_\Omega \phi^{**}(\nabla z)(x)\,dx = 0 \tag{46}$$

Consequently, the asymptotic Young measure satisfies

$$supp\,\nu^\infty \subseteq \{\phi^{**} = \phi = 0\} \tag{47}$$

(which completes the proof of (39)).

Proof: Since ϕ^{**} is convex, the functional

$$u \mapsto \int_\Omega \phi^{**}(\nabla u)\,dx$$

is sequentially lower semicontinuous with respect to weak convergence in $H_0^1(\Omega)$. By theorem 3.1 we have $u(\cdot,t) \rightharpoonup z$ in $H^1(\Omega)$ as $t \to +\infty$ and

$$\begin{aligned} 0 \leq \int_\Omega \phi^{**}(\nabla z)(x)\,dx &\leq \liminf_{t\to+\infty} \int_\Omega \phi^{**}(\nabla u)(x,t)\,dx \\ &= \lim_{t\to+\infty} \int_\Omega \phi^{**}(\nabla u)(x,t)\,dx \\ &= 0 \end{aligned}$$

because $E \in L^1(\mathbb{R}^+)$ and $\phi^{**} \geq 0$. From Jensen's inequality and (46) we obtain (47). $\square$

The energy is also minimized asymptotically locally in the sense of

Lemma 4.3 *For all $A \subseteq \Omega$, measurable*

$$\lim_{t\to+\infty} \int_A \phi^{**}(\nabla u)(x,t)\,dx = \int_A \phi^{**}(\nabla z)(x)\,dx \tag{48}$$

(but this limit is not necessarily monotone decreasing).

Proof: By (46) and by theorem 2.1(ii) we conclude that $((\phi^{**}(\nabla u)(\cdot,t))_{t\geq 0}$ is weakly sequentially precompact in $L^1(\Omega)$ and (48) follows. □

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