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Volume Preserving Mean Curvature Flow as a Limit of a Nonlocal Ginzburg-Landau Equation.

Lia Bronsard* Barbara Stoth#

Abstract. We study the asymptotic behaviour of radially symmetric solutions of the nonlocal equation

$$\varepsilon\varphi_t - \varepsilon\Delta\varphi + \frac{1}{\varepsilon}W'(\varphi) - \lambda_\varepsilon(t) = 0$$

in a bounded spherically symmetric domain $\Omega \subset \mathbf{R}^n$, where $\lambda_\varepsilon(t) = \frac{1}{\varepsilon}\fint_\Omega W'(\varphi)\,dx$, with a Neumann boundary condition. The analysis is based on "energy methods" combined with some a-priori estimates, the latter being used to approximate the solution by the first two terms of an asymptotic expansion. We only need to assume that the initial data as well as their energy are bounded. We show that, in the limit as $\varepsilon \to 0$, the interfaces move by a nonlocal mean curvature flow, which preserves mass. As a byproduct of our analysis, we obtain an L^2 estimate on the "Lagrange multiplier" $\lambda_\varepsilon(t)$. In addition we show rigorously that the nonlocal Ginzburg–Landau equation and the Cahn–Hilliard equation occur as special degenerate limits of a viscous Cahn–Hilliard equation.

Section 1: Introduction.

We consider the nonlocal reaction–diffusion equation introduced recently by Rubinstein and Sternberg [RS]

$$\begin{aligned} &\varepsilon\varphi_t - \varepsilon\Delta\varphi + \frac{1}{\varepsilon}W'(\varphi) - \lambda(t) = 0, \\ &\lambda(t) = \frac{1}{\varepsilon}\fint_\Omega W'(\varphi)\,dx \end{aligned} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbf{R}^n$, $n \geq 2$, with Neumann boundary condition

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega\times[0,T]} = 0. \tag{1.2}$$

The potential W is a bistable potential, that is $W \geq 0$ and it vanishes exactly at two points.

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The typical bistable potential is given by

$$W(\varphi) = \frac{1}{2}(1-\varphi^2)^2, \tag{1.3}$$

and we will present our results for this specific potential. However we point out that our results can be extended to the more general case.

An important property of this flow is that its mass is preserved, that is

$$\partial_t \int_\Omega \varphi(x,t)dx = 0. \tag{1.4}$$

Rubinstein and Sternberg ([RS]) introduced the nonlocal equation (1.1) as a simpler alternative to the classical Cahn–Hilliard equation [CH]

$$\varepsilon\varphi_t = \Delta(-\varepsilon\Delta\varphi + \frac{1}{\varepsilon}W'(\varphi)), \tag{1.5}$$

to model phase separation after quenching (rapid cooling) of homogeneous binary systems such as glasses and polymers. The function φ represents the difference in concentration of the binary mixture and hence is a conserved quantity. Using multiple time scale asymptotic expansions to study the behaviour of the solution to (1.1)-(1.2), Rubinstein and Sternberg [RS] obtained formally that the domain Ω is divided in regions where φ is close to the local minima of W. Moreover the interfaces $\{\Gamma_i\}$ dividing these regions evolve (in the limit $\varepsilon \to 0$) with normal velocity

$$V_i = \kappa_i - \sum \frac{1}{\sum|\Gamma_j|}\int_{\Gamma_j}\kappa_j$$

where κ_i is the sum of principal curvatures of Γ_i and $|\Gamma_i|$ is its perimeter. This is a nonlocal volume–preserving mean curvature flow. We propose to use an energy–type method to prove rigorously this picture in a certain radially symmetric setting. More specifically, we assume that Ω is a ball in $\mathbf{R}^n$ and that φ is radial with several "transitions" spheres. Equation (1.1) is already written in the time scale for which the nonlocal mean curvature flow occurs in times of order one. But by a rescaling, we see that this problem corresponds to the singular perturbation problem $\varphi_\tau - \varepsilon^2\Delta\varphi + W'(\varphi) - \varepsilon\lambda = 0$.

Next, we shall compare the two equations (1.1) and (1.5), as well as their respective asymptotic limiting flows. The Cahn–Hilliard equation is the gradient flow in $H^{-1,2}(\Omega \times (0,T))$ (cf [Fi]) for the functional

$$E_\varepsilon[\varphi] = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}W(\varphi)\,dx, \tag{1.6}$$

while the nonlocal equation (1.1) is the gradient flow in $L^2(\Omega \times [0,T])$ for the same functional (1.6) against the mass constraint (1.4). The associated (time independent) minimization problem, that is, the problem of minimizing (1.6) with a mass constraint, has been studied by Luckhaus and Modica [LM]. They obtain rigorously the first order expansion in ε of the associated Lagrange multiplier. In this context, we can interpret loosely the nonlocal term $\lambda = \lambda_\varepsilon(t)$ in (1.1) as a Lagrange multiplier. In fact, because of the Neumann boundary condition (1.2), the expression for $\lambda_\varepsilon(t)$ is exactly what is needed for the gradient flow of $E_\varepsilon[\varphi]$ to conserve mass.

The asymptotic behaviour of the solutions to (1.1) and (1.5) are very different. Formal analysis of Pego [P] suggests that the asymptotic behaviour of (1.5) is given by the so–called Mullins–Sekerka [MS] or Hele–Shaw problem:

$$\Delta u = 0 \text{ in } \Omega\backslash\Gamma, \quad u = -\kappa \text{ on } \Gamma, \quad \left[\frac{\partial u}{\partial n}\right]_\Gamma = -V,$$

where Γ is an interface, κ is the sum of principal curvatures of Γ and V is its normal velocity. This has been proved recently by Stoth [S3] in the radial case in $\mathbf{R}^n$, $n \leq 3$. Alikakos-Bates-Chen [ABC] have a convergence result in general domains assuming that the limit flow is smooth and with particular initial and boundary conditions.

Both limiting flows are nonlocal and some existence results are known for each of them. Indeed, Gage [G] has proved that a convex curve evolving by the volume constrained mean curvature flow eventually becomes a sphere with the prescribed area. Also, there are simple examples which show that non-convex surfaces may develop singularities in finite time (see e.g. [RS]). For the Mullins-Serkerka problem, Chen [C] has proved a weak, local-in-time existence result for general smooth initial manifolds, and a global existence result for curves (??) which are small perturbation of spheres.

The most striking difference between the two limiting geometric flows is the effect of small spheres. Indeed, in the radial case we can easily calculate explicitly the respective evolution laws for the interfaces. In the three dimensional case, assuming that there are two interfaces $r_2(t) < r_1(t)$, the nonlocal problem is given by (7.43)

$$\dot{r}_1 = 2\left(-\frac{1}{r_1} + \frac{r_1 - r_2}{r_1^2 + r_2^2}\right), \quad \dot{r}_2 = 2\left(-\frac{1}{r_2} - \frac{r_1 - r_2}{r_1^2 + r_2^2}\right), \tag{1.7}$$

while the Mullins-Sekerka problem is given by

$$\dot{r}_1 r_1^2 = \dot{r}_2 r_2^2, \qquad \dot{r}_1 = \frac{1}{r_1^2}\frac{r_2 - r_1}{r_2 + r_1}. \tag{1.8}$$

Therefore, as r_2 approaches 0, it is clear that $\dot{r}_1$ approaches 0 in (1.7), while it approaches $-\frac{1}{r_1^2}$ in (1.8). But, once the smallest sphere has disappeared, $\dot{r}_1$ must be zero since the mass must be preserved. This means that the flow for r_1 is strongly affected by asymptotically small spheres in the Mullins-Sekerka model. In fact, Rubinstein and Sternberg [RS] used a multiple scattering expansion known as the point interaction approximation method, to suggest that the Mullins-Sekerka problem is not the appropriate asymptotic limit of the Cahn-Hilliard equation when there are asymptotically small spheres.

There is an interesting connection between equations (1.1) and (1.5). Indeed, Rubinstein and Sternberg observed that equations (1.1) and (1.5) arise by formally taking different parameter limits ($\alpha \to 0$ and $\nu \to 0$ respectively) in the viscous Cahn-Hilliard equation $\alpha\varphi_t = \Delta(W'(\varphi) - \beta\Delta\varphi + \nu\varphi_t)$. This equation was introduced by Novick-Cohen in order to include viscous effect in the Cahn-Hilliard model [NC]. We prove these convergence results rigorously in Section 8. This suggests that by taking an appropriate choice of parameter limit in the viscous Cahn-Hilliard equation, one should recover a different limit flow with possibly better properties.

We note that the singular limit of (1.1) provides a notion of weak solution for the nonlocal mean curvature flow. However there is no uniqueness theorem in general: different sequences of ε's might produce different limits. The same approach has been used to define a weak model for mean curvature flow (cf [BK2], [DS1,2]) using the Allen-Cahn equation [AC]. In that case, Evans-Soner-Souganidis [ESS] have shown that this model coincides with the weak notion of motion by mean curvature in the sense of viscosity solutions (cf [CGG], [ES]).

We prove the convergence of the nonlocal equation (1.1) to the volume preserving mean curvature flow in a radially symmetric setting. We assume that for $\varphi = \varphi_\varepsilon$

$$\begin{aligned} &\|\varphi(\cdot,0)\|_\infty \leq C, \\ &E_\varepsilon[\varphi](0) \leq C, \\ &\left|\int_\Omega \varphi(x,0)\,dx\right| < |\Omega| - \omega, \end{aligned} \tag{1.9}$$

for some positive constant ω. The second assumption means that the initial data must have a "transition-layer structure", i.e. $\varphi_\varepsilon \approx \pm 1$. The third condition ensures that there exists at least one interface. The case of general initial data is much harder, we refer to Soner [So] for the equivalent problem for the Allen–Cahn equation. Our method is an energy-type method similar to the methods developed by Bronsard and Kohn [BK1,2], in order to study the singular limit of the Allen–Cahn equation, and the methods developed by Stoth [S1,2,3], in order to study the singular limit of the phase-field model and the Cahn-Hilliard equation.

We now describe the method in more details. We first use BV-bounds (Proposition 2.3) to obtain the existence of an L^1 limit v for a subsequence of φ_ε (Remark 2.11), and then restrict the discussion to this subsequence. In addition we show that there exists a monotone L^1 limit E_0 for $E_\varepsilon[\varphi]$ (Corollary 2.12) which is used to define time intervals on which the variation of $E_\varepsilon[\varphi]$ is uniformly small (Lemma 2.21). These results are not restricted to the radially symmetric case. In Section 4, we prove a further L^2-estimate on the Lagrange multiplier λ_ε in the radial case (Proposition 4.1), which implies the existence of a weak limit λ_0 for an approriate subsequence (Remark 4.10).

The next step is the foundation of our approach. We show that away from the origin and except at finitely many time points, φ_ε is close to the standing wave solution associated to the equation $\Delta u - W'(u) = 0$. More precisely, we obtain a locally uniform-in-time bound on $\| -\varepsilon^2|\varphi_\varepsilon'|^2 + 2W(\varphi_\varepsilon)\|_{L^\infty(R_0,1)}$ which is valid except at finitely many time points (Proposition 3.9). Since $W(\varphi_\varepsilon)$ is bounded away from zero in the transition region of φ_ε, this means that $|\varphi_\varepsilon'|$ is strictly bounded away from zero in that region. Therefore, using the Implicit Function Theorem, the level sets of φ_ε are given by Hölder-$\frac{1}{2}$ graphs $r = r_\varepsilon^i(t)$ (see (5.4)) that converge to some limits $r = \bar{r}^i(t)$ (see (5.6)). The task is to find the evolution equation satisfied by $r = \bar{r}^i(t)$.

We present the idea of the method for $\bar{r} = \bar{r}^i$. Let $z = \frac{r-r_\varepsilon}{\varepsilon}$ be a rescaling and $\Phi(z,t) = \varphi(r,t)$. The equation for Φ becomes

$$\varepsilon\partial_t\Phi - \dot{r}_\varepsilon\Phi' - \frac{1}{\varepsilon}\Phi'' - \frac{n-1}{\varepsilon z + r_\varepsilon}\Phi' + \frac{1}{\varepsilon}W'(\Phi) - \lambda_\varepsilon = 0.$$

We multiply it by $\Phi'\zeta$, where ζ is a smooth time dependent test function, and in order to localize around r_ε, we integrate over $(-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}})$ and over (t_1,t_2). This gives

$$\begin{aligned}\iint \zeta\left[\varepsilon\Phi_t\Phi' - \dot{r}_\varepsilon(\Phi')^2 - \frac{n-1}{\varepsilon z + r_\varepsilon}(\Phi')^2 - \lambda_\varepsilon\Phi'\right]\,dz\,dt \\ = \int \zeta\frac{1}{\varepsilon}\left[\frac{1}{2}(\Phi')^2 - W(\Phi)\right]_{-\frac{1}{\sqrt{\varepsilon}}}^{\frac{1}{\sqrt{\varepsilon}}}\,dt.\end{aligned}$$

Now, if $\Phi(z)$ were the expected standing wave solution $\pm\tanh(z) =: \Phi_0(z)$, this would lead to the following equation for the limit $\bar{r}$

$$-c_0\int\zeta\left[\dot{\bar{r}} + \frac{(n-1)}{\bar{r}}\right]\,dt = 2\int\zeta\nu(\bar{r})\lambda_0\,dt,$$

where $c_0 = \int_{-1}^1\sqrt{2W(\varphi)}\,d\varphi$ is the constant surface tension, and ν is the direction of the jump. Thus all interfaces $\bar{r}^i$ evolve according to $-c_0\left(\dot{\bar{r}}^i + \frac{n-1}{\bar{r}^i}\right) = 2\nu^i\lambda_0$ (Proposition 7.31). But using the mass conservation property we can calculate λ_0 explicitly in terms of $\bar{r}^i$ (Proposition 7.38), thereby deducing the equation for the limiting interface.

This formal derivation was done assuming that $\Phi = \Phi_0$ around each interface. Section 6 is devoted to establishing $H^{1,2}$ and $H^{1,\infty}$ bounds on the difference between Φ and Φ_0 (Corollary 6.37). We need a bound of order $\varepsilon^{\frac{1}{2}+s}$ for some positive s, in order to replace Φ by Φ_0 in the above equation. As expected for this type of problems, this means that we have to prove these estimates for a higher order expansion. It turns out that in our case a second order expansion is sufficient (Proposition 6.14). The main observation used in the proof is that the linearization of the nonlinear operator around the standing wave defines a strictly elliptic operator (see Berger and Fraenkel [BF] and Proposition 6.24).

We put everything together in Section 7, where we derive rigorously the equation for the limiting interfaces. There are several difficulties to overcome. The most serious one is that we cannot exclude a-priori the possibility that several interfaces r_ε^i converge to the same point as $\varepsilon \to 0$. There are two cases to distinguish. Either an odd number of interfaces collide, and this corresponds to a "true" interface or jump of the limit v, or an even number of interfaces collide, and this corresponds to a "phantom" interface, i.e. $\nu = 0$. They correspond to interfaces separating the same phase. We prove that true interfaces evolve by the nonlocal flow and that their "multiplicity" is one almost everywhere. Furthermore, we show that phantom interfaces evolve by mean curvature, but we do not characterize their multiplicity (Corollary 7.42). Making use of the properties of the nonlocal flow, we also show that all interfaces decrease and at that most two true interfaces can meet or nucleate (Corollary 7.42 and Remark 7.48). In fact there are examples where two interfaces collide and disappear in the interior of the domain (Example 4.49). It is not clear whether they continue as a phantom interface or completely disappear.

In the case $n = 2$, we note that as long as there are an even number of interfaces, the nonlocal flow is simply mean curvature flow. In other words, the mean curvature preserves area in that case.

Our estimates of Section 6 are strong enough to prove a formula for the limit energy E_0, that counts both true and phantom interfaces together with their multiplicities. From this it follows that there cannot be any nucleation in the interior if there is no nucleation at the origin. But we cannot rule this out after the first geometric singularity of the nonlocal flow (see Remark 7.50).

Finally, we note that when $n = 1$, the evolution of the interfaces is expected to be exponentially slow in ε. This can be proven easily using the energy method of [BK1] combined with the result of Grant [G]. This exponentially slow motion has already been proven rigorously for the Cahn-Hilliard equation (cf [ABF], [BH], [G], [BX]).

Section 2: Energy estimates

In this section, we derive all the energy estimates necessary for the next sections. We assume that φ_ε is a solution to (1.1) with the boundary data (1.2), that the domain $\Omega \subset \mathbf{R}^n$ is bounded with Lipschitz boundary and that

$$E_\varepsilon[\varphi](0) \leq C. \tag{2.1}$$

Here and throughout this paper C denotes a positive constant, that might vary from line to line. We show that the energy

$$E_\varepsilon[\varphi](t) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}W(\varphi)\,dx \tag{2.2}$$

is a Lyapunov functional for (1.1)-(1.2), and use this fact to obtain appropriate BV bounds, as well as some "weak" Hölder estimates on φ_ε. We then produce an L^1 limit for the solution φ_ε. In addition we use the fact that $E_\varepsilon[\varphi]$ is a monotone function to show that it is weakly compact in $BV(0,T)$ and compact in $L^1(0,T)$. Finally we ust the monotonicity of the energy to construct positive time intervals, where the variation of the energy is uniformly small in ε.

Proposition 2.3. Energy estimates.

Let $\varphi := \varphi_\varepsilon$ be a solution to (1.1) with boundary condition (1.2) and suppose that the initial data satisfy (2.1). Let g be defined via $g'(s) := \sqrt{2W(s)}$ with $g(0) := 0$, and let $0 \le s < \tau \le T$. Then the following statements hold

$$\varepsilon \int_s^\tau \int_\Omega |\partial_t \varphi|^2 \, dx \, dt + E_\varepsilon[\varphi](\tau) - E_\varepsilon[\varphi](s) = 0, \tag{2.4}$$

$$\sup_{t \in [0,T]} \int_\Omega |\nabla g(\varphi)| \, dx \le \sup_{t \in [0,T]} E_\varepsilon[\varphi](t) \le C, \tag{2.5}$$

$$\int_s^\tau \int_\Omega |\partial_t g(\varphi)| \, dx \, dt \le C\sqrt{\tau - s}. \tag{2.6}$$

Proof. First multiplying equation (1.1) by $\partial_t \varphi$ and integrating in x, it follows

$$\varepsilon \int_\Omega |\partial_t \varphi|^2 \, dx - \varepsilon \int_\Omega \partial_t \varphi \Delta \varphi \, dx + \frac{1}{\varepsilon} \int_\Omega \partial_t W(\varphi) \, dx - \lambda_\varepsilon(t) \int_\Omega \partial_t \varphi \, dx = 0.$$

Using the mass conservation property (1.5) and integrating by part, this reduces to

$$\varepsilon \int_\Omega |\partial_t \varphi|^2 \, dx + \int_\Omega \partial_t \left[\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} W(\varphi) \right] dx = 0.$$

Equation (2.4) follows from the definition of E_ε in (2.2) and integration in time from s to τ.

Next we obtain the BV estimates (2.5) and (2.6). These estimates are not new (see for example [M] or [BK]) but we include them for the sake of completeness. Using the definition of g, we have (cf [M])

$$\int_\Omega |\nabla g(\varphi)| \, dx = \int_\Omega \sqrt{2W(\varphi)} |\nabla \varphi| \, dx \le E_\varepsilon[\varphi](t).$$

Inequality (2.5) now follows from (2.4) and the initial bound on the energy (2.1). Moreover we obtain the "weak" Hölder estimate (cf [BK])

$$\begin{aligned}
\int_s^\tau \int_\Omega |\partial_t g(\varphi)| \, dx \, dt &\le \left(\int_s^\tau \int_\Omega \frac{1}{\varepsilon} W(\varphi) \, dx \, dt \right)^{\frac{1}{2}} \left(\int_s^\tau \int_\Omega \varepsilon |\partial_t \varphi|^2 \, dx \, dt \right)^{\frac{1}{2}} \\
&\le \left(\int_s^\tau E_\varepsilon[\varphi](0) \, dt \right)^{\frac{1}{2}} (E_\varepsilon[\varphi](\tau) - E_\varepsilon[\varphi](s))^{\frac{1}{2}} \\
&\le C\sqrt{\tau - s}.
\end{aligned}$$

This completes the proof of Proposition 2.3.

In particular, it follows that the functional E_ε is a Lyapunov functional for equation (1.1)–(1.2). From this fact follows the existence of a limit for φ_ε and an a-priori bound on the Lagrange multiplier $\lambda_\varepsilon(t) = \fint_\Omega W'(\varphi) \, dx$.

Corollary 2.7. *Under the same hypothesis as in Proposition 2.3, the following results hold:*

$$\varphi \in L^\infty(0, T, L^4(\Omega)), \tag{2.8}$$

$$\sup_t \lambda_\varepsilon(t) \le \frac{C}{\sqrt{\varepsilon}}, \tag{2.9}$$

$$\sup_{t,x} |\varphi(t,x)| \le C\sqrt{\varepsilon} + \sup_x |\varphi(x,0)| + 1. \tag{2.10}$$

Proof. The statement (2.8) is a direct consequence of Proposition 2.3. Inequality (2.9) follows from (2.8), since

$$\sup_t \lambda_\varepsilon = \sup_t \frac{1}{\varepsilon} \int_\Omega 2(\varphi^3 - \varphi)\, dx \le \sup_t \frac{C}{\varepsilon} \left(\int_\Omega \varphi^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega W(\varphi)\, dx\right)^{\frac{1}{2}} \le \frac{C}{\sqrt{\varepsilon}}.$$

The last estimate (2.10) is a consequence of the maximum principle and (2.9).

Remark 2.11. The energy estimates imply weak compactness for the sequence $g(\varphi_\varepsilon)$ in $BV(\Omega \times (0,T))$, so that we can choose a subsequence $g(\varphi_\varepsilon) \xrightarrow[\varepsilon\to 0]{*} g(v)$ in BV. This in turn implies that for some subsequence $\varphi_\varepsilon \to v$ in $L^1(\Omega \times (0,T))$, since g^{-1} exists, and $\varphi_\varepsilon \in L^\infty(0,T; L^4(\Omega))$. In addition, $v = \pm 1$ a.e. and it is "weakly" Hölder continuous

$$\int_\Omega |v(x,\tau) - v(x,s)|\, dx \le C\sqrt{\tau - s} \quad \text{for} \quad T \ge \tau \ge s \ge 0.$$

This Hölder continuity of v in L^1 is due to Bronsard and Kohn [BK]. Here it is a consequence of (2.6), since this estimate carries over to the limit by lower semi-continuity.

In addition, by choosing another subsequence if necessary, we may assume that the initial data $\varphi_\varepsilon(\cdot,0)$ converge in $L^1(\Omega)$ to $v(\cdot,0)$. From now on we will only consider this subsequence and we will still denote it by φ_ε. In what follows we will select still other subsequences of this one, but this does not have an impact on v.

Another important consequence of Proposition 2.3 is that $E_\varepsilon[\varphi](\cdot)$ is monotone decreasing in t and hence weakly compact in $\mathrm{BV}(0,T)$.

Corollary 2.12. *Let φ_ε be as in Proposition 2.3. Then $E_\varepsilon[\varphi_\varepsilon](\cdot)$ is weakly compact in $BV(0,T)$. Therefore for an appropriate subsequence of ε's, there exists a function $E_0(\cdot)$ such that*

$$E_\varepsilon[\varphi_\varepsilon](\cdot) \longrightarrow E_0(\cdot) \quad \textit{in } L^1(0,T) \textit{ and almost everywhere,} \tag{2.13}$$

$$\partial_t E_\varepsilon[\varphi_\varepsilon](\cdot) \overset{*}{\longrightarrow} \partial_t E_0(\cdot), \tag{2.14}$$

*where the weak * convergence is in $[C^0(0,T)]'$.*

Proof. By (2.5) $E_\varepsilon[\varphi]$ is clearly uniformly bounded in $L^1(0,T)$. Moreover the identity (2.4) implies that $E_\varepsilon[\varphi]$ is monotone decreasing and thus uniformly in $BV(0,T)$ by assumption (2.1) on the initial data. Thus we may select a subsequence of ε's as claimed.

The results of Modica [M], Modica–Mortola [MM] and Sternberg [S] show that E_ε Γ-converges to a functional E_* which is defined on BV functions by

$$E_*[v] = c_0 \int_\Omega |\nabla v|\, dx, \tag{2.15}$$

where $c_0 := \int_{-1}^1 \sqrt{2W(\varphi)}\, d\varphi$. In particular, it is easy to see that for almost all t

$$\liminf_{\varepsilon\to 0} E_\varepsilon[\varphi_\varepsilon](t) \ge E_*[v](t). \tag{2.16}$$

From Corollary 2.12, we cannot conclude that E_0 is $E_*[v]$, but we know that for almost all t, we have $E_0(\cdot) \geq E_*[v](\cdot)$.

In Corollary 2.12 we have shown that for almost all t,

$$E_\varepsilon[\varphi_\varepsilon](t) \to E_0(t).$$

Define for any $\eta > 0$ a set $N(\eta) \subset [0,T]$ as the set of all jump points of E_0 with height at least η:

$$N(\eta) := \left\{ t \mid ess \inf_{s<t} E_0(s) - ess \sup_{s>t} E_0(s) \geq \eta \right\}. \tag{2.17}$$

Then for any $\eta > 0$ the set $N(\eta)$ is finite, since E_0 is monotone decreasing in an L^1–sense:

$$E_0(t) \geq E_0(s) \quad \text{for almost every } s \geq t. \tag{2.18}$$

In fact, since $E_\varepsilon[\varphi_\varepsilon](t) \leq C$ by (2.5), it follows that

$$\# N(\eta) \leq \frac{C}{\eta}. \tag{2.19}$$

For $t_0 > 0$ we define $T_\varepsilon(\eta, t_0) > 0$ by

$$\varepsilon \int_{t_0 - T_\varepsilon(\eta,t_0)}^{t_0+T_\varepsilon(\eta,t_0)} \int_\Omega (\partial_t \varphi_\varepsilon)^2 \, dx \, dt = \eta. \tag{2.20}$$

The following lemma is very important in our approach. It is based on the fact that E_0 is monotone decreasing. It basically says that given any $t_0 \notin N(\eta)$, we can find an open interval $(t_0 - T_0(\eta, t_0), t_0 + T_0(\eta, t_0))$ on which the variation of the energy $E_\varepsilon[\varphi](\cdot)$ is uniformly small in ε.

Lemma 2.21. *Let $\varphi = \varphi_\varepsilon$ be as in Proposition 2.3. Let $0 < t_0 \notin N(\eta)$, where $N(\eta)$ is given by (2.17) and let $T_\varepsilon(\eta, t_0)$ be as in (2.20). Then there exists $T_0(\eta, t_0) > 0$ such that*

$$T_\varepsilon(\eta, t_0) > T_0(\eta, t_0) \quad \textit{for } \varepsilon \leq \varepsilon_0(\eta, t_0).$$

In particular

$$E_\varepsilon[\varphi](t_0 - T_0) - E_\varepsilon[\varphi](t_0 + T_0) = \varepsilon \int_{t_0 - T_0(\eta,t_0)}^{t_0+T_0(\eta,t_0)} \int_\Omega (\partial_t \varphi^\varepsilon)^2 \, dx \, dt \leq \eta.$$

Proof. Suppose to the contrary that $T_\varepsilon \to 0$ for some subsequence. Then using (2.4) and the monotonicity of $E_\varepsilon[\varphi_\varepsilon]$, we have for almost any $\tau > 0$

$$\begin{aligned}
0 < \eta &= \lim_{\varepsilon \to 0} \varepsilon \int_{t_0 - T_\varepsilon(\eta,t_0)}^{t_0+T_\varepsilon(\eta,t_0)} \int_\Omega (\partial_t \varphi)^2 \, dx \, dt \\
&= \lim_{\varepsilon \to 0} \left(E_\varepsilon[\varphi](t_0 - T_\varepsilon(\eta, t_0)) - E_\varepsilon[\varphi](t_0 + T_\varepsilon(\eta, t_0)) \right) \\
&\leq \lim_{\varepsilon \to 0} \left(E_\varepsilon[\varphi](t_0 - \tau) - E_\varepsilon[\varphi](t_0 + \tau) \right) \\
&= E_0(t_0 - \tau) - E_0(t_0 + \tau).
\end{aligned}$$

Thus

$$0 < \eta \le ess \inf_{s<t_0} E_0(s) - ess \sup_{s>t_0} E_0(s) < \eta,$$

by the choice of t_0.

Section 3: A first approximation

The subsequent sections will be restricted to radially symmetric solutions; without loss of generality we will assume that Ω is the unit disk in $\mathbf{R}^n$, $n \ge 2$.

The evolution for $\varphi = \varphi_\varepsilon(r,t)$ becomes in radial coordinates

$$\begin{aligned} &\varepsilon\partial_t\varphi - \varepsilon\varphi_{rr} - \frac{\varepsilon(n-1)}{r}\varphi_r + \frac{1}{\varepsilon}W'(\varphi) - \lambda_\varepsilon(t) = 0, \\ &\varphi(r,0) = \varphi_\varepsilon^0(r), \end{aligned} \tag{3.1}$$

where as explained in the introduction, we choose $W(\varphi) = \frac{1}{2}(1-\varphi^2)^2$. Moreover, since we consider the case of a Neumann boundary condition and since φ is smooth,

$$\varphi'(1,t) = 0 \quad \text{and} \quad \varphi'(0,t) = 0. \tag{3.2}$$

Thus mass is preserved. We assume the following conditions on the initial data

$$\|\varphi_\varepsilon^0\|_{L^\infty(0,1)} \le C, \tag{3.3}$$

and

$$E_\varepsilon[\varphi_\varepsilon^0] \le C. \tag{3.4}$$

In addition, we assume that there exists $\omega > 0$ such that

$$\left|\int_\Omega \varphi_\varepsilon^0\, dx\right| < |\Omega| - \omega, \tag{3.5}$$

in order to ensure that the limit problem has an interface.

The next proposition is essential to the approach used in this paper. It is used to show that, away from the origin, the solution φ_ε is a-priori close to the function $\pm q(\frac{\xi}{\varepsilon})$ where q solves

$$q_{\xi\xi} = W'(q) \qquad \text{with} \qquad q(-\infty) = -1, \quad q(\infty) = 1, \quad q(0) = 0.$$

In other words the solution φ_ε is close to the one dimensional standing wave solution $\pm q(\xi)$ associated to the equation $u_t = u_{\xi\xi} - W'(u)$, as is predicted from the formal asymptotic expansions of Rubinstein–Sternberg (cf [RS]). For the existence and properties of the standing wave solution q we refer to Aronson–Weinberger [AW] and to Fife–McLeod [FM]. When $W(\varphi) = \frac{1}{2}(1-\varphi^2)^2$, this standing wave is given by $q(\xi) = \tanh(\xi)$.

Proposition 3.6. *Let φ_ε and φ_ε^0 satisfy (3.1)–(3.5). Let $0 < R_0 < 1$, and $t_1 > t_2$. Then for any $t_2 < t < t_1$*

$$\begin{aligned} &\left\| -\frac{\varepsilon^2}{2}|\varphi'|^2 + W(\varphi)\right\|_{L^\infty(R_0,1)}(t) \\ &\qquad \le C(R_0)\left(\sqrt{\varepsilon} + \left(\varepsilon\int_{t_2}^{t_1}\int_\Omega \varphi_t^2\, dx\, dt\right)^{\frac{1}{2}} + \left(\varepsilon^3\int_\Omega \varphi_t^2(x,t_2)\, dx\right)^{\frac{1}{2}}\right). \end{aligned}$$

Proof. First we note that $\|\varphi\|_{L^\infty(\Omega\times(0,T))} \le C$ by assumption (3.3) and Corollary 2.7. Therefore, multiplying (3.1) by $\varepsilon\varphi'$, integrating over $(\eta,\rho) \subset (R_0, 1)$, using that the energy is bounded (cf (2.5)) and the bound on $\lambda_\varepsilon(\cdot)$ (cf (2.10)), it follows

$$
\begin{aligned}
\left|-\frac{\varepsilon^2}{2}|\varphi'(\rho,t)|^2 + W(\varphi(\rho,t))\right| &= \left|-\frac{\varepsilon^2}{2}|\varphi'(\eta,t)|^2 + W(\varphi(\eta,t)) - \varepsilon^2\int_\eta^\rho \varphi_t\varphi'\,dr\right. \\
&\qquad \left. +2\varepsilon^2\int_\eta^\rho \frac{1}{r}|\varphi'|^2\,dr + \varepsilon\lambda_\varepsilon(t)(\varphi(\rho,t)-\varphi(\eta,t))\right| \qquad (3.7)\\
&\le \frac{\varepsilon^2}{2}|\varphi'(\eta,t)|^2 + W(\varphi(\eta,t)) \\
&\quad + \frac{\varepsilon^2}{R_0^2}\left(\int_\eta^\rho (\varphi_t)^2 r^2\,dr\right)^{\frac12}\left(\int_\eta^\rho |\varphi'|^2 r^2\,dr\right)^{\frac12} + \frac{2\varepsilon^2}{R_0^3}\int_\eta^\rho |\varphi'|^2 r^2\,dr \\
&\quad + 2\varepsilon|\lambda_\varepsilon(t)|\,\|\varphi\|_{L^\infty(\Omega\times(0,T))} \\
&\le \frac{\varepsilon^2}{2}|\varphi'(\eta,t)|^2 + W(\varphi(\eta,t)) + C\frac{\varepsilon^{\frac32}}{R_0^2}\left(\int_{R_0}^1 (\varphi_t)^2 r^2\,dr\right)^{\frac12} \\
&\quad + C\frac{\varepsilon}{R_0^3} + C\sqrt{\varepsilon}.
\end{aligned}
$$

Next integrating in η over $(R_0, 1)$ and again using the bound on the energy (2.5), we find for $t_0 \le t \le t_1$

$$
\left|-\frac{\varepsilon^2}{2}|\varphi'(\rho,t)|^2 + W(\varphi(\rho,t))\right| \le C(R_0)\left(\sqrt{\varepsilon} + \left(\varepsilon^3\int_\Omega (\varphi_t(x,t))^2\,dx\right)^{\frac12}\right) \qquad (3.8)
$$

So we are left with estimating the last term in (3.7). For this we follow Stoth ([S1]) and consider the equation satisfied by $\partial_t\varphi$ on $\Omega\times(t_2,t_1)$

$$
\varepsilon\partial_{tt}\varphi - \varepsilon\Delta\partial_t\varphi + \frac{1}{\varepsilon}W''(\varphi)\partial_t\varphi - \partial_t\lambda_\varepsilon = 0.
$$

We multiply it by $\varepsilon\partial_t\varphi$ and integrate over $\Omega\times(t_2,\tau)$ for $\tau < t_1$ to obtain

$$
\begin{aligned}
\int_{t_2}^\tau\int_\Omega \varepsilon^2\varphi_{tt}\,\varphi_t\,dx\,dt - \int_{t_2}^\tau\int_\Omega \varepsilon^2\varphi_t\Delta\varphi_t\,dx\,dt &= -\int_{t_2}^\tau\int_\Omega W''(\varphi)(\varphi_t)^2\,dx\,dt \\
&\quad + \int_{t_2}^\tau \varepsilon\partial_t\lambda_\varepsilon\int_\Omega \varphi_t\,dx\,dt \\
&= -\int_{t_2}^\tau\int_\Omega W''(\varphi)(\varphi_t)^2\,dx\,dt,
\end{aligned}
$$

by the mass conservation property (1.4). Next we integrate by parts and find using the boundary condition (3.2) and that $W''(\varphi) = 2(3\varphi^2 - 1)$ is bounded:

$$
\begin{aligned}
&\int_\Omega \frac{\varepsilon^2}{2}|\varphi_t(x,\tau)|^2\,dx + \int_{t_2}^\tau\int_\Omega \varepsilon^2|\nabla\varphi_t|^2\,dx\,dt \\
&\qquad \le \int_\Omega \frac{\varepsilon^2}{2}|\varphi_t(x,t_2)|^2\,dx + C\int_{t_2}^\tau\int_\Omega (\varphi_t(x,t))^2\,dx\,dt.
\end{aligned}
$$

The proposition now follows from (3.8).

Now according to Lemma 2.21, we can choose T_0 small enough such that $\varepsilon\int_{t_0-T_0}^{t_0+T_0}\int_\Omega(\varphi_t)^2\,dx\,dt$ is as small as desired if $t_0\notin N(\eta)$. This means that, away from the origin, the solution φ_ε is as close as we want to the standing wave q in (t_0-T_0,t_0+T_0). This is the content of the following important consequence of Proposition 3.6 and Lemma 2.21.

Proposition 3.9 (First approximation).
Let φ_ε and φ_ε^0 satisfy (3.1) - (3.5). Let $0<R_0<1$ and $\delta>0$. Let $0\neq t_0\notin N(\eta)$ for $\eta=\eta(\delta,R_0)$ as defined below in the proof. Then there exists $T_0=T_0(\delta,R_0,t_0)>0$ and $\varepsilon_0=\varepsilon_0(\delta,R_0,t_0)>0$ such that

$$\sup_{t_0-T_0\le t\le t_0+T_0}\left\|-\frac{\varepsilon^2}{2}|\varphi'|^2+W(\varphi)\right\|_{L^\infty(R_0,1)}\le\delta^2\qquad\text{for } \varepsilon\le\varepsilon_0.$$

We then rename $N(\eta(\delta,R_0))$ to be $N(\delta,R_0)$.

Proof. Define η via $\sqrt{\eta}=\frac{\delta^2}{2C(R_0)}$, with $C(R_0)$ as in (3.7), and choose $T_0=T_0(\delta,R_0,t_0)$ to be as in Lemma 2.21. Then we use Proposition 3.6 with $t_1=t_0+T_0$ and the mean value over $t_2\in(t_0-T_0,t_0-\frac{T_0}{2})$ to obtain for $t_0-\frac{T_0}{2}\le t\le t_0+T_0$

$$\begin{aligned}
&\left\|-\frac{\varepsilon^2}{2}|\varphi'(\cdot,t)|^2+W(\varphi(\cdot,t))\right\|_{L^\infty(R_0,1)}\\
&\quad\le C(R_0)\left(\sqrt{\varepsilon}+\left(\varepsilon\int_{t_0-T_0}^{t_0+T_0}\int_\Omega\varphi_t^2\,dx\,dt\right)^{\frac12}+\left(\varepsilon^3\int_\Omega\varphi_t^2(x,t_2)\,dx\right)^{\frac12}\right)\\
&\left\|-\frac{\varepsilon^2}{2}|\varphi'(\cdot,t)|^2+W(\varphi(\cdot,t))\right\|_{L^\infty(R_0,1)}\\
&\quad\le C(R_0)\left(\sqrt{\varepsilon}+\left(\varepsilon\int_{t_0-T_0}^{t_0+T_0}\int_\Omega\varphi_t^2\,dx\,dt\right)^{\frac12}+\left(\varepsilon^3\frac{2}{T_0}\int_{t_0-T_0}^{t_0-\frac{T_0}{2}}\int_\Omega\varphi_t^2(x,t_2)\,dx\,dt_2\right)^{\frac12}\right)\\
&\quad\le C(R_0)\left(\sqrt{\varepsilon}+\frac{\delta^2}{2C(R_0)}+\varepsilon\sqrt{\frac{2}{T_0}}\right)\\
&\quad\le\delta^2,
\end{aligned}$$

for $\varepsilon\le\varepsilon_0(\delta,R_0,t_0)$. We then rename $\frac{T_0}{2}$ to be T_0.

Remark 3.10. If $t_0=0$, then the same result as Proposition 3.9 holds true with $(-T_0,T_0)$ substituted by $[0,T_0)$. A condition for this is, that $\varepsilon^3\int_\Omega\partial_t\varphi_\varepsilon^2(x,0)\,dx\to0$, which by equation (1.1) is equivalent to the condition $\varepsilon^3\int_\Omega(\Delta\varphi_\varepsilon-\frac{1}{\varepsilon^2}W'(\varphi_\varepsilon))^2(x,0)\,dx\to0$.

This proposition is crucial in our approach. It has two important consequences. The first one is that we can define the interfaces of φ_ε by showing that the level sets of φ_ε are graphs. This is done in Section 5. The second consequence is even better approximation of φ_ε by the standing wave solution associated to (1.1), namely we obtain a second order approximation for φ_ε. This will be shown in Section 6. Finally this approximation will be used in taking the weak limit of equation (3.1) to obtain the desired limiting equation in Section 7.

Section 4: Two additional estimates in the radial case

In this section we first obtain a better estimate on the Lagrange multiplier $\lambda_\varepsilon(\cdot)$. In Corollary 2.7 we have shown that the L^∞-norm of the Lagrange multiplier is of order $\frac{1}{\sqrt{\varepsilon}}$, but the formal asymptotics suggest that the Lagrange multiplier is bounded in ε. Here we prove that in fact the L^2-norm of $\lambda_\varepsilon(\cdot)$ is uniformly bounded.

An other objective of this section is to study the blow up of φ'_ε and $W(\varphi_\varepsilon)$ at the origin.

So let $\varphi = \varphi_\varepsilon$ be a solution of the radial equation (3.1) with the Neumann condition (3.2), the energy bound (3.4) and the mass condition (3.5).

Proposition 4.1. (Estimate on the Lagrange multiplier). *For λ_ε as in (1.1) we have the estimate*

$$\int_0^T |\lambda_\varepsilon(t)|^2\, dt \leq C.$$

Proof. We multiply the nonlocal differential equation (3.1) by φ', integrate over (σ,ρ) and multiply by $\rho^{n-1}\sigma^{n-1}$. This yields

$$\begin{aligned}
&\lambda_\varepsilon(t)\,(\varphi(\rho) - \varphi(\sigma))\,\rho^{n-1}\sigma^{n-1} \\
&\quad = \rho^{n-1}\sigma^{n-1}\left(\varepsilon\int_\sigma^\rho \partial_t\varphi\varphi'\, dr - \frac{\varepsilon}{2}|\varphi'|^2\Big|_\sigma^\rho + \frac{1}{\varepsilon}W(\varphi)\Big|_\sigma^\rho - (n-1)\varepsilon\int_\sigma^\rho \frac{1}{r}|\varphi'|^2\, dr\right) \\
&\quad =: G_\varepsilon(\rho,\sigma).
\end{aligned} \tag{4.2}$$

The idea now is to integrate this equality in ρ over the set where φ is close to 1 and in σ over the set where φ is close to -1. For this to give a meaningful result, it is important to prove that the measure of both these sets is bounded away form zero uniformly in t. This follows from the property that the mass is conserved and the fact that the measure of the transition layers of φ are uniformly small.

Indeed define (for η less but close to 1)

$$C := C_\varepsilon(t) := \{y \in \Omega : \eta \leq \varphi(y,t) \leq \frac{1}{\eta}\},$$

$$B := B_\varepsilon(t) := \{x \in \Omega : -\frac{1}{\eta} \leq \varphi(x,t) \leq -\eta\}.$$

We observe that these definitions and the conservation of mass imply immediately

$$\mathcal{L}^n(C) + \mathcal{L}^n(B) \leq \mathcal{L}^n(\Omega), \tag{4.3}$$

$$\eta\mathcal{L}^n(C) - \frac{1}{\eta}\mathcal{L}^n(B) \leq M_\varepsilon(t) \leq \frac{1}{\eta}\mathcal{L}^n(C) - \eta\mathcal{L}^n(B) \tag{4.4}$$

where

$$M_\varepsilon(t) := \int_\Omega \varphi(x,0)\,dx - \int_{\Omega\setminus B\cup C} \varphi(x,t)\,dx$$

is the mass, without the transition zones.
But by the energy estimate (2.5) we have $\int_{\Omega\setminus B\cup C} W(\varphi)\,dx \leq C\varepsilon$, so that

$$\left|\int_{\Omega\setminus B\cup C} \varphi(x,t)\,dx\right| \leq C\cdot\varepsilon\left(\frac{1}{W(\eta)} + \frac{1}{W(1/\eta)}\right).$$

Thus for ε sufficiently small, assumption (3.5) implies

$$|\Omega| - |M_\varepsilon(t)| > \frac{\omega}{2} > 0, \tag{4.5}$$

and

$$\mathcal{L}^n(C) + \mathcal{L}^n(B) \geq |\Omega| - \frac{\omega}{4}. \tag{4.6}$$

Now note that the equations (4.3), (4.4) and (4.6) define a polygon and the calculation of the corner points together with the above estimate (4.5) yields

$$\mathcal{L}^n(B);\ \mathcal{L}^n(C) \geq \frac{\eta^2}{1+\eta^2}\left(\frac{\omega}{4}\right) - \frac{\eta(1-\eta)}{1+\eta^2}|\Omega|, \tag{4.7}$$

if only η is close enough to 1.

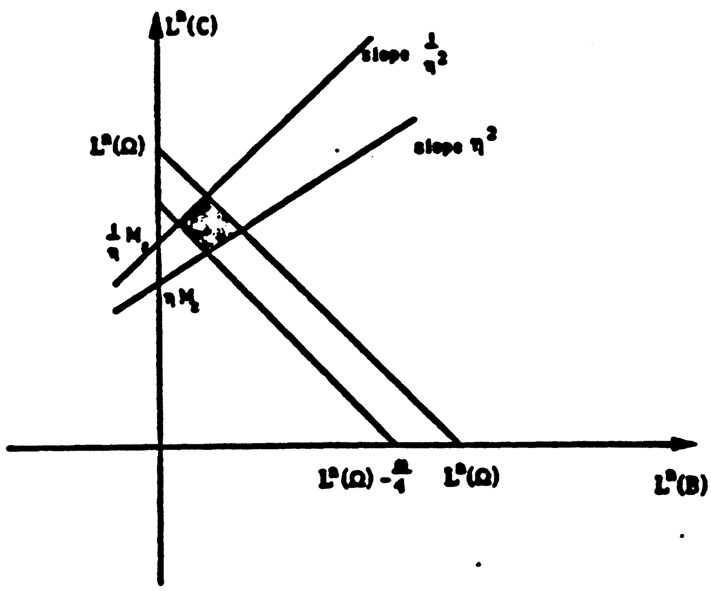

Figure 1: the case $M_\varepsilon > 0$

Thus for η close enough to 1, depending on ω, we obtain that

$$\mathcal{L}^n(B),\quad \mathcal{L}^n(C) \geq d_0 > 0 \tag{4.8}$$

for all sufficiently small ε.

Now we return to formula (4.2). We integrate it over $\rho \in B\backslash B_{R_0}$ and $\sigma \in C\backslash B_{R_0}$ and use estimate (4.8) to find (with ω_n being the measure of the sphere)

$$
\begin{aligned}
&2|\lambda_\varepsilon(t)|\,\eta\frac{1}{\omega_n^2}(d_0 - R_0^2)^2 \\
&\leq \left| \int_{B\backslash B_{R_0}} \int_{C\backslash B_{R_0}} G_\varepsilon(\rho,\sigma)\,d\rho\,d\sigma \right| \\
&\leq \varepsilon|\Omega|^2 \left(\int_\Omega |\partial_t\varphi|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |\varphi'|^2\,dx\right)^{\frac{1}{2}} + 2E_\varepsilon(\varphi) + (n-1)|\Omega|^2\varepsilon \int_{R_0}^1 r^{n-2}|\varphi'|^2\,dr \\
&\leq C(R_0)\left(|\Omega|^2\sqrt{\varepsilon}\left(\int_\Omega |\partial_t\varphi|^2\,dx\right)^{\frac{1}{2}} + 1\right). \qquad (4.9)
\end{aligned}
$$

once again applying the energy estimate (2.5) Now we choose R_0 small enough, square the inequality and integrate in time to conclude.

Remark 4.10. As a consequence of this proposition, we choose a further subsequence of ε's such that $\lambda_\varepsilon \longrightarrow \lambda_0$ in $L^2(0,T)$.

Proposition 4.11. *For $\varphi = \varphi_\varepsilon$ we have the estimate*

$$
\int_0^T \int_0^1 |g(\varphi_\varepsilon)'|r^{n-2}\,dr\,dt \leq \int_0^T \int_0^1 \left(\frac{\varepsilon}{2}|\varphi_\varepsilon'|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon)\right) r^{n-2}\,dr\,dt \leq C. \qquad (4.12)
$$

Proof. We multiply the nonlocal equation (3.1) by $(-r^{n-1}\varphi_\varepsilon')$ and integrate over $(0,s)$. This yields

$$
\begin{aligned}
\varepsilon \int_0^s \varphi_\varepsilon''\varphi_\varepsilon' r^{n-1}\,dr + (n-1)\varepsilon \int_0^s {\varphi_\varepsilon'}^2 r^{n-2}\,dr - \frac{1}{\varepsilon}\int_0^s W'(\varphi_\varepsilon)\varphi_\varepsilon' r^{n-1}\,dr \\
= -\lambda_\varepsilon \int_0^s \varphi_\varepsilon' r^{n-1}\,dr + \varepsilon \int_0^s \partial_t\varphi_\varepsilon\varphi_\varepsilon' r^{n-1}\,dr.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{n-1}{2}\varepsilon \int_0^s {\varphi_\varepsilon'}^2 r^{n-2}\,dr + \frac{n-1}{\varepsilon}\int_0^s W(\varphi_\varepsilon) r^{n-2}\,dr \\
\leq \left(\frac{\varepsilon}{2}|\varphi_\varepsilon'|^2(s) + \frac{1}{\varepsilon}W(\varphi_\varepsilon)(s) + 2|\lambda_\varepsilon|\,\|\,\varphi_\varepsilon\,\|_{L^\infty}\right) s^{n-1} \\
+ \varepsilon \int_0^s |\,\partial_t\varphi_\varepsilon\varphi_\varepsilon'\,| r^{n-1}\,dr.
\end{aligned}
$$

Now the left hand side at $s = \frac{1}{2}$ is bounded by the mean value of the right hand side taken over $s \in (1/2,1)$. Therefore the energy estimate (2.5), the L^∞-bound (2.8) and the bound on the Lagrange multiplier (4.1) give

$$
\int_0^T \int_0^{1/2} \left(\frac{\varepsilon}{2}|\varphi_\varepsilon'|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon)\right) r^{n-2}\,dr\,dt \leq C.
$$

This establishes the result, since in the interval $(\frac{1}{2}, 1)$ there is nothing to prove.

Section 5: Definition of the interfaces of φ_ε and of the limit v

In this section we present the definition and properties of the interfaces of φ_ε and of v. The definition of the interfaces is based on the fact that Proposition 3.9 implies a lower bound on $|\varphi_\varepsilon'|$ such that we can apply the Implicit Function Theorem. Indeed, let

$$\delta^2 < \frac{1}{8} \quad \text{and } 0 < Q < tanh\left[\frac{1}{2} - tanh^{-1}\frac{1}{\sqrt{3}}\right] \text{ be such that } W(Q) \geq \frac{1}{4}. \tag{5.1}$$

We will study the level sets of φ_ε, of value less than Q. This precise choice of Q is important in the ellipticity Proposition 6.24 and in the following.

According to Proposition 3.9, $\frac{\varepsilon^2}{2}|\varphi_\varepsilon'|^2 \geq \frac{1}{8}$ in the subset of $(R_0, 1) \times (t_0 - T_0, t_0 + T_0)$ defined by $|\varphi_\varepsilon| \leq Q$, since in this set $W(\varphi_\varepsilon) \geq \frac{1}{4}$. This means that φ_ε must be monotone in r on each connected component of this set.

So let φ_ε satisfy (3.1) through (3.5), and define for any $R_0 > 0$

$$A_{R_0} := \bigcup_{t_0 \notin N(\delta, R_0)} (t_0 - T_0(\delta, R_0, t_0), t_0 + T_0(\delta, R_0, t_0)), \tag{5.2}$$

where δ is a fixed constant to be chosen later. We remark that by definition A_{R_0} is open and that its complement has at most finitely many points, all of them in $N(\delta, R_0)$.

Let C_{R_0} be open and its closure still in A_{R_0}. Then $\overline{C_{R_0}}$ and hence C_{R_0} can be covered by finitely many of the $(t_0 - T_0(\delta, R_0, t_0), t_0 + T_0(\delta, R_0, t_0))$, that were used to define A_{R_0}. Thus on C_{R_0}, Proposition 3.9 implies that

$$\sup_{C_{R_0}} \| -\varepsilon^2|\varphi'|^2 + 2W(\varphi)\|_{L^\infty(R_0,1)} \leq 2\delta^2,$$

for all $\varepsilon \leq \varepsilon_0(\delta, R_0)$.

We now consider the "ε problem" in the strip $(R_0, 1) \times C_{R_0}$.

Let Q be as in (5.1). Then on $\{|\varphi_\varepsilon| \leq Q\} \cap (R_0, 1) \times C_{R_0}$, we have $\varepsilon^2|\varphi_\varepsilon'|^2 \geq \frac{1}{4}$. Thus for any $-Q < a < Q$ the set $\{\varphi_\varepsilon(r,t) = a\} \cap (R_0, 1) \times C_{R_0}$ consists of a collection of graphs $r_\varepsilon^i(\cdot, a_\varepsilon)$.

Moreover by the implicit function theorem the following identities hold

$$\begin{aligned} \partial_t\varphi_\varepsilon(r_\varepsilon^i(t,a),t) + \partial_t r_\varepsilon^i(t,a)\partial_r\varphi_\varepsilon(r_\varepsilon^i(t,a),t) &= 0, \\ \partial_a r_\varepsilon^i(t,a)\partial_r\varphi_\varepsilon(r_\varepsilon^i(t,a),t) &= 1. \end{aligned}$$

Using the co-area formula this implies an $H^{1,2}$-estimate on r_ε^i. Indeed

$$\begin{aligned} \int_{\{|\varphi_\varepsilon(\cdot,t)|<Q, r>R_0\}} \frac{|\partial_t\varphi_\varepsilon(r,t)|^2}{|\varphi_\varepsilon'(r,t)|}\,dr &= \int_{-Q}^{Q}\int_{\{\varphi_\varepsilon(\cdot,t)=a, r>R_0\}} \frac{|\partial_t\varphi_\varepsilon(r,t)|^2}{|\varphi_\varepsilon'(r,t)|^2}\,d\mathcal{H}^0\,da \\ &= \int_{-Q}^{Q}\left(\sum_i |\partial_t r_\varepsilon^i(t,a)|^2\right) da. \end{aligned}$$

Thus

$$\int_{C_{R_0}} \int_{-Q}^{Q} \left(\sum_i |\partial_t r_\varepsilon^i(t,a)|^2 \right) da\, dt \le \frac{\varepsilon}{\sqrt{\delta}} \int_{C_{R_0}} \int_{R_0}^{1} |\partial_t \varphi_\varepsilon(r,t)|^2\, dr\, dt$$
$$\le C(R_0,\delta).$$

Thus we may choose $a_\varepsilon \in (-Q,Q)$ such that (for some bigger $C(R_0,\delta)$)

$$\int_{C_{R_0}} (|\partial_t r_\varepsilon^i(t,a_\varepsilon)|^2)\, dt \le C(R_0,\delta).$$

This in turn implies that the graphs $r_\varepsilon^i(\cdot,a_\varepsilon)$ are Hölder–$\frac{1}{2}$ by imbedding.

We define the interfaces of the ε problem by

$$r_\varepsilon^i(t) := r_\varepsilon^i(t,a_\varepsilon). \tag{5.3}$$

WE note that none to these interfaces hits the fixed boundary $\partial\Omega$, because on the fixed boundary $\varphi_\varepsilon' = 0$ and thus by Proposition 3.9, the values of φ_ε have to be close to ± 1. But on the interfaces the values of φ_ε are given by a_ε and are hence uniformly away from ± 1. Thus all the interfaces exist, as long as they do not hit $r = R_0$. This allows us to introduce the notation

$$r_\varepsilon^i : I_\varepsilon^i \subset C_{R_0} \longrightarrow (R_0, 1) \qquad \text{for } i = 1, \dots, M_\varepsilon \tag{5.4}$$

with $r_\varepsilon^i > r_\varepsilon^{i+1}$ and $r_\varepsilon^i = R_0$ on $\partial I_\varepsilon^i \cap C_{R_0}$. In addition $sign(\varphi_\varepsilon(t,1) - a_\varepsilon)$ is fixed in C_{R_0}. We continue φ_ε by its boundary values to values $r > 1$.

Proposition 5.5. *The number M_ε of graphs $r_\varepsilon^i(t)$ is finite.*

Proof. By definition $\varphi_\varepsilon(r_\varepsilon^i(t),t) = a_\varepsilon$ and therefore there exist points $c_\varepsilon^i(t)$ such that

$$r_\varepsilon^i(t) < c_\varepsilon^i(t) < r_\varepsilon^{i+1}(t),$$

with the property that $\varphi_\varepsilon'(c_\varepsilon^i(t),t) = 0$. Now the estimates given by Proposition 3.9 imply

$$W(\varphi_\varepsilon(c_\varepsilon^i(t),t)) \le \delta^2,$$

and $\varphi_\varepsilon(c_\varepsilon^i(t),t)$ have opposite signs for consecutive i's. In consequence,

$$|g(\varphi_\varepsilon(c_\varepsilon^{i+1}(t),t)) - g(\varphi_\varepsilon(c_\varepsilon^i(t),t))| = |g(\varphi_\varepsilon(c_\varepsilon^{i+1}(t),t))| + g(\varphi_\varepsilon(c_\varepsilon^i(t),t))| \ge C(\delta) > 0.$$

Thus

$$\begin{aligned} C(\delta) M_\varepsilon &\le \sum_{i=1}^{M_\varepsilon} |g(\varphi_\varepsilon(c_\varepsilon^{i+1}(t),t)) - g(\varphi_\varepsilon(c_\varepsilon^i(t),t))| \\ &\le \sum_{i=1}^{M_\varepsilon} \left| \int_{c_\varepsilon^i(t)}^{c_\varepsilon^{i+1}(t)} g(\varphi_\varepsilon)'\, dr \right| \\ &\le C(R_0), \end{aligned}$$

by the energy estimate (2.5). Thus M_ε is uniformly bounded.

As a result of this proposition, for a subsequence of ε's (depending on the set C_{R_0}) the number $M_\varepsilon =: M_0$ must be constant, and for $i = 1, ..., M_0$, there exist

$$\bar{r}^i : I^i_{R_0} \to (R_0, 1],$$

$$\text{such that} \qquad r^i_\varepsilon \longrightarrow \bar{r}^i \tag{5.6}$$

weakly in $H^{1,2}(I^i_{R_0})$ and uniformly.

In view of this we define for any $R_0 > 0$ and any C_{R_0} compact in A_{R_0} the limit set

$$\Gamma := \{(\bar{t}_0, \bar{r}_0) \mid \bar{t}_0 \in C_{R_0} \text{ and } \bar{r}_0 = \bar{r}^i(\bar{t}_0) \text{ for some } i = 1, ..., M_0\}. \tag{5.7}$$

This set Γ contains the free boundary $\partial\{v = -1\}$, but it may contain more.

Next we study the "ε-problem" locally around r any $(\bar{t}_0, \bar{r}_0) \in \Gamma$ (here by locally renaming the interfaces r^i_ε):

Let m_ε be given by the property that at time $\bar{t}_0$, there are exactly m_ε graphs r^i_ε which converge to $\bar{r}_0$, i.e.

$$r^i_\varepsilon(\bar{t}_0) \to \bar{r}_0, \qquad 1 \le i \le m_\varepsilon. \tag{5.8}$$

Since all the r^i_ε are uniformly Hölder-$\frac{1}{2}$, there exists a box

$$B := (\bar{t}_1, \bar{t}_2) \times (a, b)$$

around $(\bar{t}_0, \bar{r}_0)$ that contains exactly m_ε graphs, all of them defined over the entire interval $(\bar{t}_1, \bar{t}_2)$. In addition, these graphs in B are $O(1)$ away from all other graphs. Of course, m_ε is independent of ε, if $\varepsilon \le \varepsilon_0(R_0, C_{R_0})$ is small enough.

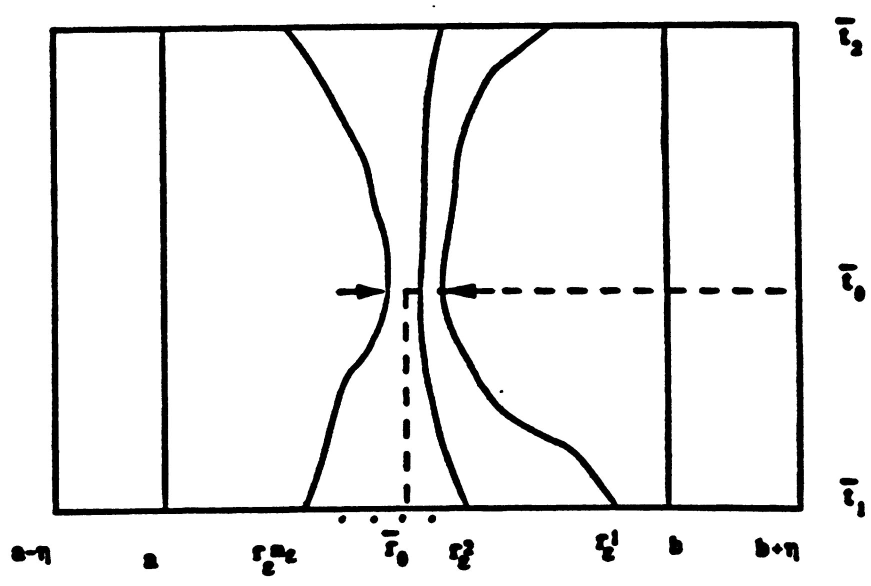

Figure 2

In conclusion, by putting together all the above results, we have the following

Proposition and Definition 5.9. The local situation.

For all $(\bar{t}_0, \bar{r}_0) \in \Gamma$ with $\bar{r}_0 \neq 1$ there exists a natural number m_0 and a number $\nu \in \{-1, 0, +1\}$ and a box $B = (\bar{t}_1, \bar{t}_2) \times (a, b) \subset C_{R_0} \times (R_0, 1)$ such that
(1) $\{\varphi = a_\varepsilon\} \cap B$ consists of m_ε graphs r_ε^i over $(\bar{t}_1, \bar{t}_2)$, which are uniformly Hölder–$\frac{1}{2}$ and with derivatives uniformly in L^2, and $r_\varepsilon^i > r_\varepsilon^{i+1}$,
(2) $m_\varepsilon = m_0$, if ε is small enough and at $\bar{t}_0$, exactly m_0 interfaces converge to r_0,
(3) $r_\varepsilon^i \to \bar{r}^i$ uniformly and $\partial_t r_\varepsilon^i \longrightarrow \partial_t \bar{r}^i$ weakly in $L^2(\bar{t}_1, \bar{t}_2)$ for $1 \leq i \leq m_0$,
(4) $\bar{r}^i(\bar{t}_0) = \bar{r}_0$ for $1 \leq i \leq m_0$,
(5) for some $\eta > 0$ with $a - \eta > R_0$ the sets $\{(\bar{t}_1, \bar{t}_2) \times (a - \eta, a]\} \cap \{\varphi^\varepsilon = a_\varepsilon\}$ and $\{(\bar{t}_1, \bar{t}_2) \times [b, b + \eta)\} \cap \{\varphi^\varepsilon = a_\varepsilon\}$ are empty,
(6)

$$\nu = \begin{cases} +1, & \text{if } \varphi_\varepsilon(\bar{t}_0, a) < a_\varepsilon \text{ and } \varphi_\varepsilon(\bar{t}_0, b) > a_\varepsilon, \\ -1, & \text{if } \varphi_\varepsilon(\bar{t}_0, a) > a_\varepsilon \text{ and } \varphi_\varepsilon(\bar{t}_0, b) < a_\varepsilon, \\ 0, & \text{otherwise}. \end{cases}$$

If $\bar{r}_0 = 1$, then $B = (\bar{t}_1, \bar{t}_2) \times (a, b) \not\subset C_{R_0} \times (R_0, 1)$, but since we continued φ_ε by its boundary values, the above definitions remain meaningful.

Due to symmetry of the argument we will later on only explicitly describe the case $\varphi_\varepsilon(\bar{t}_0, b) > a_\varepsilon$, such that ν is either $+1$ or 0. The case $\nu = 0$ corresponds to the case that the limit v has a "phantom" interface at which v "jumps" from 1 to 1 or -1 to -1, whereas the case $\nu \neq 0$ corresponds to true interfaces of v.

Section 6: A rigorous first order expansion

Once again throughout this section we assume that φ_ε satisfies (3.1) through (3.5), such that the analysis of the preceeding sections is valid.

We now have well–defined interfaces. We propose to study the solution near each interface. The final goal is to pass to the limit in equation (3.1) around each interface. For this we will need a very good approximation of the solution φ_ε in $H^{1,\infty}$. This section is devoted to obtaining this approximation. The idea is to show that the asymptotic expansion is rigorous up to second order, at least in a weak sense. We will show this using appropriate $H^{1,2}$ error estimates. However, we will not prove an approximation of φ_ε everywhere in Ω as in [S2]. Instead, with the use of a cut-off function, we only consider the approximation of φ_ε locally around the interfaces.

In this section, we restrict the discussion to the box B defined in Proposition 5.9. First we introduce a stretched variable around the biggest interface $r_\varepsilon^1(t)$ in the box B. Let

$$z := \frac{|x| - r_\varepsilon^1(t)}{\varepsilon}, \tag{6.1}$$

such that $z \in \left(\frac{-r_\varepsilon^1(t)}{\varepsilon}, \frac{1 - r_\varepsilon^1(t)}{\varepsilon}\right)$. From now on, we shall use *capital letters* for functions defined in the stretched variables and *minuscule letters* for functions written in the original variables, so that for example

$$\Phi_\varepsilon(z, t) := \varphi_\varepsilon(r, t). \tag{6.2}$$

Moreover the index ε will be dropped whenever it does not affect the clarity of the text. Then for η as in Proposition 5.9, the rescaling (6.1) maps the collection of points

$$a - \eta < a < r_\varepsilon^{m_\varepsilon} < \ldots < r_\varepsilon^2 < r_\varepsilon^1 < b < b + \eta \tag{6.3}$$

onto

$$z_- - \frac{\eta}{\varepsilon} < z_- < z_\varepsilon^{m_\varepsilon} < \ldots < z_\varepsilon^2 < z_\varepsilon^1 \, (= 0) < z_+ < z_+ + \frac{\eta}{\varepsilon}. \tag{6.4}$$

(See Figure 3.)

Now motivated by the formal analysis of [RS], we make the ansatz that Φ_ε is well approximated near $z = 0$ by

$$\Theta^\varepsilon(z,t) := \Phi_0^\varepsilon(z,t) + \varepsilon\Phi_1^\varepsilon(z,t) \quad \text{for } z \in (z_- - \frac{\eta}{\varepsilon}, z_+ + \frac{\eta}{\varepsilon}). \tag{6.5}$$

The zero–th order expansion $\Phi_0^\varepsilon(z,t)$ is given by

$$\Phi_0^\varepsilon(z,t) := \sum_{i=1}^{m_\varepsilon} \Xi_i^\varepsilon(z,t) \underbrace{\tanh((-1)^{i+1}[z - z_\varepsilon^i(t)] + \mu_\varepsilon)}_{=:\Phi_{0i}^\varepsilon(z,t)}, \tag{6.6}$$

where $\mu_\varepsilon = tanh^{-1} a_\varepsilon$ and Ξ_i is a partition of unity. More precisely, for $2 \le i \le m_\varepsilon - 1$ the function Ξ_i has support in $(\frac{z^{i+1}+z^i}{2} - 1, \frac{z^i+z^{i-1}}{2} + 1)$, while Ξ_1 has support in $(\frac{z^2}{2} - 1, \infty)$ and Ξ_{m_ε} has support in $(-\infty, \frac{z^{m_\varepsilon}+z^{m_\varepsilon-1}}{2} + 1)$. Moreover Ξ_i' has support in two disjoint intervals each of length two given typically by $(\frac{z^i+z^{i+1}}{2} - 1, \frac{z^i+z^{i+1}}{2} + 1)$.

Remark 6.7. We later prove that $|z_i - z_{i-1}|$ is uniformly bigger than 2 (cf Lemma 6.26), as a consequence of the first approximation Proposition 3.9, so that the above partition of unity is meaningful.

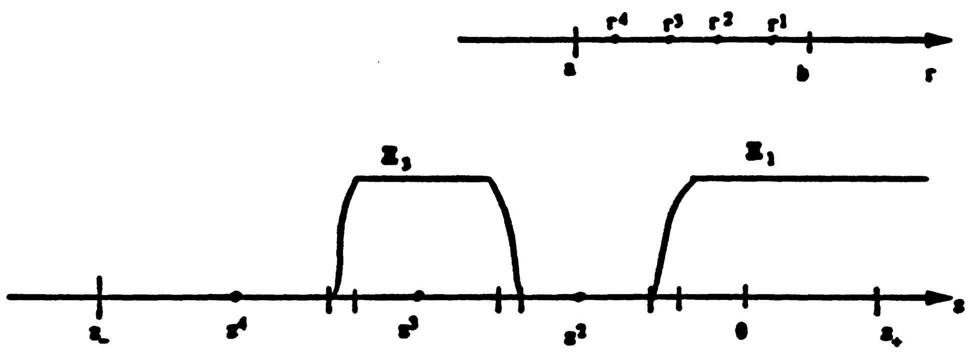

Figure 3

The first order expansion is given by

$$\Phi_1^\varepsilon(z,t) := \sum_{i=0}^{m_\varepsilon} \Xi_i^\varepsilon(z,t)\Phi_{1i}^\varepsilon(z,t), \tag{6.8}$$

where $\Phi_{1i} = \Phi_{1i}^\varepsilon(z,t)$ solves

$$\begin{cases} -\Phi_{1i}'' + W''(\Phi_{0i})\Phi_{1i} - \lambda_\varepsilon(t) = 0 \text{ in } (-\infty, z_\varepsilon^i) \cup (z_\varepsilon^i, \infty) \\ \Phi_{1i}(z_\varepsilon^i) = 0, \end{cases} \tag{6.9}$$

and $W''(\Phi) = 2(3\Phi^2 - 1)$. We note that equation (6.9) is the equation satisfied by the first order term in the asymptotic expansion of [RS].

Remark 6.10. We do not impose the differential equation for Φ_{1i} to be satisfied at $z = z_\varepsilon^i$, in order to ensure that the solution remains uniformly bounded over the entire real axis (cf Lemma 6.25). We refer to the work of Niethammer [N], who determines the expansion of the Lagrange multiplier for the radial, stationary problem with the mass constraint by the condition that the equation be satisfied in the whole of $\mathbb{R}$.

Remark 6.11. The approximation depends on the direction of the jump. Here we give the definition for the case as selected in Proposition 5.9. If to the contrary the jump direction was opposite, the tanh has to be substituted by $-$tanh.

Remark 6.12. In the formal inner expansion, one also expands $\lambda_\varepsilon(t) = \lambda_0(t) + \varepsilon\lambda_1(t) + \ldots$ (see [RS]). Here we do not do this, since we are only interested in the zeroth order term. The lowest order term will be determined later by the mass preservation property.

The rest of this section is devoted to proving that (6.5) is indeed a good approximation to Φ_ε. To this end let

$$\Psi^\varepsilon(z,t) := \Phi_\varepsilon(z,t) - \Theta^\varepsilon(z,t) \quad \text{for } \bar{t}_1 \le t \le \bar{t}_2 \tag{6.13}$$

Proposition 6.14. (Second order approximation). *Let ξ_ε be a cut-off function with*

$$\xi_\varepsilon(z,t) = \begin{cases} 1, & \text{in } (-\varepsilon^{-\alpha} + z_\varepsilon^{m_\varepsilon}(t), \varepsilon^{-\alpha}) \\ 0, & \text{in } \mathbb{R}\backslash(-\varepsilon^{-\beta} + z_\varepsilon^{m_\varepsilon}(t), \varepsilon^{-\beta}) \end{cases} \tag{6.15}$$

for $\frac{1}{2} < \alpha < \beta < 1$ so that

$$supp\, \xi(\cdot,t) \subset (z_- - \frac{\eta}{\varepsilon}, z_+ + \frac{\eta}{\varepsilon}).$$

Then for Ψ given as in (6.13), we have the following estimates

$$\int_{\bar{t}_1}^{\bar{t}_2} \int_{(-\infty, z_\varepsilon^{m_\varepsilon}) \cup (0,\infty)} (|\Psi'|^2 + |\Psi|^2)\xi^2 \, dz\, dt \le C\varepsilon^{2\beta}, \tag{6.16}$$

$$\int_{\bar{t}_1}^{\bar{t}_2} \int_{z_\varepsilon^{m_\varepsilon}}^0 (|\Psi'|^2 + |\Psi|^2)\, dz\, dt \to 0. \tag{6.17}$$

In order to prove this proposition, we first find the equation satisfied by Ψ^ε. Using the definition (6.1) for Φ_ε and equation (3.1), we find (if $z \le \frac{1-r_\varepsilon^1}{\varepsilon}$)

$$\begin{aligned} -\Phi'' + W'(\Phi) - \varepsilon\lambda_\varepsilon(t) &= -\varepsilon^2 \partial_t \varphi(\varepsilon z + r_\varepsilon^i(t), t) + \frac{2\varepsilon}{\varepsilon z + r_\varepsilon^1(t)}\Phi' \\ &=: F_\varepsilon(t,z). \end{aligned} \tag{6.18}$$

Define $F_\varepsilon(t,z)$ by $W'(\varphi_\varepsilon(t,1)) - \varepsilon\lambda_\varepsilon(t)$ for $z > \frac{1-r_\varepsilon^1}{\varepsilon}$.

The equation for Θ^ε is more complicated because of the extra terms coming from the partition of unity. To simplify the presentation, we let $\Theta_i := \Phi_{0i} + \varepsilon\Phi_{1i}$. Then we have

$$-\Theta'' + W'(\Theta) - \varepsilon\lambda_\varepsilon(t) = H_\varepsilon(z,t), \tag{6.19}$$

where

$$H_\varepsilon(z,t) := \varepsilon^2 \sum_{i=1}^{m_\varepsilon} \Xi_i \left(6\Phi_{0i}\Phi_{1i}^2 + 2\varepsilon\Phi_{1i}^3\right)$$
$$+ \sum_{i=1}^{m_\varepsilon-1} [\Theta_{i+1} - \Theta_i] \left(\Xi_i'' + 2\Xi_i\Xi_{i+1}\{(1+\Xi_i)\Theta_i^2 + (\Xi_i - 2)\Theta_{i+1}^2 + (1-2\Xi_i)\Theta_i\Theta_{i+1}\}\right)$$
$$+ 2\sum_{i=1}^{m_\varepsilon-1} [\Theta_{i+1} - \Theta_i]' \Xi_i'. \tag{6.20}$$

This formula comes from a linearization of $W'(\Phi) = -2\Phi(1-\Phi^2)$ around Φ_0. We note that this sum is only taken over two integers at a time because of the definition of Ξ_i. Also as we will see later, the last two sums are of small order (cf (6.33) and (6.34)).

Therefore combining (6.18) and (6.19) the equation for the difference is

$$-\Psi'' + W''(\Theta)\Psi = -2(3\Theta\Psi^2 + \Psi^3) + F_\varepsilon(z,t) - H_\varepsilon(z,t), \tag{6.21}$$

and it holds in

$$(z_- - \frac{\eta}{\varepsilon}, z_+ + \frac{\eta}{\varepsilon}) \backslash \{z_\varepsilon^1, ..., z_\varepsilon^{m_\varepsilon}\}, \tag{6.22}$$

with boundary values

$$\Psi^\varepsilon(z_\varepsilon^i, t) = 0 \quad \text{for } i = 1, .., m_\varepsilon. \tag{6.23}$$

Next, as in [S2], we follow an idea of Berger and Fraenkel and we show that in some sense the equation (6.21) for Ψ is uniformly strictly elliptic.

Proposition 6.24. (Ellipticity). *There exist $\zeta_1 > 0$ and $\zeta_2 > 0$, such that for $\bar{t}_1 < t < \bar{t}_2$*

$$\int(-\Psi'' + W''(\Theta)\Psi)\Psi\,\xi^2\,dz \geq \zeta_1 \int |\Psi'|^2\xi^2\,dz$$
$$+ \zeta_2 \int |\Psi|^2\xi^2\,dz$$
$$- \frac{2}{\zeta_1}\int |\Psi|^2|\xi'|^2\,dz,$$

where intergration is either over $(-\varepsilon^{-\beta} + z^{m_\varepsilon}, z^{m_\varepsilon})$ or $(0, \varepsilon^{-\beta})$ or $(z^{m_\varepsilon}, 0)$.

This proposition is very similar to the proof of Proposition 8 in [S2] and we include its proof in the appendix.

We are now left with estimating all the terms in the right hand side of (6.21). For this we find further estimates on $\Phi_\varepsilon - \Phi_0^\varepsilon$ and on Φ_1^ε. First we present a bound on Φ_1^ε, which in particular gives the estimate $\|\Theta^\varepsilon - \Phi_0^\varepsilon\|_{H^{1,\infty}(z_- - \frac{\eta}{\varepsilon}, z_+ + \frac{\eta}{\varepsilon}) \times (\bar{t}_1, \bar{t}_2)} \leq C\sqrt{\varepsilon}$.

Lemma 6.25.

$$\|\Phi_1^\varepsilon\|_{H^{1,\infty}(z_- - \frac{\eta}{\varepsilon}, z_+ + \frac{\eta}{\varepsilon}) \times (\bar{t}_1, \bar{t}_2)} \leq C\,|\lambda_\varepsilon(t)|.$$

The proof follows the same line as the proof of Lemma 7 in [S2] but we include it for the sake of completeness.

Proof. Following Berger and Fraenkel [BF], the solution Φ^{ε}_{1i} of (6.9) is given by

$$\Phi^{\varepsilon}_{1i}(z+z^{i}_{\varepsilon},t) = A(z)\int_0^z B(z)\lambda_\varepsilon(t)\,dz + B(z)\int_z^\infty A(z)\lambda_\varepsilon(t)\,dz$$

where $A(z) := 1-\tanh^2(z)$ and $B(z) := -A(z)\int_0^z \frac{1}{A^2}$. Therefore

$$|\Phi^{\varepsilon}_{1i}(z,t)| \le |\lambda_\varepsilon(t)|\left(A(z)\int_0^z |B| + |B(z)|\int_z^\infty A\right)$$
$$|(\Phi^{\varepsilon}_{1i})'(z,t)| \le |\lambda_\varepsilon(t)|\left(|A'(z)|\int_0^z |B| + |B'(z)|\int_z^\infty A\right).$$

Now since A is monotone decreasing, we have the bounds

$$|B(z)| \le \int_0^z \frac{1}{A} \le \frac{\tanh z}{1-\tanh z},$$
$$\int_0^z |B| \le \int_0^z \frac{\tanh(\cdot)}{1-\tanh(\cdot)} = \frac{\tanh z}{1-\tanh z} \le \frac{1}{1-\tanh z},$$

so that

$$A(z)\int_0^z |B| \le 1+\tanh z \le 2$$
$$|B(z)|\int_z^\infty A \le |B(z)|(1-\tanh z) \le 1.$$

Similarly, using that $A'(z) = -2A(z)\tanh z$ and that $B'(z) = -2B(z)\tanh z - \frac{1}{A(z)}$, we have

$$|A'(z)|\int_0^z |B| \le 2(1+\tanh z) \le 4$$
$$|B'(z)|\int_0^\infty A \le 2 + \frac{1}{1+\tanh z} \le 3.$$

Putting this together, the lemma follows.

Finally, before proving the second order approximation Proposition 6.14, we need one more lemma. Most of the proof of this lemma is devoted to showing that Φ is close to Φ_0 in L^∞. So as yet another consequence of Proposition 3.9, we have

Lemma 6.26. *For any $\delta > 0$ there exists $e(\delta) > 0$ and $M(\delta) > 0$ such that*

$$|z^{i}_{\varepsilon} - z^{i-1}_{\varepsilon}| \ge e(\delta) \quad \textit{and} \quad |z^{1}_{\varepsilon} - \frac{1-r^{1}_{\varepsilon}}{\varepsilon}| \ge e(\delta),$$

$$\|(\Psi^{\varepsilon}\Theta^{\varepsilon})_{-}\|_{L^\infty((x_- - \frac{2}{\varepsilon}, x_+ + \frac{2}{\varepsilon})\times(\bar t_1, \bar t_2))} \le M(\delta),$$

with $M(\delta) \to 0$ and $e(\delta) \to \infty$ as $\delta \to 0$.

Proof. Since $\Psi = \Phi - (\Phi_0 + \varepsilon\Phi_1)$ and $\varepsilon\|\Phi_1\|_\infty \le \|\lambda_\varepsilon\|_\infty \le C\sqrt{\varepsilon}$ by Lemma 6.25 and Corollary 2.7, we may substitute Ψ by $\Phi - \Phi_0$ and Θ by Φ_0 in the claim, while only making a uniformly small error. Next we show that Φ is close to Φ_0 in L^∞ by solving explicitly the ODE satisfied by Φ.

To start with, we note that the solution to the problem (with a_ε as chosen in section 5)

$$\begin{cases} |\Phi'(z)|^2 - 2W(\Phi(z)) = 0 \\ \Phi(0) = a_\varepsilon, \Phi' > 0 \end{cases}$$

is $\Phi(z) = \tanh(z + \mu_\varepsilon)$, where $\mu_\varepsilon = tanh^{-1}(a_\varepsilon)$. The first approximation Proposition 3.9 suggests that for $z_- \le z \le z_+$, we have to solve for $\Phi = \Phi_\varepsilon$

$$\begin{cases} |\Phi'(z,t)|^2 - 2W(\Phi(z,t)) = 2K(z,t) \\ \Phi(z_i^\varepsilon, t) = a_\varepsilon, \end{cases} \tag{6.27}$$

where $\|K\|_{L^\infty(z_- - \frac{\delta}{\varepsilon}, z_+ + \frac{\delta}{\varepsilon}) \times (\bar{t}_1, \bar{t}_2)} \le \delta^2$. We solve this ODE by direct integration. Indeed by the definition of a_ε we know that $W(\Phi(z_\varepsilon^i, t)) = W(a_\varepsilon) \ge 2\delta$, so that that there exist $e_\pm^i = e_\pm^i(t) > 0$ such that for $z \in (-e_-^i, e_+^i)$, we have $W(\Phi(z + z_\varepsilon^i, t)) > \delta$ and $W(\Phi(e_\pm^i + z_\varepsilon^i, t)) = \delta$. Propostion 3.9 implies that Φ' has a fixed sign in this interval. From (6.27), it then follows that

$$\frac{|\Phi_z(z + z_\varepsilon^i, t)|}{\sqrt{2W(\Phi(z,t))}} = \sqrt{1 + \frac{K(z,t)}{W(\Phi(z,t))}}.$$

Therefore integrating this for $z \in (-e_-^i, e_+^i)$, we find

$$\Phi(z + z_\varepsilon^i, t) = (-1)^{i+1} \tanh\left(\mu_\varepsilon + \int_0^z \sqrt{1 + \frac{K(\cdot,t)}{W(\Phi(\cdot,t))}}\right).$$

In consequence,

$$\begin{aligned} |\Phi(z + z_\varepsilon^i, t) - (-1)^{i+1}\tanh(z + \mu_\varepsilon)| &\le |z| \sup_{[-e,e]} \left|\sqrt{1 + \frac{K(z,t)}{W(\Phi(z,t))}} - 1\right| \\ &\le |z| \frac{\|K\|_\infty}{\delta} \frac{1}{2\sqrt{1 - \frac{\|K\|_\infty}{\delta}}} \\ &\le \delta|z|, \end{aligned} \tag{6.28}$$

for $z \in (-e_-^i, e_+^i)$. This implies in particular (because $\Phi(e_\pm^i + z_\varepsilon^i, t)$ is given in terms of δ only), that uniformly in t

$$e_\pm^i(\delta) \to \infty \quad \text{as } \delta \to 0.$$

The above argument generalizes as follows: for any $z_0 \in \{W(\Phi) > \delta\}$

$$|\Phi(z + z_0, t) \pm \tanh(z + \mu_0)| \le \delta|z|, \tag{6.29}$$

$$\mu_0 = \tanh^{-1}(\Phi(z_0, t))$$

for z in the maximal connected component of $\{W(\Phi) > \delta\}$, that contains z_0.

Now assume $a_\varepsilon < \Phi(z_0,t) < \sqrt{1-\sqrt{\delta}} < \tanh\left(\frac{1}{2\sqrt{\delta}}\right)$. Let z_1 be given by $\tanh(z_1+\mu_0) = a_\varepsilon - \sqrt{\delta}$. Then by construction $|z_1| \le \frac{1}{\sqrt{\delta}}$. Thus (6.29) implies

$$\Phi(z_1+z_0,t) \le \sqrt{\delta} + a_\varepsilon - \sqrt{\delta} = a_\varepsilon.$$

Consequently there exists z_2 with $|z_2| \le \frac{1}{\sqrt{\delta}}$, such that $\Phi(z_2+z_0,t) = a_\varepsilon$, hence $z_2+z_0 = z_\varepsilon^i$ for some i, and

$$|z_0 - z_\varepsilon^i| \le \frac{1}{\sqrt{\delta}}.$$

Of course the same argument applies for $-\tanh\left(\frac{1}{2\sqrt{\delta}}\right) < -\sqrt{1-\sqrt{\delta}} < \Phi(z_0,t) < a_\varepsilon$. Thus either $|z-z_\varepsilon^i| \le \frac{1}{\sqrt{\delta}}$ and then Φ is well approximated by a tanh as in (6.28) or $|\Phi| > \sqrt{1-\sqrt{\delta}}$.

Now, if $|z-z_\varepsilon^i| \le \frac{1}{\sqrt{\delta}}$, then $|\Phi - \Phi_0| \le \delta$, and thus $|(\Phi-\Phi_0)\Phi_0| \le 2\delta$.

If $|z-z_\varepsilon^i| > \frac{1}{\sqrt{\delta}}$, then either $\Phi > \sqrt{1-\sqrt{\delta}}$ or $\Phi < -\sqrt{1-\sqrt{\delta}}$. We describe the first case only. If $\Phi > 1$, then consequently $\Phi > \Phi_0$ and $\Phi_0 > 0$ by construction so that there is nothing to prove. If $1 \ge \Phi > \sqrt{1-\sqrt{\delta}}$, then Φ is uniformly close to Φ_0, because both are uniformly close to 1.

Here uniformly means uniformly with respect to δ, and the 'uniform closeness' is given by $M(\delta)$.

Putting all cases together proves the Lemma.

We are now ready for the proof of the second order approximation Proposition 6.14.

Proof of Proposition 6.14. Let S be $(-\varepsilon^{-\beta}+z^{m_\varepsilon}, z^{m_\varepsilon})$ or $(0,\varepsilon^{-\beta})$ or $(z^{m_\varepsilon},0)$. We multiply equation (6.21) for Ψ by $\xi^2\Psi$, where ξ is defined by (6.15), then we integrate in z and t and use Proposition 6.24 to obtain

$$\begin{aligned}
\zeta_1 \int_{\bar{t}_1}^{\bar{t}_2}\int_S |\Psi'|^2\xi^2\,dz\,dt &+ \zeta_2 \int_{\bar{t}_1}^{\bar{t}_2}\int_S |\Psi|^2\xi^2\,dz\,dt \\
&\le \frac{2}{\zeta_2}\int_{\bar{t}_1}^{\bar{t}_2}\int_S |F_\varepsilon|^2\xi^2 + |H_\varepsilon|^2\xi^2\,dz\,dt \\
&\quad + \left(\frac{\zeta_2}{4} + 6\|(\Psi\Theta)_-\|_{L^\infty}\right)\int_{\bar{t}_1}^{\bar{t}_2}\int_S |\Psi|^2\xi^2\,dz\,dt \\
&\quad + \frac{2}{\zeta_1}\int_{\bar{t}_1}^{\bar{t}_2}\int_S |\Psi|^2|\xi'|^2\,dz\,dt. \qquad (6.30)
\end{aligned}$$

Let $M(\delta)$ be as defined in Lemma 6.26 and choose δ small enough that $6M(\delta) \le \frac{\zeta_2}{4}$. Incorporating this in (6.25) yields

$$\begin{aligned}
\zeta_1 \int_{\bar{t}_1}^{\bar{t}_2}\int_S |\Psi'|^2\xi^2\,dz\,dt &+ \frac{\zeta_2}{2} \int_{\bar{t}_1}^{\bar{t}_2}\int_S |\Psi|^2\xi^2\,dz\,dt \\
&\le \frac{2}{\zeta_2}\int_{\bar{t}_1}^{\bar{t}_2}\int_S (|F_\varepsilon|^2 + |H_\varepsilon|^2)\xi^2\,dz\,dt \\
&\quad + \frac{2}{\zeta_1}\int_{\bar{t}_1}^{\bar{t}_2}\int_S |\Psi|^2|\xi'|^2\,dz\,dt. \qquad (6.31)
\end{aligned}$$

So we are left with estimating the right hand side of (6.31). Using the definition of F_ε given by (6.18), we have (if $\bar{r}_0 \neq 1$)

$$\begin{aligned}
\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\beta}+z^{m_\varepsilon}}^{\varepsilon^{-\beta}} |F_\varepsilon|^2\,\xi^2\,dz\,dt &\leq \int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\beta}+z^{m_\varepsilon}}^{\varepsilon^{-\beta}} \varepsilon^4|\partial_t\varphi(\varepsilon z + r_\varepsilon^i(t),t)|^2\,dz\,dt \\
&\quad + \int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\beta}+z^{m_\varepsilon}}^{\varepsilon^{-\beta}} \frac{4\varepsilon^2}{(\varepsilon z + r_\varepsilon^1)^2}|\Phi'|^2\,dz\,dt \\
&\leq \frac{1}{(R_0)^{n-1}}\int_{\bar{t}_1}^{\bar{t}_2}\int_0^1 \varepsilon^3|\partial_t\varphi(r,t)|^2 r^{n-1}\,dr\,dt \\
&\quad + \frac{1}{(R_0)^{n+1}}\int_{\bar{t}_1}^{\bar{t}_2}\int_0^1 \varepsilon^3|\varphi_r(r,t)|^2 r^{n-1}\,dr\,dt \\
&\leq C\varepsilon^2.
\end{aligned} \tag{6.32}$$

Moreover, we have the estimate

$$\int_{\bar{t}_1}^{\bar{t}_2}\int_{(-\varepsilon^{-\beta}+z^{m_\varepsilon},z^{m_\varepsilon})\cup(0,\varepsilon^{-\beta})} |H_\varepsilon|^2\xi^2\,dz\,dt \leq C\varepsilon^{3-\beta} \leq C\varepsilon^{2\beta}, \tag{6.33}$$

for $\beta \leq 1$. Indeed, the last two sums in the definition (6.20) of H_ε drop out since in $(-\varepsilon^{-\beta}+z^{m_\varepsilon}, z^{m_\varepsilon})$ the function $\Xi_{m_\varepsilon} \equiv 1$ and $\Xi_i \equiv 0$ for $1 \leq i \leq m_\varepsilon - 1$, while in $(0,\varepsilon^{-\beta})$ the functions $\Xi_i \equiv 0$ for $2 \leq i \leq m_\varepsilon$ and $\Xi_1 \equiv 1$. Then (6.33) follows from the fact that $\|\Phi_0\|_\infty \leq 1$, the Lemma 6.25 and the L^2 and L^∞ bounds on $\lambda_\varepsilon(t)$ (cf Proposition 4.1 and Corollary 2.7), since it yields

$$\begin{aligned}
\int_{\bar{t}_1}^{\bar{t}_2} \varepsilon^{4-\beta}\|\Phi_1^\varepsilon(\cdot,t)\|_\infty^4 + \varepsilon^{6-\beta}\|\Phi_1^\varepsilon(\cdot,t)\|_\infty^6\,dt &\leq \varepsilon^{4-\beta}\int_{\bar{t}_1}^{\bar{t}_2}|\lambda_\varepsilon(t)|^4\,dt + \varepsilon^{6-\beta}\int_{\bar{t}_1}^{\bar{t}_2}|\lambda_\varepsilon(t)|^6\,dt \\
&\leq \left(\varepsilon^{4-\beta}\frac{1}{\varepsilon} + \varepsilon^{6-\beta}\frac{1}{\varepsilon^2}\right)\int_{\bar{t}_1}^{\bar{t}_2}|\lambda_\varepsilon(t)|^2\,dt \\
&\leq C\varepsilon^{3-\beta}.
\end{aligned}$$

In addition, we claim that

$$\int_{\bar{t}_1}^{\bar{t}_2}\int_{z^{m_\varepsilon}}^{0} |H_\varepsilon|^2\,dz\,dt \to 0, \tag{6.34}$$

as $\varepsilon \to 0$. To prove this we need a better estimate on $|z^{i+1} - z^i|$ than the one we obtained in Lemma 6.26. Indeed this estimate is not strong enough to show that in the set $\{\Xi_i\Xi_{i+1} \neq 0\}$ we have $|\Theta_{i+1} - \Theta_i| \to 0$ as $\varepsilon \to 0$. However, if we go back to Proposition 3.6, formula (3.7), we can easily obtain the following bound for almost all $t \in (\bar{t}_1, \bar{t}_2)$ by using the energy estimate (2.4) and (2.5) and Proposition 4.1 and spending a little factor $\varepsilon\ln(1/\varepsilon)$

$$\|-\varepsilon^2(\varphi'(\cdot,t))^2 + 2W(\varphi(\cdot,t))\|_{L^\infty(R_0,1)} \leq C(t)\varepsilon\ln(1/\varepsilon).$$

Therefore we have in fact for almost all $t \in (\bar{t}_1, \bar{t}_2)$

$$|z^{i+1} - z^i| \to \infty \quad \text{as } \varepsilon \to 0, \tag{6.35}$$

and therefore

$$|\Theta_{i+1} - \Theta_i| \to 0 \quad \text{in } \{\Xi_i\Xi_{i+1} \neq 0\}.$$

Since Θ_i are uniformly bounded, we can now conclude that
$\int_{\bar{t}_1}^{\bar{t}_2}\int_{\{\Xi_i\Xi_{i+1}\neq 0\}}|\Theta_{i+1}-\Theta_i|^2\,dz\,dt \to 0$ as $\varepsilon\to 0$ and hence (6.34) follows.

Finally, since $|\xi'(z)|\le C\frac{1}{\varepsilon^{-\beta}-\varepsilon^{-\alpha}} < C\varepsilon^{\beta}$, the last term can be estimated as follows

$$
\begin{aligned}
C\frac{2}{\zeta_1}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\beta}+z^{m_\varepsilon}}^{\varepsilon^{-\beta}}|\Psi|^2|\xi'|^2\,dz\,dt &\le \frac{2\varepsilon^{2\beta}}{\zeta_1}\iint_{\{\xi\neq 0,1\}}|\Psi|^2\,dz\,dt\\
&\le C\frac{\varepsilon^{2\beta}}{\zeta_1}\iint_{\{\xi\neq 0,1\}}(|\Phi-\Phi_0|^2+\varepsilon^2|\Phi_1|^2)\,dz\,dt\\
&\le C\frac{\varepsilon^{2\beta-1}}{R_0^{n-1}\zeta_1}\int_{\bar{t}_1}^{\bar{t}_2}\int_{r_\varepsilon^1(t)}^{b+\eta}(\varphi(r,t)-1)^2r^{n-1}\,dr\,dt\\
&\quad + C\frac{\varepsilon^{2\beta-1}}{R_0^{n-1}\zeta_1}\int_{\bar{t}_1}^{\bar{t}_2}\int_{a+\eta}^{r_\varepsilon^{m_\varepsilon}}(\varphi(r,t)+1)^2r^{n-1}\,dr\,dt\\
&\quad + C\varepsilon^{\beta+1},
\end{aligned}
$$

where we have used that in $\{(z,t)|\xi(z,t)\neq 0,1\}$, the function Φ_0 is exponentially close to ± 1 depending if $z>0$ or $z<z^{m_\varepsilon}$, while $\varepsilon^2\|\Phi_1\|^2_{L^\infty(z_--\frac{a}{\varepsilon},z_++\frac{a}{\varepsilon})\times(\bar{t}_1,\bar{t}_2)} < C\varepsilon$. Now using the energy bound (2.5), we obtain for $\beta<1$

$$
\begin{aligned}
\frac{2}{\zeta_1}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\beta}+z^{m_\varepsilon}}^{\varepsilon^{-\beta}}|\Psi|^2|\xi'|^2\,dz\,dt &< C\varepsilon^{2\beta}\left(\frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\int_0^1(\varphi^2(r,t)-1)^2r^2\,dr\,dt\right)+C\varepsilon^{\beta+1}\\
&< C\varepsilon^{2\beta}. \qquad (6.36)
\end{aligned}
$$

The proof now follows from (6.31)-(6.36).

If $\bar{r}_0=1$, all the above arguments are valid, only those leading to (6.32) and (6.36) have to be changed. If in (6.32) we intergrate over $(-\varepsilon^{-\beta}+z_\varepsilon^{m_\varepsilon},\frac{1-r_\varepsilon^1}{\varepsilon})$, the result is the same. For $z>\frac{1-r_\varepsilon^1}{\varepsilon}$, we have $F_\varepsilon=W'(\varphi_\varepsilon(t,1))-\varepsilon\lambda_\varepsilon(t)$. But with the help of the estimate preceding (6.35) and because $\varphi_\varepsilon'(1,t)=0$, we find $F_\varepsilon^2\le C(t)\varepsilon ln(1/\varepsilon)$, and the remaining integral in (6.32) still converges to zero by Lebesgue's convergence theorem. The same reasoning holds for (6.36).

Finally we conclude with some $H^{1,\infty}$ bounds from this $H^{1,2}$ bound.

Corollary 6.37. *Let $J=J_{\text{good}}(t):=(-\varepsilon^{-\alpha}+z^{m_\varepsilon},z^{m_\varepsilon})\cup(0,\varepsilon^{-\alpha})$ if $\bar{r}_0\neq 1$ and $J=J_{\text{good}}(t):=(-\varepsilon^{-\alpha}+z^{m_\varepsilon},z^{m_\varepsilon})$ if $\bar{r}_0=1$. Then*

$$
\begin{aligned}
&\int_{\bar{t}_1}^{\bar{t}_2}\|\Psi_\varepsilon'\|^2_{H^{1,\infty}(J)}\,dt\le C\varepsilon^{2\beta},\\
&\int_{\bar{t}_1}^{\bar{t}_2}\|\Psi_\varepsilon'\|^2_{H^{1,\infty}(0,z^{m_\varepsilon})}\,dt\to 0,\\
&\int_{\bar{t}_1}^{\bar{t}_2}\|\Phi^\varepsilon(\cdot,t)-\Phi_0^\varepsilon(\cdot,t)\|^2_{H^{1,2}(-\varepsilon^{-\alpha}+z^{m_\varepsilon},\varepsilon^{-\alpha})}\,dt\to 0,\\
&\int_{\bar{t}_1}^{\bar{t}_2}\|\Phi^\varepsilon(\cdot,t)-\Phi_0^\varepsilon(\cdot,t)\|^2_{H^{1,\infty}(J)}\,dt\le C\varepsilon^{2\beta},\\
&\int_{\bar{t}_1}^{\bar{t}_2}\|\Phi^\varepsilon(\cdot,t)-\Phi_0^\varepsilon(\cdot,t)\|^2_{H^{1,\infty}(0,z^{m_\varepsilon})}\,dt\to 0.
\end{aligned}
$$

Proof. The first two results follow from Proposition 6.14 by the Sobolev imbedding theorem in $\mathbb{R}$, since Ψ satisfies the differential equation (6.21) and thus Ψ'' satisfies the same bounds as Ψ and Ψ'.

The last three results follow either from Proposition 6.14 or from the two first results of this Corollary using the by now "familiar" estimates

$$\int_{\bar{t}_1}^{\bar{t}_2} \|\varepsilon \Phi_1^\varepsilon(\cdot,t)\|^2_{H^{1,2}(-\varepsilon^{-\alpha},z^{m_\varepsilon}+\varepsilon^{-\alpha})}\,dt \le \varepsilon^2 \frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2} |\lambda_\varepsilon(t)|^2\,dt \le C\varepsilon,$$

and

$$\int_{\bar{t}_1}^{\bar{t}_2} \|\varepsilon \Phi_1^\varepsilon(\cdot,t)\|^2_{H^{1,\infty}(-\varepsilon^{-\alpha},z^{m_\varepsilon}+\varepsilon^{-\alpha})}\,dt \le \varepsilon^2 \int_{\bar{t}_1}^{\bar{t}_2} |\lambda_\varepsilon(t)|^2\,dt \le C\varepsilon^2.$$

Section 7: The limit equation

In this section, we first restrict the discussion to boxes B as defined in Proposition 5.9 and we derive the differential equation of the limiting interface (true or "phantom") given by $r = \bar{r}^i$.

So once again we assume (3.1) – (3.5) for φ_ε and hence the analysis of the preceding sections applies.

We first consider the case $\bar{r}_0 \neq 1$ and later point out the differences in the other case.

In the z-variable the nonlocal equation (3.1) becomes

$$\varepsilon\partial_t\Phi - \dot{r}^1\Phi' - \frac{1}{\varepsilon}\Phi'' - \frac{n-1}{\varepsilon z + r^1}\Phi' + \frac{1}{\varepsilon}W'(\Phi) - \lambda_\varepsilon(t) = 0. \tag{7.1}$$

Let ζ be a smooth time dependent test function with compact support in $(\bar{t}_1,\bar{t}_2)$. First, we multiply equation (7.1) by $\Phi'\zeta$ and integrate over $(-\varepsilon^{-\alpha}+z^{m_\varepsilon},\varepsilon^{-\alpha})\times(\bar{t}_1,\bar{t}_2)$,

$$\varepsilon\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\Phi_t\Phi'\,dz\,dt - \int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\dot{r}^1\Phi'^2\,dz\,dt$$
$$-\frac{1}{2\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\,((\Phi')^2)'\,dz\,dt - \frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\frac{n-1}{\varepsilon z + r^1}\Phi'^2\,dz\,dt$$
$$+\frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\,\partial_z W(\Phi)\,dz\,dt - \int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\lambda_\varepsilon\Phi'\,dz\,dt = 0.$$

This can be rewritten as follows

$$\varepsilon\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\Phi_t\Phi'\,dz\,dt - \int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\dot{r}^1\Phi'^2\,dz\,dt$$
$$-\frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\zeta\left(\frac{1}{2}(\Phi')^2 - W(\Phi)\right)\Big|_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}}\,dt$$
$$= \int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \frac{(n-1)\zeta}{\varepsilon z + r^1}|\Phi'|^2\,dz\,dt + \int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\lambda_\varepsilon\Phi'\,dz\,dt. \tag{7.2}$$

The main part of this section is to evaluate the limit of each term in (7.2) as $\varepsilon \to 0$. This will yield equation (7.14) for $\bar{r}^i$.

We start with the third term in (7.2).

Third term. We claim that

$$\int_{\bar{t}_1}^{\bar{t}_2} \frac{1}{\varepsilon}\left(\frac{1}{2}(\Phi')^2 - W(\Phi)\right)\Big|_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{z^{m_\varepsilon}} \zeta\, dt \xrightarrow{\varepsilon\to 0} 0. \tag{7.3}$$

Indeed the endpoints $-\varepsilon^{-\alpha}+z^{m_\varepsilon}$ and $\varepsilon^{-\alpha}$ are in J_{good}, so that we can apply Corollary 6.37 and replace Φ by Φ_0, since $\int_{\bar{t}_1}^{\bar{t}_2}\|\Phi-\Phi_0\|^2_{H^{1,\infty}(J)}\,dt \le C\varepsilon^{2\beta}$ and $2\beta - 1 > 0$. The claim (7.3) now follows since $\Phi_0(-\varepsilon^{-\alpha}+z^{m_\varepsilon},t) = \tanh(\pm\varepsilon^{-\alpha}+\mu^\varepsilon)$ and $\Phi_0(\varepsilon^{-\alpha},t) = \tanh(\varepsilon^{-\alpha}+\mu^\varepsilon)$.

Next, we study the convergence of the fourth term in (7.2).

Fourth term. We claim that

$$(n-1)\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\frac{1}{\varepsilon z + r^1_\varepsilon}|\Phi'|^2\,dz\,dt \xrightarrow{\varepsilon\to 0} \frac{4}{3}(n-1)\int_{\bar{t}_1}^{\bar{t}_2}\zeta\sum_{i=1}^{m_0}\frac{1}{\bar{r}^i}\,dt. \tag{7.4}$$

To prove this we first replace Φ by Φ_0. We can do this since $|\frac{1}{\varepsilon z+r^1_\varepsilon}| \le \frac{1}{R_0}$ and since by the approximation Proposition 6.14, Φ' is well approximated by Φ_0'.

Therefore we only have to consider the limit of the integral

$$\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\frac{1}{\varepsilon z + r^1_\varepsilon}|\Phi_0'|^2\,dz\,dt. \tag{7.5}$$

To find this limit, we divide the interval of integration in subintervals each of them containing one interface as follows

$$\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\frac{1}{\varepsilon z + r^1_\varepsilon}|\Phi_0'|^2\,dz\,dt = \sum_{i=1}^{m_\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\int_{b^i}^{b^{i-1}} \zeta\frac{1}{\varepsilon z + r^1_\varepsilon}|\Phi_0'|^2\,dz\,dt, \tag{7.6}$$

where $b^0 = \varepsilon^{-\alpha}$, $b^{m_\varepsilon} = -\varepsilon^{-\alpha}+z^{m_\varepsilon}$ and $b^i = \frac{z^{i+1}+z^i}{2}$, for $1 \le i \le m_\varepsilon - 1$. Then we make the change of variable $z' = z - z^i_\varepsilon$ and let again $z' = z$ so that

$$\begin{aligned}
\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} \zeta\frac{1}{\varepsilon z + r^1_\varepsilon}|\Phi_0'|^2\,dz\,dt &= \sum_{i=1}^{m_\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\int_{b^i-z^i}^{b^{i-1}-z^i} \zeta\frac{1}{\varepsilon z + r^1_\varepsilon}(\Phi_0'(z+z^i))^2\,dz\,dt \\
&= \sum_{i=1}^{m_\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\int_{\frac{z^{i+1}-z^i}{2}+1}^{\frac{z^{i-1}-z^i}{2}-1} \zeta\frac{1}{\varepsilon z + r^1_\varepsilon}(\Phi_0'(z+z^i))^2\,dz\,dt \\
&\quad + \sum_{i=1}^{m_\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}\int_{\{\Xi_i\neq 0,1\}} \zeta\frac{1}{\varepsilon z + r^1_\varepsilon}(\Phi_0'(z))^2\,dz\,dt.
\end{aligned} \tag{7.7}$$

The last term in the right hand side of (7.7) goes to 0 with ε. Indeed the set $\{\Xi_i \neq 0,1\}$ is of length 4, we have the estimate $|\frac{1}{\varepsilon z+r^1_\varepsilon}| \le \frac{1}{R_0}$ and we know that $\|\Phi_0'(z)\|_{L^\infty(\{\Xi_i\neq 0,1\})} \xrightarrow{\varepsilon\to 0} 0$ almost everywhere since by (6.35) $|z^{i+1}-z^i| \xrightarrow{\varepsilon\to 0} \infty$.

So, to prove the claim (7.4), we are left with calculating the first term in the right hand side of (7.7). For that we note that by the definition (6.6) of Φ_0, we have $\Phi_0(z+z^i) = \tanh((-1)^{i+1}z+\mu_\varepsilon)$ for $z \in \left(\frac{z^{i+1}-z^i}{2}+1, \frac{z^{i-1}-z^i}{2}-1\right)$, therefore we claim that the first term in the right hand side of (7.7) converges to

$$
\begin{aligned}
&\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i+1}-z^i}{2}+1}^{\frac{z^{i-1}-z^i}{2}-1}\frac{\zeta}{\varepsilon z+r_\varepsilon^i}(1-\tanh^2((-1)^{i+1}z+\mu_\varepsilon))^2\,dz\\
&\overset{\varepsilon\to 0}{\Longrightarrow}\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\frac{\zeta}{\bar r^i}\int_{-\infty}^{\infty}(1-\tanh^2(z))^2\,dz\,dt\\
&=\frac{4}{3}\int_{\bar t_1}^{\bar t_2}\zeta\sum_{i=1}^{m_\varepsilon}\frac{1}{\bar r^i}\,dt.
\end{aligned}
\tag{7.8}
$$

Indeed,

$$
\begin{aligned}
\left|(1-\tanh^2(\pm z+\mu_\varepsilon))^2\left(\frac{1}{\varepsilon z+r_\varepsilon^i}-\frac{1}{\bar r^i}\right)\right| &\le |(1-\tanh^2(\pm z+\mu_\varepsilon))^2|\frac{|\bar r^i-r_\varepsilon^i|+\varepsilon|z|}{|\bar r^i(\varepsilon z+r_\varepsilon^i)|}\\
&\le |(1-\tanh^2(\pm z+\mu_\varepsilon))^2|\frac{|\bar r^i-r_\varepsilon^i|}{R_0^2}\\
&\quad+|(1-\tanh^2(\pm z+\mu_\varepsilon))^2|\frac{\varepsilon|z|}{R_0^2}.
\end{aligned}
\tag{7.9}
$$

Since $|z_\varepsilon^{i+1}-z_\varepsilon^i| \overset{\varepsilon\to 0}{\longrightarrow} \infty$ almost everywhere in $(\bar t_1,\bar t_2)$ (see (6.35)), $|(1-\tanh^2(\cdot))^2| \in L^1(R)$ and $|\bar r^i - r_\varepsilon^i| \overset{\varepsilon\to 0}{\longrightarrow} 0$ uniformly in $(\bar t_1,\bar t_2)$ (cf Proposition 5.9), the integral over the first term in the right hand side of (7.9) converges to 0 with ε. For the second term, using that $|z| \le \frac{1}{\varepsilon}$, we have

$$
\begin{aligned}
\int_{\frac{z^{i+1}-z^i}{2}+1}^{\frac{z^{i-1}-z^i}{2}-1}(1-\tanh^2(\pm z+\mu_\varepsilon))^2\varepsilon|z|\,dz &\le \int_{-R}^{R}(1-\tanh^2(\pm z+\mu_\varepsilon))^2\varepsilon|z|\,dz\\
&\quad+\int_{\frac{1}{\varepsilon}>|z|>R}(1-\tanh^2(\pm z+\mu_\varepsilon))^2\varepsilon|z|\,dz\\
&\le C\varepsilon R+\int_{|z|>R}(1-\tanh^2(\pm z+\mu_\varepsilon))^2\,dz,
\end{aligned}
$$

so the claim (7.8) follows by choosing $R=\frac{1}{\sqrt{\varepsilon}}$. Therefore (7.4) follows by (7.7) and (7.8).

Next we consider the last term in (7.2).

Last term. We claim that

$$
\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-a}+z^{m_\varepsilon}}^{\varepsilon^{-a}}\zeta\lambda_\varepsilon\Phi'\,dz\,dt \overset{\varepsilon\to 0}{\longrightarrow} 2\nu\int_{\bar t_1}^{\bar t_2}\zeta\lambda_0\,dt,
\tag{7.10}
$$

where ν has been introduced in Proposition 5.9 and λ_0 in Remark 4.10. Indeed integrating by parts yields

$$
\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-a}+z^{m_\varepsilon}}^{\varepsilon^{-a}}\zeta\lambda_\varepsilon\Phi'\,dz\,dt = \int_{\bar t_1}^{\bar t_2}\zeta\lambda_\varepsilon\,\Phi|_{-\varepsilon^{-a}+z^{m_\varepsilon}}^{\varepsilon^{-a}}\,dt.
\tag{7.11}
$$

But

$$\Phi(\varepsilon^{-\alpha},t);\quad \Phi(-\varepsilon^{a}+z^{m_\varepsilon},t)\longrightarrow \pm 1 \tag{7.12}$$

uniformly as a consequence of the approximation Proposition (6.14), and since $\lambda_\varepsilon \longrightarrow \lambda_0$ in L^2, such that (7.10) is immediate. Note that we allow for $\nu=\{-1,0,1\}$.

Finally, we study the limit of the first and second term in (7.2).

First and second term. We claim that

$$\varepsilon\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-a}+z^{m_\varepsilon}}^{\varepsilon^{-a}} \zeta\Phi_t\Phi'\,dz\,dt-\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-a}+z^{m_\varepsilon}}^{\varepsilon^{-a}} \zeta\,\dot r^1\Phi'^2\,dz\,dt \stackrel{\varepsilon\to 0}{\Longrightarrow} -\int_{\bar t_1}^{\bar t_2}\frac{4}{3}\zeta\sum_{i=1}^{m_0}\partial_t\bar r^i\,dt. \tag{7.13}$$

This proof is similar to the convergence of the fourth term. We make use again of the change of variable $z'=z-z^i$ (and letting again $z'=z$):

$$\begin{aligned}
\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-a}+z^{m_\varepsilon}}^{\varepsilon^{-a}} \zeta\Phi'(\varepsilon\Phi_t-\dot r^1\Phi')\,dz\,dt &= \sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{b^i}^{b^{i-1}}\zeta\Phi'(\varepsilon\Phi_t-\dot r^1\Phi')\,dz\,dt\\
&=\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{b^i-z^i}^{b^{i-1}-z^i}\zeta\Phi'(z+z^i,t)(\varepsilon\Phi_t(z+z^i,t)-\dot r^i\Phi'(z+z^i,t))\,dz\,dt\\
&=\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i+1}-z^i}{2}}^{\frac{z^{i-1}-z^i}{2}}\varepsilon\zeta\Phi'(z+z^i,t)\Phi_t(z+z^i,t)\,dz\,dt\\
&\quad-\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i+1}-z^i}{2}}^{\frac{z^{i-1}-z^i}{2}}\zeta\dot r^i(\Phi'(z+z^i,t))^2\,dz\,dt.
\end{aligned}\tag{7.14}$$

We study the limit of each sum in (7.14). We start with the second sum. We claim that

$$\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i-1}-z^i}{2}}^{\frac{z^{i+1}-z^i}{2}}\zeta\dot r^i(\Phi'(z+z^i,t))^2\,dz\,dt \stackrel{\varepsilon\to 0}{\Longrightarrow}\int_{\bar t_1}^{\bar t_2}\frac{4}{3}\zeta\sum_{i=1}^{m_0}\partial_t\bar r^i\,dt. \tag{7.15}$$

We know from Proposition 5.9 that $\partial_t r^i\longrightarrow\partial_t\bar r^i$ weakly in $L^2(\bar t_1,\bar t_2)$. So (7.15) is true if we show that

$$\int_{\frac{z^{i+1}-z^i}{2}}^{\frac{z^{i-1}-z^i}{2}}(\Phi'(z+z^i,t))^2\,dz\longrightarrow\frac{4}{3}\quad\text{strongly in } L^2(\bar t_1,\bar t_2). \tag{7.16}$$

This amounts to showing that we can replace Φ by Φ_0 in (7.16) since $\int_{-\infty}^{\infty}(\Phi_0')^2\,dz=\int_{-\infty}^{\infty}(1-\tanh^2(z))^2\,dz=\frac{4}{3}$. But going back to the original variable z, we have

$$\begin{aligned}
\left|\int_{b^i}^{b^{i-1}}(\Phi')^2-(\Phi_0')^2\,dz\right| &\le\left(\int_{b^i}^{b^{i-1}}(\Phi'-\Phi_0')^2\,dz\right)^{\frac12}\left(\int_{b^i}^{b^{i-1}}(\Phi'+\Phi_0')^2\,dz\right)^{\frac12}\\
&\le C\left(\int_{-\varepsilon^{-a}+z^{m_\varepsilon}}^{\varepsilon^{-a}}(\Phi'-\Phi_0')^2\,dz\right)^{\frac12},
\end{aligned}$$

since $\Phi_0' \in L^2(R)$ and since by the energy estimate (2.4), we have $\int(\Phi')^2\,dz \le \varepsilon\int_\Omega |\nabla\varphi|^2\,dx \le C$. This proves (7.16), and hence (7.15), since by Corollary 6.37,

$$\left(\int_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{\varepsilon^{-\alpha}} (\Phi' - \Phi_0')^2\,dz\right)^{\frac{1}{2}} \longrightarrow 0 \quad \text{strongly in } L^2(\bar t_1, \bar t_2).$$

Finally we study the limit of the first sum in (7.14). We claim that

$$\sum_{i=1}^{m_\varepsilon} \int_{\bar t_1}^{\bar t_2} \int_{\frac{z^{i+1}-z^i}{2}}^{\frac{z^{i-1}-z^i}{2}} \varepsilon\zeta\Phi'(z+z^i,t)\Phi_t(z+z^i,t)\,dz\,dt \overset{\varepsilon\to 0}{\Longrightarrow} 0. \tag{7.17}$$

This is the most intricate estimate because of the many time dependent functions in (7.17). First, as usual, we replace Φ' by Φ_0'. This time this is possible because

$$\varepsilon\left(\int(\Phi_t)^2\,dz\,dt\right)^{\frac{1}{2}} \le C, \tag{7.18}$$

while

$$\int(\Phi' - \Phi_0')^2\,dz\,dt \longrightarrow 0, \tag{7.19}$$

(with space integration over $(-\varepsilon^{-\alpha}+z^{m_\varepsilon}, \varepsilon^{-\alpha})$). Indeed, (7.19) is part of Corollary (6.37). In order to show estimate (7.18), we proceed as in [S1] and prove the following a-priori estimate on $\partial_t\Phi$,

$$\begin{aligned}
\int|\Phi_t|^2\,dz\,dt &\le 2\int|\Phi_t - \frac{\dot r^1}{\varepsilon}\Phi'|^2\,dz\,dt \\
&\quad + \frac{2}{\varepsilon^2}\int|\dot r^1|^2|\Phi'|^2\,dz\,dt \\
&\le \frac{2}{\varepsilon R_0^{n-1}}\int_0^T\int_\Omega|\partial_t\varphi(x,t)|^2\,dx\,dt + \frac{2}{\varepsilon R_0^{n-1}}\sup_{t\in(0,T)}\int_\Omega|\nabla\varphi(x,t)|^2\,dx\int_0^T|\dot r^1|^2\,dt \\
&\le \frac{C}{\varepsilon^2}.
\end{aligned} \tag{7.20}$$

Therefore, using (7.18)–(7.19), we may substitute Φ' by Φ_0' and only have to show that

$$\varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i-1}-z^i}{2}}^{\frac{z^{i+1}-z^i}{2}} \zeta\Phi_t\Phi_0'(z+z^i,t)\,dz\,dt \overset{\varepsilon\to 0}{\Longrightarrow} 0, \tag{7.21}$$

in order to prove (7.17). We first divide the interval of integration as we did while studying the limit of the fourth term (cf (7.7)).

$$\begin{aligned}
\varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i-1}-z^i}{2}}^{\frac{z^{i+1}-z^i}{2}} \zeta\Phi_t\Phi_0'(z+z^i,t)\,dz\,dt &= \varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i-1}-z^i}{2}+1}^{\frac{z^{i+1}-z^i}{2}-1} \zeta\Phi_t\Phi_0'(z+z^i,t)\,dz\,dt \\
&\quad + \varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i-1}-z^i}{2}-1}^{\frac{z^{i-1}-z^i}{2}} \zeta\Phi_t\Phi_0'(z+z^i,t)\,dz\,dt \\
&\quad + \varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i+1}-z^i}{2}}^{\frac{z^{i+1}-z^i}{2}+1} \zeta\Phi_t\Phi_0'(z+z^i,t)\,dz\,dt.
\end{aligned} \tag{7.22}$$

Since $\varepsilon\left(\int_{\bar t_1}^{\bar t_2}\int(\Phi_t)^2\,dz\,dt\right)^{\frac12}\leq C$ by (7.18) and $\Phi_0'(z+z^i))\overset{\varepsilon\to 0}{\longrightarrow}0$ pointwise for $z\in(\frac{z^{i+1}-z^i}{2},\frac{z^{i+1}-z^i}{2}+1)$, we have

$$\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i+1}-z^i}{2}}^{\frac{z^{i+1}-z^i}{2}+1}(\Phi_0')^2\overset{\varepsilon\to 0}{\longrightarrow}0,$$

so that the last sum in the right hand side of (7.22) converge to 0 with ε. Similarly, the second sum converges to 0 with ε.

We note that in the first sum in (7.22), the essential fact is that $\Phi_0(z+z^i)=\tanh((-1)^{i+1}z+\mu_\varepsilon)$, so that in particular $\partial_t\Phi_0'=0$. Therefore integrating the first sum in (7.22) by parts in time, we obtain

$$\begin{aligned}\varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i-1}-z^i}{2}+1}^{\frac{z^{i+1}-z^i}{2}-1}&\zeta\Phi_t\Phi_0'(z+z^i,t)\,dz\,dt\\
&=-\varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{\frac{z^{i-1}-z^i}{2}+1}^{\frac{z^{i+1}-z^i}{2}-1}\partial_t\zeta\Phi\Phi_0'(z+z^i,t)\,dz\,dt\\
&\quad-\varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\zeta\Phi\Phi_0'|_{\frac{z^{i+1}+z^i}{2}+1}\frac{\dot r^{i+1}-\dot r^i}{2\varepsilon}\,dt\\
&\quad+\varepsilon\sum_{i=1}^{m_\varepsilon}\int_{\bar t_1}^{\bar t_2}\zeta\Phi\Phi_0'|_{\frac{z^{i-1}+z^i}{2}-1}\frac{\dot r^{i-1}-\dot r^i}{2\varepsilon}\,dt.\end{aligned}\tag{7.23}$$

Since $\Phi_0'(\cdot+z^i)\in L^1(R)$ and since Φ is bounded, the first sum in (7.23) converges to 0 with ε. For the second and third term, we use the fact that $\dot r^i$ converge weakly in L^2 and the fact that $\Phi_0'(\frac{z^{i+1}+z^i}{2}-1)\overset{\varepsilon\to 0}{\longrightarrow}0$ pointwise and thus in L^2. This finishes the proof of (7.21) and hence of (7.17). Combining (7.15) and (7.17) in (7.14), yields (7.13).

If $\bar r_0=1$, the only difference in strategy is, that we integrate (7.1) over $\left(\varepsilon^{-\alpha}+z^{m_\varepsilon},min(\varepsilon^{-\alpha},\frac{1-r_\varepsilon^1}{\varepsilon})\right)$. Then everything remains the same, only (7.3) has to be changed into

$$\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\frac{1}{\varepsilon}\left(\frac12(\Phi')^2-W(\Phi)\right)\Big|_{-\varepsilon^{-\alpha}+z^{m_\varepsilon}}^{min(\varepsilon^{-\alpha},\frac{1-r_\varepsilon^1}{\varepsilon})}\zeta\,dt\leq 0.$$

In summary, we have evaluated the limit of each term in (7.2). Therefore, using equation (7.2) and the limit of each term given by (7.3), (7.4), (7.10) and (7.13), we have shown that in the limit as $\varepsilon\to 0$, we obtain the equation

$$-\int_{\bar t_1}^{\bar t_2}\frac43\zeta\sum_{i=1}^{m_0}(\dot{\bar r}^i+\frac{n-1}{\bar r^i})\,dt=2\nu\int_{\bar t_1}^{\bar t_2}\zeta\lambda_0\,dt,\tag{7.24}$$

or the inequality with $\geq$, if $\bar r_0$ was in the fixed boundary of the domain. This limit was obtained in the box B defined in Proposition 5.9. We have thus proven the following

Proposition 7.25. *Let A_{R_0} be as in (5.2) and C_{R_0} be compact in A_{R_0}. Then for all $(\bar t_0,\bar r_0)\in\Gamma$ as in (5.7) there exists a box $B=(\bar t_1,\bar t_2)\times(a,b)$ as in Proposition 5.9, such that $(\bar r_0,\bar t_0)\in B$ and* *such that $\Gamma \cap B$ consists of $m_0 = m_0(\bar r_0, \bar t_0)$ Hölder–$\frac{1}{2}$ graphs $\bar r^i : (\bar t_1, \bar t_2) \to (a,b)$ with $\bar r^i(\bar t_0) = \bar r_0$. Moreover, if $\bar r_0 \neq 1$*

$$-\sum_{i=1}^{m_0} \left(\dot{\bar r}^i + \frac{n-1}{\bar r^i}\right) = \frac{3}{2}\nu(\bar r_0, \bar t_0)\lambda_0 \quad \text{in } (\bar t_1, \bar t_2), \tag{7.26}$$

and $\geq$ is true in the above, if $\bar r_0 = 1$.

The constant $m_0(\bar r_0, \bar t_0)$ can be understood as the "multiplicity" of Γ. We shall prove that $m_0 = 1$ almost everywhere in $\{\nu \neq 0\}$. For that we show first that $m_0(\bar r_0, \bar t_0) \neq 1$ in B is equivalent to $\nu(\bar r_0, \bar t_0)\lambda_0(\bar t_0) = 0$. Then, using the fact that the mass is preserved in time, we calculate explicitly λ_0, and show that $\lambda_0(\bar t_0) \neq 0$.

To show the first part of this approach, we need the following result.

Lemma 7.27. *Assume the statements of Proposition 7.25. Then if $\nu(\bar t_0, \bar r_0) \neq 0$ and $\bar r_0 \neq 1$*

$$-\left(\dot{\bar r}^1 + \frac{n-1}{\bar r^1}\right) \geq \frac{3}{2}\nu(\bar r_0, \bar t_0)\lambda_0 \geq -\left(\dot{\bar r}^{m_0} + \frac{n-1}{\bar r^{m_0}}\right) \quad \text{in } (\bar t_1, \bar t_2), \tag{7.28}$$

and if $\bar r_0 = 1$, then the second of the above inequalities holds.

Proof. Without loss of generality assume $\nu = 1$. Now, instead of integrating (7.2) over $(-\varepsilon^{-\alpha} + z^{m_\varepsilon}, \varepsilon^{-\alpha})$, we integrate only over the last branch of Φ. More precisely, let $c_\varepsilon(t)$ be the first negative point where $\Phi'(\cdot, t)$ vanishes. We note that in the same way as we proved that $|z^{i+1} - z^i| \overset{\varepsilon \to 0}{\longrightarrow} \infty$ (cf 6.35), we can show that $c_\varepsilon(t) \overset{\varepsilon \to 0}{\longrightarrow} -\infty$ pointwise. We integrate (7.2) over $(c_\varepsilon, \varepsilon^{-\alpha})$ and we use the same arguments as before to obtain the limit as $\varepsilon \to 0$. This will give an inequality for $\dot{\bar r}_1$ similar to (7.26). We only point out the main differences with the previous case. The claim (7.3) is replaced by

$$\lim_{\varepsilon \to 0} \int_{\bar t_1}^{\bar t_2} \zeta \frac{1}{\varepsilon}\left(\frac{1}{2}(\Phi')^2 - W(\Phi)\right)\Big|_{c_\varepsilon}^{\varepsilon^{-\alpha}} dt = \int_{\bar t_1}^{\bar t_2} \zeta \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} W(\Phi(c_\varepsilon))\, dt,$$

and the proof is the same as before since $\Phi'(c_\varepsilon(t), t) = 0$.

The claim (7.4) is replaced by

$$(n-1)\int_{\bar t_1}^{\bar t_2} \int_{-\varepsilon^{-\alpha}}^{c_\varepsilon} \zeta \frac{1}{\varepsilon z + \bar r^1} |\Phi'|^2\, dz\, dt \overset{\varepsilon \to 0}{\longrightarrow} \frac{4}{3}(n-1)\int_{\bar t_1}^{\bar t_2} \zeta \frac{1}{\bar r^1}\, dt,$$

and the proof is similar as above since $c_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} -\infty$.

The claim (7.10) is replaced by

$$\int_{\bar t_1}^{\bar t_2} \int_{-\varepsilon^{-\alpha}}^{c_\varepsilon} \zeta \lambda_\varepsilon \Phi'\, dz\, dt \overset{\varepsilon \to 0}{\longrightarrow} 2\int_{\bar t_1}^{\bar t_2} \zeta \lambda_0\, dt.$$

The proof must be slightly modified since we only know that $\Phi(c_\varepsilon, t) \to -1$ pointwise and thus in L^2. But this is enough to pass to the limit.

The claim (7.13) is replaced by

$$\varepsilon \int_{\bar t_1}^{\bar t_2} \int_{c_\varepsilon}^{\varepsilon^{-\alpha}} \zeta \Phi_t \Phi'\, dz\, dt - \int_{\bar t_1}^{\bar t_2} \int_{c_\varepsilon}^{\varepsilon^{-\alpha}} \zeta \dot{\bar r}^1 \Phi'^2\, dz\, dt \overset{\varepsilon \to 0}{\longrightarrow} -\int_{\bar t_1}^{\bar t_2} \frac{4}{3}\zeta \partial_t \bar r^1\, dt,$$

and the proof is the same.

So in the limit we find

$$-\int_{\bar{t}_1}^{\bar{t}_2}\frac{4}{3}\zeta(\dot{\bar{r}}^1+\frac{n-1}{\bar{r}^1})\,dt-\int_{\bar{t}_1}^{\bar{t}_2}\zeta\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}W(\Phi(c_\varepsilon))\,dt=2\int_{\bar{t}_1}^{\bar{t}_2}\zeta\lambda_0\,dt. \tag{7.29}$$

Similarly, we can integrate (7.2) over $(-\varepsilon^{-\alpha}+z^{m_\varepsilon},d_\varepsilon)$, where d_ε is the first point to the right of z^{m_ε} for which Φ' vanishes. Since $\Phi_0(-\varepsilon^{-\alpha}+z^{m_\varepsilon})\overset{\varepsilon\to 0}{\longrightarrow}-1$ and $\Phi_0(d_\varepsilon)\overset{\varepsilon\to 0}{\longrightarrow}1$, this yields in the limit

$$-\int_{\bar{t}_1}^{\bar{t}_2}\frac{4}{3}\zeta(\dot{\bar{r}}^{m_0}+\frac{n-1}{\bar{r}^{m_0}})\,dt+\int_{\bar{t}_1}^{\bar{t}_2}\zeta\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}W(\Phi(d_\varepsilon))\,dt=2\int_{\bar{t}_1}^{\bar{t}_2}\zeta\lambda_0\,dt. \tag{7.30}$$

Since $W\geq 0$ we conclude for $\zeta\geq 0$, that

$$-\int_{\bar{t}_1}^{\bar{t}_2}\frac{4}{3}\zeta(\dot{\bar{r}}^1+\frac{n-1}{\bar{r}^1})\,dt\geq 2\int_{\bar{t}_1}^{\bar{t}_2}\zeta\lambda_0\,dt,$$
$$-\int_{\bar{t}_1}^{\bar{t}_2}\frac{4}{3}\zeta(\dot{\bar{r}}^{m_0}+\frac{n-1}{\bar{r}^{m_0}})\,dt\leq 2\int_{\bar{t}_1}^{\bar{t}_2}\zeta\lambda_0\,dt.$$

This proves the Lemma.

We now come back to the global situation and the interfaces as defined by (5.6).

Proposition 7.31.

Let A_{R_0} and C_{R_0} be as in Proposition 7.25. Then the interfaces $\bar{r}=\bar{r}^i$ as defined in (5.6) evolve in their domain of existenxe $I^i_{R_0}$ according to

$$-\left(\dot{\bar{r}}+\frac{n-1}{\bar{r}}\right)=\frac{3}{2}\nu(\cdot,\bar{r})\lambda_0. \tag{7.32}$$

In addition for almost all $(\bar{t}_0,\bar{r}_0)\in\Gamma$

$$\textit{either}\quad \nu(\bar{t}_0,\bar{r}_0)\lambda_0(\bar{t}_0)=0\quad \textit{or}\quad m_0(\bar{t}_0,\bar{r}_0)=1. \tag{7.33}$$

and if $\bar{r}_0=1$, then $\nu(\bar{t}_0,\bar{r}_0)\neq 0$.

Proof. Define for any $M_0\geq k\geq j\geq 1$ the set (formally putting $\bar{r}^0=+\infty$ and $\bar{r}^{M_0+1}=-\infty$)

$$I:=\{\bar{t}_0\in C_{R_0}\mid \bar{r}^{j-1}>\bar{r}^j=\ldots=\bar{r}^k>\bar{r}^{k+1}\text{ at }\bar{t}_0\}.$$

There exists only finitely many j and k as above, and any t has to be in one of the corresponding I's. So, if we prove the claim in I, this proves the Proposition.

For any $\bar{t}_0\in I$ and $\bar{r}_0:=\bar{r}^j(\bar{t}_0)$

$$m_0(\bar{t}_0,\bar{r}_0)=k-j+1\quad\text{and}\quad \nu(\bar{t}_0,\bar{r}_0)=\pm 1,\text{ if } m_0 \text{ is odd.}$$

Note that almost everywhere in I

$$\dot{\bar{r}}^j=\ldots=\dot{\bar{r}}^k,$$

and thus Proposition 7.26 implies that almost everywhere in I (if $r^j < 1$)

$$-m_0\left(\dot{r}^j + \frac{n-1}{r^j}\right) = -\sum_{i=j}^{k}\left(\dot{r}^i + \frac{n-1}{r^i}\right) = \frac{3}{2}\nu(\cdot, r^j)\lambda_0. \tag{7.34}$$

If m_0 is even, then $\nu = 0$ almost everywhere in I and thus the differential equation (7.32) is immediate.

If m_0 is odd, then Lemma 7.27 applies, giving

$$-\left(\dot{r}^j + \frac{n-1}{r^j}\right) \geq \frac{3}{2}\nu(\cdot, r^j)\lambda_0 \geq -\left(\dot{r}^k + \frac{n-1}{r^k}\right) \quad \text{in } I.$$

But since both sides have to agree, the differential equation (7.32) is satisfied.

Finally subtracting the differential equation (7.32) from (7.34) implies

$$(1-m_0)\left(\dot{r}^j + \frac{n-1}{r^j}\right) = 0.$$

Thus either $m_0 = 1$ or $\dot{r}^j + \frac{n-1}{r^j} = 0$, the latter in turn implying that $\lambda_0 = 0$.

If $r^j = 1$ and thus $j = 1$, we have $\geq$ in (7.34) and $\dot{r}^j = 0$. Thus $-m_0(n-1) \geq \frac{3}{2}\nu(\cdot, r^j)\lambda_0$. If m_0 is even, then $\nu = 0$, which leads to a contradiction. If m_0 is odd, then by Lemma 7.27 we find $\frac{3}{2}\nu(\cdot, r^j)\lambda_0 \geq -(n-1)$. This first implies m_0=1 and then $\frac{3}{2}\nu(\cdot, r^j)\lambda_0 = -(n-1) = -(\dot{r}^j + \frac{n-1}{r^j})$. This finishes the proof.

Now we have to identify the limit Lagrange multiplier λ_0. To determine λ_0 in terms of the interfaces of v, we use the mass conservation property of the nonlocal flow, namely

$$\partial_t \int_\Omega \varphi_\epsilon(x,t)\,dx = 0, \text{ and hence } \partial_t \int_\Omega v(x,t)\,dx = 0.$$

Since these are nonlocal quantities, we have to get rid of the constraint $R_0 > 0$.

We recall that A_{R_0} is open and its complement has only finitely many points (cf (5.2)). For each C_{R_0} compact in A_{R_0} we defined interfaces $r^i : I^i_{R_0} \subset C_{R_0} \to [R_0, 1]$ for $i = 1, ..., M_0$. Putting this together we generalize the notation, so that we have interfaces

$$r^i : I^i_{R_0} \subset A_{R_0} \to (R_0, 1] \quad \text{for } i = 1, ..., M_0 \tag{7.35}$$

with the property that $r^i > r^{i+1}$ and $r^i = R_0$ on $\partial I^i_{R_0} \cap A_{R_0}$. These interfaces are locally Hölder–$\frac{1}{2}$. Let us extend the definition of r^i by R_0 to the whole of A_{R_0}. This extension is of class $H^{1,2}$. Formally we set in addition $r^i = R_0$ for $i > M_0$.

We then define for a sequence $R_0 \to 0$

$$A := \bigcap_{R_0 \to 0} A_{R_0}. \tag{7.36}$$

Then, since this is a countable intersection, the complement of A has at most countably many points. In addition we define $R^i(t) := r^i(t)$, if $t \in I^i_{R_0}$ and (with a slight abuse of notation)

$$R^i = R^i_{|A} : A \longrightarrow [0,1] \qquad \text{for } i = 1, \ldots. \tag{7.37}$$

Then in particular $I^i_{R_0} = \{R^i > R_0\}$. This definition is reasonable, since the A_{R_0} are included in each other with decreasing R_0. Moreover for any R_0 the $\bar{r}^i$ were constructed (see section 5) by selecting subsequences, and, making the appropriate choices of subsequences, we may assume that the subsequence does not depend on R_0. In addition it is possible to choose a_ε independently of R_0. Thus the limits $\bar{r}^i$ coincide on intersections of the sets $A_{R_0} \times (R_0, 1]$.

Proposition 7.38. *In the set A the limit Lagrange multiplier is given by*

$$\lambda_0 = -\frac{2}{3}\frac{(n-1)\sum \nu(R^i)(R^i)^{n-2}}{\sum |\nu(R^i)|(R^i)^{n-1}}, \tag{7.39}$$

where summation is over all interfaces of radii less than 1. If the space dimension is strictly bigger than 2, then $\lambda_0 \neq 0$. If the space dimension is equal to 2, then $\lambda_0 \neq 0$ in case of an odd number of interfaces, and identically zero in case of an even number of interfaces.

Proof. Since $\varphi_\varepsilon \to v$ in L^1 and $v = \pm 1$ almost everywhere, the same is true for

$$\chi_\varepsilon := \frac{\varphi_\varepsilon - a_\varepsilon}{|\varphi_\varepsilon - a_\varepsilon|}.$$

By construction χ_ε has interfaces r^i_ε for $i = 1, ..., M_0$ in $A_{R_0} \times (R_0, 1)$. Thus for $t \in A_{R_0}$ and $\varepsilon \to 0$

$$\begin{aligned}\int_{|x|>R_0} v\,dx &\longleftarrow \int_{|x|>R_0} \chi_\varepsilon\,dx \\ &= \pm\frac{1}{n}\left\{2\sum_{i=1}^{M_0}(-1)^{i+1}(r^i_\varepsilon)^n + (-1)^{M_0+1}(R_0)^n - 1\right\} \\ &\longrightarrow \pm\frac{1}{n}\left\{2\sum_{i=1}^{M_0}(-1)^{i+1}(\bar{r}^i)^n + (-1)^{M_0+1}(R_0)^n - 1\right\}.\end{aligned}$$

Differentiating the resulting identity yields

$$\partial_t \int_{|x|>R_0} v\,dx = \pm 2\sum_{i=1}^{M_0}(-1)^{i+1}(\bar{r}^i)^{n-1}\dot{\bar{r}}^i \qquad \text{for } t \in A_{R_0}. \tag{7.38}$$

This can be rewritten as

$$\partial_t \int_{|x|>R_0} v\,dx = \left\{\pm 2\sum_{i=1}^{M_0}(-1)^{i+1}(\bar{r}^i)^{n-1}\dot{\bar{r}}^i\right\}\chi + \mu_{R_0} \tag{7.39}$$

where χ is the characteristic function of A and μ_{R_0} is a measure with support in $(0,T)\backslash A$.

Since $\partial_t\varphi \in BV((0,T)\times\Omega)$, we know that $\partial_t \int_{|x|>R_0} v\,dx \in BV(0,T)$ uniformly bounded with respect to R_0 and thus, since $\int_{|x|>R_0} v\,dx \to \int_\Omega v\,dx$

$$\partial_t \int_{|x|>R_0} v\,dx \overset{*}{\rightharpoonup} 0 = \partial_t \int_\Omega v\,dx \quad \text{in } [C^0(0,T)]'. \tag{7.40}$$

We now use the differential equation (7.32) to calculate the limit of the sum in (7.39):

$$\sum_{i=1}^{M_0}(-1)^{i+1}(\bar{r}^i)^{n-1}\dot{\bar{r}}^i =$$

$$-\frac{3}{2}\lambda_0 \sum_{i=M_1(t)}^{M_2(t)} (-1)^{i+1}\nu(\bar{r}^i)(\bar{r}^i)^{n-1} - (n-1) \sum_{i=M_1(t)}^{M_2(t)} (-1)^{i+1}(\bar{r}^i)^{n-2},$$

where $M_2(t) := \#\{i \mid \bar{r}^i(t) > R_0\}$ and $M_1(t)$ the first index, such that the corresponding interface is less than 1.

We assume without loss of generality, that v is positive near $\partial\Omega$. Then we remark that if $\nu(\bar{r}^i) \neq 0$, then $(-1)^{i+1} = -\nu(\bar{r}^i)$, because for any i with $\nu(\bar{r}^i) = 0$ an even number of interfaces collides. In the second sum, we may substitute $(-1)^{i+1}$ by $-\nu(\bar{r}^i)$, because, either they agree, or $(\bar{r}^i)^{n-2}$ is added up an even number of times with alternating signs. Thus

$$\sum_{i=1}^{M_0}(-1)^{i+1}(\bar{r}^i)^{n-1}\dot{\bar{r}}^i =$$

$$-\frac{3}{2}\lambda_0 \sum_{i=M_1(t)}^{M_2(t)} |\nu(\bar{r}^i)|(\bar{r}^i)^{n-1} - (n-1) \sum_{i=M_1(t)}^{M_2(t)} \nu(\bar{r}^i)(\bar{r}^i)^{n-2}.$$

We now want to pass to the limit $R_0 \to 0$. This is possible due to the following bounds. Since the jumps of v are exactly given by the interfaces R^i, with $\nu(R^i) \neq 0$, we have the formula

$$\sup_{(0,T)} \int_\Omega |\nabla v| = \sup_A \omega_n \sum_{i=1}^{\infty} |\nu(R^i)|(R^i)^{n-1} \leq C.$$

For the second sum we apply Proposition 4.11. By lower semicontinuity the estimate (4.12) carries over to the limit $g(v) = \frac{4}{3}v$ and thus

$$\int_0^T \int_0^1 |v'|r^{n-2}\, dr\, dt = \int_A \sum_{i=1}^{\infty} |\nu(R^i)|(R^i)^{n-2}\, dt \leq C.$$

So we know that all the sums converge absolutely as $R_0 \to 0$ and by the monotone convergence Theorem this convergence (after multiplication with χ as in (7.39)) is in $L^1(0,T)$

$$\left\{\sum_{i=1}^{M_0}(-1)^{i+1}(\bar{r}^i)^{n-1}\dot{\bar{r}}^i\right\}\chi$$

$$\xrightarrow{R_0\to 0} \left\{-\frac{3}{2}\lambda_0 \sum_{i=M_1(t)}^{\infty} |\nu(R^i)|(R^i)^{n-1} - (n-1) \sum_{i=M_1(t)}^{\infty} \nu(R^i)(R^i)^{n-2}\right\}\chi \qquad (7.41)$$

This, together with (7.39) and (7.40), implies

$$\mu_{R_0} \xrightarrow{*} \mu = 0$$

and

$$\frac{3}{2}\lambda_0 \sum_{i=M_1(t)}^{\infty} |\nu(R^i)|(R^i)^{n-1} + (n-1) \sum_{i=M_1(t)}^{\infty} \nu(R^i)(R^i)^{n-2} = 0 \quad \text{almost everywhere in } A.$$

We now summarize the results of this section in

Corollary 7.42.
Let A be given by (7.36) and R^i by (7.37). Then for any $0 < R_0 < 1$ the evolution of the interfaces is governed by

$$\left(\dot{R}^i + \frac{n-1}{R^i}\right) = \nu(\cdot, R^i) \frac{(n-1)\sum_{j=1}^{\infty} \nu(\cdot, R^j)(R^j)^{n-2}}{\sum_{j=1}^{\infty} |\nu(\cdot, R^j)|(R^j)^{n-1}} \quad \text{in } I^i_{R_0}. \tag{7.43}$$

Furthermore

$$\dot{R}^i \leq 0 \quad \text{in } I^i_{R_0}. \tag{7.44}$$

The Lagrange multiplier λ_0 is uniformly bounded and changes sign at most at finitely many points. If the space dimension is strictly bigger than 2 or if the space dimension is 2 and there is an odd number of interfaces, then

$$\nu(t, R^i(t)) \neq 0 \quad \text{implies} \quad m_0(t, R^i(t)) = 1 \quad \text{in } I^i_{R_0}. \tag{7.45}$$

Locally in $A_{R_0} \times (R_0, 1)$ the free boundary $\partial\{v = -1\}$ of v either is a graph or consists of exactly two graphs which meet or nucleate. In the first case the free boundary of v is locally given by one of the graphs R^j and the corresponding $\nu(R^j)$ is constant and nonzero while none of the other graphs R^k are close. In the second and third cases the free boundary is given by exactly two true interfaces R^j and R^k that either meet and then form a single "phantom" interface or that nucleate from a single "phantom" interface.

Proof. By Proposition 7.31 we know that in $I^i_{R_0}$ the evolution equation is satisfied, and by Proposition 7.38 we have a formula for the Lagrange multiplier. Combining both Propositions we see

$$\dot{R}^i \leq 0 \Longleftrightarrow \sum_j \nu(R^j)\nu(R^i)R^i(R^j)^{n-2} \leq \sum_j |\nu(R^j)|(R^j)^{n-1},$$

where summation is over all interfaces of radii less than 1. But, if k is the smallest index such that the corresponding interface has radius less than 1 and has nonzero ν, then

$$\begin{aligned}\sum_j \nu(R^j)\nu(R^i)R^i(R^j)^{n-2} &\leq \begin{cases} |\nu(R^i)|R^i(R^k)^{n-2}, & \text{if } i \leq k \\ 0, & \text{otherwise}\end{cases} \\ &\leq (R^k)^{n-1} \leq \sum_j |\nu(R^j)|(R^j)^{n-1}.\end{aligned}$$

This proves that $\dot{R}^i \leq 0$ in $I^i_{R_0}$ for any $R_0 > 0$.

Since mass is preserved there exists an $R_{\min}$, given in terms of the initial mass, such that $R^1 > R_{\min}$. Thus by the above $\dot{R}^1 \leq 0$ in $I^1_{R_{\min}} = A_{R_{\min}}$ and R^1 jumps at most at finitely many points. This implies that the denominator in the formula for λ_0 is bounded below, and since the nominator is an alternating sum, λ_0 is uniformly bounded.

In $A_{R_{min}} \cap \{R^1 = 1\}$ the differential equation for R^1 is $-(n-1) = \frac{3}{2}\nu(R^1)\lambda_0$. Substituting the formula (7.39) for λ_0 into this yields

$$\nu(R^1)\sum_j \nu(R^j)(R^j)^{n-2} = \sum_j |\nu(R^j)|(R^j)^{n-1},$$

where summation is again over all interfaces whose radii are less than 1. But we observe that the left hand side of this equality is non positive because the first non vanishing coefficient is negative. Since the right hand side is strictly positive, the set $A_{R_{min}} \cap \{R^1 = 1\}$ must have measure zero. Since R^1 is monotone decreasing and continuous in $A_{R_{min}}$, this implies that $R^1 < 1$ in $A_{R_{min}}$, and thus the sign of the limit order parameter v is fixed in any connected component of $\{(t,r) \mid t \in A_{R_{min}} \text{ and } r > R^1(t)\}$. As a consequence we first obtain an improved formula for λ_0

$$\lambda_0 = -\frac{3}{2}(n-1)\frac{\sum_j \nu(R^j)(R^j)^{n-2}}{\sum_j |\nu(R^j)|(R^j)^{n-1}},$$

where summation now ranges from 1 to ∞. Furthermore, since the sign of λ_0 is given by the first nonzero ν in the above sum, the sign of λ_0 is constant in any connected part of $A_{R_{min}}$.

Now suppose $t \in A_{R_0}$ for some R_0. Fix $R^j(t)$ and $R^k(t)$ with $j < k$ and assume that there exists for both $(R^j(t), t)$ and $(R^k(t), t)$ a neighborhood, such that in these neighborhoods $\nu(R^j) = \nu_j$ and $\nu(R^k) = \nu_k$ are constant and nonzero. Thus by the differential equation

$$\begin{aligned} \dot{R}^j - \dot{R}^k &= \frac{n-1}{R^k} - \frac{n-1}{R^j} + \lambda_0(\nu_k - \nu_j) \\ &> \lambda_0(\nu_k - \nu_j), \end{aligned} \tag{7.46}$$

and hence the distance between R^j and R^k increases if $\lambda_0(\nu_k - \nu_j) \geq 0$. This is in particular true if $\nu_k = \nu_j$.

In the case that the dimension of the space $n \geq 3$, we know from (7.33) and Proposition 7.38 that for almost every $t \in A_{R_0}$, the condition $\nu(t, R^j(t)) \neq 0$ implies $m_0 = 1$. By the definition of m_0, this implies that for almost all t and all points $(R^j(t), t)$ with $R^j(t) > R_0$ and $\nu(t, R^j(t)) \neq 0$ there exists a box B such that the set $\Gamma \cap B$ consists exactly of the corresponding R^j. Thus the observation (7.46) implies that the distance between all interfaces R^k and R^j increases if they have the same normal, and hence such interfaces can never meet. As a consequence we find that if two R^j and R^k with nonvanishing normal meet at some point t, then $\nu_j - \nu_k$ is either in the set $\{2, 0\}$ or in the set $\{0, -2\}$ in a whole neighborhood of the meeting point and consequently does not change sign. Thus (7.46) shows that either $\dot{R}^j - \dot{R}^k \geq 0$ or $\dot{R}^j - \dot{R}^k \leq 0$ in the whole neighborhood of the meeting point, because the sign of λ_0 is fixed and $\frac{n-1}{R^k} - \frac{n-1}{R^j}$ is Lipschitz continuous in the neighborhood. Meeting is only possible in the second case. Integrating the inequality yields $(R^j - R^k)(\tau) \leq 0$ for τ bigger than the meeting time. On the other hand by the ordering we know $(R^j - R^k)(\tau) \geq 0$, thus equality follows after the meeting point. A similar result holds for nucleation. Thus at most two true interfaces can nucleate from a "phantom" interface and then stay apart or at most two true interfaces can meet and then form a single "phantom" interface. These are the only possibilities of geometric singularities in A_{R_0}.

Of course "phantom" interfaces cannot meet each other, because they move by mean curvature. Moreover they can only meet any of the true interfaces at one of the above meeting points because otherwise the density m_0 would be bigger than 1 on a true interface.

Similar arguments apply for $n = 2$. Indeed in that case the Lagrange multiplier has locally either a fixed sign, or is identically zero. In the first case, the same analysis as above applies because the density m_0 of interfaces with non vanishing normal is 1 again. In the second case, all the interfaces move by mean curvature and cannot meet anyway. Thus the proof of Corollary 7.42 is complete.

Remark 7.47. The evolution equation is the radial version of the expected limiting nonlocal geometric problem

$$V_i - k_i = -\frac{1}{\sum_j |\Gamma_j|} \sum_j \int_{\Gamma_j} k_j \, ds,$$

where V_i is the normal velocity and k_i is the sum of the principal curvatures of the interface Γ_i.

Note that in the right hand side of (7.43), we sum only over interfaces R^i such that $\nu(R^i) \neq 0$. These interfaces represent exactly the free boundary of v. If $\nu(R^i) = 0$, these "phantom" interfaces are not seen by the limit v and they correspond to collapsing ε–interfaces. They do not have an impact on the evolution of the other interfaces.

Remark 7.48. The limit v satisfies the weak Hölder continuity estimate in Remark 2.11. For true interfaces it implies that at the points of $N(R_0)$ (see (2.17) and Proposition 3.9 for the definition) only one of the following behaviours of the graphs R^j is possible.

As seen in the proof of Corollary 7.42 at most two true interfaces can meet or nucleate at a point $t \in N(R_0)$. Two meeting true interfaces can only continue as true interfaces accross $t \in N(R_0)$ if λ_0 changes sign at this time point, and this can only happens if a true interface nucleates from the fixed boundary.

Any true interface that does not meet with another one at times in $N(R_0)$ continues as a single true interface across this time point.

When there exists a continuation, it has to be of class Hölder–$\frac{1}{2}$ and thus the differential equation is satisfied across that point.

Example 7.49. Colliding interfaces are generic to the nonlocal flow. Indeed if two interfaces are sufficiently close to each other initially, sufficiently close with respect to their distance to the others and to their own size, then they have to meet before the smaller of them has time to shrink to zero.

Let $n = 3$ and assume that there are three initial interfaces. Then their evolution, as long as they all exist, is governed by

$$-(\dot{R}^i + \frac{n-1}{R^i}) = -(-1)^{i+1}(n-1)\mu_0$$

where

$$\mu_0 = \frac{R^1 - R^2 + R^3}{(R^1)^2 + (R^2)^2 + (R^3)^2}.$$

Therefore

$$\dot{R}^3 \geq -\frac{n-1}{R^3}$$

and thus $(R^3)^2(t) \geq -2(n-1)t + (R^3)^2(0)$. Thus R^3 cannot dissapear at the origin, as long as $t \leq t_{\max} = \frac{1}{2(n-1)}(R^3)^2(0)$.

Since mass is preserved the biggest interface is bounded below by some number $R_{\min}$, which is such that the mass of a ball of radius $R_{\min}$ equals the initial mass:

$$(R_{\min})^3 = ((R^1)^3 - (R^2)^3 + (R^3)^3)(0).$$

Since all interfaces are decreasing, this implies the following bound for μ_0:

$$\mu_0 \geq \frac{R_{\min} - R^2(0)}{((R^1)^2 + (R^2)^2 + (R^3)^2)(0)}.$$

Subtracting the equation for the second and third interface results in

$$\dot{R}^2 - \dot{R}^3 = \frac{n-1}{R^3} - \frac{n-1}{R^2} - 2(n-1)\mu_0 \leq -2(n-1)\mu_0.$$

Now suppose the interfaces R^2 and R^3 do not meet before R^3 vanishes. Then we can integrate the above inequality, use the bound for μ_0 and evaluate the result at $t_{\max}$ to find

$$0 \leq (R^2 - R^3)(t_{\max}) \leq -\frac{R_{\min} - R^2(0)}{((R^1)^2 + (R^2)^2 + (R^3)^2)(0)} + (R^2 - R^3)^2(0).$$

If $(R^2 - R^3)^2(0)$ is small, then $R_{\min}$ is approximately $R^1(0)$ and it is clearly possible to choose initial data for R^1 that contradicts the above inequality. Thus R^2 and R^3 have to meet before the smaller one had time to dissapear. After the meeting point the evolution becomes stationary and $R^1 = R_{\min}$.

Remark 7.50. Our estimates of section 6 are strong enough to prove that

$$E_0(t) = \frac{4}{3}\sum m_0(t, R^i(t))(R^i(t))^{n-1}.$$

Thus jumps in the energy correspond to either jumps in interfaces or jumps in density. If we impose that initially no interfaces with vanishing normal exist and that there are only finitely many interfaces initially, then by the maximum principle for the ε–equation, no interfaces can nucleate from the origin as long as the smallest of the initial interfaces has not disappeared. Thus up to that time all interfaces have nonvanishing normal and consequently density 1 and are continuous.

Section 8: The viscous Cahn–Hilliard equation

Here we consider the viscous Cahn–Hilliard equation in $\Omega_T := \Omega \times (0, T)$ as introduced by Novick–Cohen [NC]

$$\alpha\partial_t\varphi - \Delta u = 0, \tag{8.1}$$

$$u = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}W'(\varphi) + \nu\partial_t\varphi. \tag{8.2}$$

Imposing Neumann–zero boundary conditions for both u and φ and applying the usual techniques one obtains the equations

$$\frac{1}{\alpha}\int_\Omega |\nabla u|^2\,dx + \partial_t E_\varepsilon(\varphi) + \nu\int_\Omega |\partial_t\varphi|^2\,dx = 0 \tag{8.3}$$

and

$$\int_\Omega \partial_t\varphi\,dx = 0 \quad \text{and} \quad \fint_\Omega u\,dx = \frac{1}{\varepsilon}\fint_\Omega W'(\varphi)\,dx. \tag{8.4}$$

We want briefly show, how this equation relates both to the Cahn–Hilliard and the nonlocal Allen–Cahn equation.

Let us first consider the limit $\nu \to 0$, keeping all the other parameters fixed. Formally the limit problem is the standard Cahn–Hilliard equation. If the initial energy is uniformly bounded in ν, this can be shown rigorously. First we note that the estimate (8.3) immediately gives weak compactness in $L^2(\Omega_T)$ for $\nabla\varphi$ and a bound of φ in $L^4(\Omega_T)$. But since the equation is nonlinear it is important to have strong compactness in $L^1(\Omega_T)$ for φ. To this end we note that the first equation (8.1) implies

$$\alpha\,\|\,\partial_t\varphi\,\|_{L^2(H^{-1,2})} = \|\,\nabla u\,\|_{L^2(\Omega_T)} \le E_\varepsilon(\varphi)(0) \tag{8.5}$$

and thus (with $E_\varepsilon(\varphi)(0) \le C$)

$$\alpha\,\|\varphi(\cdot,\cdot-h)-\varphi\,\|_{L^2(H^{-1,2})} \le C\cdot h. \tag{8.6}$$

Now interpolating between $L^2(H^{-1,2})$ and $L^2(H^{1,2})$ yields

$$\|\,\varphi(\cdot,\cdot+h)-\varphi\,\|_{L^1(\Omega_T)} \le Ch^{\frac{1}{2}}. \tag{8.7}$$

This implies strong compactness for φ in $L^1(\Omega_T)$.
Of course the estimate for ∇u of (8.3) together with the bound of its mean value by (8.4) imply weak compactness in $L^2(\Omega_T)$ for both u and ∇u. This convergence is strong enough to pass to the limit $\nu \to 0$ in the viscous Cahn–Hilliard equation (8.1) and (8.2).

If the limit $\alpha \to 0$ is considered, the limit problem will be the non-local equation. To see this, once again asssume that the energy is uniformly bounded in α. Then the estimate (8.3) yields the L^1–compactness of the order parameter φ immediately, and the weak $L^2(\Omega_T)$-compactness of u and ∇u is again obtained from (8.3) and the mean value condition (8.4). This compactness allows to pass to the limit in (8.1) and (8.2). The limit of (8.1) gives $\Delta u = 0$ for the limit, and thus u is a constant. But then (8.4) yields the correct formula for u and we find the non-local equation in the limit.

So we see that both the non-local equation and the Cahn–Hilliard equation occur as special degenerate limits of the viscous Cahn–Hilliard equation.

Appendix: Ellipticity of the linearized Allen–Cahn equation

Here we give the proof of the ellipticity Proposition (6.24).

Proof of Proposition 6.24. Let S be one of the sets of integration as in Proposition 6.24. We start integrating the left hand side of the claimed estimate by parts. This results in

$$\begin{aligned}(LHS) &= \int_S (-\Psi'' + W''(\Theta)\Psi)\Psi\xi^2\,dz \\ &= \int_S (|\Psi'|^2 + W''(\Theta)\Psi^2)\,\xi^2\,dz + 2\int_S \Psi'\Psi\xi'\xi\,dz\end{aligned} \tag{A.1}$$

Here we should point out that Ψ is not necessarily smooth at $z = z_\epsilon^i$. But sind Ψ vanishes at these points, the integration by parts is nevertheless valid. Now we note that $W''(\Theta) = 2(3\Theta^2 - 1)$, and in the set where this function is strictly positive nothing is to prove. Careful study of Θ will show, that the measure of the set where W'' is not strictly positive is small enough, such that the integral of Ψ^2 over this set can still be controled by the integral of Ψ'^2.

So we introduce for any $a > 0$ the set

$$I_\epsilon := \{W''(\Theta) < 2a\} \cap (-\epsilon^{-\beta} + z_\epsilon^{m_\epsilon}, \epsilon^{-\beta}). \tag{A.2}$$

We want to estimate the diameter of any connected component of I_ϵ. By definition (6.5) and (6.6)

$$\Theta(z) = \sum_i \Xi_i(z) tanh((-1)^i(z - z_\epsilon^i) + \mu_\epsilon) + \epsilon \Phi_1^\epsilon(z)$$

with

$$\| \Phi_1 \|_{L^\infty(-\epsilon^{-\beta}+z_\epsilon^{m_\epsilon}, \epsilon^{-\beta})} \leq C\| \lambda_\epsilon \| \leq \frac{C}{\sqrt{\epsilon}} \quad \text{uniformly in time}$$

by Corollary 2.7 and Lemaa 6.25.
Since $W''(\Theta) < 2a \Leftrightarrow |\Theta| < \sqrt{\frac{a+1}{3}}$ we thus find

$$\begin{aligned}
I_\epsilon &\subset \{ \sum_i \Xi_i \,|\, tanh((-1)^i(z - z_\epsilon^i) + \mu_\epsilon) \,|\, \leq \sqrt{\frac{a+1}{3}} + C\epsilon \} \\
&\subset \bigcup_i \{ \,|\, tanh((-1)^i(z - z_\epsilon^i) + \mu_\epsilon) \,|\, \leq \sqrt{\frac{a+1}{3}} + C\epsilon \} \\
&\subset \bigcup_i \{ |z - z_\epsilon^i| \leq tanh^{-1}\left(\sqrt{\frac{a+1}{3}} + C\epsilon\right) + |\mu_\epsilon| \} \\
&=: \bigcup_i I^i.
\end{aligned} \tag{A.3}$$

We return to (A.1). We continue to estimate as follows

$$\begin{aligned}
(LHS) \geq &\int_S |\Psi'|^2 \xi^2 \, dz + 2a \int_S \Psi^2 \xi^2 \, dz \\
&- (2 + 2a) \int_{I_\epsilon} \Psi^2 \xi^2 \, dz + 2 \int_S \Psi' \Psi \xi' \xi \, dz.
\end{aligned} \tag{A.4}$$

But, using $\Psi(z_\epsilon^i) = 0$, gives

$$\begin{aligned}
\int_{I_\epsilon} \Psi^2 \xi^2 \, dz &= \sum_i \int_{I^i \setminus I^{i-1}} \xi^2 \Psi^2 \, dz \\
&= \sum_i \int_{I^i \setminus I^{i-1}} \xi^2 \left(\int_{z_\epsilon^i}^z \Psi' \right)^2 dz \\
&\leq \sum_i \int_{I^i \setminus I^{i-1}} |\Psi'|^2 \frac{1}{2} |I^i| \\
&\leq \left(tanh^{-1}\left(\sqrt{\frac{a+1}{3}} + C\epsilon\right) + tanh^{-1} Q \right) \int_S |\Psi'|^2 \xi^2 \, dz
\end{aligned} \tag{A.5}$$

by (A.3) and since $|\mu_\varepsilon = |tanh^{-1} a_\varepsilon| \le tanh^{-1} Q$. Note in addition that by construction $\bigcup I^i \subset \{\xi = 1\}$. We put this into (A.4) and continue to estimate

$$(LHS) \ge \underbrace{\left(1 - (2+2a)\left(tanh^{-1}\left(\sqrt{\frac{a+1}{3}} + C\varepsilon\right) + tanh^{-1} Q\right)\right)}_{=:c_0(a,\varepsilon)} \int_S |\Psi'|^2 \xi^2 \, dz$$
$$+ 2a \int_S \Psi^2 \xi^2 \, dz + 2 \int_S \Psi' \Psi \xi' \xi \, dz. \qquad \text{(A.6)}$$

The number c_0 turns out to be positive if $\varepsilon = 0$ and $a = 0$ by the choice of Q in (5.1), and thus there exist positive ε_0 and a_0 such that

$$0 < c_0(\varepsilon_0, a_0) < c_0(\varepsilon, a_0)$$

for all $\varepsilon < \varepsilon_0$.

This proves the proposition with $\zeta_1 = \frac{1}{2} c_0(\varepsilon_0, a_0)$ and $\zeta_2 = 2a_0$, if we still use the Hölder estimate for $\int \Psi' \Psi \xi' \xi$.

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