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Remarks on Optimal Design Problems

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REMARKS ON OPTIMAL DESIGN PROBLEMS

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Abstract: One discusses some problems of optimal design for a diffusion equation, when the functional to minimize contains a term in $|\mathrm{grad}(u)|^2$.

Introduction

A little more than 20 years ago, following the discovery by François MURAT of some counterexamples [1a-1b], we set up the mathematical basis of a new method for solving some optimal design problems with distributed parameters [2]. The lack of existence of solutions for some optimal design problems had actually been discovered earlier by Konstantin LUR'E [3], but his own attempts for finding a solution were not so successful, I was told, until he had read [2]. Our analysis relied on an extension of some earlier work of Sergio SPAGNOLO [4a-4b], extension which became later described as homogenization, following the terminology introduced by Ivo BABUŠKA [5]. However, if many applications in Engineering that I. BABUŠKA had in mind did use periodic structures, it would not have been logical for us to impose any restriction of any kind for the minimizing sequences trying to find the optimal designs that we were seeking and we had worked in the same general framework that S. SPAGNOLO had considered, and called G-convergence as a reminder that it was related to the convergence of GREEN functions. We actually developed a slightly more general framework, that we called H-convergence [6], but the main difficulty that we had to overcome was to find a characterization of the limits that we were considering. In discovering some work of Enrique SANCHEZ-PALENCIA [7a-7b], we had understood that our problem was identical to that of describing effective properties of mixtures, and that the first bounds that we had obtained were indeed the classical elementary ones derived from variational principles.

In order to go further, I devised a method [8] using ideas from the theory of compensated compactness that I had also developed with F. MURAT, but as the method required the use of some adapted functionals that had to be guessed, it was only a few years after that I found a good choice for characterizing the isotropic conducting mixtures obtained by mixing in given proportions two isotropic conductors, a result identical with the classical bounds derived by Zvi HASHIN & S. SHTRIKMAN [9]. Using the same functionals that I had found, the characterization of anisotropic conducting mixtures using two isotropic conductors in given proportions was then obtained with F. MURAT and presented at a meeting at NYU in June 1981, while the detailed proof was only written two years after [10], even after we had finally written our first article on our original method [11]; ironically, the technical characterization that we had obtained was not really needed for concluding in usual cases, as one could succeed in using only the classical cruder bounds.

In brief, the prototype of problems that can be covered by our original method is to consider first a diffusion equation in a bounded open subset Ω of R^N

$$-\mathrm{div}\Big(a\,\mathrm{grad}(u)\Big) = f \text{ in } \Omega, \quad u \in H^1_0(\Omega), \tag{1}$$

with $f \in H^{-1}(\Omega)$, the scalar coefficient a satisfying

$$0 < \alpha \le a(x) \le \beta < \infty \text{ a.e. in } \Omega. \tag{2}$$

One considers then a control problem where the coefficient a is constrained and one wants to minimize a functional J of the type

$$J(a) = \int_\Omega F\Big(x, a(x), u(x)\Big)\,dx, \tag{3}$$

with F satisfying suitable regularity and growth conditions. Interpreting a as a conductivity, a natural situation is that one has at one's disposal finite amounts $\kappa_1, \ldots, \kappa_m$ of isotropic conductors with conductivity

$\gamma_1, \ldots, \gamma_m$ and the constraints imposed on a are that a must only take some of the values $\gamma_j, j = 1, \ldots, m$, with

$$meas\{x : a(x) = \gamma_j\} \le \kappa_j, \text{ for } j = 1, \ldots, m, \tag{4}$$

assuming of course that $meas(\Omega) \le \sum_{j=1}^{m} \kappa_j$.

In the preceding situation, minimizing sequences a^n are such that a subsequence $a^{n'}$ H-converges to an effective (symmetric) tensor A_{eff}, and (2) is replaced by

$$\alpha|\xi|^2 \le (A_{eff}(x)\xi.\xi) \le \beta|\xi|^2 \text{ a.e. } x \in \Omega, \text{ for all } \xi \in R^N. \tag{5}$$

In order to be more precise, one needs to use the local amounts of each of the materials, i.e. write

$$a^n(x) = \sum_{j=1}^{m} \chi_j^n(x)\gamma_j, \tag{6}$$

where the functions χ_j^n are characteristic functions of disjoint measurable subsets of Ω, and assume that

$$\chi_j^{n'} \rightharpoonup \theta_j \text{ in } L^\infty(\Omega) \text{ weak } \star, \tag{7}$$

and then (5) can be improved into the following simple variational bounds

$$a_-(x)|\xi|^2 \le (A_{eff}(x)\xi.\xi) \le a_+(x)|\xi|^2 \text{ a.e. } x \in \Omega, \text{ for all } \xi \in R^N, \tag{8}$$

where a_- and a_+ are the harmonic and arithmetic averages

$$\frac{1}{a_-(x)} = \sum_{j=1}^{m} \frac{\theta_j}{\gamma_j}, \quad a_+(x) = \sum_{j=1}^{m} \theta_j\gamma_j, \text{ a.e. } x \in \Omega. \tag{9}$$

Of course one has

$$0 \le \theta_j, \text{ a.e. } x \in \Omega, j = 1, \ldots, m, \quad \sum_{j=1}^{m} \theta_j = 1, \text{ a.e. } x \in \Omega, \quad \int_\Omega \theta_j dx \le \kappa_j, j = 1, \ldots, m, \tag{10}$$

but, apart from the case $m = 2$, the characterization of the admissible set of $(A_{eff}, \theta_1, \ldots, \theta_m)$ is not known. For $m = 2$, let $\lambda_j(x)$ be the eigenvalues of $A_{eff}(x)$; then the simple constraint (8) means

$$a_-(x) \le \lambda_j(x) \le a_+(x) \text{ a.e. } x \in \Omega, j = 1, \ldots, N, \tag{11}$$

and for obtaining the characterization one must add two more inequalities, for which one assumes $\gamma_1 < \gamma_2$:

$$\sum_{j=1}^{m} \frac{1}{\lambda_j(x) - \gamma_1} \le \frac{1}{a_-(x) - \gamma_1} + \frac{N-1}{a_+(x) - \gamma_1}, \text{ a.e. } x \in \Omega, \tag{12$_a$}$$

$$\sum_{j=1}^{m} \frac{1}{\gamma_2 - \lambda_j(x)} \le \frac{1}{\gamma_2 - a_-(x)} + \frac{N-1}{\gamma_2 - a_+(x)}, \text{ a.e. } x \in \Omega. \tag{12$_b$}$$

In the case where all the λ_j are equal, $(12)_a$-$(12)_b$ correspond to the classical HASHIN-SHTRIKMAN's bounds.

Independently of finding the unknown characterization in the general case, the sequence $u^{n'}$ converges in $H_0^1(\Omega)$ weak to the solution u of

$$-\text{div}\Big(A_{eff}\text{grad}(u)\Big) = f \text{ in } \Omega, \quad u \in H_0^1(\Omega), \tag{13}$$

and

$$\lim_{n'\to\infty} J(a^{n'}) = \sum_{j=1}^{m} \int_\Omega \theta_j(x) F\Big(x, \gamma_j, u(x)\Big)\,dx, \tag{14}$$

if each function $F(x, \gamma_j, u)$ is a CARATHÉODORY function satisfying a suitable growth condition (in order to discover the best possible growth condition one must of course use SOBOLEV's imbedding theorem, but one must also have a good understanding of how the critical exponent in MEYERS's regularity theorem depends upon the regularity of f and the smoothness of the boundary of Ω, and that matter is far from being completely understood).

One has obtained then a relaxed problem of the original one: the control is $(A_{eff}, \theta_1, \ldots, \theta_m)$, the state u is defined by (13), the cost function J^* is given by the formula

$$J^*(A_{eff}, \theta_1, \ldots, \theta_m) = \sum_{j=1}^{m} \int_\Omega \theta_j(x) F\Big(x, \gamma_j, u(x)\Big)\,dx, \tag{15}$$

but there is a difficulty: although $\theta_1, \ldots, \theta_m$ satisfy (10) and A_{eff} satisfies (8)-(9), there are some additional but yet unknown extra conditions on A_{eff}, and therefore the precise set of controls is not known, although there are some partial results, for example those obtained by Robert KOHN and Graeme MILTON [12].

The miracle is that it does not matter here because a solution will be obtained by considering the larger set of controls satisfying (8)-(9)-(10) and observing that a solution of that problem will be given by a symmetric tensor A_{eff} having $a_-(x)$ and $a_+(x)$, or only one of these numbers, as eigenvalues; as such a tensor can be obtained by a simple layering, one finds then that an optimal control of the enlarged class actually belongs to the otherwise unknown admissible set of controls, and the precise knowledge of what this admissible set is appears then to be irrelevant.

In the same visit at the COURANT Institute where I had found a way to use my earlier method – which Graeme MILTON popularized a few years later as the "translation method" – and where I had learned from George PAPANICOLAOU about the classical HASHIN-SHTRIKMAN bounds for isotropic mixtures, I had also mentioned to Robert KOHN our approach to optimal design using homogenization, as he was planning to work on similar questions. In the interesting work he did then with Gilbert STRANG [13a-13b-13c], the approach was necessarily different from ours because homogenization questions in linearized elasticity are much harder than for a diffusion equation, and because the analogous problem would not have been to consider a functional like (3) where only u occurs, but a functional where grad(u) occurs and very little is known in that case; they succeeded by cleverly changing a realistic problem of elasticity into a manageable one, in order to benefit from another type of miracle than the one I have described. As R. KOHN justly pointed out to me later, the miracle in our situation would usually not occur if one considered a functional of the type

$$J(a) = \sum_{k=1}^{p} \int_\Omega F_k\Big(x, a(x), u_k(x)\Big)\,dx, \tag{16}$$

where u_k is the solution of

$$-\mathrm{div}\Big(a\,\mathrm{grad}(u_k)\Big) = f_k \text{ in } \Omega, \quad u_k \in H_0^1(\Omega), \tag{17}$$

for some linearly independent $f_k \in H^{-1}(\Omega)$, $k = 1, \ldots, p$, and $p > 1$.

A new problem

As a step towards considering more realistic problems of optimal design, one considers now the problem (1)-(2) with the purpose of minimizing the cost function

$$K(a) = \int_\Omega \Big(|\mathrm{grad}(u)|^2 + (\mathrm{grad}(u).b)dx + F\Big(x, a(x), u(x)\Big)\Big)\,dx,$$

with $b \in \left(L^2(\Omega)\right)^N$, or equivalently

$$K(a) = \int_\Omega \Big(|\text{grad}(u-v)|^2 + F\big(x, a(x), u(x)\big) \Big) dx, \tag{18}$$

for some $v \in H_0^1(\Omega)$.

For the case of (18) with $F = 0$, we had already discussed the question twenty years ago in connection with an abstract result of Michael EDELSTEIN [14], which I had improved then; I want to describe that improvement here, as I believe it has only appeared in print in the way Marie-Françoise BIDAUT mentioned it in her thesis [15].

Theorem 1: Let C be a nonempty (strongly) closed subset of a HILBERT space H. Then there exists a dense G_δ subset K_0 of H such that for any $x \in K_0$, the minimizing sequences $c_n \in C$ of the function $c \mapsto ||x - c||$ are CAUCHY sequences. In particular the subset of points of H with a unique projection on C contains a dense G_δ subset, as it contains K_0.

Proof: Recall that a G_δ subset of a topological space is a countable intersection of open subsets of this space. BAIRE's theorem asserts that a countable intersection of dense open subsets of a complete metric space is dense, so that a countable intersection of dense G_δ subsets of a complete metric space is still a dense G_δ subset. As a corollary, one sees that for a countable number of closed subsets C_i of a HILBERT space H, there does exist a dense G_δ subset K_0 of H having for each one of the C_i the property mentioned in the Theorem.

For $x \in H$ let $d(x; C)$ denote the distance of x to C and for a bounded subset Z of H let $diam(Z)$ denote its diameter. For $\varepsilon > 0$, let

$$C(x, \varepsilon) = \{c \in C : ||x - c|| \le d(x; C) + \varepsilon\}, \tag{19}$$

and for $\alpha > 0$ let

$$K_\alpha = \Big\{x \in H : diam\big(C(x, \varepsilon)\big) \le \alpha \text{ for some } \varepsilon > 0\Big\}, \tag{20}$$

and finally let

$$K_0 = \cap_{\alpha > 0} K_\alpha. \tag{21}$$

By definition, if $x \in K_\alpha$ and c_n is a minimizing sequence of $||x - c||$ on C, then for any $\varepsilon > 0$, c_n enters $C(x, \varepsilon)$ for large n and therefore $\limsup_{n,m\to\infty} ||c_n - c_m|| \le \alpha$. By construction of K_0, if $x \in K_0$ and c_n is a minimizing sequence of $||x - c||$ on C, $\limsup_{n,m\to\infty} ||c_n - c_m|| \le \alpha$ for any $\alpha > 0$, i.e. c_n is a CAUCHY sequence. Theorem 1 will follow from the fact that each K_α is open and dense: K_0 is indeed a G_δ subset of H as the intersection is the same if one restricts the intersection to a sequence of α tending to 0, and it is dense by BAIRE's theorem.

That each K_α is open follows from the simple remark that if $||x - y|| \le \frac{\varepsilon}{3}$, then $d(y, C) \le d(x, C) + \frac{\varepsilon}{3}$ and $C(y, \frac{\varepsilon}{3}) \subset C(x, \varepsilon)$: indeed if $||y - z|| \le d(y, C) + \frac{\varepsilon}{3}$ then $||x - z|| \le d(y, C) + \frac{2\varepsilon}{3} \le d(x, C) + \varepsilon$. If then $x \in K_\alpha$ and $\varepsilon > 0$ is such that $diam(C(x, \varepsilon)) \le \alpha$, then for $||x - y|| \le \frac{\varepsilon}{3}$ one finds that $diam(C(y, \frac{\varepsilon}{3})) \le \alpha$, so that $y \in K_\alpha$.

For proving the density of K_α, one follows the first step of M. EDELSTEIN's proof (valid for uniformly convex spaces). Taking an arbitrary point of H that one may assume to be 0 by translation, one must find arbitrarily near 0 some points of K_α. If $0 \in C$, then $0 \in K_0 \subset K_\alpha$ for any $\alpha > 0$, so one may assume that $\delta = d(0; C) > 0$. For any β such that $0 < \beta \le 2\delta$, one chooses $z \in C$ such that $\delta \le ||z|| < \delta + \beta$, and one defines e as the unit vector $\frac{z}{||z||}$; the claim is then that for $x = \eta e$ with $\frac{\beta}{2} \le \eta \le \delta$ one has $x \in K_\lambda$ with $\lambda^2 = \frac{8\beta\delta^2}{\eta}$, so that by choosing $\beta = \frac{1}{n^2}$ and $\eta = \frac{1}{n}$ one obtains a sequence of points x_n converging to 0 with values λ_n tending to 0 and therefore $x_n \in K_\alpha$ for n large enough. As $d(x; C) \le ||x - z|| = ||z|| - \eta < \delta + \beta - \eta$, one looks at the points $c \in C$ such that $||x - c|| \le \delta + \beta - \eta$ and this corresponds to looking at $C(x, \varepsilon)$ with $\varepsilon = \delta + \beta - \eta - d(x; C) > 0$;

one wants to show that these c satisfy $||\delta e - c||^2 \le \frac{2\beta\delta^2}{\eta}$, giving the desired upper bound λ for the diameter of $C(x,\varepsilon)$; one has $(\delta+\beta-\eta)^2 \ge ||\eta e - c||^2 = \frac{\eta}{\delta}||\delta e - c||^2 + (1-\frac{\eta}{\delta})||c||^2 + \eta^2 - \eta\delta$, and using $||c||^2 \ge \delta^2$, one deduces that $\frac{\eta}{\delta}||\delta e - c||^2 \le 2\beta\delta + \beta^2 - 2\beta\eta \le 2\beta\delta$, which is the desired inequality.

If one takes $F = 0$ in the functional (18), then the minimization of K consists in projecting v onto the subset of $u \in H_0^1(\Omega)$ obtained when a runs through the set of admissible controls, and Theorem 1 will apply and give the existence and uniqueness of the projection of v for v belonging to a G_δ dense subset of $H_0^1(\Omega)$ if one can show that when a varies the subset of the corresponding u is closed in $H_0^1(\Omega)$; this is indeed the case in dimension 1.

Theorem 2: Let $\Omega = (0,L)$, i.e. $N = 1$, and let a run through an admissible set of controls $\mathcal{A}$ satisfying (2) and having the following property: if a sequence $a_n \in \mathcal{A}$ converges to b in $L^1(\omega)$ strong for a measurable subset ω of $(0,L)$, then there exists $a \in \mathcal{A}$ such that $a = b$ on ω. Then for v belonging to a dense G_δ subset of $H_0^1((0,L))$, the function K given by (18) with $F = 0$ attains its minimum on $\mathcal{A}$ and all the optimal a correspond to the same u.
Proof: One shows that the hypothesis implies that the subset of $u \in H_0^1((0,L))$ satisfying (1) is closed when a spans $\mathcal{A}$.

Let a sequence $a_n \in \mathcal{A}$ be such that the corresponding solutions u_n converge strongly to u_∞ in $H_0^1((0,L))$. One extracts a subsequence a_m such that $a_m \rightharpoonup a_+$ in $L^\infty(0,L)$ weak $\star$ and $\frac{1}{a_m} \rightharpoonup \frac{1}{a_-}$ in $L^\infty(0,L)$ weak $\star$, so that one has $a_m \frac{du_m}{dx} \rightharpoonup a_+ \frac{du_\infty}{dx}$ in $L^2(0,L)$ weak, but in dimension 1 one always have $a_{eff} = a_-$ and $a_m \frac{du_m}{dx} \rightharpoonup a_- \frac{du_\infty}{dx}$ in $L^2(0,L)$ strong. One deduces then that on the subset ω of $(0,L)$ where $\frac{du_\infty}{dx} \ne 0$ one has $a_- = a_+$ and that implies that $a_m \to a_+$ in $L^1(\omega)$ strong (or equivalently in $L^p(\omega)$ strong for any $p < \infty$); the hypothesis implies then that there exists $a \in \mathcal{A}$ such that $a = a_+$ on ω, and as $\frac{du_\infty}{dx} = 0$ outside ω one has $a \frac{du_\infty}{dx} = a_+ \frac{du_\infty}{dx}$ a.e. in $(0,L)$ and u_∞ is then associated to that a, so the subset of $u \in H_0^1((0,L))$ is closed.

Proposition 3: The set $\mathcal{A}$ of measurable functions taking only some of the values $\gamma_j, j = 1,\dots,m$ and satisfying (4) does satisfy the condition of Theorem 2.
Proof: Indeed let $a^n \in \mathcal{A}$ converge to b in $L^1(\omega)$ strong. First one notices that on ω the function b only takes some of the values $\gamma_j, j = 1,\dots,m$ and that

$$meas\{x \in \omega : a^n(x) = \gamma_j\} \to meas\{x \in \omega : b(x) = \gamma_j\} \text{ for } j = 1,\dots,m. \tag{22}$$

As in (6)-(7), writing $a^n = \sum_{j=1}^{m} \chi_j^n(x)\gamma_j$ and extracting a subsequence such that $\chi_j^{n'} \rightharpoonup \theta_j$ in $L^\infty((0,L)\setminus\omega)$ weak $\star$, one finds that

$$meas\{x \in (0,L)\setminus\omega : a^{n'}(x) = \gamma_j\} \to \int_{(0,L)\setminus\omega} \theta_j(x)dx \text{ for } j = 1,\dots,m, \tag{23}$$

so that (22)-(23) implies

$$meas\{x \in \omega : b(x) = \gamma_j\} + \int_{(0,L)\setminus\omega} \theta_j(x)dx \le \kappa_j \text{ for } j = 1,\dots,m, \tag{24}$$

as each $a^n \in \mathcal{A}$. One decomposes then $(0,L)\setminus\omega$ as the union of m disjoint measurable subsets as follows

$$(0,L)\setminus\omega = \cup_{1\le j\le m}\omega_j \text{ with } meas(\omega_j) = \int_{(0,L)\setminus\omega} \theta_j(x)dx \text{ for } j = 1,\dots,m, \tag{25}$$

and this is indeed possible as $\sum_{j=1}^{m} \theta_j = 1$. The function a^* given by the formula

$$a^*(x) = b(x) \text{ when } x \in \omega, \quad a^*(x) = \gamma_j \text{ when } x \in \omega_j, \text{ for } j = 1, \dots, m, \tag{26}$$

satisfies the desired properties as it belongs to $\mathcal{A}$ because of (24) and it does coincide with b on ω.

Remark 4: As Theorem 2 gives uniqueness of u when v is chosen in a suitable dense G_δ subset of $H_0^1(0,L)$ (depending upon f), it is natural to wonder if this implies uniqueness for a. If there are two functions a_1, a_2 satisfying (2) such that $-\frac{d}{dx}\left(a_j(x)\frac{du}{dx}\right) = f$ in $(0,L)$ for $j = 1,2$, one deduces that $a_j(x)\frac{du}{dx} = F(x) + C_j$ in $(0,L)$, where $F \in L^2(0,L)$ satisfies $\frac{dF}{dx} = f$ in $(0,L)$ and C_j are constants, and $\int_0^L \frac{F+C_j}{a_j}dx = 0$ for $j = 1,2$. If $\frac{du}{dx} = 0$ on a subset of $(0,L)$ of positive measure, then one can change a_1 in any way one likes on this subset and one obtains a different solution a_2, but this cannot happen if

$$meas\{x \in (0,L) : F(x) = C\} = 0 \text{ for every } C \in R. \tag{27}$$

Assuming (27), nonuniqueness of a might still occur in the case where F is discontinuous, but is impossible if one assumes F to be continuous. Indeed as $C_1 = C_2$ implies $a_1 = a_2$ because of (27), one may assume that $C_1 \neq C_2$ and one deduces from $(a_2 - a_1)\frac{du}{dx} = C_2 - C_1$ in $(0,L)$, that $(\beta - \alpha)\left|\frac{du}{dx}\right| \geq |C_2 - C_1|$ in $(0,L)$, so that $\frac{du}{dx}$ avoids an interval around 0 on the entire interval $(0,L)$, but this is impossible if F is continuous as the function $F + C_1$ must change sign because $\int_0^L \frac{F+C_1}{a_1}dx = 0$ and therefore there must exist $x_0 \in (0,L)$ where $F(x_0) + C_1 = 0$ and that forces $a_1(x)\frac{du}{dx}$ to be small near x_0.

In dimension $N \geq 2$, the set of $u \in H_0^1(\Omega)$ corresponding to an admissible control $a \in \mathcal{A}$ might not be (strongly) closed in $H_0^1(\Omega)$, and a natural question then is that of characterizing the strong closure of that subset. Obviously if $a^n \in \mathcal{A}$ with the corresponding sequence of solutions u^n converging strongly to u^∞ in $H_0^1(\Omega)$, one can extract a subsequence $a^{n'}$ converging to a_+ in $L^\infty(\Omega)$ weak $\star$ and as $a^{n'}\text{grad}(u^{n'})$ converges to $a_+\text{grad}(u^\infty)$ in $(L^2(\Omega))^N$ weak, one deduces that u^∞ satisfies the equation $-\text{div}(a_+\text{grad}(u^\infty)) = f$ in Ω. Conversely if a sequence $a^n \in \mathcal{A}$ converges to a_+ in $L^\infty(\Omega)$ weak $\star$ and if $w \in H_0^1(\Omega)$ is the solution of $-\text{div}(a_+\text{grad}(w)) = f$ in Ω, can one deduce that w belongs to the (strong) closure of the subset of $u \in H_0^1(\Omega)$ corresponding to some $a \in \mathcal{A}$? The following result gives an answer to this question.

Theorem 5: Let $N \geq 2$. Assume that Ω has a smooth boundary so that MEYERS's regularity theorem holds and let $f \in W^{-1,p}(\Omega)$ for some $p > 2$. Let $\mathcal{A}$ be the set of measurable functions taking only some of the values $\gamma_j, j = 1, \dots, m$ and satisfying (4), and let Z be the subset of $H_0^1(\Omega)$ of solutions u of (1) when $a \in \mathcal{A}$. Then the (strong) closure of Z in $H_0^1(\Omega)$ is the set of solutions $w \in H_0^1(\Omega)$ of

$$-\text{div}\Big(a_+\text{grad}(w)\Big) = f \text{ in } \Omega, \tag{28}$$

with a_+ described by (9)-(10).

Proof: The proof will make use of some strong convergence results valid in the case of locally layered media.

Let ω be an open subset of Ω, and consider a sequence $a^n \in \mathcal{A}$ which is of the form $a^n(x_1)$ on ω, and assume that $a^n \rightharpoonup a_+$ and $\frac{1}{a^n} \rightharpoonup \frac{1}{a_-}$ in $L^\infty(\omega)$ weak $\star$; then if a^n H-converges on Ω (which depends upon what a^n does outside ω), the sequence $u^n \in H_0^1(\Omega)$ of solutions of $-\text{div}(a^n\text{grad}(u^n)) = f$ in Ω converges to u^∞ in $H_0^1(\Omega)$ weak, and in ω some strong convergence hold, namely $a^n\frac{\partial u^n}{\partial x_1}$ converges strongly in $L^2(\omega)$ to

$a_-\dfrac{\partial u^\infty}{\partial x_1}$ and $\dfrac{\partial u^n}{\partial x_j}$ converges strongly in $L^2(\omega)$ to $\dfrac{\partial u^\infty}{\partial x_j}$, for $j = 2,\dots,N$; if one happens to have $\dfrac{\partial u^\infty}{\partial x_1} = 0$ in ω one deduces that $\mathrm{grad}(u^n)$ converges strongly to $\mathrm{grad}(u^\infty)$ in $\left(L^2(\omega)\right)^N$. As the usual argument of strong convergence holds in $L^2_{loc}(\omega)$ [16], one uses MEYERS's regularity theorem for deducing that the strong convergence actually holds in $L^2(\omega)$.

Let a_+ satisfy (9)-(10) and let w be the solution of (28); as one may assume that $p < p_c$, the critical exponent for MEYERS's regularity theorem, one has $w \in W_0^{1,p}(\Omega)$. Let $\varepsilon > 0$ and let $w_\varepsilon \in W_0^{1,p}(\Omega)$ be piecewise affine with $||w - w_\varepsilon||_{W_0^{1,p}(\Omega)} \le \varepsilon$, so that $-\mathrm{div}\left(a_+\mathrm{grad}(w_\varepsilon)\right) = f_\varepsilon$ in Ω with $||f_\varepsilon - f||_{W^{-1,p}(\Omega)} \le C\varepsilon$. One will construct a sequence $a_\varepsilon^n \in \mathcal{A}$ such that the sequence $u_\varepsilon^n \in W_0^{1,p}(\Omega)$ of solutions of $-\mathrm{div}\left(a_\varepsilon^n\mathrm{grad}(u_\varepsilon^n)\right) = f_\varepsilon$ in Ω converges in $H_0^1(\Omega)$ strong to w_ε, so that for $n \ge n(\varepsilon)$ one will have $||u_\varepsilon^n - w_\varepsilon||_{W_0^{1,p}(\Omega)} \le \varepsilon$. On the other hand the sequence $u^n \in W_0^{1,p}(\Omega)$ of solutions of $-\mathrm{div}\left(a_\varepsilon^n\mathrm{grad}(u^n)\right) = f$ in Ω satisfies $||u_\varepsilon^n - u^n||_{W_0^{1,p}(\Omega)} \le C\varepsilon$ (where as usual C denotes various constants); by using the subsequence $a_\varepsilon^{n(\varepsilon)}$, one sees that the corresponding solutions $u^{n(\varepsilon)}$ will converge to w in $H_0^1(\Omega)$ strong as ε goes to 0.

As w_ε is piecewise affine, Ω is the finite union of open sets ω_i on which $\mathrm{grad}(w_\varepsilon)$ is a constant vector e_i plus a region near the boundary where w_ε is identically 0, plus a set of measure 0. On each ω_i, one constructs the sequence a_ε^n by imposing three conditions: that it depends only upon $(x.e_i')$ with e_i' perpendicular to e_i so that $(\mathrm{grad}(w_\varepsilon).e_i') = 0$ on ω_i, that it uses an amount of material with conductivity γ_k equal to $\int_{\omega_i}\theta_k dx$, and that it converges in $L^\infty(\omega_i)$ weak $\star$ to a_+, which is indeed possible because of (4). After having done so on each ω_i and after having extracted a subsequence which H-converges on Ω, the first part of the argument shows that u_ε^n converges strongly to w_ε in $H_0^1(\Omega)$.

As minimizing the distance to v on a set $Z \in H_0^1(\Omega)$ or on its (strong) closure in $H_0^1(\Omega)$ is the same, one deduces the

Corollary 6: If $N \ge 2$, if $f \in W^{-1,p}(\Omega)$ and if Ω is smooth enough for MEYERS's regularity theorem to hold, then for v belonging to a G_δ dense subset of $H_0^1(\Omega)$, the function K given by (18) with $F = 0$ is such that minimizing sequences a^n correspond to strongly convergent sequences u^n in $H_0^1(\Omega)$, and the original minimization problem is equivalent to minimizing the same functional on the larger set of controls a_+ satisfying (9)-(10).

Optimality conditions

As for $N \ge 2$ the new minimization problem corresponds to a convex set of controls, one can then deduce some simple necessary conditions of optimality. The mapping $a \mapsto u$ is analytic from $L^\infty(\Omega)$ into $H_0^1(\Omega)$ and its derivative is the mapping $\delta a \mapsto \delta u$ given implicitly by

$$-\mathrm{div}\Big(a\mathrm{grad}(\delta u) + (\delta a)\mathrm{grad}(u)\Big) = 0 \text{ in } \Omega, \quad \delta u \in H_0^1(\Omega), \tag{29}$$

and the derivative of the functional K with respect to a is the mapping $\delta a \mapsto \delta K$ defined by

$$\delta K = 2\int_\Omega \Big(\mathrm{grad}(u - v).\mathrm{grad}(\delta u)\Big)\,dx. \tag{30}$$

In order to eliminate δu and only use δa, one introduces the "adjoint state" p given by the equation

$$-\mathrm{div}\Big(a\mathrm{grad}(p) + \mathrm{grad}(u - v)\Big) = 0 \text{ in } \Omega, \quad p \in H_0^1(\Omega), \tag{31}$$

and one obtains

$$\delta K = -2\int_\Omega \Big(a\mathrm{grad}(p).\mathrm{grad}(\delta u)\Big)\,dx = 2\int_\Omega \Big((\delta a)\mathrm{grad}(p).\mathrm{grad}(u)\Big)\,dx. \tag{32}$$

If a_0 is an optimal solution one must have $\delta K \ge 0$ for all admissible variations δa, and because a belongs to a convex set it means that

$$b \mapsto \int_\Omega b\Big(\mathrm{grad}(p).\mathrm{grad}(u)\Big)\,dx \text{ attains its minimum on } \mathcal{A}^* \text{ at } a_0, \tag{33}$$

where $\mathcal{A}^*$ denotes the set of measurable functions which are weak $\star$ limits in $L^\infty(\Omega)$ of a sequence $a^n \in \mathcal{A}$, i.e. the set of functions a_+ described by (9)-(10).

Remark 7: The necessary condition of optimality (33) imposes some constraints on a_0 which can be described using the level sets of the function G defined by

$$G = \Big(\text{grad}(p).\text{grad}(u)\Big), \tag{34}$$

and for describing them, one assumes that $\gamma_1 < \gamma_2 < \ldots < \gamma_m$, and one denotes $\Omega_-, \Omega_0, \Omega_+$ the subsets of Ω where $G < 0, G = 0, G > 0$.

On Ω_-, a_0 must use first the material of conductivity γ_m on the part of Ω_- where G is the most negative, and if $meas(\Omega_-) > \kappa_m$, a_0 must then use the material of conductivity γ_{m-1} on the remaining part of Ω_- where G is the most negative, and so on.

On Ω_+, a_0 must use first the material of conductivity γ_1 on the part of Ω_+ where G is the most positive, and if $meas(\Omega_+) > \kappa_1$, a_0 must then use the material of conductivity γ_2 on the remaining part of Ω_+ where G is the most positive, and so on.

On Ω_0, a_0 can use whatever materials are left after the attributions to Ω_- and to Ω_+.

Of course, as in our original method [2], one hopes to derive stronger necessary conditions of optimality by comparing the candidate for optimality to some more general homogenized materials. The difficulty here is that if a sequence a^n H-converges to A_{eff}, so that u^n converges weakly in $H_0^1(\Omega)$ to the solution u of (13), one must be able to compute the weak limit of $|\text{grad}(u^n)|^2$ in order to compute the limit of $J(a^n)$, and that limit cannot be determined from the knowledge of u and A_{eff}, but it can be determined from the knowledge of correctors P^n as $\text{grad}(u^n) - P^n\text{grad}(u)$ converges strongly to 0.

In proving Theorem 5, one has actually considered a case where the correctors can be computed easily, the case of locally layered materials. In the particular case of an open subset ω of Ω where a^n only depends upon x_1, and using MEYERS's regularity theorem for showing that nothing wrong happens near the boundary of ω, one can use on ω the correctors P^n defined by

$$P_{11}^n = \frac{1}{a^n}, \quad P_{jj}^n = 1 \text{ for } j \geq 2, \quad P_{ij}^n = 0 \text{ for } i \neq j. \tag{35}$$

In the situation considered in Theorem 5, one always had $P^n\text{grad}(u) = \text{grad}(u)$, although P^n was different from I.

Remark 8: In order to obtain more stringent necessary conditions of optimality, one needs to identify more situations where A_{eff} and $\lim_n J(a^n)$ can both be computed. As before, let $\mathcal{A}$ be the set of measurable functions taking only some of the values $\gamma_j, j = 1, \ldots, m$ and satisfying (4) and $\mathcal{A}^*$ the set of measurable functions which are weak $\star$ limits in $L^\infty(\Omega)$ of a sequence $a^n \in \mathcal{A}$, i.e. the set of functions a_+ described by (9)-(10). One will consider now a sequence of functions $a^n \in \mathcal{A}^*$ corresponding to a locally layered medium.

If Ω is decomposed into a finite union of open subsets ω^j plus a subset of measure 0, and if e^j is a unit vector, one considers a sequence $a^n \in \mathcal{A}^*$ such that a^n only depends upon $(x.e^j)$ on ω^j. If the sequence a^n satisfies

$$a^n \rightharpoonup a_+, \quad \frac{1}{a^n} \rightharpoonup \frac{1}{a_-}, \quad \frac{(a_-)^2}{(a^n)^2} \rightharpoonup \lambda \text{ in } L^\infty(\Omega) \text{ weak } \star, \tag{36}$$

then a_+, a_- and λ only depend upon $(x.e^j)$ and $\lambda \geq 1$ on ω^j, and a^n H-converges to A_{eff} such that

$$A_{eff}(x).e^j = a_-(x)e^j, \quad A_{eff}(x).e' = a_+(x)e' \text{ for every vector } e' \text{ orthogonal to } e^j, \text{ a.e. in } \omega^j. \tag{37}$$

Moreover one can compute correctors P^n on ω^j by a formula similar to (35), so that $(grad(u^n).e') \to (grad(u).e')$ in $L^2(\omega^j)$ strong if e' is orthogonal to e^j, and $a^n(grad(u^n).e^j) \to a_-(grad(u).e^j)$ in $L^2(\omega^j)$ strong; one deduces that

$$\lim_n \int_{\omega^j} |grad(u^n - v)|^2 dx = \int_{\omega^j} \Big(|grad(u-v)|^2 + (\lambda - 1)|(grad(u).e^j)|^2\Big) dx. \tag{38}$$

The next step is to identify the possible values of (a_+, a_-, λ) corresponding to all possible sequences $a^n \in \mathcal{A}^*$. As $a^n = \sum_{i=1}^m \theta_i^n \gamma_i$, one must understand the constraints on the functions θ_i^n, for $i = 1, \ldots, m$: there are pointwise constraints on θ_i^n, i.e. $\theta_i^n \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m \theta_i^n = 1$, as well as integral constraints, i.e. $\int_\Omega \theta_i^n dx \leq \kappa_i$ for $i = 1, \ldots, m$.

In order to characterize all the possible weak $\star$ limits in $L^\infty(\Omega)$, one must identify the closed convex hull $\mathcal{K}^*$ of the subset $\mathcal{K}$ of R^{m+3} defined by

$$\mathcal{K} = \Big\{k \in R^{m+3} : k_i \geq 0, i = 1, \ldots, m, \quad \sum_{i=1}^m k_i = 1, \quad k_{m+1} = \sum_{i=1}^m k_i \gamma_i, \quad k_{m+2} = \frac{1}{k_{m+1}}, \quad k_{m+3} = k_{m+2}^2\Big\}. \tag{39}$$

Then the desired limits (a_+, a_-, λ) are obtained by considering a function k with values in $\mathcal{K}^*$ such that $\int_\Omega k_i dx \leq \kappa_i$ for $i = 1, \ldots, m$, and by writing $a_+ = k_{m+1}$, $\frac{1}{a_-} = k_{m+2}$ and $\frac{\lambda}{a_-^2} = k_{m+3}$.

In such a general framework, it is not clear yet how to carry on all the necessary computations, so in order to simplify the computations, one restricts now the analysis to the *very simple case* where one mixes two materials with *no volume constraints*, i.e. $\gamma_1 = \alpha$, $\gamma_2 = \beta$ and $\kappa_1 = \kappa_2 = meas(\Omega)$. In that case, the simple necessary conditions of optimality that were described in Remark 7 express that $a_0 = \alpha$ when $G > 0$ and $a_0 = \beta$ when $G < 0$.

In this simpler setting, one use $\theta = \theta_1$ as variable and $\theta_2 = 1 - \theta$, and the constraints are $0 \leq \theta \leq 1$. Instead of $\mathcal{K}$ one may consider the curve $\mathcal{K}'$ of R^3 defined by

$$\mathcal{K}' = \Big\{k \in R^3 : k_1 = \theta \in [0,1], \quad k_2 = \frac{1}{\beta - \theta(\beta - \alpha)}, \quad k_3 = \Big(\frac{1}{\beta - \theta(\beta - \alpha)}\Big)^2\Big\}, \tag{40}$$

and the preceding complete analysis requires the identification of its closed convex hull $\mathcal{K}^{*\prime}$. In order to avoid some tedious calculations, it is worth noticing now that what one actually wants from this computation is the smallest possible value of λ in terms of a_+ and a_- defined by (36), and that will only be necessary near the curve where $a_- = a_+$, the characterization of the pairs (a_+, a_-) being easy and given by

$$\alpha \leq a_+ \leq \beta, \quad \frac{\alpha\beta}{\alpha + \beta - a_+} \leq a_- \leq a_+ \text{ a.e. } x \in \Omega. \tag{41}$$

Remark 9: Because one wants to test the optimality of an element $a \in \mathcal{A}^*$, one wants to compare it to a case where a_+ and a_- are near a. One chooses $b \in \mathcal{A}^*$, and for $0 < \varepsilon < 1$ and ε tending to 0 one considers $a_+ = (1-\varepsilon)a + \varepsilon b$, $\frac{1}{a_-} = \frac{1-\varepsilon}{a} + \frac{\varepsilon}{b}$, $\frac{\lambda}{a_-^2} = \frac{1-\varepsilon}{a^2} + \frac{\varepsilon}{b^2}$, i.e. after some easy computations

$$a_+ = a + \varepsilon(b - a), \quad a_- = a + \varepsilon a \frac{b-a}{b} + O(\varepsilon^2), \quad \lambda - 1 = \varepsilon \frac{(b-a)^2}{b^2} + O(\varepsilon^2), \tag{42}$$

so that the respective derivatives computed at $\varepsilon = 0$ are

$$\frac{da_+}{d\varepsilon} = b - a, \quad \frac{da_-}{d\varepsilon} = b - a - \frac{(b-a)^2}{b}, \quad \frac{d(\lambda - 1)}{d\varepsilon} = \frac{(b-a)^2}{b^2}. \tag{43}$$

The computation of the derivative of the functional with respect to ε will start like (29), but with

$$\frac{\delta a}{\delta \varepsilon} = (b-a)I - \frac{(b-a)^2}{b} e^j \otimes e^j \text{ on } \omega^j, \tag{44}$$

which will be used for computing a term analog to (32) in the derivative $\frac{\delta K}{\delta\varepsilon}$, but there is another term corresponding to the terms with the coefficient $(\lambda-1)$ in (38). The new necessary conditions of optimality will then be that for any $b \in \mathcal{A}^*$ one has $\frac{\delta K}{\delta\varepsilon} \geq 0$ where

$$\frac{\delta K}{\delta\varepsilon} = \sum_j \int_{\omega^j} \Big(2(b-a)\Big(\mathrm{grad}(p).\mathrm{grad}(u)\Big) - 2\frac{(b-a)^2}{b}(\mathrm{grad}(p).e^j)(\mathrm{grad}(u).e^j) + \frac{(b-a)^2}{b^2}|(\mathrm{grad}(u).e^j)|^2\Big)\,dx. \tag{45}$$

Remark 10: Of course taking b near a makes $(b-a)^2$ negligible compared to $(b-a)$, and one finds then the necessary conditions of Remark 7, expressed in terms of $G = \big(\mathrm{grad}(u).\mathrm{grad}(p)\big)$, i.e. $G \geq 0$ where $a = \alpha$, $G = 0$ where $\alpha < a < \beta$ and $G \leq 0$ where $a = \beta$. Now one learns more from (45) because by varying the decomposition of Ω into open subsets ω^j as well as the vectors e^j, a more stringent necessary condition of optimality is that for any $b \in \mathcal{A}^*$ and every measurable function e taking values on unit vectors one has

$$\int_\Omega \Big(2(b-a)\Big(\mathrm{grad}(p).\mathrm{grad}(u)\Big) - 2\frac{(b-a)^2}{b}(\mathrm{grad}(p).e)(\mathrm{grad}(u).e) + \frac{(b-a)^2}{b^2}|(\mathrm{grad}(u).e)|^2\Big)\,dx \geq 0. \tag{46}$$

If $\alpha < a < \beta$ on a subset ω of Ω with positive measure, one already knows that one must have $G = 0$ a.e. on ω, but the new necessary conditions of optimality assert that one must actually have

$$|\mathrm{grad}(u)|.|\mathrm{grad}(p)| = 0 \text{ a.e. where } \alpha < a < \beta. \tag{47}$$

Indeed as $G = 0$ on ω one must have $|(\mathrm{grad}(u).e)|^2 \geq 2b(\mathrm{grad}(p).e)(\mathrm{grad}(u).e)$ a.e. on ω for every unit vector e, and this implies that on the subset of ω where $\mathrm{grad}(u) \neq 0$ one has $\mathrm{grad}(p) = \mu\,\mathrm{grad}(u)$ for some measurable function μ which must then satisfy $\mu \leq \frac{1}{2\beta}$, and as $G = 0$ this can only happen if $\mu = 0$.

If $a = \alpha$ on a subset ω_α of Ω with positive measure, one already knows that one must have $G \geq 0$ a.e. on ω_α, but the new necessary conditions of optimality assert that one must also have

$$2b^2G - 2b(b-\alpha)(\mathrm{grad}(p).e)(\mathrm{grad}(u).e) + (b-\alpha)|(\mathrm{grad}(u).e)|^2 \geq 0$$

a.e. on ω_α for every unit vector e. On the subset of ω_α where $U = |\mathrm{grad}(u)| \neq 0$ and $P = |\mathrm{grad}(p)| \neq 0$, minimizing in e gives a solution in the plane spanned by $\mathrm{grad}(u)$ and $\mathrm{grad}(p)$; if η is the angle between $\mathrm{grad}(u)$ and $\mathrm{grad}(p)$ and θ the angle between $\mathrm{grad}(u)$ and e, then one has

$$2b^2UP\cos(\eta) - 2b(b-\alpha)UP\cos(\theta)\cos(\theta-\eta) + (b-\alpha)U^2\cos^2(\theta) \geq 0.$$

This inequality has the form $A + B\frac{1+\cos(2\theta)}{2} + C\sin(2\theta) \geq 0$ with $A = 2b^2UP\cos(\eta)$, $B = (b-\alpha)U^2 - 2b(b-\alpha)UP\cos(\eta)$, $C = -b(b-\alpha)UP\sin(\eta)$; minimizing in θ gives $A + \frac{B}{2} \geq \sqrt{(\frac{B}{2})^2 + C^2}$, i.e. $2A + B \geq 0$ and $A^2 + AB \geq C^2$, the first inequality being a consequence of $\cos(\eta) \geq 0$ and the second inequality is $2b^2UP\cos(\eta)\big((b-\alpha)U^2 + 2b\alpha UP\cos(\eta)\big) \geq b^2(b-\alpha)^2U^2P^2\sin^2(\eta)$, equivalent to

$$2\cos(\eta)\Big((b-\alpha)U^2 + 2b\alpha UP\cos(\eta)\Big) \geq (b-\alpha)^2UP\sin^2(\eta),$$

which is true for $b \in [\alpha, \beta]$ if it is true for $b = \alpha$ and for $b = \beta$, which finally gives $G \geq 0$ and

$$P\Big((\beta-\alpha)^2\sin^2(\eta) - 4\beta\alpha\cos^2(\eta)\Big) \leq 2\cos(\eta)(\beta-\alpha)U,$$

and, adding the possibility that $U = 0$ or $P = 0$, this can be written as

$$(\beta-\alpha)^2|\mathrm{grad}(u)|^2|\mathrm{grad}(p)|^2 - (\beta+\alpha)^2\Big(\mathrm{grad}(u).\mathrm{grad}(p)\Big)^2 \leq 2(\beta-\alpha)\Big(\mathrm{grad}(u).\mathrm{grad}(p)\Big)|\mathrm{grad}(u)|^2 \text{ on } \omega_\alpha. \tag{48}$$

Notice that the condition is automatically satisfied if $(\beta-\alpha)^2\sin^2(\eta)\le 4\beta\alpha\cos^2(\eta)$, i.e. if the angle between grad(u) and grad(p) is small enough.

If $a=\beta$ on a subset ω_β of Ω with positive measure, one already knows that one must have $G\le 0$ a.e. on ω_β, but the new necessary conditions of optimality assert that one must also have

$$2b^2G-2b(b-\beta)(\text{grad}(u).e)(\text{grad}(p).e)+(b-\beta)|(\text{grad}(u).e)|^2\le 0$$

a.e. on ω_β for every unit vector e. On the subset of ω_β where $U=|\text{grad}(u)|\ne 0$ and $P=|\text{grad}(p)|\ne 0$, maximizing in e gives a solution in the plane spanned by grad(u) and grad(p); if η is the angle between grad(u) and grad(p) and θ is the angle between grad(u) and e, then one has

$$2b^2UP\cos(\eta)-2b(b-\beta)UP\cos(\theta)\cos(\theta-\eta)+(b-\beta)U^2\cos^2(\theta)\le 0.$$

This inequality has the form $A+B\dfrac{1+\cos(2\theta)}{2}+C\sin(2\theta)\le 0$ with $A=2b^2UP\cos(\eta)$, $B=(b-\beta)U^2-2b(b-\beta)UP\cos(\eta)$, $C=-b(b-\beta)UP\sin(\eta)$; maximizing in θ gives $A+\dfrac{B}{2}+\sqrt{(\frac{B}{2})^2+C^2}\le 0$, i.e. $2A+B\le 0$ and $A^2+AB\ge C^2$, the first inequality being a consequence of $\cos(\eta)\le 0$ and the second inequality is $2b^2UP\cos(\eta)((b-\beta)U^2+2b\beta UP\cos(\eta))\ge b^2(b-\beta)^2U^2P^2\sin^2(\eta)$, equivalent to

$$2\cos(\eta)\Big((b-\beta)U^2+2b\beta UP\cos(\eta)\Big)\ge(b-\beta)^2UP\sin^2(\eta),$$

which is true for $b\in[\alpha,\beta]$ if it is true for $b=\alpha$ and for $b=\beta$, which finally gives $G\le 0$ and

$$P\Big((\beta-\alpha)^2\sin^2(\eta)-4\beta\alpha\cos^2(\eta)\Big)\le -2\cos(\eta)(\beta-\alpha)U,$$

and, adding the possibility that $U=0$ or $P=0$, this can be written as

$$(\beta-\alpha)^2|\text{grad}(u)|^2|\text{grad}(p)|^2-(\beta+\alpha)^2\Big(\text{grad}(u).\text{grad}(p)\Big)^2\le -2(\beta-\alpha)\Big(\text{grad}(u).\text{grad}(p)\Big)|\text{grad}(u)|^2 \text{ on } \omega_\beta. \tag{49}$$

Notice that the condition is automatically satisfied if $(\beta-\alpha)^2\sin^2(\eta)\le 4\beta\alpha\cos^2(\eta)$, i.e. if the angle between grad(u) and grad(p) is near enough to π.

Remark 11: Although one had first focused attention on the special case of the functional K given by (18) with $F=0$, for which Theorem 1 could be used, it is important to notice that Theorem 5 characterizes a strong closure of u and is therefore independent of any functional that one wants to minimize, and that what has been said for optimality conditions applies then to many other functionals.

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Note: A forthcoming book, edited by R. KOHN and published by Birkhaüser, will contain translations into English of some of the early references on optimal design which were originally published in Russian or in French ([2], [6], [8] and [11]).

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